NBER WORKING PAPER SERIES

A TRANSACTIONS BASED MODEL OF THE MONETARY TRANSMISSION MECHANISM: PART 1

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Working Paper No. 973

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge MA 02138

September 1982

The research reported here is part of the NBER's research programs in Economic Fluctuations and in Financial Markets and Monetary Economics. Any opinions expressed are those of the authors and not those of the National Bureau of Economic Research.

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ABSTRACT

What are the effects of open market operations? How do these differ from money falling from heaven? We propose a new explanation of how open market operations can change real and nominal interest rates which emphasizes three often mentioned but seldom explicitly articulated features of actual monetary economies: i) going to the bank is costly so that people will tend to bunch cash withdrawals, ii) people don't all go to the bank simultaneously and, because of these, iii) at any instant of time agents hold different amounts of cash. We show that these considerations imply that an open market purchase of a bond for fiat money will drive down nominal and real interest rates, lead to a delayed positive price response, and have damped persistent effects on both prices and nominal interest rates if agents have logarithmic utility of consumption. We assume output is exogenous, so that the model can shed only indirect light on the relationship between money and aggregate output.

The model has emphasized how a change in the money supply affects the spending decision of those agents making withdrawals at the time of an open market operation. Considerations of intertemporal substitution imply that the real rate must decline to induce these agents to consume more. Because this new money is spent gradually, prices will rise slowly and reach their steady state level long after the interval of time between trips to the bank.

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July 1982

A Transactions Based Model of the Monetary Transmission Mechanism: Part I

by

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I. Introduction

What are the effects of open market operations? How do these differ from money falling from heaven? We propose a new explanation of how open market operations can change real and nominal interest rates which emphasizes three often mentioned but seldom explicitly articulated features of actual monetary economies i) going to the bank is costly so that people will tend to bunch cash withdrawals ii) people don't all go to the bank simultaneously and, because of these iii) at any instant of time agents hold different amounts of cash. We show that these considerations imply that an open market purchase of a bond for fiat money will drive down nominal and real interest rates, lead to a delayed positive price response, and have damped persistent effects on both prices and nominal interest rates if agents have logarithmic utility of consumption. We assume output is exogenous, so that the model can shed only indirect light on the relationship between money and aggregate output.

The model is a hybrid of the type suggested by Clower (1967) which assumes that agents require cash in advance of expenditures, and the partial equilibrium inventory theoretic models of Tobin (1956) and Baumol (1952) which stress that transaction costs necessitate that money withdrawals be periodic, so that it will not be optimal for agents to go to the bank at each instant.

* Research supported by NSF grant SES-8112036.

Formally, we assume that money withdrawals are staggered, so that a fixed fraction of the population makes a withdrawal each period. While it would certainly be preferable to make the time between trips to the bank an endogenous choice variable as in the Tobin-Baumol model, analysis of this consideration outside of steady states is too complicated, so that we choose this simpler formulation. (See Jovanovic (1982) for the analysis of the steady state time between trips to the bank in a model similar in structure to ours). As we hope will be clear, letting the time between trips to the bank adjust to changes brought about by open market operations would not substantially alter our conclusion regarding the non-neutrality of money.

Although the timing of withdrawals is fixed, the size of these withdrawals is endogenously determined. All income receipts are assumed to accrue as interest earning deposits, so that bank withdrawals are the only source of cash for consumers. The Clower cash-in-advance constraint implies that withdrawals for each agent equal his planned nominal spending over the ensuing time interval before his next withdrawal. Planned nominal spending is determined by the possibility of intertemporal consumption substitution and thus is influenced by expected prices and future nominal interest rates. The model is completely deterministic and we assume that those expectations are in fact realized; we attribute to agents perfect foresight.

We assume that only consumers hold stocks of fiat money. The consumers use money to buy goods from firms. We assume that firms deposit their cash receipts instantly into the interest bearing accounts of their various claimants and hold no money themselves. Similarly, under certainty it is difficult to see why banks would hold cash at positive interest rates; we assume they hold none. Under this formulation the money stock is held exclusively by consumers to finance spending before their next withdrawal. Equilibrium requires that the flow of cash into the bank at each period equal agents' desired withdrawals. The flow into the bank consists of the nominal value of firms' receipts (nominal GNP) plus any changes in nominal money introduced by open market operations.

An essential feature of our model is the fact that it is optimal for people to take time to run down their cash balances. Thus if there is to be a steady flow of money out of the bank, there will have to be a steady rate at which people run out of money. Thus the cross sectional distribution of money holdings at a given point in time cannot be degenerate. If everyone holds exactly the same amount of money then they will exhaust at the same day. On the dates when they don't exhaust there would be no one to hold the (non interest) bearing money which flows from the stores to the banks. Thus it is impossible to have everyone exhaust at the same time.

Under this formulation it is straightforward to see how an open market purchase differs from a transfer to each agent proportional to his existing nominal balances. When the money supply increases through an open market operation agents at the bank must be induced to hold the whole of the increase and thus a disproportionately larger share of the stock of money. This is because most of the people who are not at the bank (i.e., those people who have not yet exhausted their cash balances) will not find it optimal to go to the bank and withdraw extra cash, until they exhaust their current cash. Thus, the share of nominal spending attributable to agents at the bank must rise. To induce agents at the bank to withdraw and hence consume more, banks must lower the real and nominal interest rates. Since this new money is spent gradually over the interval before the next trip to the bank, the price level rises gradually through time, even though prices are completely flexible. This scenario contrasts with a proportional transfer which would raise all nominal prices by the same percentage and thus have no real effects.

The types of wealth redistribution associated with open market operations are novel features of the analysis. We emphasize that the new money withdrawal is financed by running down other asset holdings of equal nominal value; there is no direct benefit bestowed on the recipients of the new money. Rather, wealth is redistributed through two indirect channels. The first channel involves the asymmetry of existing nominal holdings of money. Since those agents not currently at the bank have more money than those at the bank (who have none), the inflation induced by money creation redistributes wealth from those not at the bank to those at the bank. The second channel is more subtle and focus on the redistribution arising from interest rate changes. We assume that all current period receipts accrue as interest earning deposits. Thus those agents not at the bank implicitly lend their current period income to those making withdrawals. A decline in interest rates redistributes wealth from creditors to debtors, which enhances the relative position of those making withdrawals when interest rates decline as a result of open market purchases.

The paper is organized as follows: Section 2 outlines the model. Section 3 discusses the properties of the steady state equilibrium. Section 4 considers the effects of both a one time proportional transfer and of an open market operation for the case of logarithmic utility. This assumption simplifies the analysis by making consumption independent of the real interest rate. (In Part 2, the results are extended to the case where future consumption is an increasing function of the real interest rate). The Fifth section presents conclusions and relates our model to other models. The Appendix outlines a continuous time version of our model.

2. Model

2A. The Flow of Money

We assume that there is a market where interest earning assets can be bought and sold in exchange for money (a non-interest bearing asset). We call this market a bank and assume that there is a fixed transaction cost of "going to the bank", i.e. of converting an interest bearing asset into money. We assume that there is another market, called "stores", where consumers buy goods with money. Goods can only be bought with money.

Given the fixed transactions cost of going to the bank, consumers will not find it optimal to go to the bank for each transaction. Instead they will go to the bank occasionally and withdraw a stock of money which they then use up over time by purchasing goods. We assume that consumers' expenditures and receipts are not perfectly synchronized. In order to generate a non-degenerate cross-sectional distribution of money holdings we want the nonsyncronization to imply that it is optimal for a consumer to have a stock of money which is a declining function of the length of time since his last trip to the bank. This characteristic is most easily developed in a model where all of the consumer's receipts arrive at the bank, so that the only source of cash consumers have is via withdrawals from the bank.

Since the "bank" is a shorthand for an asset market it makes no sense to imagine that the "bank" holds a stock of money when the nominal interest rate is positive. If some consumers are making withdrawals from the bank, then other consumers must be making deposits which exactly equal the withdrawals. We assume that the flow of money into the bank is being generated by the expenditures of consumers at the stores, and the stores instantaneously deposit their receipts at the bank. More precisely, we consider an economy where output is exogenous and consumers own the stores which own the economy's endowment of goods. As money comes into the stores, the stores deposit the money in consumers' bank accounts (i.e. they purchase interest bearing securities for their owners).

