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ON NON-UNIQUENESS IN RATIONAL EXPECTATIONS MODELS: AN ATTEMPT AT PERSPECTIVE

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ABSTRACT

Many macroeconomic models involving rational expectations give rise to an infinity of solution paths, even when the models are linear in all variables. Some writers have suggested that this non-uniqueness constitutes a serious weakness for the rational expectations hypothesis. One purpose of the present paper is to argue that the non-uniqueness in question is not properly attributable to the rationality hypothesis but, instead, is a general feature of dynamic models involving expectations. It is also argued that there typically exists, in a very wide class of linear rational expectations models, a single solution that excludes "bubble" or "bootstrap" effects--ones that occur only because they are arbitrarily expected to occur. A systematic procedure for obtaining solutions free from such effects is introduced and discussed. In addition, this procedure is used to interpret and reconsider several prominent examples with solution multiplicities, including ones developed by Fischer Black and John B. Taylor.

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I. Introduction

Numerous writers have commented, in recent years, on the multiplicity of solution paths in linear macroeconomic models involving rational expectations. A far-from-complete listing might include Black (1974), Blanchard (1979), Burmeister (1980a, 1980b), Brock (1975), Flood and Garber (1980a, 1980b), Gourieroux, Laffont, and Monfort (1979), Sargent and Wallace (1973), Shiller (1978), and Taylor (1977). While some of these authors have avoided general conclusions, others have suggested that this characteristic non-uniqueness constitutes a serious weakness for the rational expectations hypothesis.[1/] In fact, Gourieroux, Laffont, and Monfort have gone so far as to suggest that, because of the infinity of solution paths, "...the usefulness of the R.E. hypothesis would then seem rather doubtful..." (p. 2) and that the hypothesis "...clearly does not have yet the status of a theory" (p. 3).

Quite different opinions exist, of course. In a recent survey paper, Barro (1981, p. 54) remarks that "...it is interesting to note the divergent reactions to these types of uniqueness problems. One set of opinion regards these problems as symptomatic of the inadequacy of equilibrium analysis and even as evidence that private markets require government intervention. Another set regards them as empirically irrelevant intellectual curiosities, which will eventually be disposed of by deeper theoretical arguments."[2/]

The first purpose of the present paper is to argue that the multiplicity of solutions does not reflect any particular weakness of the rational expectations hypothesis, but that "deeper theorizing" is unlikely to dispose of the problem. Instead, the paper suggests that the non-uniqueness is simply an inescapable aspect of dynamic models involving expectations, one which is not basically attributable to the rationality assumption.

But, for this line of argument to be viable, there must exist some well-defined procedure that will single out a particular rational expectations solution in each of the class of models at hand--in this case, complete and internally consistent linear models. Otherwise, if a single solution were not selected in some way, the models would not be usable. The second major purpose of the paper, accordingly, is to describe such a procedure. The one proposed begins with the faimilar technique of undetermined coefficients, but augments that technique with the proviso that a minimal set of state variables be employed in agents' forecasting rules. That requirement has been utilized in many studies but usually without explicit acknowledgement of its role. The adoption of this one provision is not, as it happens, sufficient to yield a unique solution in all cases. Consequently, a second proviso--one that requires solution formulae to be valid for all admissable parameter values--is introduced. With these two provisos a unique solution can be obtained, the paper argues, for each of a wide class of linear models. The recommended procedure does not, it should be noted, rely upon any assumption or condition concerning dynamic stability of the system.

The third major purpose of the paper is to reconsider, using the suggested procedure, solution multiplicities featured in the well-known papers of Taylor (1977) and Black (1974). It is shown that these multiplicities are avoided by means of the suggested procedure and that the solutions singled out are sensible, i.e., do not possess peculiar or abberational properties. Similar conclusions are shown also to apply to the money/growth models of the type developed by Tobin (1965). Finally, some properties of the procedure are investigated by means of simple examples.

It should be said at the outset that the models considered are "macroeconomic", in the sense that underlying maximization problems for individual agents are not described. This approach would not be satisfactory if solution multiplicities never arose in cogent equilibrium models with maximizing agents. But such is not the case. If agents have infinite planning horizons, transversality conditions may rule out explosive paths, such as those discussed in Section II. They will not, however, eliminate multiplicities of the type described by Taylor (1977) and discussed in Sections III and IV; this has been demonstrated by Calvo (1979). And if agents have finite planning horixons, multiple solutions can arise as they do in overlapping generations models--see Wallace (1979). Thus, the need for some approach such as that taken in this paper evidently cannot be eliminated by restriction of attention to models consistent with competitive markets and maximizing agents.

