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MEASURING LONG-RUN RISK

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ABSTRACT

We characterize and measure a long-run risk return tradeoff for the valuation of financial cash flows that are exposed to fluctuations in macroeconomic growth. This tradeoff features components of financial cash flows that are only realized far into the future but are still reflected in current asset values. We use the recursive utility model with empirical inputs from vector autoregressions to quantify this relationship; and we study the long-run risk differences in aggregate securities and in portfolios constructed based on the ratio of book equity to market equity. Finally, we explore the resulting measurement challenges and the implied sensitivity to alternative specifications of stochastic growth.

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1 Introduction

Applied time series analysts have studied extensively how macroeconomic aggregates respond in the long run to underlying economic shocks. For instance, Cochrane (1988) used time series methods to measure the importance of permanent shocks to output and Blanchard and Quah (1989) advocated using restrictions on long run responses to identify economic shocks and measure their importance. The unit root contributions measured by macroeconomists are a source of long-run risk that should be reflected in the valuation of cash flows. Financial market prices are by nature forward looking, and thus provide information about how risk averse investors value the stochastic growth components of macroeconomic and financial time series. This paper develops and applies methods for incorporating the asset valuation of cash flows with stochastic growth into macroeconomic analyses.

We investigate the valuation of hypothetical and actual financial cash flows with stochastic growth components. We exploit the fact that transient components of cash flows have negligible contributions to value in the long run. For stochastic growth processes with payoffs far into the future, valuation turns out to be dominated by a single pricing component. We characterize this dominant component by exploiting a formulation developed in Hansen and Scheinkman (2005). This component isolates value movements due to long-run cash-flow variability and gives a well defined risk adjustment. By changing growth processes, and hence long-run risk exposure, we delineate a long-run risk-return tradeoff.

The methods we use to characterize and measure long-run risk are complementary to those developed by Campbell and Shiller (1988). Our analysis is motivated in part by recent research seeking to construct *cash flow betas*, [*e.g.*, see Bansal, Dittmar, and Lundblad (2005) and Campbell and Vuolteenaho (2003)]; but our interpretation and justification for such objects is novel. Previous work uses return- or dividend-based measures of long-run cash-flow risk to isolate risk components of one-period returns. Long-run cash flow risk is only a partial contributor to one-period risk, however. Our interest in the stochastic growth components of cash flows leads us naturally to the study of risk exposure extrapolated into the distant future. We aim to elucidate risk adjustments in present value models and to characterize formally a long-run notion of the risk-return relation.

The valuation of cash flows reflects expected growth, discounting and riskiness. The value of cash flows in the distant future declines as the horizon increases at a rate that is approximately constant. When this decay rate is small, future cash flows have a durable contribution to current values. A small decay rate in the contribution to value reflects in part cash flow growth, however. Since dividend growth rates projected far into the future are approximately constant, there is a well defined adjustment for cash-flow growth. By adding the dividend growth rate to the value decay rate, we extract a risk-adjusted discount rate. The risk adjustment comes from two sources. One is the direct random fluctuation in the growth rates of the cash flow, and the other is the riskiness that is imputed by the valuation of this cash flow.

Value decompositions of the type just described require a specific economic model and empirical inputs to characterize the growth and riskiness of cash flows. The calculations in

this paper are based on a well specified, albeit highly stylized, model. Following Epstein and Zin (1989b), Weil (1990), Tallarini (1998), Bansal and Yaron (2004) and many others, we use a recursive utility framework of Kreps and Porteus (1978). For these preferences, the intertemporal composition of risk matters to the decision maker. Changing the time of information revelation regarding intertemporal consumption lotteries affects the implied preference ordering. As emphasized by Epstein and Zin (1989b), these preferences also offer a convenient and appealing way to break the preference link between risk aversion and intertemporal substitution. Furthermore, Bansal and Yaron (2004) showed that predictable components in consumption growth can amplify the risk premia in security market prices. We study how long-run risk depends on intertemporal substitution, on risk aversion and on the predictable components to consumption growth.

In addition to an economic model, our value decompositions require statistical inputs that quantify long-run stochastic growth in macroeconomic variables, particularly in consumption. The decompositions also require knowledge of the long-run link between stochastic cash flows and the macroeconomic risk variables. These components of financial risk cannot be fully diversified and hence require nontrivial risk adjustments. The long-run nature of these risks adds to the statistical challenges just as it does in the related macroeconomic literature.

As many prior studies have done, we choose to study log linear vector autoregressive (VAR) models of consumption and cash flows. These models are designed to accommodate dynamics in a convenient yet flexible way. Our focus on long-run risk deliberately stretches the VAR methods beyond their ability to capture transient dynamics. This leads us to explore the resulting empirical challenges. How sensitive are risk-measures to details in the specification of the time series evolution? How accurately can we measure these components? When should we expect these components to play a fundamental role in valuation? In addition to providing a long-run valuation counterpart to the familiar risk-return tradeoff, this paper examines the sensitivity of the measurements to estimation and model uncertainty.

As we have just described, our paper uses a well posed economic model of valuation in conjunction with statistical inputs to make valuation assessments that pertain to growth rate risk. In addition to our substantive interest in such risk, there is a second and perhaps more speculative for featuring the long run in our analysis. Highly stylized economic models like the ones we explore here are typically misspecified when examined with full statistical scrutiny. Behavioral biases or transactions costs, either economically grounded or metaphorical in nature, challenge the high frequency implications of such models. Valuation implications over longer horizons may be less sensitive to misspecification, although this remains to be demonstrated formally.¹ Misspecified models continue to be used by applied economists because of their analytical tractability and conceptual simplicity. Characterizing valuation implications that dominate over long time horizons helps us understand better when such models provide useful approximations. For instance, it helps us to determine when transient implications are important and when long-run implications dominate.

¹Analogous reasoning led Daniel and Marshall (1997) to use an alternative frequency decomposition of the consumption Euler equation.

In section 2 we use a finite state Markov chain to illustrate our methods. We follow this with a formal discussion of our methodology in section 3. In section 4 we use the recursive utility model to show why the intertemporal composition of risk might matter to an investor. In section 5 we identify important aggregate shocks that affect consumption in the long run. Section 6 constructs the implied measures of the risk-return relation for portfolio cash flows. Section 7 explores the valuation sensitivity of alternative specifications of the long-run statistical relationship between consumption and portfolio cash flows. Section 8 concludes.

2 Markov chain model

In section 3 we develop a general framework for characterizing long-run risk. A feature of this framework is that multi-period claims are conveniently priced by iterating on valuation operators, and long-run risk is measured by the limiting behavior of these operators. Prior to this development, we illustrate our techniques using Markov chains and the associated matrix operations. In the case of a discrete-state Markov chain, iterating operators is accomplished by raising appropriately constructed matrices to powers. This naturally leads us to explore the eigenvalues and eigenvectors of the matrices used in valuation.

At time t , a discounted cash flow is given by:

$$\frac{P_t}{D_t} = E \left[\sum_{j=1}^{\infty} \left(\prod_{\tau=1}^j S_{t+\tau, t+\tau-1} \right) \frac{D_{t+j}}{D_t} \middle| x_t \right] \quad (1)$$

where $\{D_{t+j} : j \geq 0\}$ is a stochastic cash flow process with price P_t at date t and $S_{t+1,t}$ is a stochastic discount factor process between date t and date $t+1$ and is assumed to be strictly positive. Since it varies across states, this discount factor provides both a time discount and a risk adjustment.

Suppose that the dynamics of cash flows and the stochastic discount factor are determined by an N state Markov chain. State n of this Markov chain is denoted \mathbf{x}_n , and the probabilities of transiting from one state to another are given by:²

$$\mathbf{a}_{m,n} = \text{Prob}(x_{t+1} = \mathbf{x}_n | x_t = \mathbf{x}_m) .$$

To evaluate discounted sum (1), we scale $\mathbf{a}_{m,n}$ by two objects. The first is the stochastic or state-dependent discount factor $\mathbf{s}_{m,n}$ for the next period's state \mathbf{x}_n conditioned on the current state being \mathbf{x}_m .³ The specification of this discount factor comes from an underlying economic model. In section 4 we develop the recursive utility model as an example of a stochastic discount factor model.

²We assume that the resulting probability matrix is irreducible. That is, for some integer τ , the entries of \mathcal{A}^τ are strictly positive, where \mathcal{A} is formed from the $\mathbf{a}_{m,n}$'s.

³This transformation of the probabilities is familiar from asset pricing where the "risk-neutral" distribution is obtained from the pricing model and the objective distribution. We do not, however, rescale the discount factors to behave as probabilities.

The second object in our scaling is a stochastic growth factor that captures the long-run growth in the cash flows. The value of this growth factor between current state x_m and future state x_n is $d_{m,n}$ and is assumed to be positive. We study growing cash flows with a multiplicative representation

$$D_{t+j} = D_{t+j}^* \psi(x_{t+j})$$

where D_{t+j}^* is a reference growth process that is initialized at one and an additional term that is a function of the Markov state. The multiplicative increment D_{t+1}^*/D_t^* is a time invariant positive function of x_{t+1} and x_t and has a constant expectation conditioned on x_t denoted by $\exp(\epsilon)$. Given the finite number of states, we represent the value of the reference growth factor between current state x_m and future state x_n as the positive number $d_{m,n}$ restricted so that $\sum_m d_{m,n} a_{m,n} = \exp(\epsilon)$ is independent of n .⁴ The level variable $\psi(x_t)$ is represented as an N -dimensional column vector f that gives the values of the function for each of the N Markov states.

It is critical to our evaluation of the long-run risk of cash flows to consider alternative stochastic growth specifications. For example, suppose that all of the entries of the matrix \mathcal{A} are positive and consider the specification:

$$d_{m,n} = \begin{cases} \frac{\exp(\epsilon)-1}{a_{\ell,n}} + 1 & \text{if } n = \ell \\ 1 & \text{otherwise} \end{cases} . \quad (2)$$

This corresponds to a stochastic growth specification that features Markov state ℓ . By changing ℓ and ϵ , we explore changes in the risk exposure of alternative growth trajectories.

The two scaling objects lead us to a new matrix \mathcal{P} with entries:

$$p_{m,n} = a_{m,n} s_{m,n} d_{m,n} .$$

We use this matrix \mathcal{P} to compute and decompose the valuation (1) by payoff horizon j . Term j in the infinite sum is given by:

$$E \left[\left(\prod_{\tau=1}^j S_{t+\tau, t+\tau-1} \right) \frac{D_{t+j}}{D_t} \middle| x_t = \mathbf{x}_m \right] = e_m (\mathcal{P})^j f$$

where e_m is an N -dimensional row vector of zeros with a one in the m^{th} column. The contributions to value at different payoff horizons are determined by the properties of this matrix raised to the power j and the vector f .

Raising a matrix to a power preserves the eigenvectors. Eigenvalues are altered but in a straightforward way. The original eigenvalues are raised to the same power as the matrix. There is *principal* eigenvalue, $\exp(-\nu)$, that is positive and a corresponding eigenvector, f^* , with positive entries.⁵ The principal eigenvalue has the largest magnitude among all eigenvalues of \mathcal{P} , and as a consequence it dominates the evaluation of \mathcal{P}^j for large values of j .

⁴As we will see, this restriction is essentially a convenient normalization.

⁵This is known from the Frobenius-Perron theory of matrices.

To illustrate the influence on valuation of the dominate eigenvalue and eigenvector, for simplicity suppose that \mathcal{P} has unique eigenvalues. Let Λ be a diagonal matrix with the eigenvalues on the diagonal and $\exp(-\nu)$ as the upper left element. Further let T be a matrix with the corresponding eigenvectors as columns. Then:

$$\mathcal{P}^j = T\Lambda^jT^{-1} .$$

In this decomposition, the first column of T is the column eigenvector, f^* , and the first row of T^{-1} is the row eigenvector, g^* , corresponding to the eigenvalue $\exp(-\nu)$. Since $\exp(-\nu)$ is the dominant eigenvalue

$$\lim_{j \rightarrow \infty} \exp(\nu j)\Lambda^j = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} ,$$

and

$$\lim_{j \rightarrow \infty} \exp(\nu j)\mathcal{P}^j = f^*g^* .$$

Hence for any f

$$\lim_{j \rightarrow \infty} \exp(\nu j)(\mathcal{P})^j f = (g^* f)f^* .$$

As the valuation horizon gets large, the vector of values are approximately proportional to f^* , provided of course that $(g^* f)$ is not zero. The specific choice of f does not alter the limiting distribution of values. Moreover, when f has nonnegative entries and at least one strictly positive entry,

$$\lim_{j \rightarrow \infty} -\frac{1}{j}[\log(\mathcal{P})^j f] = \nu \mathbf{1}_N$$

where $\mathbf{1}_N$ is an N -dimensional column vector of ones. Thus ν is the asymptotic decay rate of the valuation series (1).

As is familiar from the Gordon growth model, the decay rate ν is influenced by two factors: the asymptotic (risk adjusted) discount rates and the asymptotic growth rates in cash flows. When cash flows grow faster, values decay slower. Thus to produce a risk adjusted discount rate, we need to adjust ν for dividend growth. To measure this, we form a matrix \mathcal{G} with entries $\mathbf{a}_{m,n}\mathbf{d}_{m,n}$. By assumption this matrix has one as its dominant eigenvector, and its dominant eigenvalue is the average growth factor $\exp(\epsilon)$. As a consequence, the asymptotic cash flow growth rate is ϵ , and the implied discount rate is $\epsilon + \nu$.

This discount rate includes of an adjustment for long-run risk. As we change the stochastic growth specification $\mathbf{d}_{m,n}$, we alter the implied risk-adjusted discount rate giving rise to a long-run risk return relation. For instance, as we alter ℓ and ϵ in example (2) we alter the long-run risk adjusted discount rates. In the next section we explore a risk-return counterpart for an economy that has normally distributed shocks as building blocks instead of discrete Markov states.

3 Long run risk in a log-linear economy

In the remainder of the paper we use linear Markov processes instead of Markov chains. We do this so that we can explore temporal dependence in a more flexible manner. To support this application, we extend the approach just described by replacing matrices with operators that integrate over continuous states.

The state of the economy is given by a vector x_t which evolves according to a first-order vector autoregression:

$$x_{t+1} = Gx_t + Hw_{t+1}. \quad (3)$$

The matrix G has strictly stable eigenvalues (eigenvalues with absolute values that are strictly less than one), and $\{w_{t+1} : t = 0, 1, \dots\}$ is iid normal with mean zero and covariance matrix I . The stochastic discount factor is linked to this state vector by:

$$s_{t+1,t} = \mu_s + U_s x_t + \xi_0 w_{t+1} \quad (4)$$

3.1 Dominant Eigenfunction and Valuation Decay Rate

Consider a reference stochastic growth process modeled as the exponential of a random walk with drift:

$$D_t^* = \exp \left[\zeta t + \sum_{j=1}^t \pi w_j \right].$$

Using this growth process we introduce a transient or stationary component to produce the cash flow:

$$D_t = D_t^* \psi(x_t). \quad (5)$$

Pricing of D_t requires valuation of both the transient and growth components. The implications of the growth component for valuation and risk are invariant to the choice of the transitory component ψ . This specification allows us to focus on the growth rate risk exposure as parameterized by π . Changes in valuation, as we alter π , give a characterization of long run risk. In addition to characterizing this risk, we examine how important this long-run component is to overall value.

The counterpart to the matrix \mathcal{P} used in section 2 is the one-period *valuation operator* given by:

$$\mathcal{P}\psi(x) = E(\exp[s_{t+1,t} + \zeta + \pi w_{t+1}] \psi(x_{t+1}) | x_t = x). \quad (6)$$

We call this a valuation operator because it assigns values to cash flows constructed with alternative functions ψ but with the same growth component. The function ψ plays the role of the vector f for the Markov chain economy. Formally, we view this operator as mapping functions of the Markov state into functions of the Markov state. In particular, it is well defined for functions that are bounded functions of the Markov state, although it is well defined for other functions as well.

Multi-period prices can be inferred from this one-period pricing operator through iteration. The value of a date $t + j$ cash flow (5) is given by:

$$D_t [\mathcal{P}^j \psi(x_t)].$$

The notation \mathcal{P}^j denotes the application of the one-period valuation operator j times, which is the counterpart to raising a matrix to the j th power.

When the cash flow process is a dividend process, the date t price-dividend ratio is:

$$\frac{P_t}{D_t} = \frac{\sum_{j=1}^{\infty} \mathcal{P}^j \psi(x_t)}{\psi(x_t)} \quad (7)$$

provided that $\psi(x_t)$ is strictly positive. The term

$$\frac{\mathcal{P}^j \psi(x_t)}{\psi(x_t)}$$

is the contribution of the date $t + j$ cash flow to the price-dividend ratio at time t . The price dividend ratio is given by the sum of these objects.