In the above description of the economy there is a flow of money from all consumers to stores, then to banks and then to consumers who have just run out of cash, so they are at the bank to make a withdrawal. We can mostly easily illustrate the circular flow of money in a discrete time example. Suppose that it is optimal for consumers to withdraw enough cash to last for exactly two "periods" of consumption. We assume that spending during a period can take place only out of cash held at the beginning of the period. At a given point in time some consumers have one period to go before they return to the bank to make a withdrawal, while other consumers have just been to the bank so they have two periods to go before making a withdrawal. Thus at a given point in time there is a cross-sectional distribution of money (M_t^a, M_t^b), where M_t^a is the amount of money held at the end of period t by those people who will go to the bank at the end of $t+1$ (i.e. those people who find it optimal to exhaust their cash balances before the end of $t+1$); where M_t^b is the amount of money held by people who have just been to the bank and will again go to the bank at the end of $t+2$. The above discrete representation of the economy is designed to capture the following property which any continuous time model must have. In continuous time, if some consumers are reducing their cash balances then other consumers must be increasing their cash balances. Thus the flow of money into stores must be offset by a flow of withdrawals from banks (recalling the assumption that stores immediately transfer money to banks). Hence in a continuous time model, all consumers could not be exhausting their cash at the same time (since their expenditures must have the effect of increasing some consumers' cash balances.) It is analytically convenient to deal with a discrete time model (see Appendix A for a discussion of the continuous time model, and further comments on the cross-sectional distribution of cash balances.) In order to capture the idea that a consumer's reduction in cash balances must offset by another consumer's increase in cash we have divided consumers into the above groups: consumers of type b have just increased their cash balances, while those of type a have decreased theirs. At period $t+1$ type a will increase its balances while type b will decrease its balances, and so on.

2B. Consumers' Optimization Problem

We begin with a discussion of the optimization problem faced by a consumer of type a, i.e., one who will find it optimal to go to the bank at the end of period 1. Let the cash balances posessed by this consumer at the end of date 0 be denoted by M_0^a. Let c_t denote consumption at time t. The consumers objective is to maximize discounted utility

$$\sum_{t=1}^{\infty} \beta^{t-1} U(c_t) \tag{2.1}$$

where $0 < \beta < 1$, subject to a wealth constraint and a constraint which states that only money can purchase goods. We assume that the cost of going to the bank is such that it is optimal for this consumer to go to the bank at the end of every odd numbered day, i.e. $t = 1,3,5,\ldots$. Let R_i denote one plus the nominal interest rate earned on assets between the end of date i and the end of i+1. Let $\alpha_1 \equiv 1$, $\alpha_t \equiv R_1 . R_2 . \ldots . R_{t-1}$. The present value of the consumers withdrawals from the bank is given by

$$\sum_{t \text{ odd}} \frac{M_t^a}{\alpha_t} .$$

Let p_t denote the price of goods in terms of money at date t. Then the Clower constraints on expenditures are given by

$$\text{(2.2)} \qquad p_1 c_1 \leq M_o^a$$

$$p_t c_t + p_{t+1} c_{t+1} \leq M_{t-1}^a \qquad t = 2,4,6 \ldots\ldots$$

A consumer has the option not to spend all his money. If the constraint in (2.2) is not binding so that the end of t+1, $M_{t-1}^a - (p_t c_t + p_{t+1} c_{t+1})$ is brought back to the bank, then wealth equal to this quantity divided by α_{t+1} is generated. Thus, the consumer's wealth constraint is

$$\text{(2.3)} \qquad p_1 c_1 + \sum_{t \text{ odd}} \frac{M_t^a}{\alpha_t} - \sum_{t \text{ odd}} \frac{M_t^a - (p_{t+1} c_{t+1} + p_{t+2} c_{t+2})}{\alpha_{t+2}} \leq M_o^a + W_o^a$$

where W_o^a is the value of the consumer's non-money wealth. The consumer's problem is to choose a consumption plan $\underline{c} \equiv (c_1, c_2, c_3, \ldots)$ and money holdings $\underline{M}^a = (M_1^a, M_2^a, \ldots)$ to maximize (2.1) subject to (2.2) and (2.3) taking M_o^a, W_o^a, prices and interest rates as given.

Note that we have not modelled the consumer's decision about when to go to the bank. He is being sent to the bank every other period. We could add the decision about when to go to the bank by reducing the consumer's wealth by some discounted transaction cost every time he decides to go to the bank. In a steady state (i.e. where prices and interest rates are constant) the consumer would then pick some fixed interval of time between trips to the bank. We simply define the length of our period to be half of that length of time. Unfortunately, it is probably true that, out of the steady state, the consumer will not find it optimal to have a fixed interval of time between trips to the bank. (However if the consumer faces discrete choices of periods between trips to the bank, then there will be an interval of price and interest rate paths "near" the steady state where the consumer will not change his frequency of trips to the bank as the economy moves away from the steady state due to say a "small" open market operation.) Since we are taking the time between trips to the bank as exogenous there is no point in keeping track of the transaction cost of going to the bank. Thus it does not appear explicitly in (2.1) or (2.3).

Returning to the consumer's optimization problem, we assume that $u'(0) = \infty$, $u'(c) > 0$, $u''(c) < 0$ and derive necessary and sufficient conditions for a maximum. Note that for both the type a and type b consumers' problems to have a solution, it is necessary that the price of a bond which pays $1 forever to be finite, i.e.

$$\sum_{t=1}^{\infty} \frac{1}{\alpha_t} < \infty \quad . \tag{2.4}$$

Note that $\alpha_t \geq 1$ since negative nominal interest rates make no sense.

Let γ be the Lagrange multiplier associated with the constraint in (2.3). Let λ_t be the multiplier associated with the t^{th} constraint in (2.2), then:

(2.5a) $$u'(c_1) = (\lambda_1 + \gamma)p_1$$

(2.5b) $$\left.\begin{aligned} \beta^j u'(c_{j+1}) &= (\lambda_{j+1} + \frac{\gamma}{\alpha_{j+2}})p_{j+1} \\ \beta^{j+1} u'(c_{j+2}) &= (\lambda_{j+1} + \frac{\gamma}{\alpha_{j+2}})p_{j+2} \end{aligned}\right\} \qquad j = 1,3,5,\ldots$$

(2.5c) $$\lambda_{j+1} = \frac{\gamma}{\alpha_j} - \frac{\gamma}{\alpha_{j+2}} \qquad j = 1,3,5\ \ldots$$

Equation (2.5c) immediately implies that if interest rates are always positive then $\lambda_j > 0$ and hence all the constraints in (2.2) for $t > 1$ must be binding. This fact is intuitively clear. In a world of certainty with positive interest rates a consumer would never withdraw more money from the bank than he plans to consume before his next trip to the bank. Thus (2.2) is replaced by

(2.6a) $$p_1 c_1 \leq M_o^a$$

(2.6b) $$p_t c_t + p_{t+1} c_{t+1} = M_{t-1}^a \qquad t = 2,4,6,\ldots\ldots$$

Manipulation of (2.5b) yields

$$u'(c^a_{j+1}) = \beta\, u'(c^a_{j+2}) \frac{p_{j+1}}{p_{j+2}} \qquad j = 1,3,5,\ldots. \tag{2.7}$$

$$u'(c^a_{j+1}) = \beta^2\, u'(c^a_{j+3}) \frac{p_{j+1}}{p_{j+3}} R_j\, R_{j+1} \qquad j = 1,3,5,\ldots. \tag{2.8}$$

where we have added an a superscript for future reference to denote a's optimal policy.

Consider the problem:

$$\max_{c_1, c_2} \quad u(c_1) + \beta\, u(c_2) \quad \text{s.t} \quad p_1 c_1 + p_2 c_2 = M \ . \tag{2.9}$$

Note that (2.6b) and (2.7) imply that for an optimal choice of M^a_{t-1}, the consumer choose c_t and c_{t+1} to maximize two period discounted consumption subject to the constraint that he exhaust his money. It is convenient to let the solution to (2.9) for the second period consumption be denoted by

$$c_2 = \phi\left(\frac{p_1}{p_2}, \frac{M}{p_2}\right) .$$

Hence if M^a_{t-1} is optimal for the consumer

$$c^a_{t+1} = \phi\left(\frac{p_t}{p_{t+1}}, \frac{M^a_{t-1}}{p_{t+1}}\right) \qquad t = 2,4,6,\ldots \tag{2.10}$$

$$c^a_t = \frac{M^a_{t-1} - p_{t+1}\, \phi\left(\frac{p_t}{p_{t+1}}, \frac{M^a_{t-1}}{p_{t+1}}\right)}{p_t} \qquad t = 2,4,6,\ldots \tag{2.11}$$

Note that (2.10) states that the consumer's savings for the last period before going to the bank is a function of the rate of return to savings p_t/p_{t+1} and his real spendable wealth M^a_{t-1}/p_{t+1} . Recall that between trips to the bank cash is held as savings for one period with a rate of return equal to the rate of deflation. For future reference note that if utility is homothetic, i.e., $u(c) = \frac{c^{1-A}}{1-A}$, then

$$\phi(X,Y) = Y \frac{\beta^{1/A}}{\beta^{1/A} + X^{\frac{A-1}{A}}} \quad . \tag{2.12a}$$