II. Basic Considerations

Let us begin the discussion by considering the so-called Cagan model of inflation, which was utilized in the papers by Black (1974), Burmeister (1980b), Flood and Garber (1980b), Shiller (1978), Sargent-Wallace (1973), and many others.[3/] Thus we imagine an economy with constant output, wealth, and real interest rate, and with the following aggregate money demand function:

$$(1) \quad m_t - p_t = \gamma + \alpha({}_tp^e_{t+1} - p_t) + u_t \quad , \qquad \alpha < 0.$$

Here m_t and p_t are logs of the money stock and the price level, while ${}_tp^e_{t+1}$ is the value of p_{t+1} expected as of period t. The stochastic disturbance term u_t is generated by a white noise process; u_t is independent of past values of all variables. The money creation process is for the moment taken to be exogenous and autoregressive, viz.,

$$(2) \quad m_t = \mu_0 + \mu_1 m_{t-1} + e_t \quad , \qquad -1 < \mu_1 < 1.$$

The driving shock, e_t, is generated by a white noise process that is independent of the u_t process.

Now we examine the behavior of p_t in this economy under the assumption that expectations are rational, i.e., that ${}_tp^e_{t+1} = E_tp_{t+1}$, where $E_tp_{t+1} \equiv E(p_{t+1}|\Phi_t)$ with $\Phi_t = \{m_t, m_{t-1}, \ldots, p_t, p_{t-1}, \ldots\}$. The equations of the model are of course used in computing the conditional mathematical expectations. To obtain a solution to the model (1), (2), we begin by writing p_t as a linear function of the predetermined "state variables" m_t, u_t, and the constant 1:

(3) $p_t = \pi_0 + \pi_1 m_t + \pi_2 u_t$.

For appropriate real values of the coefficients π_0, π_1, π_2, the expectational variable $E_t p_{t+1}$ will then be given by

(4) $E_t p_{t+1} = \pi_0 + \pi_1 E_t m_{t+1} = \pi_0 + \pi_1(\mu_0 + \mu_1 m_t)$.

To evaluate the π's, substitute (3) and (4) into (1) and obtain

(5) $m_t = \gamma + (1-\alpha)[\pi_0 + \pi_1 m_t + \pi_2 u_t] + \alpha[\pi_0 + \pi_1(\mu_0 + \mu_1 m_t)] + u_t$.

But this of course implies identities in 1, m_t, and u_t as follows:

(6) $0 = \gamma + (1-\alpha)\pi_0 + \alpha(\pi_0 + \pi_1\mu_0)$

$1 = (1-\alpha)\pi_1 + \alpha\pi_1\mu_1$

$0 = (1-\alpha)\pi_2 + 1$.

And these are easily solved for

(7) $\pi_0 = -\gamma - \dfrac{\alpha\mu_0}{1-\alpha(1-\mu_1)}$

$\pi_1 = \dfrac{1}{1-\alpha(1-\mu_1)}$

$\pi_2 = \dfrac{-1}{1-\alpha}$.

Thus, with π_0, π_1, and π_2 given by (7), expression (3) provides a solution for p_t in terms of the predetermined variables and disturbances that appear in the model.

That this solution is not unique may be shown as follows. Suppose that expectations are given, not by (4), but by

(4') ${}_t p^e_{t+1} = \pi_0 + \pi_1(\mu_0 + \mu_1 m_t) + \pi_3\psi^{t+1}$

where π_3 and ψ are constants. Then it can be verified that p_t will obey

$$(3') \quad p_t = \pi_0 + \pi_1 m_t + \pi_2 u_t + \pi_3 \psi^t \quad ,$$

with π_0, π_1, and π_2 still given by (7), provided that[4/]

$$(8) \quad \psi = (\alpha-1)/\alpha \quad .$$

But that implies that ${}_tp^e_{t+1} = E_t p_{t+1}$ for any value of π_3. Thus (3') and (7) provide an infinity of rational expectations solutions, the "natural" solution (3) being a special case in which $\pi_3 = 0$.

Now the multiplicity of solutions just obtained is indeed unfortunate. But does it arise because of some peculiar deficiency of the rational expectations hypothesis? To answer that question, and place the non-uniqueness problem in perspective, one must necessarily consider other hypotheses concerning expectation formation. Historically, of course, single-variable distributed lag representations for ${}_tp^e_{t+1}$ in terms of $p_t, p_{t-1}, \ldots$ were routinely used before rational expectations came on the scene. Thus it was typically assumed, in both theoretical and applied analysis, that

$$(9) \quad {}_tp^e_{t+1} = \omega_0 p_t + \omega_1 p_{t-1} + \omega_2 p_{t-2} + \cdots$$