As in section 2 we study the limiting behavior of these components by constructing dominant eigenfunctions and eigenvalues of the pricing operator \mathcal{P} . The dominant eigenfunction of is a positive function ϕ that solves the equation:

$$\mathcal{P}\phi = \exp(-\nu)\phi,$$

where $\exp(-\nu)$ is the eigenvalue corresponding to ϕ . The eigenfunction is only well defined up to scale. A solution exists to this equation of the form $\phi = \exp(-\omega x)$. A simple application of the formula for the (conditional mean) of a lognormal implies that

$$(U_s - \omega G)x + \mu_s + \zeta + \frac{|-\omega H + \pi + \xi_0|^2}{2} = -\nu - \omega x.$$

Solving for ω and ν results in:

Theorem 1. *Suppose that the state of the economy evolves according to (3) and the stochastic discount factor is given by (4), then*

a) *the dominant eigenfunction, ϕ , of the one-period valuation operator (6) is a scale multiple of $\exp(-\omega x)$ where*

$$\omega \doteq -U_s(I - G)^{-1}.$$

b) *the dominant eigenvalue is $\exp(-\nu)$ where*

$$\nu \doteq -\mu_s - \zeta - \frac{|\pi - \pi^*|^2}{2}$$

and

$$\pi^* \doteq -\xi_0 - U_s(I - G)^{-1}H.$$

Recall that the left eigenvector of a matrix is the right eigenvector of its transpose. The analogue to the left eigenvector of matrix \mathcal{P} of section 2 is the eigenfunction of the adjoint of the operator \mathcal{P} , where the adjoint is the operator equivalent of a transpose. In appendix B we show that this eigenfunction, φ , is a scale multiple of $\exp(-\omega^*x)$. As shown by Hansen and Scheinkman (2005), whenever $E(\psi\varphi)$ and $E(\varphi\phi)$ are well defined and finite:

$$\lim_{j \rightarrow \infty} \exp(\nu j) \mathcal{P}^j \psi(x) = \frac{E(\varphi\psi)}{E(\varphi\phi)} \phi(x). \quad (8)$$

Thus when $E(\varphi\psi) > 0$,

$$\lim_{j \rightarrow \infty} \frac{\log[\mathcal{P}^j \psi(x)]}{j} = -\nu.$$

This calculation gives us an asymptotic decay rate for the contribution to total value of the cash flow at time $t + j$. The decay rate depends on both cash flow growth through the specification of π and ζ , and on the economic value associated with that growth. It does not depend on the particular function ψ that dictates the transient contribution to cash flows. The eigenfunction ϕ is dominant as it gives the limiting state dependence of the values as reflected in formula (8). Thus the pair (ν, ϕ) measures how long-run prospects about dividends contribute to value. The ψ contribution is transient and does not alter the asymptotic decay rate or the relative values across states.

Since we are interested in cash flows with transient components, we shall also define operators to measure the expected cash-flow growth and the resulting limiting behavior. Let

$$\mathcal{G}\psi(x) = E[\exp(\zeta + \pi w_{t+1}) \psi(x_{t+1}) | x_t = x].$$

By iterating on this *growth operator*, we can study expected cash flow growth over multi-period horizons. In particular, the expected value of the cash flow (5) is:

$$D_t [\mathcal{G}^j \psi(x_t)].$$

The asymptotic cash-flow growth is characterized by an analogous eigenfunction-eigenvalue pair. A straightforward calculation shows that the dominant eigenfunction of \mathcal{G} is one and that

$$\mathcal{G}\psi = \exp\left(\zeta + \frac{|\pi|^2}{2}\right) \psi$$

for $\psi = 1$.⁶ Thus

$$\epsilon = \zeta + \frac{|\pi|^2}{2}. \quad (9)$$

is the implied asymptotic expected rate of growth for the cash flow.

In what follows we will motivate the study of $\epsilon + \nu$ as a risk adjusted discount rate. This sum depends on π but not on ζ , the deterministic trend.

⁶When the martingale approximation for the cash flow has heteroskedastic increments, this calculation ceases to have a trivial solution.

3.2 Long-run returns and dominant eigenfunctions

Consider first a security with dividend process of the form (5) where the dominant eigenfunction ϕ is used in place of ψ . Using the eigenvalue property and (7), this security has a constant price-dividend ratio given by:

$$\frac{P_t}{D_t} = \frac{\exp(-\nu)}{1 - \exp(-\nu)}. \quad (10)$$

Thus the dividend-price ratio depends directly on the valuation decay rate ν . As in the Gordon growth model, the factor $\exp(-\nu)$ includes both a pure discount factor (adjusted for risk) and a dividend growth factor. The implied asymptotic discount rate is $\nu + \epsilon$ since the asymptotic dividend growth factor for dividends with long-run risk is: $\exp(\epsilon)$.

The (gross) return to holding this security from time t to $t + 1$ is given by:

$$R_{t,t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} = \frac{P_{t+1}}{P_t} \exp(\nu) = \exp(\nu) \exp(\zeta + \pi w_{t+1}) \frac{\phi(x_{t+1})}{\phi(x_t)}. \quad (11)$$

The logarithm of this return has two components: a cash flow component: $\zeta + \pi w_{t+1}$ determined by the reference growth process and a valuation component $\nu + \log \phi(x_{t+1}) - \log \phi(x_t)$ determined by the dominant eigenvalue and eigenfunction. In what follows we will refer to this constructed return as a valuation return associated with a cash flow with risk vector π .

The return to buying this security and reinvesting the dividends for k periods is given by the product of these one-period returns:

$$R_{t,t+k}^k = \exp(\nu k) \exp \left[\zeta k + \pi \sum_{j=1}^k w_{t+j} \right] \frac{\phi(x_{t+k})}{\phi(x_t)}.$$

Variation in the logarithm of this return will be dominated by direct cash flow contribution as k gets large because the variance of a random walk grows linearly in k while $\log \phi(x_{t+k}) - \log \phi(x_t)$ does not. Moreover, the logarithm of the expected return yields the following simplification in the limit:⁷

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log E (R_{t,t+k}^k | \mathcal{F}_t) = \nu + \lim_{k \rightarrow \infty} \frac{1}{k} \log \mathcal{G}^k \phi(x_t) = \nu + \epsilon. \quad (12)$$

In this sense we view $\nu + \epsilon$ is an expected rate of return.

We use this result to study the valuation of a more general security with a transient component of cash flows, ψ , that is different from the dominant eigenfunction. As part of a valuation decomposition, consider a security with an initial payoff k periods into the future. Using result (8) the date t value of the payoff to the cash flow (5) at time $t + j$ is approximately:

$$\exp(-\nu j) \exp \left[\zeta(t) + \pi \sum_{\tau=1}^t w_\tau \right] \frac{E[\varphi(x_t) \psi(x_t)]}{E[\varphi(x_t) \phi(x_t)]} \phi(x_t).$$

⁷Strictly speaking this requires that $E\phi\kappa$ is finite where κ is the eigenfunction of the adjoint of the operator \mathcal{G} .

for large j . Adding over horizons $j \geq k$ for some large k gives the price of the constructed security as:

$$\hat{P}_t^k = \frac{\exp(-\nu k) E[\varphi(x_t)\psi(x_t)]}{1 - \exp(-\nu) E[\varphi(x_t)\phi(x_t)]} \exp \left[\zeta(t) + \pi \sum_{\tau=1}^t w_\tau \right] \phi(x_t)$$

Except for a scale factor, the dominant eigenfunction approximates variation in the valuation over time as a function of the Markov state x_t . Changing the transient component changes the scale factor. The approximate one-period return on this security is:

$$\frac{\hat{P}_{t+1}^{k-1}}{\hat{P}_t^k} = \exp(\nu) \exp(\zeta + \pi w_{t+1}) \frac{\phi(x_{t+1})}{\phi(x_t)}.$$

This is equal to (11), the one-period return to holding a claim on cash flows where the transient component is equal to the dominant eigenfunction.

Characterizing the dependence of $\nu + \epsilon$ on π gives a long-run risk return relation. The vector π dictates how the cash flow weights on the underlying shocks and $\nu + \epsilon$ gives the implied expected rate of return.⁸

Theorem 2. *Suppose that the state of the economy evolves according to (3) and the stochastic discount factor is given by (4), then the expected rate of return (12) is:*

$$\epsilon + \nu = \zeta^* + \pi^* \cdot \pi$$

where

$$\begin{aligned} \pi^* &\doteq -\xi_0 - U_s(I - G)^{-1}H \\ \zeta^* &\doteq -\mu_s - \zeta - \frac{\pi^* \cdot \pi^*}{2}. \end{aligned}$$

Proof. This result follows immediately from the characterization of ν given in theorem 1 and of ϵ in (9). \square

The term π^* is the price of exposure to long-run risk of cash flows as measured by π . The logarithm of a stochastic discount fact over horizon k is

$$\sum_{j=1}^k s_{t+j, t+j-1}.$$

⁸By setting $\pi = 0$, we obtain a benchmark return that is the long-run counterpart to the riskfree return. Alvarez and Jermann (2001) study of the holding period returns to long-horizon discount bonds. The approximate one-period return is:

$$\exp(\nu) \frac{\phi(x_{t+1})}{\phi(x_t)}$$

constructed using the $\pi = 0$ and $\zeta = 0$ for the associated dominant eigenvalue and function. In this case, ϵ is zero by construction. They compare this return to the maximal growth rate return of Bansal and Lehmann (1997) to infer long run properties of the stochastic discount factor.

As k gets large the long-run response to the shock vector w_{t+1} converges to $-\pi^* w_{t+1}$. Thus the long-run risk price vector has a simple characterization in this economy. It is the negative of the response coefficients for the long-run log stochastic discount factor to the underlying shocks. Our empirical aim is to measure π^* and study its consequences. To do this we need a model for $s_{t+1,t}$. We turn to this task in the next section.

4 Stochastic discount factor

There remains considerable controversy within the asset pricing literature about the construction of an economically meaningful model of a stochastic discount factor. We find it useful to focus on a recursive utility model that, by design, leads to tractable restrictions on economic time series. This model is rich enough to help us examine return heterogeneity as it relates to long-run risk and to understand better the intertemporal values of equity.

4.1 Preferences

We follow Kreps and Porteus (1978), Epstein and Zin (1989b) and Weil (1990) in choosing to examine recursive preferences. As we will see below, this specification of preferences provides a simple justification for examining the temporal composition of risk in consumption.

In our specification of these preferences, we use a CES recursion:

$$V_t = [(1 - \beta)(C_t)^{1-\rho} + \beta \mathcal{R}_t(V_{t+1})^{1-\rho}]^{\frac{1}{1-\rho}}. \quad (13)$$

The random variable V_{t+1} is the continuation value of a consumption plan from time $t + 1$ forward. The recursion incorporates the current period consumption C_t and makes a risk adjustment $\mathcal{R}_t(V_{t+1})$ to the date $t + 1$ continuation value. We use a CES specification for this risk adjustment as well:

$$\mathcal{R}_t(V_{t+1}) \doteq \left[E(V_{t+1})^{1-\theta} \mid \mathcal{F}_t \right]^{\frac{1}{1-\theta}}$$

where \mathcal{F}_t is the current period information set. The outcome of the recursion is to assign a continuation value V_t at date t .

The preferences provide a convenient separation between risk aversion and the elasticity of intertemporal substitution [see Epstein and Zin (1989b)]. For our purposes, this separation allows us to examine the effects of changing risk exposure with modest consequences for the risk-free rate. When there is perfect certainty, the value of $1/\rho$ determines the *elasticity of intertemporal substitution* (EIS). A measure of risk aversion depends on the details of the gamble being considered. As emphasized by Kreps and Porteus (1978), with preferences like these intertemporal compound consumption lotteries cannot necessarily be *reduced* by simply integrating out future information about the consumption process. Instead the timing of information can have a direct impact on preferences and hence the intertemporal composition of risk matters. As we will see, this will be reflected explicitly in the equilibrium asset prices

that we characterize. On the other hand, the aversion to simple wealth gambles is given by θ .

To analyze growth, we scale the continuation values in (13) by consumption:

$$\frac{V_t}{C_t} = \left[(1 - \beta) + \beta \mathcal{R}_t \left(\frac{V_{t+1}}{C_{t+1}} \frac{C_{t+1}}{C_t} \right)^{1-\rho} \right]^{\frac{1}{1-\rho}}.$$

Since consumption and continuation values are positive, we find it convenient to work with logarithms instead. Let v_t denote the logarithm of the continuation value relative to the logarithm of consumption, and let c_t denote the logarithm of consumption. We rewrite recursion (13) as

$$v_t = \frac{1}{1-\rho} \log \left((1 - \beta) + \beta \exp \left[(1 - \rho) \mathcal{Q}_t (v_{t+1} + c_{t+1} - c_t) \right] \right), \quad (14)$$

where \mathcal{Q}_t is:

$$\mathcal{Q}_t(v_{t+1}) = \frac{1}{1-\theta} \log E \left(\exp \left[(1 - \theta) v_{t+1} \right] \mid \mathcal{F}_t \right).$$

We will use this recursion to solve v_t from an infinite horizon model.

4.2 Shadow Valuation

Consider the shadow valuation of a given consumption process. The utility recursion gives rise to a corresponding valuation recursion and implies stochastic discount factors used to represent this valuation. In light of the intertemporal budget constraint, the valuation of consumption in equilibrium coincides with wealth.

The first utility recursion (13) is homogeneous of degree one in consumption and the future continuation utility. Use Euler's Theorem to write:

$$V_t = (MC_t)C_t + E \left[(MV_{t+1})V_{t+1} \mid \mathcal{F}_t \right] \quad (15)$$

where

$$\begin{aligned} MC_t &= (1 - \beta)(V_t)^\rho (C_t)^{-\rho} \\ MV_{t+1} &= \beta(V_t)^\rho [\mathcal{R}_t(V_{t+1})]^{\theta-\rho} (V_{t+1})^{-\theta} \end{aligned}$$

The right-hand side of (15) measures the shadow value of consumption today and the continuation value of utility tomorrow.

Let consumption be numeraire, and suppose for the moment that we value claims to the future continuation value V_{t+1} as a substitute for future consumption processes. Divide both sides of (15) by MC_t and use marginal rates of substitution to compute shadow values. The shadow value of a claim to a continuation value is priced using $\frac{MV_{t+1}}{MC_t}$ as a stochastic discount factor. Thus a claim to next period's consumption is valued using

$$S_{t+1,t} = \frac{MV_{t+1}MC_{t+1}}{MC_t} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{V_{t+1}}{\mathcal{R}_t(V_{t+1})} \right)^{\rho-\theta} \quad (16)$$

as a stochastic discount factor. There are two (typically highly correlated) contributions to the stochastic discount factor in formula (16). One is the direct consumption growth contribution familiar from the Rubinstein (1976), Lucas (1978) and Breeden (1979) model of asset pricing. The other is the continuation value relative to its risk adjustment. The contribution is forward-looking and is present only when ρ and θ differ.

Given the homogeneity in the recursion used to depict preferences, equilibrium wealth is given by $W_t = \frac{V_t}{MC_t}$. Substituting for the marginal utility of consumption, the wealth-consumption ratio is:

$$\frac{W_t}{C_t} = \frac{1}{1 - \beta} \left(\frac{V_t}{C_t} \right)^{1 - \rho}.$$

Taking logarithms, we find that

$$\log W_t - \log C_t = -\log(1 - \beta) + (1 - \rho)v_t \tag{17}$$

When $\rho = 1$ we obtain the well known result that the wealth consumption ratio is constant.

A challenge in using this model empirically is to measure the continuation value, V_{t+1} , which is linked to future consumption via the recursion (13). When $\rho \neq 1$, one approach is to use the relationship between wealth and the continuation value, $W_t = V_t/MC_t$ to construct a representation of the stochastic discount factor based on consumption growth and the return to a claim on future wealth. In general this return is unobservable. An aggregate stock market return is sometimes used to proxy for this return as in Epstein and Zin (1989a), for example; or other components can be included such as human capital with assigned market or shadow values (see Campbell (1994)).

In this investigation, like that of Restoy and Weil (1998) and Bansal and Yaron (2004), we base the analysis on a well specified stochastic process governing consumption. In contrast to this literature, we feature the role of continuation values to accommodate $\rho = 1$. In fact we begin by studying the case of logarithmic intertemporal preferences ($\rho = 1$) and then explore approximations in the parameter ρ . It is well understood that $\rho = 1$ leads to substantial simplification in the equilibrium prices and returns [*e.g.* see Schroder and Skiadas (1999).]

4.3 The special case in which $\rho = 1$

We use the $\rho = 1$ specification as a benchmark. Campbell (1996) argues for less intertemporal substitution and Bansal and Yaron (2004) argue for more. We will explore such deviations subsequently. The $\rho = 1$ case is convenient when consumption has a log linear time series evolution because of the resulting continuation value is linear in the state variables.