Further if utility is logarithmic, i.e., A = 1, then

$$\phi(X,Y) = Y \frac{\beta}{1+\beta} \equiv Y \bar{\phi} \qquad \text{if} \quad u(c) = \ln c. \tag{2.12b}$$

An almost identical analysis can be applied to a consumer of type b, i.e. a consumer who last went to the bank at the end of time zero and is thus 2 periods away from his next trip to the bank. He maximizes $\sum_{t=1}^{\infty} \beta^{t-1} u(c_t)$ subject to

$$p_1 c_1 + p_2 c_2 + \sum_{\substack{t=2,4\\6,\text{etc}}} \frac{M^b_t}{\alpha_t} \leq M^b_o + W^b_o \tag{2.13a}$$

$$p_1 c_1 + p_2 c_2 \leq M^b_o \tag{2.13b}$$

$$p_t c_t + p_{t+1} c_{t+1} = M^b_{t-1} \qquad t = 3,5,7,\ldots. \tag{2.13c}$$

where we use the assumption that interest rates are strictly positive to conclude that all money is spent after the first withdrawal. Further

(2.14a) $$c^b_{t+1} = \phi\left(\frac{p_t}{p_{t+1}}, \frac{M^b_{t-1}}{p_{t+1}}\right) \qquad t = 3,5,7,\ldots\ldots.$$

(2.14b) $$c^b_t = \frac{M^b_{t-1} - p_{t+1}\, c^b_{t+1}}{p_t} \qquad t = 3,5,7,\ldots\ldots.$$

(2.14c) $$u'(c^b_t) = \beta^2\, u'(c^b_{t+2})\, \frac{p_t}{p_{t+2}}\, R_{t-1}R_t \quad t = 3,5,7,\ldots\ldots$$

Before moving to the definition of equilibrium we analyze the initial wealth W^a_o, W^b_o of consumers. As we noted earlier consumers own stores. The stores own the economy's endowment of goods. Let y_t denote the real value of the endowment at t. Then the stores' revenue at t is $p_t y_t$. Consumers receive a share s^a, s^b of these revenues via the purchase of bonds in the consumer's name, e.g. $s^a p_t y_t$ is deposited in the type a consumer's interest bearing bank account at the end of t. The shares of s^a and s^b are the initial endowments of firms owned by the two types of consumers. Thus the present value of revenues deposited with consumers is given by

$$s^a \sum_{t=1}^{\infty} \frac{p_t y_t}{\alpha_t}, \qquad s^b \sum_{t=1}^{\infty} \frac{p_t y_t}{\alpha_t} \qquad .$$

Government bonds are another asset which consumers can hold. There are also tax liabilities to pay the interest on the bonds.

If there are government bonds of value B outstanding, and lump sum taxes are shared equally, then the present value of tax liabilities to each consumer is $\frac{B}{2}$. Let B^a, B^b be the initial endowment of government bonds held by a and b respectively. Hence the initial wealth of consumers net of tax liabilities is given by

$$W_o^a = s^a \sum_{t=1}^{\infty} \frac{p_t y_t}{\alpha_t} + B^a - \frac{B}{2} \quad ; \tag{2.15a}$$

$$W_o^b = s^b \sum_{t=1}^{\infty} \frac{p_t y_t}{\alpha_t} + B^b - \frac{B}{2} \quad . \tag{2.15b}$$

Note that in the statement of the consumer's optimization problem we did not give consumers a demand for bonds. This is because in a world of perfect certainty where consumers live forever, their lifetime consumption opportunities are exactly described by present value formulae. A consumer is indifferent as to how many bonds he purchases. The purchase of a government bond by a consumer can be exactly offset by private borrowing. That is, a consumer can keep his money-consumption plans constant and borrow a dollar to buy a dollars worth of government bonds. The interest on the government bond is used to pay the private loan, and nothing real has changed. We have introduced government bonds into the model to facilitate the analysis of open market operations.

2C. The Definition of Equilibrium

For expositional convenience we will be concerned with equilibria where interest rates after date 1 are strictly positive, and where a perpetuity has a finite price i.e., (2.4) holds. This allows us to use the fact that all money withdrawn from the bank will be spent before a return to the bank. An equilibrium is a price sequence $\underline{p} \equiv (p_1,p_2,p_3,..)$ and an interest rate sequence $\underline{R} = (R_1,R_2,...)$ such that when all consumers choose their consumption and money holdings optimally, all markets clear (for all time).

Let $\underline{c}^a(\underline{p},\underline{R},M_o^a,W_o^a)$, $\underline{M}^a(\underline{p},\underline{R},M_o^a,W_o^a) \equiv (M_1^a,M_2^a....)$ denote the optimal consumption and money holding sequence for a consumer of type a, i.e. the maximizer of (2.1) subject to (2.3),(2.6) and (2.15a). Similarly let $\underline{c}^b(\underline{p},\underline{R},M_o^b,W_o^b)$ and $M^b(\underline{p},\underline{R},M_o^b,W_o^b)$ be optimal for the type b consumer. Note that $\underline{y} = (y_1,y_2,...)$, s^a,s^b,B^a,B^b are exogenous.

Let M_t^s be the total money supply at the end of period t. An equilibrium price and interest rate sequence $\underline{p},\underline{R}$, from the given initial distribution of wealth and money is a solution to:

$$(2.16) \qquad c_t^a(\underline{p},\underline{R},M_o^a,W_o^a) + c_t^b(\underline{p},\underline{R},M_o^b,W_o^b) = y_t \qquad t = 1,2,3,\ldots .$$

$$(2.17) \qquad M_t^a(\underline{p},\underline{R},M_o^a,W_o^a) + M_t^b(\underline{p},\underline{R},M_o^b,W_o^b) = M_t^s \qquad . \ t = 1,2,3,\ldots..$$

A considerable simplification of these conditions is possible. In particular we proceed to eliminate the interest rates and derive an equation which relates the price path to that of output and the money supply.

Note that for $t = 2, 4, 6,\ldots$ $M_t^a = M_{t-1}^a - p_t c_t^a$ since the money holdings of type a at the end of t is composed of the withdrawal from the bank at the end of $t-1$ minus spending during period t. Hence (2.17) implies that

$$M_{t-1}^{a} - p_t c_t^{a} + M_t^{b} = M_t^{s} \qquad t = 4,6,8,\ldots \tag{2.18a}$$

If $t = 4,6,8,\ldots$, then $M_{t-1}^{b} = M_{t-2}^{b} - p_{t-1}c_{t-1}^{b} = p_t c_t^{b}$. Hence (2.17) implies that

$$M_{t-1}^{a} + p_t c_t^{b} = M_{t-1}^{s} \qquad t = 4,6,8,\ldots \tag{2.18b}$$

Subtract (2.18b) from (2.18a) and use (2.16) to derive

$$M_t^{b} = p_t y_t + M_t^{s} - M_{t-1}^{s} \qquad t = 4,6,8,\ldots \tag{2.19a}$$

A similar argument shows that

$$M_t^{a} = p_t y_t + M_t^{s} - M_{t-1}^{s} \qquad t = 3,5,7,\ldots \tag{2.19b}$$

Equations (2.19a) and (2.19b) give the flow equilibrium in the money market. The left hand side of each equation is the money withdrawn from the bank at the end of period t. The right hand side gives the money flowing into the bank at time t. Inflows are composed of the value of expenditures $p_t y_t$ plus the value of monetary injections $M_t^s - M_{t-1}^s$. The monetary injections are used to buy bonds at the "bank" (i.e., the asset market). An open market operation increases the money flowing into the bank and this necessitates an increase in withdrawals to maintain money market equilibrium.

Substitution of (2.19b) into (2.17), and using the fact that for $t = 3,5,7,..$ $M_t^b = p_{t+1}c_{t+1}^b$, (2.14) and (2.19a) yields

$$(2.20) \qquad p_t y_t + M_t^s - M_{t-1}^s + p_{t+1}\,\phi\left(\frac{p_t}{p_{t+1}}, \frac{p_{t-1}y_{t-1} + M_{t-1}^s - M_{t-2}^s}{p_{t+1}}\right) = M_t^s \ .$$

The above substitution makes (2.20) correct for $t = 3,5,7$. A similar substitution of (2.19a), etc. into (2.17) shows that (2.20) is true for $t = 4,6,8,...$ as well. Equation (2.20) is another statement of the stock equilibrium for money in (2.17). The term $p_t y_t + M_t^s - M_{t-1}^s$ is the money held at the end of t by those who go to the bank at the end of t. The next term is the money holdings of those who go to the bank at the end of $t+1$. Their money holdings are exactly enough to finance their consumption during period $t+1$, namely $p_{t+1}c_{t+1}$.