with restrictions such as $\omega_j \geqq 0$ and $\sum_{j=0}^{\infty} \omega_0 = 1$ often added. The most popular special case of (9) was, of course, the adaptive expectations scheme for the inflation rate,

$$(10) \quad {}_tp^e_{t+1} - p_t = (1-\beta) \sum_{j=0}^{\infty} (p_{t-j} - p_{t-j-1})\beta^j = \frac{(1-\beta)(1-L)p_t}{1-\beta L}$$

where $0 < \beta < 1$ and L is the lag operator, $L^n x_t = x_{t-n}$ for $n = 0, \pm 1, \pm 2, \ldots$. Let us then consider the behavior of p_t under this alternative expectational hypothesis. From (10) and (1) it is straightforward to find that

$$(11) \quad p_t = -\gamma + \frac{1}{1+\alpha-\alpha\beta} \sum_{j=0}^{\infty} [m_{t-j} - \beta m_{t-j-1} - u_{t+j} + \beta u_{t+j+1}]\theta^j$$

where $\theta \equiv (\beta+\alpha-\alpha\beta)/(1+\alpha-\alpha\beta)$, $0 < \theta < 1$, which is one way to write the solution to the system (1), (10).

Suppose now that we add the term $\pi_3 \phi^{t+1}$ to the adaptive expectations representation of ${}_tp^e_{t+1}$, just as we did in the case with rational expectations. Thus the system becomes (1) and

$$(10') \quad {}_tp^e_{t+1} - p_t = (1-\beta) \sum_{j=0}^{\infty} (p_{t-j} - p_{t-j-1})\beta^j + \pi_3 \phi^{t+1} \quad .$$

The solution then turns out to be the same as (11) but with an additional term, namely,

$$(12) \quad \frac{\pi_3 \alpha(\beta-\phi)}{1+\alpha-\alpha\beta} \phi^t \sum_{j=0}^{\infty} (\theta/\phi)^j$$

which will be finite provided that $|\theta/\phi| < 1$. So again there is an infinity of solutions. In fact, there are "more" solutions than in the rational expectations case, in the sense that ϕ is only required to satisfy $|\theta/\phi| < 1$, rather than the more stringent condition (8).

Of course one does not actually have to go through these last manipulations to show that extraneous terms can be added to adaptive or other distributed-lag expectational representations, as well as to RE representations. Indeed, many sorts of terms can be added to expectational representations that are not constrained to make expectations stochastically consistent with actual behavior. Evidently, therefore, rational expectations representations, which _are_ thus constrained, will accommodate fewer types of extraneous terms than distributed-lag representations. So the fact that RE solutions are not unique can hardly be considered a weakness for the RE hypothesis when each specific competing hypothesis--and therefore the general competing hypothesis of nonrationality--suffers from non-uniqueness to a greater degree. Instead, the appropriate conclusion seems to be that <u>any</u> dynamic system with expectational variables will have multiple solutions if extraneous terms are permitted to influence expectations.

At this point some readers may wish to ask precisely what is "extraneous" about the term ψ^{t+1} in the RE equation (4'). The answer is that it is extraneous in the well-defined and non-trivial sense that 1, m_t, and u_t constitute a complete set of state variables--i.e., a set that provides a solution for all admissible parameter values. The additional state variable ψ^t can be included, but it is unnecessary and is not suggested by the model.[5/] In particular, it appears in none of the non-expectational structural equations. Thus, the component $\pi_3\psi^{t+1}$ enters the solution solely because it is (arbitrarily) expected to do so--only because it affects agents' expectations of p_{t+1}. This component is not formally inconsistent with rational expectations, but it seems just as arbitrary to include it under that assumption as it would be under the adaptive (or any other) expectational hypothesis.[6/]

The foregoing does not imply, it should be added, that it is never appropriate to introduce extraneous state variables, for some interesting questions may be directly concerned with such variables. The notion of speculative bubbles can be formalized in such terms, for example, and it may be of interest to consider whether such bubbles have existed during particular historical episodes.[7/] But the possibility of doing so does not serve to discredit the rational expectations hypothesis. Nor does it render that hypothesis unusable for ordinary applications, in which the investigator wishes to rule out bubble or "bootstrap" effects--ones that occur only because they are expected to occur.

To this point, however, we have not shown that it is generally true that only one solution exists when extraneous terms are eliminated from the expectational representation. Certainly it is _not_ true that all examples of solution multiplicity involve trend-like terms such as ψ^{t+1}. In the next section, accordingly, we turn to examples of a different type and to more general cases.