The $\rho = 1$ limit in recursion (14) is:

$$\begin{aligned} v_t &= \beta \mathcal{Q}_t(v_{t+1} + c_{t+1} - c_t) \\ &= \frac{\beta}{1 - \theta} \log E(\exp[(1 - \theta)(v_{t+1} + c_{t+1} - c_t)] | \mathcal{F}_t). \end{aligned} \tag{18}$$

The stochastic discount factor in this special case is:

$$S_{t+1,t} \equiv \beta \left(\frac{C_t}{C_{t+1}} \right) \left[\frac{(V_{t+1})^{1-\theta}}{\mathcal{R}_t(V_{t+1})^{1-\theta}} \right].$$

Recursion (18) is used by Tallarini (1998) in his study of *risk sensitive* business cycles and asset prices.

Notice that the term of $S_{t+1,t}$ associated with the risk-aversion parameter θ satisfies

$$E \left[\frac{(V_{t+1})^{1-\theta}}{\mathcal{R}_t(V_{t+1})^{1-\theta}} \middle| \mathcal{F}_t \right] = 1.$$

This term can thus be thought of as distorting the probability distribution. The presence of this distortion reflects a rather different interpretation of the parameter θ . Anderson, Hansen, and Sargent (2003) argue that this parameter may reflect investor concerns about not knowing the precise riskiness that investors must confront in the marketplace instead of incremental risk aversion applied to continuation utilities. Under this view, the original probability model is viewed as a statistical approximation, but investors are concerned that this model may be misspecified. This alternative interpretation is germane to our analysis because we will explore sensitivity of our measurements to the choice of θ . Changing the interpretation of θ alters what we might view as reasonable values of this parameter. Instead of focusing on the intertemporal composition of risk as in the Kreps and Porteus (1978) formulation, under this view we are lead to consider potential misspecifications in probabilities that most challenge investors.

To make our formula for the marginal rate of substitution operational, we need to compute V_{t+1} using the equilibrium consumption process. Suppose that the first-difference of the logarithm of equilibrium consumption is given by:

$$c_{t+1} - c_t = \mu_c + U_c x_t + \gamma_0 w_{t+1}.$$

This representation implies an impulse response function for consumption where the date t shock w_t adds $\gamma_j w_t$ to consumption growth at date $t + j$. The response vector is:

$$\gamma_j = \begin{cases} \gamma_0 & \text{if } j = 0 \\ U_c G^{j-1} H & \text{if } j > 0 \end{cases}$$

For this lognormal consumption growth process, the solution for the continuation value is

$$v_t = \mu_v + U_v x_t$$

where

$$\begin{aligned} U_v &\doteq \beta U_c (I - \beta G)^{-1}, \\ \mu_v &\doteq \frac{\beta}{1 - \beta} \left[\mu_c + \frac{(1 - \theta)}{2} \gamma(\beta) \cdot \gamma(\beta) \right], \end{aligned}$$

and $\gamma(\beta)$ is the discounted impulse response:

$$\gamma(\beta) = \sum_{j=0}^{\infty} \beta^j \gamma_j = \gamma_0 + \beta U_c (I - G\beta)^{-1} H.$$

The logarithm of the stochastic discount factor is:

$$s_{t+1,t} = \mu_s + U_s x_t + \xi_0 w_{t+1}$$

where

$$\begin{aligned} \mu_s &= -\delta - \mu_c - \frac{(1-\theta)^2 \gamma(\beta) \cdot \gamma(\beta)}{2} \\ U_s &= -U_c \\ \xi_0 &= -\gamma_0 + (1-\theta)\gamma(\beta). \end{aligned}$$

The stochastic discount factor includes both the familiar contribution from contemporaneous consumption plus a forward-looking term that discounts the impulse responses for consumption growth. For instance, the price of payoff $\phi(w_{t+1})$ is given by:

$$E[\exp(s_{t+1})\phi(w_{t+1})|\mathcal{F}_t] = E[\exp(s_{t+1})|\mathcal{F}_t] \frac{E[\exp(s_{t+1})\phi(w_{t+1})|\mathcal{F}_t]}{E[\exp(s_{t+1})|\mathcal{F}_t]}$$

The first term is a pure discount term and the second is the expectation of $\phi(w_{t+1})$ under the so-called risk neutral probability distribution. The logarithm of the first term is:

$$\log E[\exp(s_{t+1})|\mathcal{F}_t] = -\delta - \sum_{j=0}^{\infty} \gamma_{j+1} w_{t-j} - (1-\theta)\gamma(\beta) \cdot \gamma_0 + \frac{\gamma_0 \cdot \gamma_0}{2},$$

which is minus the yield on a discount bond. The w_{t+1} coefficient on the innovation to the logarithm $s_{t+1,t}$ of the stochastic discount factor is

$$-\gamma_0 + (1-\theta)\gamma(\beta).$$

This vector is also the mean of the normally distributed shock w_{t+1} under the risk-neutral distribution.

The adjustment $-\gamma_0$ is familiar from Hansen and Singleton (1983) and the term $(1-\theta)\gamma(\beta)$ is the adjustment for the intertemporal composition of consumption risk implied by the Kreps and Porteus (1978) specification of recursive utility. Large values of the risk parameter θ enhance the importance of this component. This latter effect is featured in the analysis of Bansal and Yaron (2004). Under the alternative interpretation suggested by Anderson, Hansen, and Sargent (2003), $|(1-\theta)\gamma(\beta)|$ is a measure of model misspecification that investors cannot identify because the misspecification is disguised by shocks that impinge on investment opportunities.

Our interest is in the long-run consequences for cash flow risk. As we discussed in section 3, consider the valuation of alternative securities that are claims to the cash flows with permanent components πw_{t+1} . The valuations of these components are dominated by a single factor. Applying theorem 1 the dominant valuation factor is invariant to both the risk aversion parameter θ and cash-flow risk exposure parameter π . It is the exponential of the discounted conditional expectation of consumption growth rates. From theorem 2, the long-run cash-flow risk price is:

$$\begin{aligned}\pi^* &= \gamma_0 + U_c(I - G)^{-1}H + (\theta - 1)\gamma(\beta) \\ &= \gamma(1) + (\theta - 1)\gamma(\beta)\end{aligned}$$

where $\gamma(1)$ is the cumulative growth rate response or equivalently the limiting consumption response in the infinite future. The comparison between one-period and long-run risk prices is informative. The long-run risk price uses the long run consumption response vector $\gamma(1)$ in place of γ_0 , but the recursive utility contribution remains the same. As the subjective discount factor β tends to unity, $\gamma(\beta)$ converges to $\gamma(1)$, and hence the long-run risk price is approximately $\theta\gamma(1)$.

Given the unitary EIS, wealth in this economy is proportional to consumption

$$W_t = \frac{C_t}{1 - \beta}.$$

As noted by Rubinstein (1976) and Gibbons and Ferson (1985), we may use the return on the wealth portfolio as a proxy for the consumption growth rate. Although these papers do not study the recursive utility counterpart, the tight link between consumption and wealth applies without regard to the risk aversion parameter θ . The return on a claim to wealth is:

$$R_{t+1}^w = \frac{W_{t+1}}{\frac{\beta C_t}{1 - \beta}} = \frac{C_{t+1}}{\beta C_t}.$$

Thus

$$r_{t+1}^w = c_{t+1} - c_t - \log \beta.$$

This leads Campbell and Vuolteenaho (2003) and Campbell, Polk, and Vuolteenaho (2005) to use a market wealth return as a proxy for consumption growth and to measure γ_0 and $\gamma(\beta)$ from impulse response functions of wealth returns to shocks characterize one-period risk. Although we will directly measure the consumption responses to shocks, an alternative for us would be to infer $\gamma(1)$ and $\gamma(\beta)$ from wealth return responses.

4.4 Intertemporal substitution ($\rho \neq 1$)

Approximate characterization of equilibrium pricing for recursive utility have been produced by Campbell (1994) and Restoy and Weil (1998) based on a log-linear approximation of budget constraints. In what follows we use a distinct but related approach. We follow

Kogan and Uppal (2001) by approximating around an explicit equilibrium computed when $\rho = 1$ and varying the parameter ρ .

We start with a first-order expansion of the continuation value:

$$v_t \approx v_t^1 + (\rho - 1)Dv_t^1$$

where v_t^1 is the continuation value for the case in which $\rho = 1$. Recall that the logarithm of the continuation value/consumption ratio is:

$$\begin{aligned} v_{t+1}^1 &= U_v x_{t+1} + \mu_v \\ &= U_v H w_{t+1} + U_v G x_t + \mu_v. \end{aligned}$$

In appendix A, we show that

$$Dv_{t+1}^1 = -\frac{1}{2}x_{t+1}'\Upsilon_{dv}x_{t+1} + U_{dv}x_{t+1} + \mu_{dv}$$

where formulas for Υ_{dv} , U_{dv} and μ_{dv} are given in appendix A.

The corresponding expansion for the logarithm of the stochastic discount factor is:

$$s_{t+1,t} \approx s_{t+1,t}^1 + (\rho - 1)Ds_{t+1,t}^1.$$

where

$$Ds_{t+1,t}^1 = \frac{1}{2}w_{t+1}'\Theta_0w_{t+1} + w_{t+1}'\Theta_1x_t + \vartheta_0 + \vartheta_1x_t + \vartheta_2w_{t+1}.$$

Formulas for Θ_0 , Θ_1 , ϑ_0 , ϑ_1 and ϑ_2 are also given in appendix A.

Finally we use the stochastic discount factor expansion to determine how the decay rate ν of the dominant eigenfunction changes with ρ . This calculation makes use of the formula:

$$\frac{d\nu}{d\rho} \Big|_{\rho=1} = -\frac{E \left[Ds_{t+1,t}^1 \exp(s_{t+1,t}^1 + \pi w_{t+1}) \phi(x_{t+1}) \varphi(x_t) \right]}{\exp(-\nu) E[\phi(x_t) \varphi(x_t)]}$$

where ϕ and φ are the eigenfunctions for the $\rho = 1$ valuation operator. The details of the implementation of these formulas are given in appendix B. This derivative will depend on the assumed cash flow growth process. Since the asymptotic growth rate of cash flows does not depend on ρ this same calculation can be used to study the sensitivity of the valuation rate of return to changes in ρ .

5 Measuring Long-Run Consumption Risk

As in much of the empirical literature in macroeconomics, we use vector autoregressive (VAR) models to identify interesting aggregate shocks and estimate $\gamma(z)$. In our initial model we let consumption be the first element of y_t and corporate earnings be the second element:

$$y_t = \begin{bmatrix} c_t \\ e_t \end{bmatrix}.$$

Our use of corporate earnings in the VAR is important for two reasons. First, it is used as a predictor of consumption and an additional source of aggregate risk.⁹ For example, changes in corporate earnings potentially signal changes in aggregate productivity which will have long-run consequences for consumption. Second, corporate earnings provide a broad-based measure of the ultimate source of the cash flows to capital. The riskiness of the equity claims on these cash flows provides a basis of comparison for the riskiness of the cash flows generated by the portfolios of stocks that we consider in section 6.

The process $\{y_t\}$ is presumed to evolve as a VAR of order ℓ . In the results reported subsequently, $\ell = 5$. The least restrictive specification we consider is:

$$A_0 y_t + A_1 y_{t-1} + A_2 y_{t-2} + \dots + A_\ell y_{t-\ell} + B_0 = w_t, \quad (19)$$

The vector B_0 two-dimensional, and the square matrices $A_j, j = 1, 2, \dots, \ell$ are two by two. The shock vector w_t has mean zero and covariance matrix I . We normalize A_0 to be lower triangular with positive entries on the diagonals. Form:

$$A(z) \doteq A_0 + A_1 z + A_2 z^2 + \dots + A_\ell z^\ell.$$

We are interested in a specification in which $A(z)$ is nonsingular for $|z| < 1$. Given this model, the discounted response of consumption to shocks is given by:

$$\gamma(\beta) = (1 - \beta) u_c A(\beta)^{-1}$$

where $u_c' \doteq [1 \ 0]$.

For our measure of aggregate consumption we use aggregate consumption of nondurables and services taken from the National Income and Product Accounts. This measure is quarterly from 1947 Q1 to 2002 Q4, is in real terms and is seasonally adjusted. We measure corporate earnings from NIPA and convert this series to real terms using the implicit price deflator for nondurables and services.

Following Hansen, Heaton, and Li (2005), we consider two specifications of the evolution of y_t . In one case the model is estimated without additional restrictions, and in the other we restrict the matrix $A(1)$ to have rank one:

$$A(1) = \alpha [1 \ -1].$$

where the column vector α is freely estimated. This parameterization imposes two restrictions on the $A(1)$ matrix. We refer to the first specification as the *without cointegration* model and second as the *with cointegration* model.

The second system imposes a unit root in consumption and earnings, but restricts these series to grow together. In this system both series respond in the same way to shocks in

⁹Whereas Bansal and Yaron (2004) also consider multivariate specification of consumption risk, they seek to infer this risk from a single aggregate time series on consumption or aggregate dividends. With flexible dynamics, such a model is not well identified from time series evidence. On the other hand, while our shock identification allows for flexible dynamics, it requires that we specify *a priori* the important sources of macroeconomic risk.

the long run. Specifically, the limiting response of consumption and earnings to a shock at date 0 is the same. Since the cointegration relation we consider is prespecified, the *with cointegration* model can be estimated as a vector autoregression in the first-difference of the log consumption and the difference between the log earnings and log consumption.

In our analysis, we will not be concerned with the usual shock identification familiar from the literature on structural VAR's. This literature assigns structural labels to the underlying shocks and imposes *a priori* restrictions to make this assignment. While we have restricted A_0 to be lower triangular, this is just a normalization. This restriction leads to the identification of two shocks, but other shock configurations with an identity as a covariance matrix can be constructed by taking linear combinations of the initial two shocks we identify. Sometimes we will construct two uncorrelated shocks in a different manner. One is temporary, formed as a linear combination of shocks that has no long run impact on consumption and corporate earnings. The second is permanent which effects consumption and earnings equally in the long run. This construction is much in the same spirit as Blanchard and Quah (1989). Our primary interest is the intertemporal composition of consumption risk and not the precise labels attached to individual shocks.

We report impulse responses for estimates of the VAR with and without the cointegration restriction in figure 1. When cointegration is imposed, corporate earnings relative to consumption identifies an important long-run response to both shocks. The long-run impact of the first *consumption shock* is twice that of the impulse on impact. While the second *earnings shock* is normalized to have no immediate impact on consumption, its long-run impact is sizeable. We demonstrated in the recursive utility model, that the geometrically weighted average response of consumption to the underlying shocks and the limiting response are the two components of the long-run cash flow risk price π^* . As the subjective discount rate converges to zero, these two components become equal. Moreover, π^* is approximately equal to θ times the long-run consumption response. As we verify below, this rough approximation is quite accurate for our calculations.

Notice from the impulse responses in figure 1, that when the cointegration restriction is not imposed, the estimated long-run consumption responses are substantially smaller. The imposition of the cointegration restriction is critical to locating an important low frequency component in consumption. Moreover, in the absence of this restriction, the overall feedback from earnings shocks to consumption is substantially weakened. The earnings shocks have little impact on consumption for the *without cointegration* specification.

Using the cointegration specification, we explore the statistical accuracy of the estimated responses. Following suggestions of Sims and Zha (1999) and Zha (1999), we impose Box-Tiao priors on the coefficients of each equation and simulate histograms for the parameter estimates. This provides approximation for Bayesian posteriors with a relatively diffuse (and improper) prior distribution. These "priors" are chosen for convenience, but they give us a simple way to depict the sampling uncertainty associated with the estimates.