Equation (2.20) is a second order difference equation in p_t. To get some initial conditions we consider (2.16) and (2.17) for $t = 1$ and $t = 2$. We have <u>not</u> shown that with interest rates positive, consumers will exhaust their initial money holdings before arriving at the bank for the first time. Whether this is optimal or not depends on the benefit of savings in the form of money from date 1 to date 2 p_1/p_2 for the type a consumer, and $\frac{p_1}{p_3}$ for the type b consumer. If rates of return are high enough relative to their rate of time preference then they may not spend all their money until after their first trip to the bank. Some simplification is achieved if we restrict ourselves to equilibria where it is optimal for them to exhaust. In this case

$$(2.21) \qquad p_1 c_1^a = M_o^a \qquad p_1 c_1^b + p_2 c_2^b = M_o^b \quad .$$

Using an argument similar to that given in the derivation of (2.20) we conclude that

$$(2.22a) \qquad p_1 y_1 + M_1^s - M_o^s + p_2\, \phi\left(\frac{p_1}{p_2}, \frac{M_o^b}{p_2}\right) = M_1^s$$

$$(2.22b) \qquad p_2 y_2 + M_2^s - M_1^s + p_3\, \phi\left(\frac{p_2}{p_3}, \frac{p_1 y_1 + M_1^s - M_o^s}{p_3}\right) = M_2^s \quad .$$

Nominal interest rates can be derived as a function of the price path as follows. Use (2.14c), the fact that $c_t^b = y_t - c_t^a$, (2.10), and (2.19b) to derive

$$(2.23a) \qquad u'(y_t - c_t^a) = \beta^2 u'(y_{t+2} - c_{t+2}^a)\, \frac{p_t}{p_{t+2}}\, R_{t-1} R_t \qquad t = 3,5,7,\ldots$$

$$(2.23b) \qquad c_t^a = \phi\left(\frac{p_{t-1}}{p_t}, \frac{p_{t-2} y_{t-2} + M_{t-2}^s - M_{t-3}^s}{p_t}\right) \qquad t = 3,5,7,\ldots..$$

Similarly use (2.8), $c_t^a = y_t - c_t^b$, (2.14a) and (2.19a) to derive :

$$(2.23c) \qquad u'(y_t - c_t^b) = \beta^2 u'(y_{t+2} - c_{t+2}^b)\, \frac{p_t}{p_{t+2}}\, R_{t-1} R_t \qquad t = 2,4,6,8,\ldots$$

$$(2.23d) \qquad c_t^b = \phi\left(\frac{p_{t-1}}{p_t}, \frac{p_{t-2} y_{t-2} + M_{t-2}^s - M_{t-3}^s}{p_t}\right) \qquad t = 4,6,8,\ldots$$

In (2.23b) and (2.23d) we assume that (2.21) holds, otherwise t must start two periods later. Under (2.21) the initial values of c_t^a and c_t^b are given by

$$(2.24) \qquad c_2^b = \phi \left(\frac{p_1}{p_2}, \frac{M_o^b}{p_2} \right) .$$

Given the price path $\underline{p}$ and (2.21), then (2.23c) determines two period interest rates R_1R_2, R_3R_4, $R_5R_6, \ldots$, while (2.23a) determines R_2R_3, R_4R_5, $R_6R_7, \ldots$, etc. Thus given R_1, all interest rates are determined.

In summary, if consumers always spend all their money before arriving at the bank, then (2.20) and (2.23) govern the path of interest rates and prices as a function of p_1, R_1 and the exogenous variables M_t^s, y_t, M_o^a, M_o^b, etc. That is, (2.20) to (2.23) give necessary conditions for equilibrium. The converse is also true. In Part 2, it is shown that under particular assumptions, there is a unique p_1 which can satisfy (2.20) and (2.22). Given the unique price path so generated, c_t^a and c_t^b are given by (2.23b) and (2.23d). All two period interest rates are then determined by (2.23a) and (2.23c). We show below that R_1 is chosen so that consumers' wealth constraints are satisfied. The equilibrium generated must be checked to be sure that all interest rates are positive and that consumers want to exhaust their initial money balances. A sufficient condition for the latter is that

$$(2.26a) \qquad u'(c_1^a) \geq \beta\, u'(c_2^a) \frac{p_1}{p_2}$$

$$(2.26b) \qquad u'(c_2^b) > \beta\, u'(c_3^b) \frac{p_2}{p_3}$$

Implicit in the statement that the consumers' wealth constraints are satisfied is the condition that the price path generated leads to perpetuities having a positive price, i.e. (2.4) is satisfied.

2D. Equilibrium With Logarithmic Utility

The case where $u(c) = \log c$ is very easy to analyze. Recall from (2.12) that $\phi(X,Y) = \bar{\phi}Y$, where $\bar{\phi} \equiv \beta \div (1+\beta)$. Thus (2.20) is a first order difference equation with solution, for $t > 2$:

$$(2.27a) \qquad p_t y_t = (-\bar{\phi})^{t-2} p_2 y_2 + \sum_{j=1}^{t-2} [M_{j+1}^s (1-\bar{\phi}) + \bar{\phi} M_j^s] \, (-\bar{\phi})^{t-2-j} \quad .$$

Use (2.22) to derive

$$(2.27b) \qquad p_1 y_1 = M_o^s - M_o^b \bar{\phi} \quad , \quad p_2 y_2 = M_1^s (1-\bar{\phi}) + \bar{\phi}^2 M_o^b \ .$$

Equation (2.27) gives prices as a function of the sequence of money supplies. Given all prices equation (2.23) determines all two period interest rates. As we noted earlier R_1 remains to be determined by the budget constraint.

Let $\qquad x_t \equiv R_t R_{t+1} \qquad t = 1,2,\ldots$.

Then the type a consumer's expenditures are given by

$$p_1 c_1^a + p_2 c_2^a + p_3 c_3^a + \frac{p_4 c_4^a + p_5 c_5^a}{x_1} + \frac{p_6 c_6^a + p_7 c_7^a}{x_1 x_3} + \ldots \ ,$$

which is clearly determined given prices. The consumers wealth W_o^a is given by

$$W_o^a = s^a \left(p_1 y_1 + \frac{p_2 y_2}{R_1} + \frac{p_3 y_3}{x_1} + \frac{p_4 y_4}{R_1 x_2} + \frac{p_5 y_5}{x_1 x_3} + \ldots \right) \quad .$$

Thus given prices such that $p_t y_t > 0$ at some even date there will exist a unique R_1 such that budget balance occurs for a consumer of type a.

This R_1 along with the prices generated by (2.27) and x_t generated by (2.23) form an equilibrium. It will be clear from our later discussion that (2.26) is satisfied for a range of values of M_o^a and M_o^b near the steady state which is to be defined below. Thus since (2.27) gives a unique price sequence after date 2, we have found a unique equilbrium price and interest rate path for all dates assuming that initial money balances are exhausted before the first arrival at the bank.

3. The Steady State Distribution of Cash Balances

As is clear from the last Section the model finds a price path from the initial distribution of money and the path of monetary injections. It is worthwhile to consider the initial distribution of cash balances, which when there are no new monetary injections, maintains itself over time. If $M_t^s \equiv M$, eq. (2.19) implies that the distribution of cash will be constant if and only if spending $p_t y_t$ is constant. If y_t is not constant then the real interest rate will not in general be constant so we will be unable to maintain spending constant. Hence we assume that y_t is constant, i.e., $y_t = y$.

We can use (2.20) to get an equation of the relationship between prices, money and output when all three are constant:

$$y + \phi(1,y) = \frac{M}{p} \quad . \tag{3.1}$$

Thus in the steady state money is neutral in the sense that a proportional increase in total money raises prices by the same proportion (recall that output y, is exogenous). From (2.19) the money holdings of someone who makes a withdrawal at the end of t is given by M_t^a for t odd, M_t^b for t even:

$$M_1^a = M_o^b = py = \frac{y}{y + \phi(1,y)} M \quad . \tag{3.2}$$

The money holdings at the end of t of someone who goes to the bank at the end of $t+1$, M_1^b must satisfy $M_1^b + M_1^a = M$. Hence we have

$$M_1^b = M_o^a = \frac{\phi(1,y)}{y + \phi(1,y)} \; M. \tag{3.2b}$$

Note that $M_o^a = M_1^b = M_2^a = M_3^b = M_4^a =$ etc. and $M_o^b = M_1^a = M_2^b = M_3^a$ etc. in the steady state.

It is important to note that in the steady state $M_o^a < M_o^b$. That is, someone who has two periods to go before his trip to the bank has more money than someone who has only one period to go. To see this note that $\phi(1,y)$ is the consumption of someone is in the last period before his trip from the bank, and $y - \phi(1,y)$ is the consumption of someone who is in the first period after his trip to the bank. The assumption that $u'(0) = \infty$ implies that $y - \phi(1,y) > 0$. Hence

$$M_1^b < M_1^a, \quad M_o^a < M_o^b, \quad \text{etc.} \tag{3.3}$$

Equation (3.3) will be a key ingredient in showing that if the economy begins in a steady state, then an open market operation is not neutral (even with homothetic preferences).