In the model of Hansen and Singleton (1983), it is the immediate innovation in consumption that matters for pricing one-period securities. Figure 2 gives a histogram for the standard deviation of this estimate. In other words it gives the histogram for the estimate of

Impulse Response of Consumption and Earnings

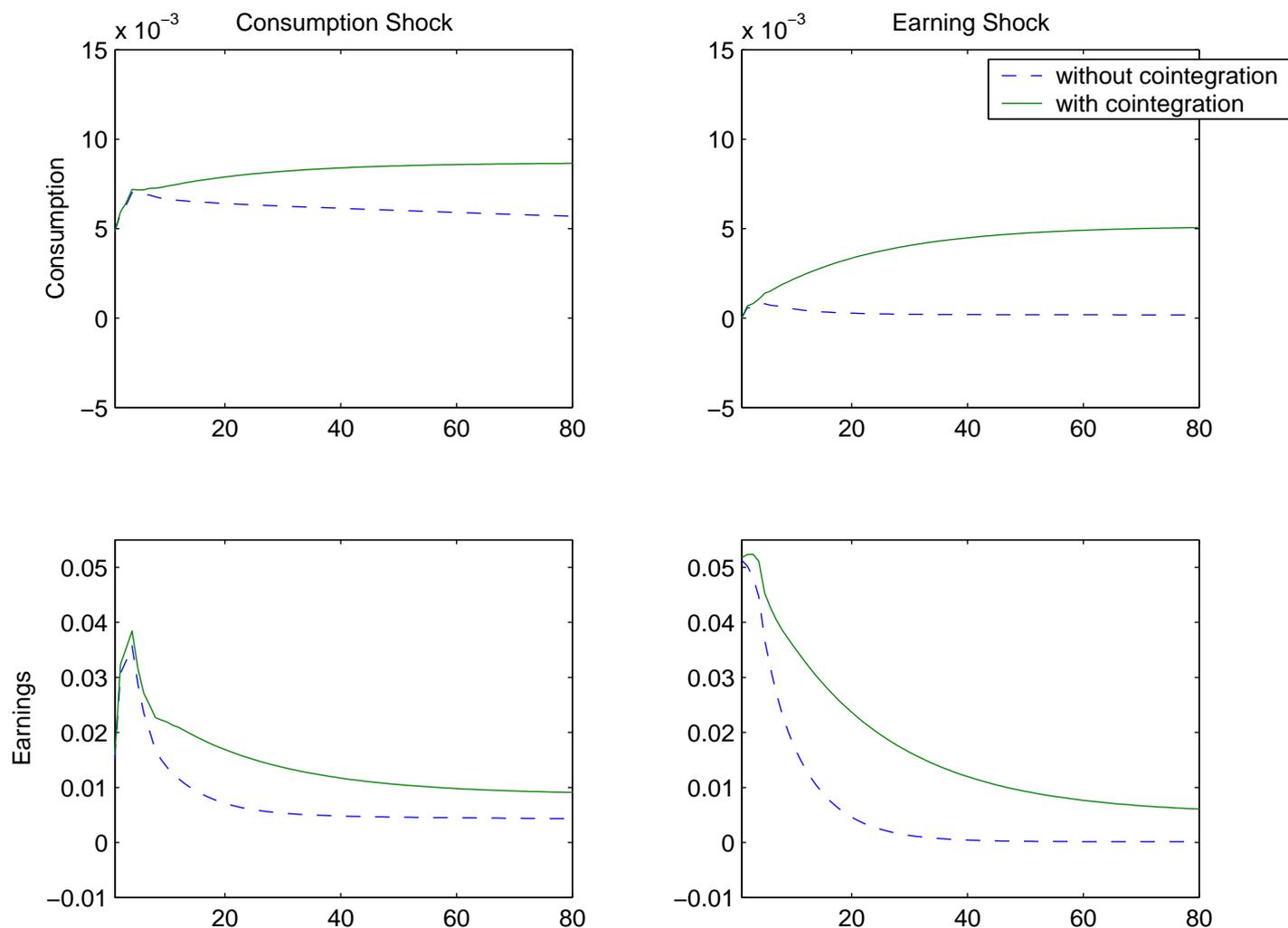


Figure 1: The impulse responses without imposing cointegration were constructed from a bivariate VAR with entries c_t, e_t . These responses are given by the dashed lines $---$. Solid lines $---$ are used to depict the impulse responses estimated from a cointegrated system. The impulse response functions are computed from a VAR with $c_t - c_{t-1}$ and $c_t - e_t$ as time series components.

the $(1, 1)$ entry of A_0 . Recall that it is the long-run response that is of interest for this paper. Thus we also report the histogram for a long-run response using the permanent-transitory decomposition just described. Figure 2 also gives a histogram for the long-run consumption response to a long-run shock. The permanent shock is normalized to have unit standard deviation, so that we can compare magnitudes across the long-run and short run responses.

As might be expected, the short run response estimate is much more accurate than the long-run response. Notice that the horizontal scales of histogram differ by a factor of ten. In particular, while the long-run response is centered at a higher value and it also has a substantial right tail. Consistent with the estimated impulse response functions, the median long-run response is about double that of the short-term response. In addition nontrivial probabilities are given to substantially larger responses.¹⁰ Thus from the standpoint of sampling accuracy, the long-run response could be even more than double that of the immediate consumption response.

The cointegrated specification with a known cointegrating coefficient imposes a restriction on the VAR. To explore the statistical plausibility of this restriction, we free up the cointegration relation by allowing consumption and earnings to have different long-run responses. To assess statistical accuracy we simulate the posterior distribution for the cointegrating coefficient imposing a Box-Tiao prior for each VAR conditioned on the cointegrating coefficient. The resulting histogram is depicted in figure 3. For sake of computation, we used a uniform prior over the interval $[-2, 2]$ for the cointegrating coefficient. This figure suggests that the *balanced growth* coefficient of unity is plausible.¹¹

Next we use these VAR estimates to measure long-run risk components of aggregate consumption. In table 1 we report long-run expected rates of return to holding a claim to aggregate consumption. In this case π and ζ of section 3 are equal to one and zero respectively. We explore sensitivity as we alter θ , and display derivatives with respect to the intertemporal substitution parameter ρ . We compare expected rates of return to those of implied by consumption and those implied by a long-run riskless rate of return (long bond). This latter return is used as the reference point for computing expected excess returns and it is the long-run riskless return considered by Alvarez and Jermann (2001).

As is evident from this table, the implied differences in expected returns across securities are small even when θ is as large as twenty. The derivatives of the returns with respect to ρ are large while the derivatives of the excess returns are small. According to the derivatives, increasing ρ by ε adds over three times ε percentage points to the expected rates of return. While larger values of ρ increase long-run riskless return rate, this increase can be offset

¹⁰The accuracy comparison could be anticipated in part from the literature on estimating linear time series models using a finite autoregressive approximation to an infinite order model (see Berk (1974)). The *on impact* response is estimated at the parametric rate, but the *long-run* response is estimated at a considerably slower rate that depends on how the approximating lag length increases with sample size. Our histograms do not confront the specification uncertainty associated with approximating an infinite order autoregressions, however.

¹¹The model with cointegration imposes two restrictions on the matrix $A(1)$. Twice the likelihood ratio for the two models is 5.9. The Bayesian information or Schwarz criterion selects the restricted model.

Approximate Posterior Distributions for Responses

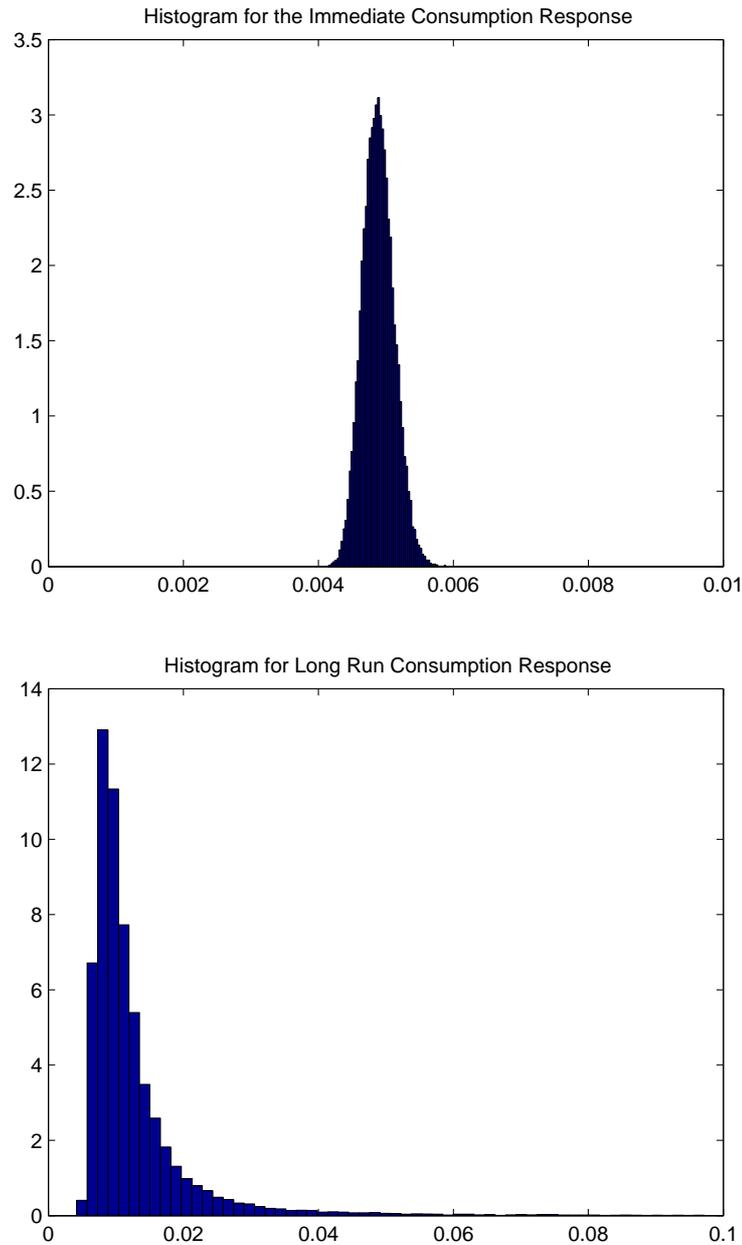


Figure 2: The top panel gives the approximate posterior for the immediate response to consumption and the bottom panel the approximate posterior for the long-run response of consumption to the permanent shock. The histograms have sixty bins with an average bin height of unity. They were constructed using using Box-Tiao priors for each equation. Vertical axes are constructed so that on average the histogram height is unity.

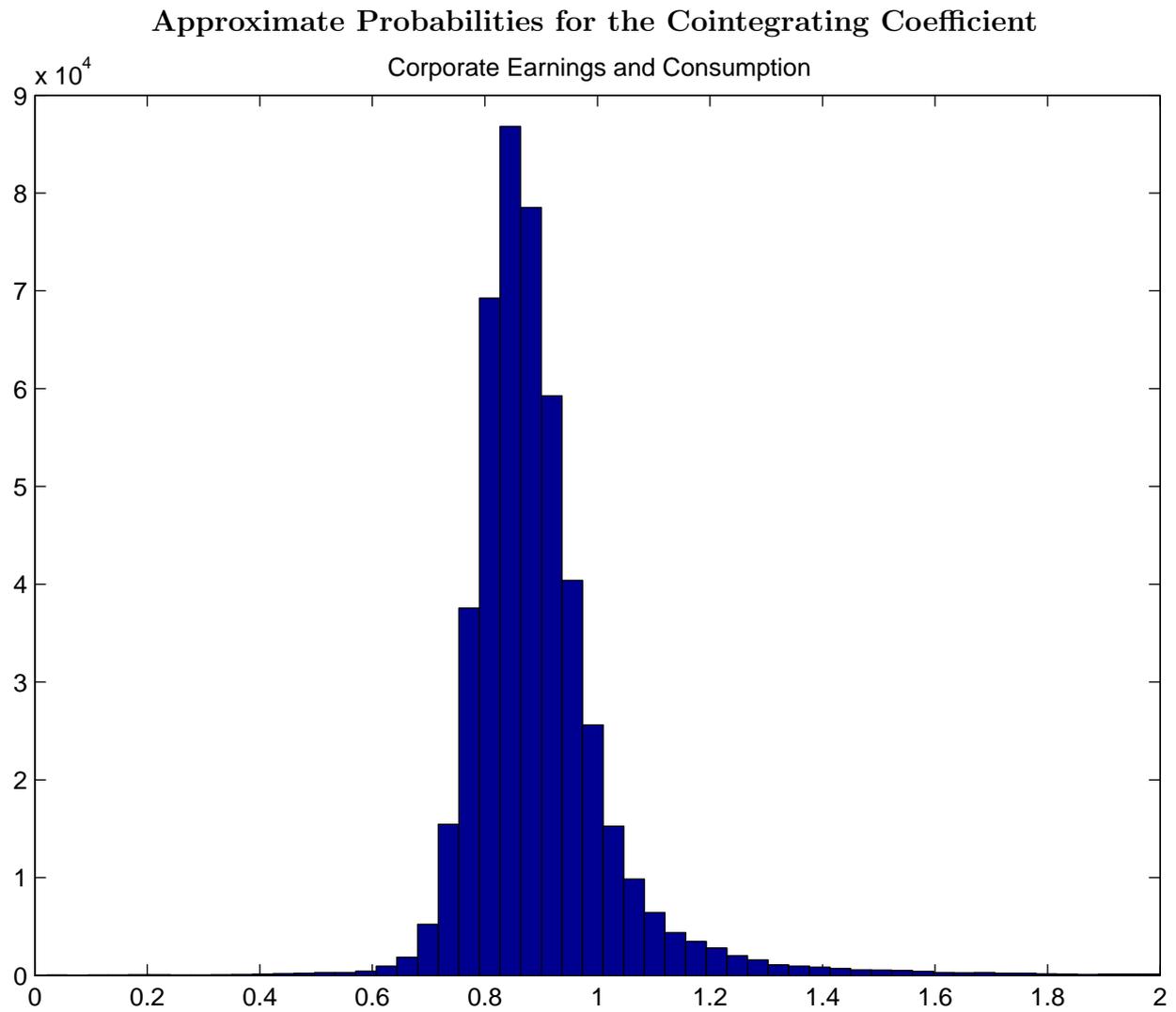


Figure 3: Box-Tiao priors are imposed on the regression coefficients and innovation variances conditioned on the cointegrating coefficient. Posterior probabilities are computed by simulating from a Markov chain constructed from the conditional likelihood function.

Valuation Returns for Aggregate Consumption

Portfolio	Return	Excess Return	Return Derivative	Excess Return Derivative
$\theta = 1$				
consumption	6.58	.05	3.51	.00
long bond	6.54	0	3.51	0
$\theta = 5$				
consumption	6.58	.19	3.43	-.01
long bond	6.39	0	3.44	0
$\theta = 20$				
consumption	6.58	.77	3.13	-.04
long bond	5.81	0	3.17	0

Table 1: The excess returns are measured relative to the return on the long horizon discount bond. The derivative entries in columns four and five are computed with respect to ρ and evaluated at $\rho = 1$.

by simultaneously reducing δ .¹² The expected excess returns to valuation are essentially proportional to θ . Quadrupling θ ($\theta = 5$ to $\theta = 20$), approximately quadruples the numbers in the “Excess Return” column. This approximation is to be expected. The proportionality would be exact if β were unity. Overall, the long-run rate of return heterogeneity is small, even when risk aversion parameter is set to a large number. As we will see this same conclusion holds when we use aggregate stock market dividends instead of consumption as the cash flow measure. This should be anticipated from the cointegration of consumption and corporate cash flows.

¹²Of course there is a limit to this reduction, when δ is restricted to be positive.

6 Long-Run Cash Flow Risk in Portfolios

The work of Bansal, Dittmar, and Lundblad (2005) and Campbell and Vuolteenaho (2003) related measures of long-run cash flow risk to one period returns. Our aim is different, but complementary to their study. In section 3 we derived the limiting factor risk prices π^* for exposure to long-run risk. We also characterized the limiting contribution to price-dividend ratios for alternative specifications of stochastic growth. These relations in conjunction with an economic model like that of section 4 allow us to compute the limiting valuation and risk adjustments. Our task in this section is to measure the long-run risk exposure of the cash flows from some portfolios familiar from financial economics and to consider the implied heterogeneity in values and the risk premia of valuation returns.

We consider financial cash flows that may not grow proportionately with consumption as in Campbell and Cochrane (1999), Bansal, Dittmar, and Lundblad (2005), Lettau, Ludvigson, and Wachter (2004), and others. This is germane to our empirical application because the sorting method we use in constructing portfolios can induce permanent differences in dividend growth. For this reason we allow cash flows or dividends to risky securities to be *levered* claims on consumption in the long run. Consistent with our use of VAR methods, we consider a log-linear model of cash flow growth:

$$d_{t+1} - d_t = \mu_d + U_d x_t + \iota_0 w_{t+1}.$$

where d_t is the logarithm of the cash flow. This growth rate process has a moving-average form:

$$d_{t+1} - d_t = \mu_d + \iota(L)w_{t+1}.$$

where

$$\iota(z) = \sum_{j=0}^{\infty} \iota_j z^j$$

and

$$\iota_j = \begin{cases} \iota_0 & \text{if } j = 0 \\ U_d G^{j-1} H & \text{if } j > 0 \end{cases}$$

6.1 Martingale approximation

In section 3, we considered benchmark growth processes that were geometric random walks with drift. This leads us to construct a martingale approximation to the cash flow process. In this approximation, log dividend process is the sum of a martingale and the first difference of a stationary process. Marginal approximations are commonly used in establishing central limit approximations (*e.g.* see Gordin (1969) or Hall and Heyde (1980)), and are not limited to log-linear processes.¹³ We write

$$d_{t+1} - d_t = \mu_d + \iota(1)w_{t+1} + U_d^* x_{t+1} - U_d^* x_t$$

¹³For scalar linear time series, it coincides with the decomposition of Beveridge and Nelson, but it is applicable much more generally.

where

$$\begin{aligned}\iota(1) &= \iota_0 + U_d(I - G)^{-1}H \\ U_d^* &= U_d(I - G)^{-1}\end{aligned}$$

Thus d_t has a growth rate μ_d and a martingale component with increment: $\iota(1)w_t$. To relate this to development in section 3, $\iota(1) = \pi$ and $\mu_d = \zeta$. We will fit processes to cash flows to obtain estimates of $\iota(1)$ and μ_d . The long-run relation between dividends and consumption is essential in determining valuation returns. For example, if consumption and dividends are cointegrated then

$$\iota(1) = \lambda\gamma(1), \quad \mu_d = \lambda\mu_c$$

where λ is the cointegration coefficient.

So far, we have considered a cash flow process with log-linear dynamics. Suppose instead we consider a *share model* as in Santos and Veronesi (2001). The discrete-time version of such a model can be depicted as:

$$D_t = C_t\Psi(x_t)$$

where $\Psi(x_t)$ is restricted to be between zero and one and gives the dividend share of aggregate consumption. Thus

$$d_t - d_{t-1} = c_t - c_{t-1} + \log \Psi(x_t) - \log \Psi(x_{t-1})$$

By construction this share model assumes that log consumption and log dividends share the same stochastic growth, so that the long-run dividend risk is the same as that of consumption. The counterpart to $\iota(1)$ is the long-run consumption response $\gamma(1)$. While physical claims to resources may satisfy balanced growth restrictions, financial claims of the type we investigate need not. Share models are not attractive models of the cash flows we consider unless the share process is allowed to have a very pronounced low frequency component.

6.2 Empirical Specification of Dividend Dynamics

We identify dividend dynamics and, in particular, the martingale component $\iota(1)$ using VAR methods. Consider a VAR with three variables: consumption, corporate earnings and dividends (all in logarithms). Consumption and corporate earnings are modelled as before in a cointegrated system. We use the cointegrated system because it identifies a long-run consumption risk component that is distinct from the one-step-ahead forecast error of consumption. In addition to consumption and earnings, we include in sequence the dividend series from each of the five book-to-market portfolios and from the market. Thus the same two shocks as were identified previously remain shocks in this system because consumption and corporate earnings remain an autonomous system. An additional shock is required to account for the remaining variation in dividends beyond what is explained by consumption and corporate earnings.

Formally, we append a dividend equation

$$A_0^*y_t^* + A_1^*y_{t-1} + A_2^*y_{t-2} + \dots + A_\ell^*y_{t-\ell} + B_0^* = w_t^*, \quad (20)$$

to equation system (19). In this equation the vector of inputs is

$$y_t^* \doteq \begin{bmatrix} y_t \\ d_t \end{bmatrix} = \begin{bmatrix} c_t \\ e_t \\ d_t \end{bmatrix}$$

and the shock w_t^* is scalar with mean zero and unit variance. This shock is uncorrelated with the shock w_t that enters (19). The third entry of A_0^* is normalized to be positive. We refer to (20) as the dividend equation, and the shock w_t^* as the dividend shock. As in our previous estimation, we set $\ell = 5$. We presume that this additional shock has a permanent impact on dividends by imposing the linear restriction:

$$A^*(1) = [\alpha^* \quad -\alpha^* \quad 0].$$

In the next section we will explore sensitivity to alternative specifications of long-run stochastic growth in the cash flows.

A stationary counterpart to this log level specification can be written in terms of the $c_t - c_{t-1}, e_t - c_t, d_t - d_{t-1}$. We estimated the VAR using these transformed variables with four lags of the growth rate variables and five lags of the logarithmic differences between consumption and earnings.

6.3 Book to Market Portfolios

We use five portfolios constructed based on a measure of book equity to market equity, and characterize the time series properties of the dividend series as it covaries with consumption and earnings. We follow Fama and French (1993) and construct portfolios of returns by sorting stocks according to their book-to-market values. We use a coarser sort into 5 portfolios to make our analysis tractable. In addition we use the value-weighted CRSP return for our “market” return.

Summary statistics for these portfolios are reported in table 2. In the row labeled “1-period Exp. Return,” we report the predicted quarterly gross returns to holding each portfolio in annual units. The expected returns are constructed based on a separate VAR for each portfolio with inputs:

$$y_t^* = \begin{bmatrix} c_t - c_{t-1} \\ e_t - c_t \\ r_t \end{bmatrix},$$

where r_t is the logarithm of the gross return of the portfolio. We impose the restriction that consumption and earnings are not Granger caused by the returns. One-period expected gross returns are calculated conditional on being at the mean of the state variable implied by the VAR. In the row labelled “Long-Run Return,” we also report the logarithm of the dominant eigenvalue of the operator \mathcal{G} implied by the VAR and where the compound returns are used as cash flows. This gives a long-run average rate of return predicted by the VAR, when dividend proceeds are continually reinvested in the respective portfolios. These results are also reported in annual units.

Properties of Portfolios Sorted by Book-to-Market

	Portfolio					Market
	1	2	3	4	5	
1-period Exp. Return (%)	7.91	8.32	9.86	10.61	12.69	8.42
Long-Run Return (%)	8.16	7.97	9.96	10.27	12.15	8.40
Avg. B/M	0.32	0.62	0.84	1.12	2.00	0.79
Avg. P/D	49.8	33.3	27.4	24.3	25.5	33.6

Table 2: Data are quarterly from 1947 Q1 to 2002 Q4 for returns and annual from 1947 to 2001 for B/M ratios. Returns are converted to real units using the implicit price deflator for nondurable and services consumption. Average returns are converted to annual units using the natural logarithm of quarterly gross returns multiplied by 4. “Avg. B/M” for each portfolio is the average portfolio book-to-market over the period computed from COMPU-STAT. “Avg. P/D” gives the average price-dividend for each portfolio where dividends are in annual units.

Notice that the portfolios are ordered by average book to market values where portfolio 1 has the lowest book-to-market value and portfolio 5 has the highest. Both one-period and long-run average returns generally follow this sort. For example, portfolio 1 has much lower average returns than portfolio 5. It is well documented that the differences in these average returns are not explained by exposure to contemporaneous covariance with consumption.

In this section we are particularly interested in the behavior of dividends from the constructed portfolios. The constructed dividend processes accommodate changes in the classification of the primitive assets and depend on the relative prices of the new and old asset in the book-to-market portfolios. Monthly dividend growth for each portfolio are constructed from the gross returns to holding each portfolio with and without dividends. Using the initial price-dividend ratio for the series, these growth rates are used to construct monthly dividend levels. Dividends on a quarterly basis are constructed as an accumulation of the monthly dividends during the quarter. Our measure of quarterly dividends in quarter t is then constructed by taking an average of the logarithm of dividends in quarter t and over the previous three quarters $t-3$, $t-2$ and $t-1$. This last procedure removes the pronounced seasonal in dividend payments. Details of this construction are given in Hansen, Heaton, and Li (2005), which follows the work of Bansal, Dittmar, and Lundblad (2005).

We estimate $\iota(1)$ from the VAR inclusive of portfolio dividends which gives us a measure of π . We then explore the limiting valuation and rates of returns using the eigenvector methods described in section 3. Table 3 gives long-run average rates of return for the five book-to-market portfolios. Again we explore formally sensitivity to the risk aversion parameter θ

and report derivatives with respect to the intertemporal elasticity parameter ρ .

Qualitatively, portfolio one has low long-run cash flow covariation with consumption relative to portfolio five. This results in qualitatively larger risk adjustments for the high book-to-market portfolios. Complementary to many other asset pricing studies, differences in the average rates of return on long-run valuation securities are small except for large values of the risk aversion parameter θ , say $\theta = 20$. In contrast to aggregate securities, the implied heterogeneity in the valuation returns are now substantial when θ is large. For the reasons we gave earlier, changing θ alters the expected excess returns almost proportionately.

Recall from table 2 that one-period and long-horizon reinvestment expected returns are similar for each portfolio. The valuation return rates that we compute only achieve comparable dispersion for large values of θ , say $\theta = 20$. While the valuation return rates in table 3 are lower, common changes in these rates can be achieved by simply altering the subjective discount factor β . While the three return concepts are distinct, the valuation returns explore the return heterogeneity attributed to payoffs far into the future. The valuation returns are the ones that are directly linked to cash flow risk exposure.

As with the aggregate returns, derivatives with respect to ρ are similar across securities so that modest movements in ρ have little impact on the excess long-run returns.

It is also of interest to study the implied logarithm of the price/dividend ratio decomposed and scaled by horizon. These are reported in figure 4. The lower panel of this figure depicts the dividend growth rate by horizon. The figures are computed assuming that the Markov state is set to its unconditional mean. The limiting values in these plots are inputs into the valuation return calculations. Thus these figures tell us over what horizons do the limits become good approximations.

When $\theta = 1$, the risk adjustments are very small and the value decomposition is a direct reflection of the dividend growth. Moreover, the values for portfolio one are dominated by those for portfolio five across all horizons. This is in direct conflict with the price/dividend ratios reported in table 2. It is only with high values of the risk aversion parameter θ that the value decomposition for the low book to market portfolios eventually exceed those of the high book to market portfolios.

Relation (10) maps these values of $-\nu$ into a corresponding long run or tail notion of a price-dividend ratio. When $\theta = 1$ the implied tail price-dividend ratios (in annual units) for portfolios 1 and 5 are 22.5 and 102.2 respectively. When $\theta = 20$, however, these values are 45.1 and 21.2, respectively, which are roughly comparable to the numbers in table 2.¹⁴

Finally, it can take many time periods for the valuations to approximate their limiting values. Even fifteen years (sixty quarters) is not be long enough to approximate well the limit in some cases. Thus the expected rates of return to valuation do indeed rely on extrapolating the implied consumption/dividend dynamics very far into the future.

The price-dividend decomposition include expected growth and expected return contri-

¹⁴Since these are limiting concepts, they need not match the average in the data even when the model is empirically plausible.

Valuation Returns for Portfolios

Portfolio	Return	Excess Return	Return Derivative	Excess Return Derivative
$\theta = 1$				
1	6.48	-.06	3.51	.00
2	6.55	.01	3.51	.00
3	6.67	.13	3.52	.01
4	6.71	.17	3.52	.01
5	6.78	.22	3.53	.02
market	6.60	.06	3.52	.00
$\theta = 5$				
1	6.11	-.27	3.45	.01
2	6.42	.03	3.43	.00
3	7.01	.61	3.40	-.04
4	7.18	.79	3.41	-.03
5	7.43	1.03	3.40	-.04
market	6.69	.30	3.43	-.01
$\theta = 20$				
1	4.74	-1.07	3.23	.07
2	5.94	0.13	3.16	.00
3	8.25	2.44	2.97	-.20
4	8.95	3.14	2.98	-.19
5	9.90	4.09	2.92	-.24
market	7.00	1.19	3.09	-.07

Table 3: The excess returns are measured relative to the return on a long horizon discount bond. The derivative entries in columns four and five are computed with respect to ρ and evaluated at $\rho = 1$.

Value and Growth Decompositions for Two Portfolios

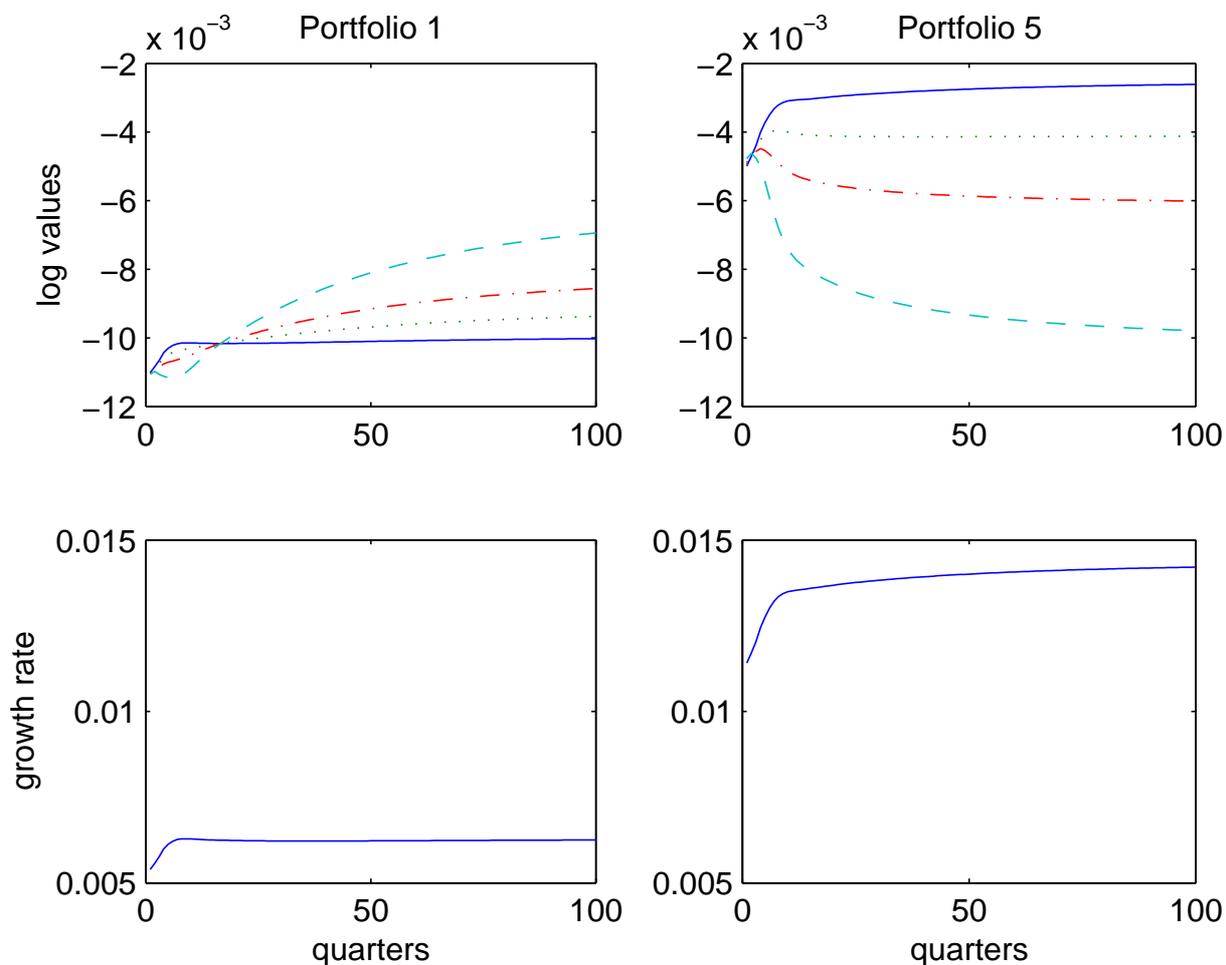


Figure 4: In the top two panels, the $_$ curve is computed using $\theta = 1$, the \dots curve assumes $\theta = 5$, the $-.$ curve assumes $\theta = 10$ and the $--$ curve assumes that $\theta = 20$.

butions. We form the expected excess rate of return by horizon by computing:

$$\frac{400}{\tau} [\log(\mathcal{P}^\tau \mathbf{1}) - \log(\mathcal{P}^\tau \psi) + \log(\mathcal{G}^\tau \psi)].$$

As τ gets arbitrarily large, the limits converge to the corresponding expected excess return limits given in tables 1 and 3. Figure 5 shows how these expected excess rates of returns change with θ and with modest movements in ρ for different values of τ . Consistent with our characterization of the limit points, small changes in ρ have little impact on this decomposition. We only consider values of ρ close to unity because the approximation we employ is local to $\rho = 1$. While expected rates of return for portfolio five and the market increase with horizon, those of portfolio one eventually decrease. The portfolio excess rates of return are more responsive to changes in θ than the market return, consistent with the limiting calculations in tables 1 and 3.

6.4 Statistical Accuracy

We consider sampling uncertainty in some of inputs used for long run risk. Recall that these inputs are based in part extrapolation of VAR systems fit to match transition dynamics. As in the related macroeconomics literature, we expect a substantial degree of sampling uncertainty. We now quantify how substantial this is for our application.

When $\rho = 1$, the expected excess returns are approximately equal to:

$$\theta \gamma(1) \cdot \pi.$$

We now investigate the statistical accuracy of $\gamma(1) \cdot \pi$ for the five portfolios, and for the difference between portfolios one and five. The vector π is measured using $\iota(1)$. In table 4 we report the approximate posterior distribution for $\gamma(1) \cdot \pi$ computed using an approach advocated by Sims and Zha (1999) and Zha (1999) based on Box-Tiao priors. While there is a considerable amount of statistical uncertainty in these risk measures, there are important differences the expected excess value returns between portfolios one and five.