Note that consumption is the same for each individual in every other period. Thus from (2.23) the two period interest factor is

$$R_1R_2 = R_2R_3 = \ldots = \beta^{-2} . \tag{3.4}$$

This determines all interest factors as a function of the initial factor R_1. If the economy begins with the steady state distribution of money (M_o^a, M_o^b) as in (3.2), then consumers of type b begin with more money. Hence if they get the same share of output (i.e. $s^a = s^b$) and have the same amount of government debt $(B^a = B^b)$, they will have different total wealths.

But, for the distribution of cash balances to maintain itself over time the consumers must have the same spending pattern. This is achieved via the nominal rate in the first period taking on a value which permits the financing of steady state expenditures. Thus, the present value of a type a consumer's expenditure is given by

$$p_1c_1 + \sum_{i \text{ even}}^{\infty} \frac{p_ic_i+p_{i+1}c_{i+1}}{\alpha_{i-1}} = P\left(\phi(1,y) + y + \frac{y}{R_1R_2} + \frac{y}{R_1R_2R_3R_4} + \ldots\right) = P\,\phi(1,y) + P\,\frac{y}{1-\beta^2}$$

The total wealth of the consumer is, from (2.15) and (3.2)

$$M_o^a + s^a \sum_{i=1}^{\infty} \frac{p_iy_i}{\alpha_i} + B^a - \frac{B}{2} = M_o^a + B^a - \frac{B}{2} + s^aPy\left(1 + \frac{1}{R_1} + \frac{1}{R_1R_2} + \frac{1}{R_1R_2R_3} + \ldots\right)$$

$$= P\phi(1,y) + B^a - \frac{B}{2} + s^aPy\,(1+\beta^2+\beta^4+\ldots) + \frac{s^aPy}{R_1}(1+\beta^2+\beta^4\ldots)$$

(3.5) $$M_o^a + W_o^a = P\phi(1,y) + B^a - \frac{B}{2} + \frac{Pys^a}{1-\beta^2}\left(1 + \frac{1}{R_1}\right)$$

Thus equilibrium requires that

(3.6) $$B^a - \frac{B}{2} + \frac{s^aPy}{1-\beta^2}\left(1 + \frac{1}{R_1}\right) = \frac{Py}{1-\beta^2}\ .$$

For example if $B^a = \frac{B}{2}$, then $R_1 = \frac{s^a}{1-s^a} \equiv \frac{s^a}{s^b}$.

Thus if the two types of traders have the same net government debt and the same share of the firms, the initial nominal interest rate is zero!! This peculiarity arises out of the fact that a steady state requires that the two types be in a symmetric position. However, if we start the system at a steady state distribution of money we automatically put the type a consumer in a worse position when $s^a = s^b$ and $B^a = B^b$. There are three sorts of differences among the two type when $B^a = B^b$ and $s^a = s^b$, i.e. when $W_o^a = W_o^b$. First, the type b consumer will have a higher present value of consumption. This is because

$$y - \phi(1,y) > \phi(1,y) \quad , \tag{3.7}$$

which can be seen from (2.7), (since prices are constant, marginal utility of consumption declines at the rate of time preference from the time of arrival at the bank until the next return). Thus, consumption alternates between the two types with type b getting the high consumption first. A second important difference between them is that when $W_o^a = W_o^b$, the type b consumer begins with a higher wealth since $M_o^b > M_o^a$ by (3.3). The third and most important difference is that a type b trader will earn one period's interest on his wealth before returning to the bank. This last effect is related to the other two. This can be summarized by looking at the difference between wealth and the present value of expenditures in equilibrium for the two types:

$$M_o^a - p_1 c_1^a \qquad\qquad + W_o^a - P\left(y + \frac{y}{R_1 R_2} + \frac{y}{R_1 R_2 R_3 R_4} + \ldots.\right) \quad ;$$

$$M_o^b - (p_1 c_1^b + p_2 c_2^b) + W_o^b - P\left(\frac{y}{R_1} + \frac{y}{R_1 R_2 R_3} + \frac{y}{R_1 R_2 R_3 R_4 R_5} + \ldots\right) .$$

Recall that all the two period interest factors $R_j R_{j+1} = \beta^{-2}$. Therefore the present value of expenditures not financed by initial cash balances for a is R_1 times the corresponding quantity for b. Thus the extra amount of money which b holds has the effect of allowing him an extra period of interest before he must withdraw money from the bank to finance his future consumption. Clearly the above two quantities can equal zero only if $R_1 = 1$, which destroys b's advantage.

We are trying to model a timeless steady state. We don't have some particular initial date in mind for starting the economy. Thus there is no particular date at which the interest rate is 0. The only interest rate which will maintain itself for all time including the initial date satisfies $R = \beta^{-1}$. From (3.6) this will be the case when

$$B^a = B^b \quad \text{if} \quad s^a(1 + \beta) = 1, \tag{3.8}$$

which implies $s^a > .5$, since $\beta < 1$. Alternatively if $s^a = s^b = .5$ then

$$B^a - \frac{B}{2} = \left(\frac{py}{1+\beta}\right)\frac{1}{2} > 0.$$

In summary the steady state will involve type b holding more initial money but less of the initial share of other nominal assets than the type a consumer. It is essential to note for what follows that the steady state involves type a holding more interest earning assets than type b. Thus, when we analyze an open market operation which increases the price level in an unanticipated way at date 1, wealth will be redistributed from b to a. That can be looked at from two points of view. First, since b holds more money, inflation yields him a real wealth loss. Second this wealth loss is transferred to the type a consumer via a lower nominal interest rate to finance his withdrawals of money at date 1. That is, it is helpful to think of b as lending the wealth to use from the end of date 1 to the end of date 2 to finance part of a's withdrawal from the bank at the end of 1.

These points will be reviewed after analyzing the effect of an open market operation.

Before proceeding to open market operations, we note that in the steady state all consumers find their initial cash in advance constraint strictly binding. That is, the inequalities in (2.26) will be strictly binding when $\beta < 1$. This means that we can consider small perturbations about the steady state which maintain strict inequality in (2.26). Thus, in what follows we will implicitly be restricting ourselves to small enough perturbations that the initial cash in advance constraint is binding.

4A. The Neutrality of Helicopter Monetary Injections

It is of some expository value to consider the effect of a monetary injection which creates a proportional increase in everyone's money holdings. Since some consumers are not at the bank, the government must literally deliver cash to them. (In a continuous time model only a small fraction dt of consumers will exhaust their money holdings between t and $t+dt$. Thus, the government will have to deliver cash to almost everyone in the economy.) Formally this involves considering what new equilibrium will correspond to initial money endowments $(1+k)M_o^a$, $(1+k)M_o^b$. That is, the government increases everyone's cash balances by k percent at time zero and from then on there are no monetary injections.

We will show that it is an equilibrium for all prices p_t to be k percent higher than they would have been had there been no monetary injection, and that all interest rates and consumptions are unchanged. In the case where $B^a \neq B/2$ it is important to clarify our assumption about taxes. We assume that real taxes are kept constant, and the real value of government debt is held constant. It is convenient to assume that the real value of debt is held constant by a proportional increase in B^a and B^b. Under these assumptions a $k\%$ increase in prices with constant interest rates will raise $B^a-B/2$ by k percent. If we do not keep the real net debt constant for each individual then the open market operation will redistribute wealth due to a change in the tax burden relative to the value of endowed goverment debt. In this case the analysis given below will require a slight modification, as indicated after the proof of the simple case.

To see that helicopter injections are neutral note from (2.7) and

(2.8) that the consumers' first order condition for optimal consumption will still be satisfied at the old consumption levels when prices are multiplied by (1+k), and interest rates are unchanged. Note that the initial cash in advance constraints (2.2) and (2.13b) will be satisfied at the old consumption levels when M_o^a, M_o^b, p_1 and p_2 are multiplied by 1+k. Thus, the only thing left which must be verified is that consumers can afford their old allocation at the new prices. As can be seen from (2.15) nominal wealth rises by k percent. Hence at the old consumption levels the present value of nominal spending is unchanged. Thus if the old allocations formed an equilibrium, then they will still be an equilibrium with p_t multiplied by 1+k, and interest rates unchanged.