Excess Return Decompositions by Horizon

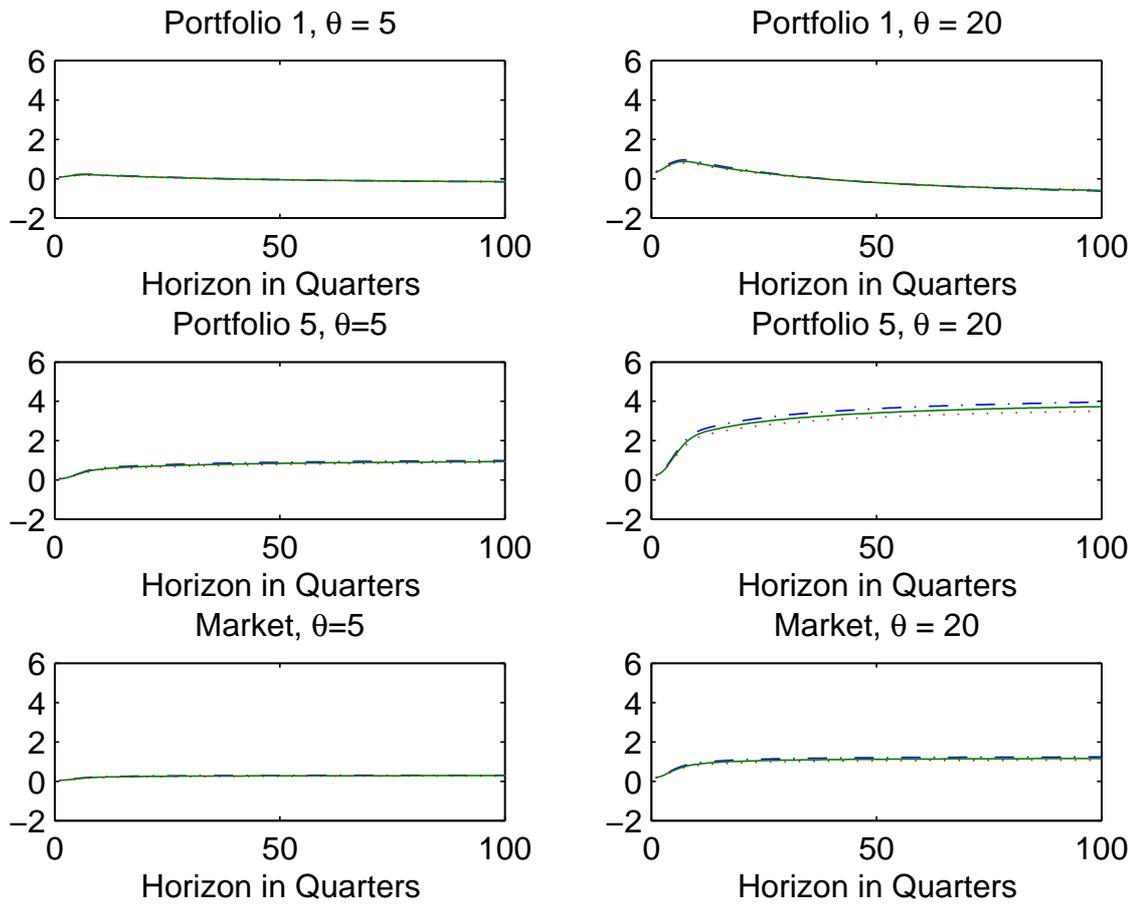


Figure 5: In the top two panels, the $_$ curves impose $\rho = 1$, the $\cdot\cdot$ curves impose $\rho = .5$ and the $-\cdot$ curves impose $\rho = 1.5$. The curves for $\rho \neq 1$ were computed using linear approximation around the point $\rho = 1$.

Accuracy of Risk Measures

Portfolio	Quantile				
	.05	.25	.5	.75	.95
1	-.63	-.13	-.05	-.01	.10
2	-.19	-.03	.01	.04	.22
3	.01	.06	.12	.28	1.42
4	.04	.10	.17	.32	1.46
5	.04	.12	.21	.42	1.88
market	-.01	.03	.06	.12	.58
5-1	.05	.15	.27	.55	2.41

Table 4: Quantiles were computed by simulating 100,000 times using Box-Tiao priors. The quantiles were computed using only simulation draws for which the absolute values of the eigenvalues were all less than .999. The fraction of accepted draws ranged from .986 to .987. The quantiles were computed using VAR's that included consumption, corporate earnings and a single dividend series with one exception. To compute quantiles for the 5 – 1 row, dividends for both portfolios were included in the VAR.

7 Alternative Models of Cash Flow Growth

Our calculations so far have been based on one model of cash flow growth. We now explore some alternative specifications used in other research and check for sensitivity. All of these specifications allow for the dividends from financial portfolios to have distinct growth components from consumption. The evidence for growth differences has been documented in a variety of different places and is evident in figure 6. This figure give the time series trajectories of the logarithms of portfolio dividends relative to aggregate consumption. Notice that the first three portfolios appear to grow slower than consumption, and even market dividends display this same pattern. In contrast, portfolios four and five show more pronounced growth than consumption.

7.1 Dividend Dynamics

In the previous section, we identified dividend dynamics and, in particular, the martingale component $\iota(1)$ using VAR methods. We used a VAR with three variables: consumption, corporate earnings and dividends (all in logarithms). Consumption and earnings were restricted to have the same long-run response to permanent shocks. We now consider two alternative specifications of dividend growth to assess sensitivity to model specification.

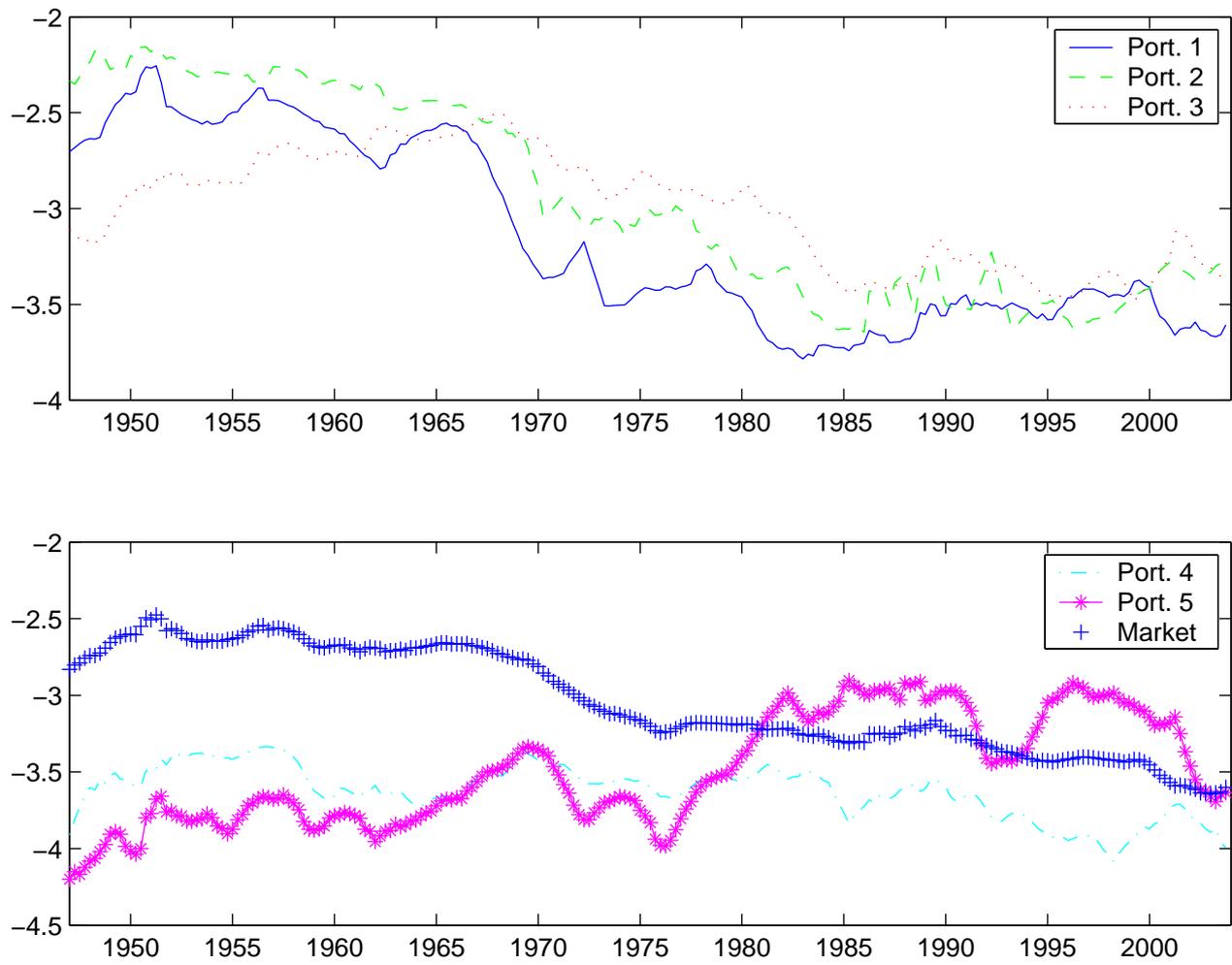


Figure 6: *Log of Ratios of Portfolio Dividends to Consumption*

Both are restrictions on the equation:

$$A_0^* y_t^* + A_1^* y_{t-1} + A_2^* y_{t-2} + \dots + A_\ell^* y_{t-\ell} + B_0^* + B_1^* t = w_t^*,$$

where the shock w_t^* is scalar with mean zero and unit variance and uncorrelated with the shock vector w_t that enters (19). The third entry of A_0^* is normalized to be positive. As in our previous estimation, we set $\ell = 5$.

7.2 Cointegration

The first specification restricts that the trend coefficient B_1^* equal zero, and is the model used by Hansen, Heaton, and Li (2005). Given our interest in measuring long-run risk, we measure the permanent response of dividends to the permanent shock. While both consumption and corporate earnings are restricted to respond to permanent shocks in the same manner, the dividend response is left unconstrained. We let λ^* denote the ratio of the long-run dividend response to the long-run consumption response. We measure this for each of the five portfolios. In this case we allow the matrix:

$$\begin{bmatrix} A(1) & 0 \\ A^*(1) \end{bmatrix}$$

to have rank two where

$$A^*(z) \doteq \sum_{j=0}^{\ell} A_j^* z^j.$$

The cointegrating vector $(1, 1, \lambda^*)$ is in the null space of this rank two matrix. For this model, the vector π is

$$\pi = \iota(1) = \lambda^* \gamma(1) \tag{21}$$

and $\zeta = \mu_d = \lambda^* \mu_c$.

The second specification includes a time trend by freely estimating B_1^* . A model like this, but without corporate earnings, was used by Bansal, Dittmar, and Lundblad (2005). We refer to this as the time trend specification. In this model the time trend introduces a second source of dividend growth. While π is constructed as in model (21), $\mu_d = \zeta$ is now left unrestricted.

The role of specification uncertainty is illustrated in the impulse response figure 7. This figure features the responses of portfolio one and five to a permanent shock. For each portfolio, the measured responses obtained for each of the three growth configurations are quite close up to about three to four years and then they diverge. Both portfolios initially respond positively to this shock with peak responses occurring in about seven time periods. The response of portfolio one is much larger in this initial phase. The limiting responses differ substantially depending on the growth configuration that is imposed in estimation. The estimated response of portfolio one is eventually negative when time trends are included or an additional stochastic growth factor is included. The time trend model leads to lower

limits for both portfolios. It is interesting, however, that the long-run differential responses between portfolio one and five are approximately the same for the time trend model and the dividend growth model.¹⁵

To better understand the importance of alternative growth configurations, figure 8 plots both the level of dividends for portfolios one and five and the fitted values implied by the “aggregate” innovations to consumption and corporate earnings alone. Results are reported for all three growth configurations. The presence of a deterministic trend in a log levels specification allows the VAR model to fit the low frequency movements of dividends for portfolio 1 much better than either of the other two models.¹⁶ In contrast the fitted values are quite similar across growth configurations for portfolio 5.

¹⁵Bansal, Dittmar, and Lundblad (2005) use their estimates with a time trend model as inputs into a cross sectional return regression. While estimation accuracy and specification sensitivity may challenge these regressions, the consistency of the ranking across methods is arguably good news, as emphasized to us by Ravi Bansal. As is clear from our previous analysis, we are using the economic model in a more formal way than the running of cross-sectional regressions.

¹⁶Results for portfolio 2 are very similar to those for portfolio 1.

Impulse Response Functions for Two Portfolios

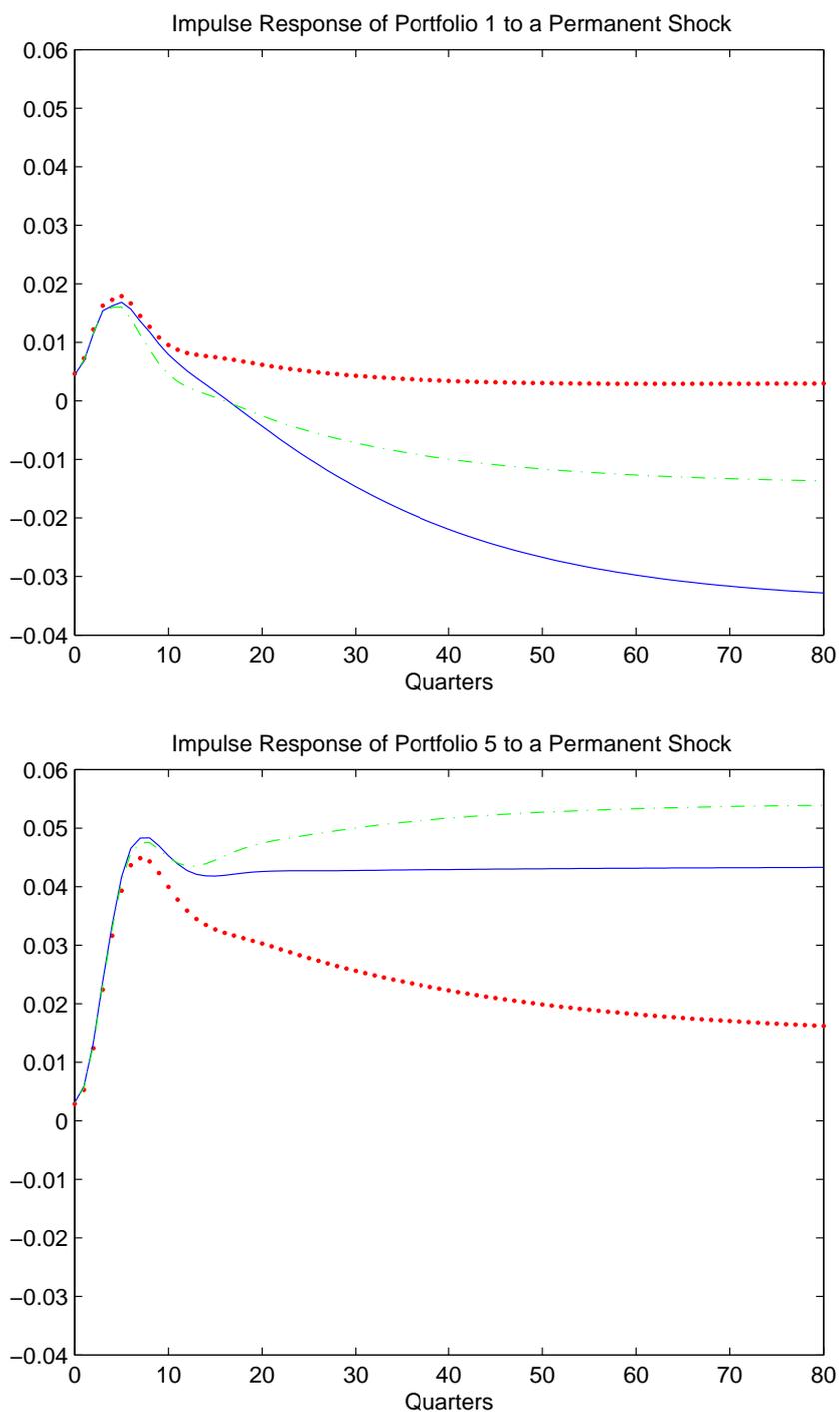


Figure 7: The \cdots curve is generated from the level specification for dividends; the --- is generated from the level specification with time trends included; and the -- -- curve is generated from the first difference specification.