Now consider the case where a wealth redistribution occurs due to the real tax burden changing for individuals. Again multiply all prices by 1+k and keep all 2-period interest rates constant. (i.e., keep $R_1R_2, R_2R_3, \ldots$ all constant). If all consumptions are held constant, then all first order conditions in (2.7),(2.8) are satisfied. There is one problem with maintaining this as an equilibrium, namely if R_1 is unchanged then either type a or type b consumers will be violating their wealth constraint, since the value of total wealth does not increase by k% for each consumer. As we noted earlier the first order conditions involve only two period interest rates. All interest rates are determined once R_1 is chosen. We noted that R_1 is chosen so that consumers can afford the consumption path assigned to them by the first order conditions. Thus we can choose a new R_1 so that the old level of consumption is affordable by say type a. (Note that if markets clear and an allocation is affordable by type a then it is affordable by type b). Hence an open market operation which redistributes wealth between type a and type b consumer will be neutral except that the one period interest rate changes--all two period rates will be unchanged. This effect is not likely to be of great importance. It is difficult to see why there would be significantly different real tax burdens associated with differences in the day individuals go to the bank!

Note that the above neutrality results hold independent of whether the economy begins at the steady state cross sectional distribution of cash balances.

4.B. The Non-Neutrality of Open Market Operations

We consider the effect of an open market operation given that the economy is in a steady state before the operation. Further, the operation is a once and for all unanticipated event. An open market operation involves the purchase or sale of government bonds for money. In a purchase of bonds the government buys bonds on the asset market (i.e. at the "bank"). The sellers of the bonds will have more money. Consumers who are at the bank or who are not at the bank are assumed to be able to sell bonds costlessly to the government. (Recall the term "a consumer not at the bank at time t" means a consumer who has not yet depleted his cash balances.) When a consumer who is not at the bank sells the bond for cash, it is optimal for him to immediately convert the cash back to bonds. Thus equilibrium will involve only those people who are at the bank at time t holding the cash. That is, we assume that the major transaction cost involves transportation to the bank to increment cash balances. Thus a small open market operation will not make it optimal for a consumer who has not yet exhausted his cash balances to go to the bank and increment them.

If there is a transaction cost of converting bonds into money, as well as a cost of transportation to the asset market, then those consumers who have not yet exhausted their cash balances will not find it optimal to sell bonds to the government. For if they did so they would have to sell bonds for cash which would have to be converted back into bonds and then converted back again into cash when they exhaust their money balances Thus, in this case also, it is an equilibrium for only those consumers at the bank to increase their money holding in response to the open market operation. Note that we assume that the open market operation is sufficiently small so that it is still optimal for consumers to wait two periods before returning to the bank after their last withdrawal.

Similar remarks apply to an open market sale of bonds by the government. Here it is important to note that cash flows into the bank from the stores that sell goods (i.e., firms are purchasing bonds for the consumers that own the firms on the asset market). The open market operation is sufficiently small to keep consumers who have not yet exhausted their cash balances from going to the bank. Thus in equilibrium the consumers who have just exhausted their cash balances will find it optimal to withdraw less money when the government sells bonds.

We begin with a formal proof that an open market operation is not neutral and then proceed to analyze dynamic effects associated with an open market operation. First it is important to do some accounting. Recall that B^a and B^b represent the nominal value of the government debt that a and b are endowed with. Before the open market operation $B^a + B^b = B$ where B is the total present value of the tax liabilities associated with the open market operation, which equals the total stock of government bonds. If an open market operation occurs at the end of period 1, which is announced at the beginning of period 1, there is no automatic increase in individuals' endowments of government bonds or money. Prices and interest rates adjust so that individuals find it optimal to hold the new money and bonds. However, there is an automatic change in individuals' tax liabilities just after the announcement. If the increase in the stock of bonds is ΔB, then the total present value of tax liabilities rises by ΔB. Again it is convenient to assume that the distribution of the tax burden does not change so that each individual's tax liability goes up by $\frac{\Delta B}{2}$. We also assume that $B^a = B^b$ for convenience. Note that an open market operation means that

$$(4.1) \qquad \Delta B + M_1^s - M_o^s = 0,$$

where $$M_o^s = M_o^a + M_o^b \quad .$$

From period 1 on, the total money supply will be constant, $M_t^s = M_1^s$, $t \geq 1$.

As in Section 4.a consider a k percent increase in the money supply, i.e. $M_1^s = (1+k)M_o^s$. We first show that if (p_t, R_t, c_t^a, c_t^b) is the equilibrium corresponding to $k = 0$, with the initial cash in advance constraints strictly binding, then it <u>cannot</u> be an equilibrium for all prices to rise by k percent, and for all two period interest rates and consumption to be unchanged. This is an immediate consequence of the cash in advance constraint for consumers of type a and b: $\bar{p}_1 c_1^a = M_o^a$, $\bar{p}_1 c_1^b + \bar{p}_2 c_2^b = M_o^b$, where the "bars" above prices indicate post-announcement prices. Since cash on hand is unchanged, if prices increase by k percent, (i.e. $\overline{p}_1 = (1+k)p_1$ after the announcement) then it is not feasible for c_1^a to be unchanged. It might be thought that this is some initial period effect which disappears, but that is quite wrong. If prices rise by $k\%$ above what they would have been with $k = 0$, then c_1^a must be lower than it would have been and c_1^b must be higher, since $c_1^a + c_1^b = y_1$. Thus c_2^b must be lower than it would have been. Further, from equation (2.19b) :

$$\overline{M}_1^a = \overline{p}_1 y_1 + \Delta M, \tag{4.2}$$

where we recall that (2.19b) holds for $t = 1$, when the cash in advance constraint is binding. Equation (4.2) implies that when prices increase by k percent the monetary withdrawal of the type a consumer increases by more than $k\%$. Recall also that $\overline{c}_2^b$ is lower than c_2^b, hence a's consumption c_2^a must rise to $\overline{c}_2^a$, and from (2.7) c_3^a must also rise. This is financed by the $\overline{M}_1^a$ being larger than $(1+k)M_1^a$ when $\overline{p}_1 = (1+k)p_1$. It follows that $\overline{c}_3^b < c_3^b$, and thus $\overline{c}_4^b > c_4^b$ because with $\overline{p}_2 = (1+k)p_2$ and $\overline{M}_2^b = \overline{p}_2 y_2$ by (2.19a), nominal spending $p_3 c_3^b + p_4 c_4^b$ must rise by $k\%$. A consequence of c_4^b rising is that c_4^a must fall. But using (2.8) the fall in c_4^a and the rise in c_2^a must imply that the nominal interest factor $R_1 R_2$ falls. This not only shows that it is not an equilibrium for all prices to increase by $k\%$ keeping interest rates and consumption constant, but illustrates how monetary shocks can persist. Of course, it need not be an equilibrium for $\overline{p}_t = (1+k)p_t$, but the fall in the interest rate $R_1 R_2$ to induce the type a consumers to hold the extra money is a property that the true equilibrium must have. The fact that it is not an equilibrium for all prices to increase by k% can be seen by recalling that (2.22a) will be a necessary condition for equilibrium when the initial cash in advance constraint is binding. Recall also that when preferences are homothetic $\phi(X,Y) = \phi(X,1)Y$. Thus (2.22a) will not hold if p_1 and p_2 are multiplied by $1+k$.

The basic reason that an open market operation is not neutral is that the people who are at the bank at the time of the operation (i.e the people who hold no money) must be induced to hold a disproportionate share of the monetary injection. Further those consumers who are holding money at the time of the monetary injection cannot increase their total spending before returning to the bank. Thus they have to respond to any increased prices by a reduction in real consumption.

These points can be made transparent by examining the case of logarithmic utility. In Section 2.D the equilibrium price path for this case was derived. The essential point to recall is that with logarithmic utility spending depends only on nominal money balances and not the real return to money. Therefore our money stock equilibrium equation (2.20) becomes for $t \geq 2$:

$$p_t y + \bar{\phi} M^w_{t-1} = (1+k)M \quad , \tag{4.3}$$

for the case where there is a k% open market operation at the end of period 1, $M_t = (1+k)M$ for $t \geq 1$, and M^w_{t-1} denotes the money withdrawn from the bank at the end of t-1. Note that $p_t y$ is the money flowing into the bank after date 1, so it is the end of period money holdings of those people who go to the bank at time t. People who go to the bank at t-1 spend

$(1-\overline{\phi})M^w_{t-1}$ during period t, so their end of period t money holdings are given by $\overline{\phi}\, M^w_{t-1}$. Recall from Section 3 that the steady state satisfies

$$(4.4) \qquad py + \overline{\phi}\, M^b_o = M \quad .$$

From equation (2.27b) we see that $p_1 = p$, that is since $c^a_1 = M^a_o/p_1$ and $c^b_1 = (1-\overline{\phi})\frac{M^b_o}{p_1}$ and $\overline{\phi}$ is a constant, supply equals demand at the old price. Next consider M^w_1, the money withdrawn by type a at date 1. From (2.19b) $M^w_1 = p_1y + \Delta M = py + kM = py + (1+\overline{\phi})kpy$ where the last equality uses (3.2a). Therefore

$$(4.5) \qquad M^w_1 > (1+k)\tilde{M}^a_1 = (1 + k)\tilde{M}^b_o \quad ,$$

when we use a "~" to denote the steady state values. Equation (4.5) states that the nominal money withdrawal rises by more than k%.
As stated earlier this is because the monetary injection must be held by only the people who have just exhausted their money balances and are thus at the bank.

Now consider date 2. Since $M^a_1 > (1+k)\tilde{M}^a_1$, it must be the case that at the end of date 2 type b holds disproportionately less of the monetary injection. At the end of date 2 type b's holdings involve the money flowing into the bank at date 2: p_2y. Thus from (4.3) - (4.5)

$$(4.6) \qquad M^w_2 = p_2y < (1+k)\tilde{M}^b_o = (1+k)py \quad .$$

Next consider date 3. Since the withdrawal at date 2 has risen by less than k%, equation (4.3) implies that total spending at date 3 must rise by more than k%, since otherwise the total stock of money held would not have risen by k%. Thus

$$M_3^w = p_3 y > (1+k)\tilde{M}_o^b = (1+k)py \tag{4.7}$$

It is easy to see that this argument repeats itself with prices more than k% higher than the old steady state at odd dates and less than k% higher on even dates. These oscillation are damped and the price sequence converges to $(1+k)p$ as can be seen from (2.27a) which in our case is

$$p_t = (1+k)p + (-\bar{\phi})^{t-2}(p_2 - (1+k)p) \quad . \tag{4.8}$$

Note that for β close to one $\bar{\phi} \approx .5$. So the oscillations will damp out rapidly.

In Part 2 it is shown that the above results do not depend on logarithmic utility: if future consumption $\phi\left(\frac{p_t}{p_{t+1}}, \frac{M_{t-1}}{p_{t+1}}\right)$ is an increasing function of the rate of return $\frac{p_t}{p_{t+1}}$, then there will be a damped oscillatory response of prices to the increase in money.

4C. The Effect of an Open Market Operation on Interest Rates

As we showed below an open market operation must cause prices and interest rates to move in such a way that consumers at the bank are temporarily willing to hold more than their steady state share of money. The cost of holding money as opposed to bonds to the consumer who is at the bank is the two period nominal interest rate. The cost of holding (i.e., withdrawing) money to increase current consumption is related to the two period real rate of interest. Thus we should expect that the two period real and nominal rates to fall. Indeed for the logarithmic case this is true, as we show next. This is extended to a more general case in Part 2.

Use (2.8) for the log case to derive

$$\beta^2 R_1 R_2 = \frac{p_4 c_4^a}{p_2 c_2^a} = \frac{(1-\bar{\phi})M_3^a}{(1-\bar{\phi})M_1^a} = \frac{p_3 y}{p_1 y+\Delta M} . \tag{4.9}$$

Use (2.19) and the definition of the steady state to derive

$$p_3 y = M^s - \bar{\phi}M_2^b = M^s - \bar{\phi}p_2 y = M^s - \bar{\phi}(M^s - \bar{\phi}M_1^a) .$$

Thus

$$\beta^2 R_1 R_2 = \frac{M(1+k)(1-\bar{\phi})}{py + \Delta M} + \bar{\phi}^2 = (1-\bar{\phi}^2)\left[\frac{1 + k}{1 + k(1+\bar{\phi})}\right] + \bar{\phi}^2 \cdot 1 . \tag{4.10}$$

The right hand side of (4.10) is a convex combination of 1 and a term less than 1. Hence

$$\beta^2 R_1 R_2 < 1.$$

For $t > 1$

$$(4.11)\qquad \beta^2 R_t R_{t+1} = \frac{M^w_{t+2}}{M^w_t} = \frac{p_{t+2}}{p_t} \quad .$$

Recall that from (4.8) p_t oscillates with declining magnitude of oscillation. Therefore p_{t+2}/p_t will fall below the steady state value of $\beta^2 R_t R_{t+1}$ which is unity. Note that as p_t converges to $(1+k)p$, $\beta^2 R_t R_{t+1}$ converges to its old value of unity.

Finally, the initial interest rate R_1-1 must fall. As we noted in Section 3 the initial interest rate is chosen so that the wealth constraint holds. We choose the steady state with lump sum taxes to pay interest and where (3.8) holds. Recall that in the steady state $M^b_o > M^a_o$ and $s^a > s^b$. When prices rise the type b consumer takes a capital loss on his initial money endowment. Recall that the type b consumer in effect lends money to the type a consumer, since the type a goes to the bank and makes a withdrawal at the end of date 1 while b waits until the end of date 2. The increase in wealth to the type a consumer will appear as a drop in the interest rate R_1-1 in his implicit cost of going to the bank one period earlier.

The above statements can be proved by direct calculation for the case of logarithmic utility. To see this use (4.9) and (4.11) to obtain:

$$(4.12a)\qquad R_t = \frac{p_{t+1}}{p_t}\; R_1\; \frac{p_1 y+\Delta M'}{p_2 y} \qquad t = 3,5,7,\dots \quad .$$

$$(4.12b)\qquad R_t = \frac{p_{t+1}}{p_t}\; \frac{1}{\beta^2 R_1}\; \frac{p_2 y}{p_1 y+\Delta M} \qquad t = 2,4,6,\dots \quad .$$

The present value of the expenditures of a type a consumer can be found using (4.12), (2.6b), and (2.19b) to satisfy:

$$(4.13)\quad p_1c_1^a + \sum_{t=2,4,\ldots} \frac{M^a_{t-1}}{\alpha_{t-1}} = p_1c_1^a + \Delta M + p_1y + (p_1y+\Delta M)\frac{\beta^2}{1-\beta^2} .$$

Recalling that (3.8) holds in the steady state, $B^a = B^b = B/2$, that for an open market operation $\Delta M + \Delta B = 0$, and that each person's new tax liability becomes $(B + \Delta B) \div 2$, we can compute the wealth of the type a consumer:

$$(4.14)\quad M_o^a + B^a - \frac{B+\Delta B}{2} + s^a \sum_{t=1}^{\infty} \frac{p_t y_t}{\alpha_t} = M_o^a + \frac{\Delta M}{2} + s^a \left\{ p_1y + \frac{p_2y}{R_1}\frac{1}{(1-\beta^2)} + (p_1y+\Delta M)\frac{\beta^2}{1-\beta^2} \right\} .$$

Equate the present value of expenditure (4.13) to wealth in (4.14) and use the facts that $\Delta M \equiv kM$, (3.8), (3.1), (3.2), $\phi(1,y) \equiv [\beta\div(1+\beta)]y \equiv \overline{\phi}y$, $M_o^a = p_1c_1^a$, to derive

$$(4.15)\qquad R_1 \equiv \left[1 + \frac{k}{1+\beta^2}\right] \div \left[\beta + k(1+2\beta)\left(\frac{1+\beta^2}{2} - \frac{\beta^2}{1+\beta}\right)\right] .$$

Write $R_1 \equiv \beta^{-1} + e$. Note that

$$\text{sign}(e) = \text{sign}\left(\beta + \frac{k\beta}{1+\beta^2} - \beta + k(1+2\beta)\left(\frac{1+\beta^2}{2} - \frac{\beta^2}{1+\beta}\right)\right)$$

$$\text{sign}(e) = \text{sign}(k)\text{sign}(2\beta(1+\beta) - (1+2\beta)(1+\beta^2)(1+\beta-\beta^2+\beta^3)) .$$

The assumption $0 < \beta < 1$ implies that $\text{sign}(e) = -\text{sign}(k)$. Hence an increase in the money supply causes the initial one period nominal rate to fall below its steady state value of β^{-1}. Since $p_2 > p_1$, the real rate also falls.

5. Conclusion

The principal analytic result of this paper is that open market operations can have real effects. We have shown that a monetary expansion will lead to a temporary reduction in both real and nominal interest rates and lead to a gradual increase in prices. The model gives analytic support to the notion that money matters in the short run, but not in the long run when prices adjust proportionally to money changes. These conclusions are similar to those of traditional Keynesian analysis which, unlike our model assumes some sort of short run price stickiness. However, in our model the distinction between "short" and "long" run is that the distribution of money holdings is taken as exogenous in the short run, but endogenous in the long run.

The model has emphasized how a change in the money supply affects the spending decision of those agents making withdrawals at the time of an open market operation. Considerations of intertemporal substitution imply that the real rate must decline to induce these agents to consume more. Because this new money is spent gradually, prices will rise slowly and reach their steady state level long after the interval of time between trips to the bank. A natural question is how long this transition period is likely to be. A "period" in our model corresponds to one half the length of time between trips to the bank for a representative consumer. In the U.S., average money holding is sufficient to purchase about 1/7 (the inverse of income velocity) year's worth of GNP. If spending occurs at a constant rate, then the average holding of money is exactly one half the total expenditures between trips to the bank, so that the representative consumer goes to the bank every 2/7 of a year, or about 15 weeks. Hence the length of a period is $7\frac{1}{2}$ weeks. In our model prices first reach (and exceed) their steady state level after 2 periods, so the model indicates that the transition period during which prices rise in response to a monetary injection is about 15 weeks.