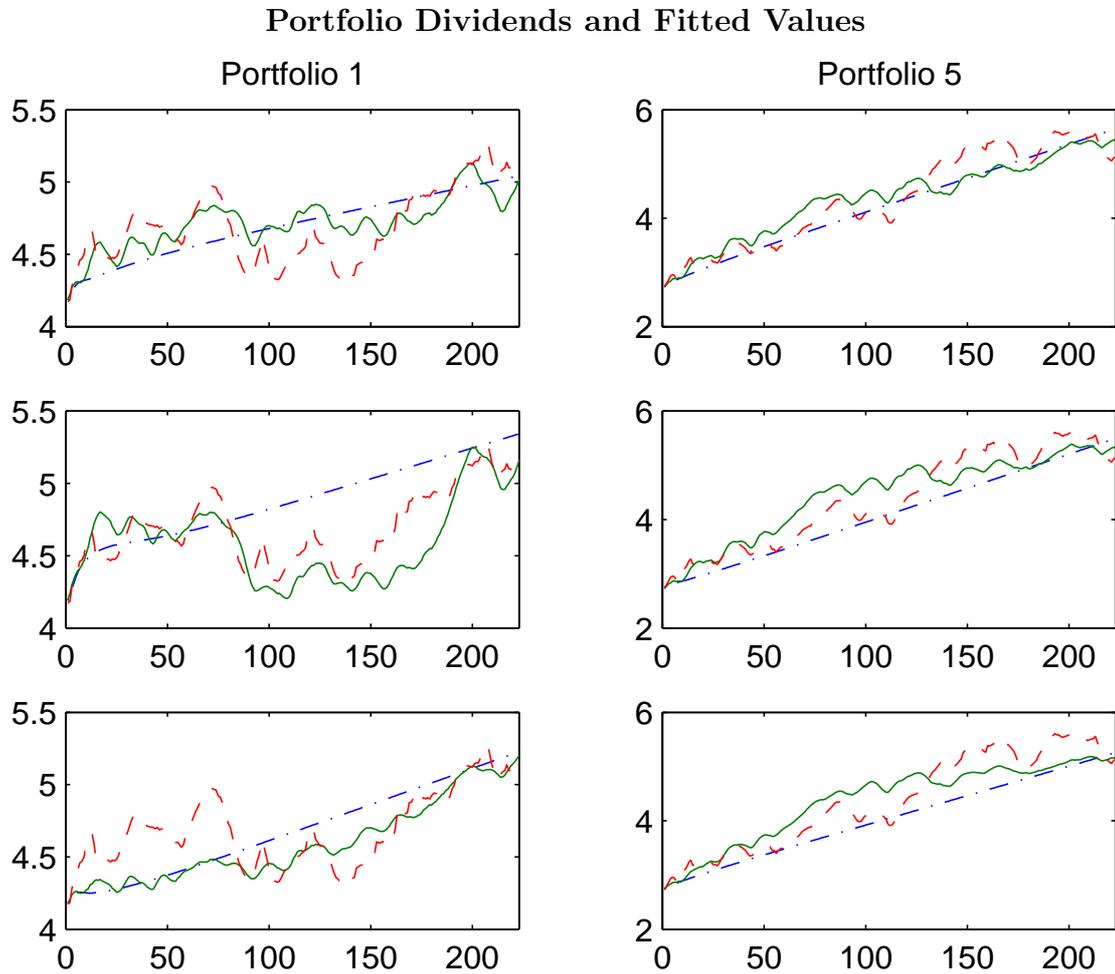


Figure 8: Dashed lines $--$ display the data. Solid lines $—$ are the fitted values based on consumption shocks alone. Dot-dashed lines $-·-$ are fitted values with all shocks set to zero. Row one gives results for the cointegrated model without time trends, row two for the cointegrated model with time trends, and row three for the model in which an additional unit root is imposed on the dividend evolution.

Up until now, we have taken the linear cointegration model with time trends literally. Is it realistic to think of these as deterministic time trends in studying the economic components of long-run risk? We suspect not. While there may be important components to the cash flows for portfolios 1 and 2 that are very persistent, it seems unlikely that these are literally deterministic time trends known to investors. Within the statistical model, the time trends for these portfolios in part offset the negative growth induced by the cointegration. We suspect that the substantially negative estimates of λ^* probably are not likely to be the true limiting measures of how dividends respond to consumption and earnings shocks. While the long-run risks associated with portfolios one and two look very different from that of portfolio five, a literal interpretation of the resulting cointegrating relation is hard to defend.

There is a potential pitfall in estimation methods that conditioned on initial data points as we have here. Sims (1991) and Sims (1996) warn against the use of such methods because the resulting estimates might over fit the the initial time series, ascribing it to a transient component far from the trend line. As Sims argues,

... that the estimated model implies that future deviations as great as the initial deviation will be extremely rare.

This impact is evident for portfolio one as seen in figure 8. This figure includes trajectories simulated from the initial conditions alone. When the time trend is included, the *deterministic* simulation tracks well the actual dividend data for the first few years. There is sharp upward movement in the initial phase of this deterministic simulation when a time trend is included in the dividend evolution. The increase is much more muted when time trends are excluded.¹⁷ In contrast, this phenomenon is not present in deterministic simulation for portfolio 5. Instead the deterministic trajectory is very similar across the three time series models.

In summary, while there is intriguing heterogeneity in the long run cash flow responses and implied returns, the implied risk measures are sensitive to the growth specification as is the case in the related macroeconomics literature. Given the observed cash flow growth, it is important to allow for low frequency departures from a balanced growth restriction. The simple cointegration model introduces only one free growth parameter for each portfolio, but results in a modest amount of return heterogeneity. The time trend growth models impose additional sources of growth. The added flexibility of the time trend specification may presume too much investor confidence in a deterministic growth component, however. The dividend growth specification that we used in our previous calculations, while *ad hoc*, presumes this additional growth component is stochastic and is a more appealing specification to us.

7.3 Adding Price Information

In the specifications we have considered so far, we have ignored any information for forecasting future consumption that might be contained in asset prices. Our model of asset pricing

¹⁷Again portfolio 2 behaves similarly to portfolio 1.

implies a strict relationship between cash flow dynamics and prices so that price information should be redundant. Prices, however, may reveal additional components to the information set of investor and hence a long-run consumption risk that cannot be identified from cash flows. For these reasons we consider an alternative specification of the VAR where we include consumption, corporate earnings, dividends as well as prices.

Parker and Julliard (2004) argue that it is the differential ability of the returns to growth and value portfolios in forecasting future consumption that is an important feature in the data. We therefore include dividends and prices for portfolios one and five simultaneously in this analysis. We continue to impose a unit root in consumption and the restriction that consumption and corporate earnings are cointegrated. We allow each dividend series to have its own stochastic growth path, but the prices of each portfolio are assumed to be cointegrated with their corresponded dividends. Finally, to assess the ability of portfolio prices to forecast future consumption we relax the assumption that consumption and corporate earnings are not Granger caused by portfolio cash flows or prices.

Figure 9 reports results for excess returns by horizon as in figure 4. The general character of the results are not changed. For large values of θ the model predicts substantial differences between portfolio excess returns at long-horizon. The exact patterns are different when prices are included, however. For example the excess returns to portfolio one, when $\theta = 20$, are larger at long horizons. Further there is more sensitivity to the parameter ρ when $\theta = 20$.

Excess Return Decompositions by Horizon

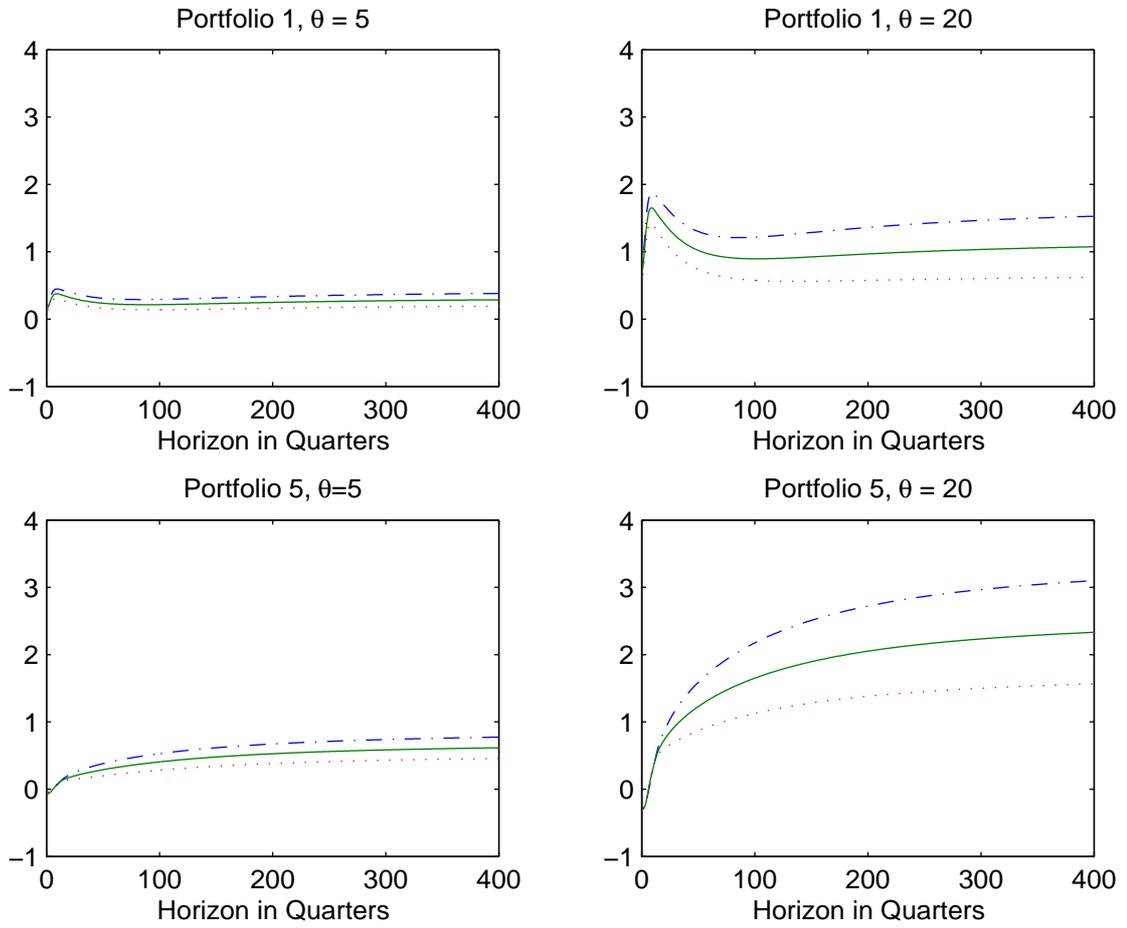


Figure 9: In the top two panels, the $_$ curves impose $\rho = 1$, the $\cdot\cdot$ curves impose $\rho = .5$ and the $-.$ curves impose $\rho = 1.5$. The curves for $\rho \neq 1$ were computed using linear approximation around the point $\rho = 1$.

8 Conclusion

Growth-rate variation in consumption and cash flows have important consequences for asset valuation. The methods on display in this paper formalize the long-run contribution to value of the stochastic components of discount factors and cash flows and quantify the importance of macroeconomic risk. We used these methods to isolate features of the economic environment that have important consequences for long-run valuation and heterogeneity across cash flows. We made operational a well defined notion of long-run valuation risk, and we studied the measurement accuracy of the inputs needed to characterize the implied risk-return tradeoff.

The recursive utility model features the intertemporal composition of risk. We used the joint evolution of consumption and corporate earnings to identify long-run risk components similar to those featured by Bansal and Yaron (2004). While the long-run risk component shares measurement challenges familiar from the macroeconomics VAR literature, we showed formally how these shocks are transmitted into risk-adjusted asset values by deriving a long-run tradeoff for macroeconomic growth rate risk. We then used portfolios constructed as in the asset pricing literature to investigate how financial cash flows relate to underlying macroeconomic growth rate risk. We found that the stochastic growth of low book-to-market portfolios has negligible covariation with consumption while the growth of high book-to-market portfolios has positive covariation. For these differences to be important quantitatively in our long-run risk-return calculations, investors must be either highly risk averse or highly uncertain about the probability models that they confront.¹⁸

In this paper we used an *ad hoc* VAR model to identify shocks. In contrast to VAR methods, an explicit valuation model is a necessary ingredient for our analysis; and thus we analyzed the valuation implications through the lens of a commonly used consumption-based model. There are important reasons for extending the scope of our analysis in future work, and the methods we described here are amenable to such extensions. One next step is to add more structure to the macroeconomic model, structure that will sharpen our interpretation of the sources of long-run macroeconomic risk. While the recursive utility model used in this paper has a simple and usable characterization of how temporal dependence in consumption growth alters risk premia, other economic models have interesting implications for the intertemporal composition of risk, including models that feature habit persistence (*e.g.* Constantinides (1990), Heaton (1995), and Sundaresan (1989)) and models of staggered decision-making (*e.g.* see Lynch (1996) and Gabaix and Laibson (2002).)

The model we explore here focuses exclusively on time variation in conditional means. Temporal dependence in volatility can be an additional source of long-run risk. Time variation in risk premia can be induced by conditional volatility in stochastic discount factors.¹⁹ While the direct evidence from consumption data for time varying volatility is modest, the implied evidence from asset pricing for conditional volatility in stochastic discount fac-

¹⁸This latter conclusion can be made precise by using detection probabilities in the manner suggested by Anderson, Hansen, and Sargent (2003).

¹⁹It can also be induced by time variation in risk exposure.

tors is intriguing. For instance, Campbell and Cochrane (1999) and others argue that risk prices vary over the business cycle in ways that are quantitatively important. Lettau and Wachter (2005) explore a long-run risk characterization that features the consequences of heteroskedasticity using an *ad hoc* but arguably flexible stochastic discount factor model.²⁰

While the methods we have proposed aid in our understanding of asset-pricing models, they also expose measurement challenges in quantifying the long-run risk-return tradeoff. Important inputs into our calculations are the long-run riskiness of cash flows and consumption. As we have shown, these objects are hard to measure in practice. Statistical methods typically rely on extrapolating the time series model to infer how cash flows respond in the long-run to shocks. This extrapolation depends on details of the growth configuration of the model, and in many cases these details are hard to defend on purely statistical grounds. While volatility can induce an additional source of risk, it also poses the additional challenge of how to measure this volatility in a flexible way that can be extrapolated reliably. Also there is pervasive statistical evidence for growth rate changes or breaks in trend lines, but this statistical evidence is difficult to use directly in models of decision-making under uncertainty without some rather specific ancillary assumptions about investor beliefs. Many of the statistical challenges that plague econometricians presumably also plague market participants. Naive application of rational expectations equilibrium concepts may endow investors in these models with too much knowledge about future growth prospects.

There are two complementary responses to the measurement and modeling conundrums. One is to resort to the use of highly structured, but easily interpretable, models of long-run growth variation. The other is to exploit the fact that asset values encode information about long-run growth. To break this code requires a reliable economic model of the long-run risk-return relation. While we explored one model-based method for extracting economic characterizations of this relation, we had to use a high risk aversion parameter to produce heterogeneity in the dominant valuation components of portfolio cash flows. Unfortunately, as of yet there is not an empirically well-grounded, and economically relevant model of asset pricing to use in deducing investor beliefs about the long-run from values of long-lived assets. There remain important challenges in modeling investor sentiments about shocks to long-run macroeconomic growth.

²⁰Lettau and Wachter (2005) challenge the long-run implications of the Campbell and Cochrane (1999) formulation.

A ρ Approximation

In this appendix we give the first order approximation in ρ for our economy following in part Kogan and Uppal (2001). The economy we study is different from that Kogan and Uppal (2001), but they suggest that extensions in the directions that interest us would be fruitful. While Campbell and Viceira (2002) (chapter 5) show the close connection between approximation around the utility parameter $\rho = 1$ and approximation around a constant consumption-wealth ratio for portfolio problems, there are some interesting differences in our application. Moreover, $\rho = 1$ is ruled out in the parameterization of recursive utility considered by Restoy and Weil (1998) and others because of their use of the return-based Euler equation.

A.1 Continuation values

We compute the first-order expansion:

$$v_t \approx v_t^1 + (\rho - 1)Dv_t^1$$

where v_t^1 is the continuation value for the case in which $\rho = 1$. We construct an appropriate recursion for Dv_t^1 by expanding the logarithm and exponential functions in (14) and including up to second-order terms in \mathcal{Q}_t . The approximate recursion is:

$$v_t \approx \beta \left[\mathcal{Q}_t(v_{t+1} + c_{t+1} - c_t) + (1 - \rho)(1 - \beta) \frac{\mathcal{Q}_t(v_{t+1} + c_{t+1} - c_t)^2}{2} \right].$$

Then

$$v_t^1 = \beta \mathcal{Q}_t(v_{t+1}^1 + c_{t+1} - c_t),$$

which is the $\rho = 1$ exact recursion and

$$\begin{aligned} Dv_t^1 &= -\beta(1 - \beta) \frac{\mathcal{Q}_t(v_{t+1}^1 + c_{t+1} - c_t)^2}{2} + \beta \tilde{E}(Dv_{t+1}^1 | \mathcal{F}_t) \\ &= -\frac{(1 - \beta)(v_t^1)^2}{2\beta} + \beta \tilde{E}(Dv_{t+1}^1 | \mathcal{F}_t) \end{aligned} \tag{22}$$

where \tilde{E} is the distorted expectation operator associated with the density

$$\frac{(V_{t+1}^1)^{1-\theta}}{E \left[(V_{t+1}^1)^{1-\theta} | \mathcal{F}_t \right]}.$$

For the log-normal model of consumption, this distorted expectation appends a mean to the shock vector w_{t+1} . The distorted distribution of w_{t+1} remains normal, but instead of mean zero, it has a risk adjusted mean of $(1 - \theta)\gamma(\beta)$. The derivative Dv_t^1 is negative because it is the (distorted) expectation of the sum of negative random variables.