This empirical issue is complicated by the fact that the monetary intermediation channels in the actual economy are more intricate than those of our model. Our estimate is likely to underestimate the duration of the transition period to the extent that firms hold idle balances and do not instantly transmit their proceeds to the bank. However, this estimate is too long if some consumers receive cash payments directly, without having to make withdrawals.

There are other reasons to suspect that monetary inpulses will have a more delayed response than suggested by the model. If the time between trips to the bank were made endogenous, rather than the fixed interval assumed in the model, then it could be imagined that the decline in nominal interest rates would induce a longer time before the next return, as the cost of holding money goes down. In this case the new money would be spent more gradually than the case presented in this paper, and the price rise would be slower and longer. Similarly, the rate of spending for the recipients of the new money would not have to rise as much; the decline in real rates would be lower than the model's conclusion.

Some of the model's conclusions are very different from those of earlier theories. The Clower cash in advance constraint makes current prices less sensitive to anticipation of future money than suggested by the analysis of Sidrauski (1976) which assumes that money provides services, much as a consumer durable. This is because in our model the rise in current spending associated with a rise in anticipated inflation is limited by the cash in advance constraint. For example, in the extreme case of logarithmic utility, the current price level is unaffected by anticipated future monetary injections (see eq. (2.27a)). This extreme result is a consequence not only of the assumption that the current money supply puts an upper bound on spending, but also on the assumption that the time between trips to the bank is fixed. A sufficiently large anticipated inflation will cause people to go to the bank sooner and hence spending will become more sensitive to anticipated inflation.

Our model, where all consumers live forever, and in which bonds can coexist with money, should be contrasted with consumption-loan models of "money". In many of the consumption-loan models money and bonds cannot coexist and what is called "money" could as easily be called "bonds" (see Bewley (1980) for an example of this approach, and references to other work which uses "bonds and "money" interchangeably). This is to be contrasted with the approach of Grandmont and Younes (1972) which implicitly uses a Clower constraint in a consumption loan model framework. However, they do not discuss the tradeoffs between bonds and money and the effects of an open market operation. Their model and others which use the Clower constraint such as Lucas (1980), assume (implicitly) that all individuals engage in trade intermediated by money during a "period". They do not analyze what happens during the "period". We emphasize that all individuals cannot be decreasing their money holdings at the same time during this "period". A model in which bonds and money coexist without the assumption of a Clower constraint appears in Bryant and Wallace (1979) and Sargent and Wallace (1980). Jovanovic (1982) considers a general equilibrium transaction demand for money model very close to the one we consider. However, he only analyzes steady states, and helicopter monetary injections.

The fact that people hold money for the sole purpose of spending it implies that money will flow through the economy from individuals to stores to banks and then back to individuals. A snapshot of the economy will reveal some consumers who have just made a withdrawal -- thus holding a large amount of money, and some customers who are about to make a withdrawal -- thus holding a small amount of money. The fact that money flows is the necessary dynamic counterpart of the fact that at an instant of time the cross-sectional distribution of money holdings must not be degenerate. This feature distinguishes our model, and is the source of the dynamic effect on prices and interest rates which we show to be a necessary consequence of an open market operation.

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Appendix

The Continuous Time Formulation

An essential feature of our model is that in a steady state monetary equilibrium people hold different amounts of money. In particular those people who recently went to the bank will be holding more money than those people who went to the bank at an earlier date. It is easy to show that in a continuous time model, if money flows into the bank at a uniform rate, then steady state equilibrium must involve a uniform arrival of people at the bank to make withdrawals. Since everyone does not arrive at the bank at the same time equilibrium will involve a non-degenerate cross-sectional distribution of cash balances.

The above ideas are easily formalized if we fix the time which elapses between trips to the bank at one unit, so that someone who goes to the bank at t also goes to the bank at dates $t-1$ and $t+1$. This one unit of time is taken as exogenous. A consumer who at time zero has money holdings M_o at time zero chooses a date of initial exhaustion of his money t_o and then exhausts at $t_o + 1$, $t_o + 2, \ldots$. His maximum problem must solve

$$\max_c \int_{t_o+n}^{t_o+n+1} u(c)e^{-\beta t}dt \qquad \text{subject to}$$

(A1) $$\dot{M}(t) = - P(t)c(t) \qquad t_o + n < t < t_o + n + 1$$

$$n = 0,1,2,\ldots \quad .$$

(A2) $$M(t_o + n) = 0 \qquad n = 0,1,2,\ldots,$$

(A3) $$M(t) \geq 0$$

and subject to a wealth constraint as in the text. Note that (A1) and (A3) are equivalent to the Clower constraint. Eq. (A2) is the condition that the consumer exhausts his money before arriving at the bank. In an equilibrium model with positive nominal rates it is easy to derive (A2) as an optimal policy for individuals.

It is convenient to assume logarithmic utility. In this case a necessary condition for a consumer to be an optimum is that $e^{\beta t}P(t)c(t)$ is a constant between trips to the bank. The constant is determined so that the money withdrawn at t_o, $M(t_o)$, exhausts at $t_o + 1$. Let $M(t,t_o)$ denote the stock of money held at $t \varepsilon [t_o,t_o+1]$ of someone who goes to the bank at t_o. Then the above two remarks imply:

(A4) $$M(t,t_o) = M(t_o)\,\frac{e^{-\beta} - e^{-\beta(t-t_o)}}{e^{-\beta} - 1} \quad .$$

Let $\dot{M}^s dt$ be the size of the monetary injection which occurs via an open market operation between t and $t+dt$. The stock of money flowing into the bank between t and $t+dt$ is composed of the monetary injection plus spending flowing into stores $\dot{M}^s dt + P(t)ydt$, where $P(t)ydt$ is the value of spending during t to $t+dt$. Recall that output is fixed at a flow rate of y and the supply of goods must equal the demand for goods. Unless the nominal interest rate is zero the money withdrawn from the bank during the period x to $x+dx$ must equal the stock which has flowed in:

$$\text{(A5)} \qquad M(x)dx = \dot{M}^s(x) + p(x)ydx \quad .$$

Finally if $M^s(t)$ is the stock of money at time t, then (with a positive interest rate) this must be held by consumers. The total stock of money held is composed of the money held by individuals who go to the bank at the various dates between $t-1$ and t. That is, every individual is characterized by the date of his last trip to the bank. Thus supply = demand for money is given by:

$$\text{(A6)} \qquad M^s(t) = \int_{t-1}^{t} M(t,x)dx = \int_{t-1}^{t} [\dot{M}^s(x)+p(x)y]\left[\frac{e^{-\beta} - e^{-\beta(t-x)}}{e^{-\beta} - 1}\right] dx,$$

for $t \geq 2$. Note that (A5) gives the equilibrium money holdings of someone who has already been to the bank. Differentiation of (A6) yields

$$\text{(A7)} \qquad p(t)y = \beta M + \frac{\beta e^{-\beta t}}{1-e^{-\beta}} \int_{t-1}^{t} e^{\beta x}[\dot{M}^s(x) + p(x)y]dx \qquad t \geq 2$$

Given an initial price path between $t = 1$ and $t = 2$, (A7) can be used to generate the price path for all time. The initial price path is determined by the initial distribution of money as in the text for the discrete time case.

We are concerned with the steady state initial distribution of money. Assume for simplicity that $\dot{M}^s(x) \equiv 0$. The right hand side of equation (A5) states that money flows into the bank at a constant rate when $p(x) \equiv p$. Therefore withdrawals must occur at a constant rate, $M(x) \equiv M^w$.

A person who is x units of time from his last withdrawal has money holdings given by (A4):

$$M_o(x) \equiv M(t_o + x, t_o) = M^w \frac{e^{-\beta} - e^{-\beta x}}{e^{-\beta} - 1} \quad . \tag{A8}$$

But this must give the initial distribution of cash balances in the steady state!! That is, there are a continuum of traders labelled by $x \varepsilon [0,1)$. Person x must begin with cash equal to $M_o(x)$, and person x will arrive at the bank at $t = 1-x$. This must be the case because we have shown that with $P(x)$ constant, $M(x)$ is constant. Thus the arrival rate of people at the bank must be uniform in time (since the money flowing in is time homogeneous). Once we know that the arrival rate is uniform, (A8) gives the initial distribution of money which would lead to a uniform arrival rate. Thus, as we noted in the text it is not an equilibrium for everyone to hold the same amount of money.