Recall that the evolution in the log linear Markov economy is

$$\begin{aligned}x_{t+1} &= Gx_t + Hw_{t+1} \\c_{t+1} - c_t &= \mu_c + U_c z_t + \gamma_0 w_{t+1}.\end{aligned}$$

We showed in the text that

$$v_t^1 = U_v x_t + \mu_v$$

where $U_v =$ and $\mu_v =$. Thus

$$(v_t^1)^2 = (U_v x_t)' U_v x_t + 2\mu_v U_v x_t + (\mu_v)^2.$$

Then from (22),

$$Dv_t^1 = -\frac{1}{2}x_t' \Upsilon_{dv} x_t + U_{dv} x_t + \mu_{dv}$$

where

$$\begin{aligned}\Upsilon_{dv} &= \frac{(1-\beta)}{\beta} U_v' U_v + \beta G' \Upsilon_{dv} G \\U_{dv} &= -\frac{(1-\beta)}{\beta} \mu_v U_v - \beta(1-\theta)\gamma(\beta) H' \Upsilon_{dv} G + \beta U_{dv} G \\ \mu_{dv} &= -\frac{(1-\beta)}{2\beta} (\mu_v)^2 - \frac{\beta(1-\theta)^2}{2} \gamma(\beta) H' \Upsilon_{dv} H \gamma(\beta)' \\ &\quad + \beta(1-\theta) U_{dv} H \gamma(\beta)' - \frac{\beta}{2} \text{trace}(H' \Upsilon_{dv} H) + \beta \mu_{dv}.\end{aligned}\tag{23}$$

The first equation in (23) is a Sylvester equation and is easily solved. Given Υ_{dv} , the solution for U_{dv} is:

$$U_{dv} = -(I - \beta G')^{-1} \left[\frac{1-\beta}{\beta} \mu_v U_v + \frac{\beta(1-\theta)}{2} G' \Upsilon_{dv} H \gamma(\beta)' \right],$$

and given Υ_{dv} and U_{dv} the solution for μ_{dv} is:

$$\mu_{dv} = \frac{-\frac{(1-\beta)}{2\beta} (\mu_v)^2 - \frac{\beta(1-\theta)^2}{2} \gamma(\beta) H' \Upsilon_{dv} H \gamma(\beta)' + \beta(1-\theta) U_{dv} H \gamma(\beta)' - \frac{\beta}{2} \text{trace}(H' \Upsilon_{dv} H)}{1-\beta}$$

A.2 Wealth expansion

When ρ is different from one, the wealth-consumption ratio is not constant. A first-order expansion of the continuation value implies a second-order expansion of the consumption-wealth ratio. This can be seen directly from (17):

$$\begin{aligned}\log W_t - \log C_t &= -\log(1-\beta) + (1-\rho) [v_t^1 + (\rho-1) Dv_t^1] \\ &= -\log(1-\beta) - (\rho-1)v_t^1 - (\rho-1)^2 Dv_t^1.\end{aligned}$$

The term v_t^1 is very similar (but not identical to) the term typically used when taking log-linear approximations.²¹ By construction, the second-order term adjusts the wealth consumption ratio in a manner that is symmetric about $\rho = 1$ and it is positive.

²¹In log-linear approximation the discount rate in this approximation is linked to the mean of the wealth consumption ratio. In the ρ expansion, the subjective rate of discount is used instead.

A.3 Stochastic discount factor

Finally, consider the first-order expansion of the logarithm of the stochastic discount factor:

$$s_{t+1,t} \approx s_{t+1,t}^1 + (\rho - 1)Ds_{t+1,t}^1.$$

Recall that the log discount factor is given by:

$$\begin{aligned} s_{t+1,t} &= -\delta - \rho(c_{t+1} - c_t) + (\rho - \theta)[v_{t+1} + c_{t+1} - \mathcal{Q}_t(v_{t+1} + c_{t+1})] \\ &= -\delta - \rho(c_{t+1} - c_t) + (\rho - \theta)[v_{t+1} + c_{t+1} - c_t - \mathcal{Q}_t(v_{t+1} + c_{t+1} - c_t)] \end{aligned}$$

Differentiating with respect to ρ gives:

$$\begin{aligned} Ds_{t+1,t}^1 &= -(c_{t+1} - c_t) + [v_{t+1}^1 + c_{t+1} - c_t - \mathcal{Q}_t(v_{t+1}^1 + c_{t+1} - c_t)] \\ &\quad + (1 - \theta) [Dv_{t+1}^1 - \tilde{E}(Dv_{t+1}^1 | \mathcal{F}_t)] \\ &= v_{t+1}^1 - \frac{1}{\beta}v_t^1 + (1 - \theta) [Dv_{t+1}^1 - \tilde{E}(Dv_{t+1}^1 | \mathcal{F}_t)]. \end{aligned}$$

Note that

$$\begin{aligned} v_{t+1}^1 - \frac{1}{\beta}v_t^1 &= U_v x_{t+1} - \frac{1}{\beta}U_v x_t + \left(1 - \frac{1}{\beta}\right)\mu_v \\ &= U_v \left(G - \frac{1}{\beta}I\right)x_t + \left(1 - \frac{1}{\beta}\right)\mu_v + U_v H w_{t+1}. \end{aligned}$$

and

$$\begin{aligned} Dv_{t+1}^1 - \tilde{E}(Dv_{t+1}^1 | \mathcal{F}_t) &= -\frac{1}{2}(H w_{t+1})' \Upsilon_{dv} H w_{t+1} - (H w_{t+1})' [\Upsilon_{dv} G x_t - U_{dv}] \\ &\quad + \frac{1}{2}(1 - \theta)^2 \gamma(\beta) H' \Upsilon_{dv} H \gamma(\beta)' + (1 - \theta) \gamma(\beta) H' [\Upsilon_{dv} G x_t - U_{dv}] \\ &\quad + \frac{1}{2} \text{trace}(H' \Upsilon_{dv} H) \end{aligned}$$

Combining these expressions we obtain:

$$Ds_{t+1,t}^1 = \frac{1}{2}w_{t+1}'\Theta_0 w_{t+1} + w_{t+1}'\Theta_1 x_t + \vartheta_0 + \vartheta_1 x_t + \vartheta_2 w_{t+1}$$

where

$$\begin{aligned} \Theta_0 &= (\theta - 1)H' \Upsilon_{dv} H \\ \Theta_1 &= (\theta - 1)H' \Upsilon_{dv} G \\ \vartheta_0 &= \left(1 - \frac{1}{\beta}\right)\mu_v + \frac{1}{2}(1 - \theta)^3 \gamma(\beta) H' \Upsilon_{dv} H \gamma(\beta)' - (1 - \theta)^2 \gamma(\beta) H' U_{dv}' + \frac{1 - \theta}{2} \text{trace}(H' \Upsilon_{dv} H) \\ \vartheta_1 &= U_v \left(G - \frac{1}{\beta}I\right) + (\theta - 1)^2 \gamma(\beta) H' \Upsilon_{dv} G \end{aligned}$$

$$\vartheta_2 = (1 - \theta)U_{dv}H + U_vH$$

The mean under the risk neutral measure for w_{t+1} is

$$[I + (\rho - 1)(\theta - 1)H'\Upsilon_{dv}H]^{-1} [-\rho\gamma(0) + (\rho - \theta)\gamma(\beta) + (\rho - 1)(\theta - 1)(U_{dv} - H'\Upsilon_{dv}Gx_t)].$$

This mean can be interpreted as the negative of a risk premia. A component of this mean is the undiscounted (by the risk free rate) price an investor is willing to pay for contingent claim to the corresponding component of the shock w_{t+1} . In a continuous time approximation, this formula simplifies to:

$$-\rho\gamma(0) + (\rho - \theta)\gamma(\beta) + (\rho - 1)(\theta - 1)(U_{dv} - H'\Upsilon_{dv}Gx_t).$$

B Eigenfunction results

B.1 Eigenfunction for the adjoint operator

Guess an eigenfunction of the form:

$$\log \varphi(x) = -\omega^*x$$

then this eigenfunction should satisfy the equation:

$$E[\exp[s_{t+1,t} + \pi w_{t+1}]\varphi(x_t)|x_{t+1}] = \exp(-\nu^*)\varphi(x_t).$$

where $\exp(-\nu^*)$ is the eigenvalue associated with eigenfunction $\psi^*(x_t)$, and we will show later ν and ν^* are the same.

First compute the reverse time evolution of x_t ,

$$x_t = G^*x_{t+1} + H^*w_t^*.$$

where w_t^* is a multivariate standard normal, independent of x_{t+1} .

The matrix G^* can be inferred by standard least squares formulas:

$$\Sigma G'\Sigma^{-1} = G^*.$$

and the matrix H^* can be inferred by factoring:

$$\Sigma - G^*\Sigma G^{*'}.$$

where Σ is the unconditional variance-covariance matrix of x_t .

Write:

$$w_{t+1} = (H'H)^{-1}H'(x_{t+1} - Gx_t) = (H'H)^{-1}H'[(I - GG^*)x_{t+1} - GH^*w_t^*].$$

Thus:

$$s_{t+1,t} + \pi w_{t+1} = \xi_0^* + \xi_1^* x_{t+1} + \xi_2^* w_t^*$$

for

$$\begin{aligned}\xi_0^* &= \mu_s \\ \xi_1^* &= U_s G^* + \xi_0 (H' H)^{-1} H' (I - G G^*) \\ \xi_2^* &= -\xi_0 (H' H)^{-1} H' G H^* + U_s H^*.\end{aligned}$$

Then the adjoint problem solves:

$$E(\exp[\xi_0^* + \xi_1^* x_{t+1} + \xi_2^* w_t^* - \omega^* (G^* x_{t+1} + H^* w_t^*)] | x_{t+1}) = \exp(-\nu^*) \exp(-\omega^* x_{t+1})$$

This problem has the same formal structure as the initial eigenvector problem. The solution is

$$\omega^* = \xi_1^* (G^* - I)^{-1} = U_c (I - G^*)^{-1}.$$

The negative logarithm of the eigenvalue is

$$\nu^* = -\xi_0^* - \frac{|\xi_2^* - \omega^* H^*|^2}{2},$$

and it can be easily shown that ν and ν^* are the same.

B.2 Eigenvalue derivative

Let q denote the stationary density for x_t . This vector is normally distributed with mean zero and covariance matrix:

$$\Sigma = \sum_{j=0}^{\infty} (G^j) H H' (G^j)',$$

which can be computed easily using a doubling algorithm.

We use the relation:

$$\exp(-\nu) = \frac{E[\exp(s_{t+1,t} + \pi w_{t+1}) \phi(x_{t+1}) \varphi(x_t)]}{E[\phi(x_t) \varphi(x_t)]}.$$

Write

$$D s_{t+1,t}^1 = \frac{1}{2} w_{t+1}' \Theta_0 w_{t+1} + w_{t+1}' \Theta_1 x_t + \vartheta_0 + \vartheta_1 x_t + \vartheta_2 w_{t+1}.$$

Then

$$\frac{d \exp(-\nu)}{d \rho} \Big|_{\rho=1} = \frac{E[D s_{t+1,t}^1 \exp(s_{t+1,t}^1 + \pi w_{t+1}) \phi(x_{t+1}) \varphi(x_t)]}{E[\phi(x_t) \varphi(x_t)]},$$

and hence

$$\frac{d \nu}{d \rho} \Big|_{\rho=1} = - \frac{E[D s_{t+1,t}^1 \exp(s_{t+1,t}^1 + \pi w_{t+1}) \phi(x_{t+1}) \varphi(x_t)]}{\exp(-\nu) E[\phi(x_t) \varphi(x_t)]}. \quad (24)$$

We take three steps to compute this eigenvalue derivative

B.2.1 Step one: computing the denominator

We must compute:

$$E[\phi(x_t)\varphi(x_t)] = \int \exp[-(\omega + \omega^*)x]q(x)dx$$

From the lognormal formula, this is

$$\exp\left[\frac{(\omega + \omega^*)\Sigma(\omega + \omega^*)'}{2}\right].$$

B.2.2 Step two: computing the numerator

We have already evaluated the denominator, but it remains to compute the numerator:

$$E\left[Ds_{t+1,t}^1 \exp(s_{t+1,t}^1 + \pi w_{t+1}) \phi(x_{t+1})\varphi(x_t)\right]$$

We do so by applying the Law of Iterated Expectations, and first computing:

$$E\left[Ds_{t+1,t}^1 \exp(s_{t+1,t}^1 + \pi w_{t+1}) \phi(x_{t+1})\varphi(x_t)|x_t\right].$$

Note that

$$\begin{aligned} & s_{t+1,t}^1 + \pi w_{t+1} + \log[\phi(x_{t+1})] + \log[\varphi(x_t)] \\ = & \xi_0 + (\xi_1 - \omega G - \omega^*)x_t + (\xi_2 - \omega H)w_{t+1} \\ = & \left[\xi_0 + \frac{|\xi_2 - \omega H|^2}{2} - (\omega + \omega^*)x_t\right] + \left[(\xi_2 - \omega H)w_{t+1} - \frac{|\xi_2 - \omega H|^2}{2}\right]. \end{aligned}$$

We use the second term in the square brackets to change the shock distribution. In particular, we change the mean of w_{t+1} from zero to $[(\xi_2 - \omega H)]$. Thus

$$\begin{aligned} & E\left[Ds_{t+1,t}^1 \exp(s_{t+1,t}^1 + \pi w_{t+1}) \phi(x_{t+1})\varphi(x_t)|x_t\right] \\ = & \exp\left[\xi_0 + \frac{|\xi_2 - \omega H|^2}{2} - (\omega + \omega^*)x_t\right] \\ & \times \left[\frac{1}{2}\text{trace}(\Theta_0) + (\xi_2 - \omega H)' \Theta_1 x_t + \vartheta_0 + \vartheta_1 x_t + \vartheta_2(\xi_2 - \omega H)\right] \\ = & \exp(-\nu) \exp[-(\omega + \omega^*)x_t] \\ & \times [(1/2)\text{trace}(\Theta_0) + (1/2)(\xi_2 - \omega H)' \Theta_0 (\xi_2 - \omega H) \\ & + (\xi_2 - \omega H)' \Theta_1 x_t + \vartheta_0 + \vartheta_1 x_t + \vartheta_2(\xi_2 - \omega H)] \end{aligned}$$

Next we compute the unconditional expectation. Again we change probability distributions. To simplify the calculation, we adopt a change in measure. We change the mean of x_t from normal mean zero and covariance matrix Σ to normal with mean

$$\mu_x^* \doteq -\Sigma(\omega + \omega^*)'$$

and covariance Σ . Using this transformation we find that

$$\begin{aligned}
& E \left[D s_{t+1,t}^1 \exp \left(s_{t+1,t}^1 + \pi w_{t+1} \right) \phi(x_{t+1}) \varphi(x_t) | x_t \right] \\
= & \exp(-\nu) \exp \left(\frac{\mu_x^{*\prime} \Sigma^{-1} \mu_x^*}{2} \right) \\
& \times \left[\frac{1}{2} \text{trace}(\Theta_0) + \frac{1}{2} (\xi_2 - \omega H)' \Theta_0 (\xi_2 - \omega H) \right. \\
& \left. + (\xi_2 - \omega H)' \Theta_1 \mu_x^* + \vartheta_0 + \vartheta_1 \mu_x^* + \vartheta_2 (\xi_2 - \omega H) \right].
\end{aligned}$$

B.2.3 Step three: combining results

We compute the right-hand side of (24) by combining numerator and denominator terms:

$$\begin{aligned}
\frac{d\nu}{d\rho} \Big|_{\rho=1} = & -\frac{1}{2} \text{trace}(\Theta_0) - \frac{1}{2} (\xi_2 - \omega H)' \Theta_0 (\xi_2 - \omega H) \\
& - (\xi_2 - \omega H)' \Theta_1 \mu_x^* - \vartheta_0 - \vartheta_1 \mu_x^* - \vartheta_2 (\xi_2 - \omega H).
\end{aligned}$$

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