III. Solution Procedure

Implicit in the foregoing argument, that the existence of multiple solutions should not be regarded as particularly awkward for the rational expectations hypothesis, is a presumption that it is possible in all cases to single out a certain unique solution that does not include components that enter only because they are expected to enter. More specifically, it is presumed that there is a solution procedure which singles out these special "bubble-free" solutions. The procedure in question begins, of course, with the method of undetermined coefficients, which was introduced into the RE literature by Lucas (1972). But this method itself will not, as we have seen, eliminate all multiplicities. Also necessary is the proviso or requirement that a minimal set of state variables--one without extraneous variables--be employed. To make this requirement precise, let us now define a minimal set of state variables as one from which it is impossible to delete (i.e., attach a π-coefficient of value zero to) any single variable, or group of variables, while continuing to obtain a solution valid for all admissible parameter values. In the model (1) (2), for example, a minimal set is 1, m_t, u_t: none of these can be deleted from the solution equation (3).[8/] Adding the extraneous variable $[(\alpha-1)/\alpha]^t$ results by contrast in a non-minimal set; the coefficient on this variable can be set equal to zero in (4') and a solution obtained nevertheless.

But even with a minimal set of state variables there is an apparent difficulty: in many applications the identities, such as (6), which relate solution coefficients [i.e., the π's in (3)] to basic structural parameters, will be nonlinear. Thus, there may be two or more values indicated by the method for the coefficients. It is possible, neverthless, to choose between (or among) these π-values by an extension of the minimal state variable strategy.[9/]

To illustrate this possibility, let us consider an example that again uses the Cagan demand function (1), but which relates money creation to past values of the price level as follows:

$$(13) \quad m_t = \rho_0 + \rho_1 \rho_{t-1} + e_t \quad , \qquad |\rho_1| < 1.$$

Then m_t is not exogenous and the minimal set of state variables is 1, p_{t-1}, u_t, and e_t. Consequently, the solution equation for p_t will be of the form

$$(14) \quad p_t = \pi_0 + \pi_1 p_{t-1} + \pi_2 u_t + \pi_3 e_t \quad .$$

The representation of the expectational variable is, accordingly,

$$(15) \quad E_t p_{t+1} = \pi_0 + \pi_1 p_t = \pi_0 + \pi_1(\pi_0 + \pi_1 p_{t-1} + \pi_2 u_t + \pi_3 e_t) \quad .$$

Combining (1), (13), and (15) yields

$$(16) \quad \rho_0 + \rho_1 p_{t-1} + e_t = \gamma + \alpha[\pi_0 + \pi_1 \pi_0 + \pi_1^2 p_{t-1} + \pi_1 \pi_2 u_t + \pi_1 \pi_3 e_t]$$
$$+ (1-\alpha)(\pi_0 + \pi_1 p_{t-1} + \pi_2 u_t + \pi_3 e_t) + u_t \quad ,$$

so the implied identities are

$$(17) \quad \begin{aligned} \rho_0 &= \gamma + \alpha\pi_1\pi_0 + \quad 0 \\ \rho_1 &= \alpha\pi_1^2 + (1-\alpha)\pi_1 \\ 1 &= \alpha\pi_1\pi_3 + (1-\alpha)\pi_3 \\ 0 &= \alpha\pi_1\pi_2 + (1-\alpha)\pi_2 + 1 \quad . \end{aligned}$$

Now the second of these identities yields

$$(18) \quad \pi_1 = \frac{\alpha-1 \pm \sqrt{(\alpha-1)^2 + 4\alpha\rho_1}}{2\alpha}$$

so there are two possibilities for π_1. To choose between them, however, let us consider the special case in which $\rho_1 = 0$. Then p_{t-1} would not appear in the system and so would not be included in the minimal set of state variables. In that case, then, the value of π_1 would be zero. But examination of (18) shows that the zero root would be obtained if the positive square root were used in the quadratic formula (since $\alpha-1 < 0$). This indicates that the positive square root is generally appropriate, i.e., for non-zero values of ρ_1, which conclusion provides a unique solution for π_1. The remaining equations (17) can then be solved for unique values for π_0, π_2, and π_3.

At this point it might be asked: is there any methodological principle that justifies the foregoing conclusion that the positive square root is appropriate for $\rho_1 \neq 0$ in (18)? As it happens, it appears that an intelligible principle is in fact available. In particular, the inappropriate solution can be eliminated by adopting the requirement that the expression defining π_1 (or any such solution coefficient) must be valid for all admissible values of the structural parameters. Then the negative square root would be ruled out provided that the admissible parameter values for ρ_1 include zero--as they are specified to do in (13).

The type of reasoning employed in the previous paragraph can, I believe, be used quite generally to eliminate apparent multiplicities that result from non-linear identities analogous to (17).[10/] The point is that the identities are nonlinear in precisely those cases in which the minimal set of state variables includes lagged values of (some of) the model's endogenous variables. Thus there always exists the possibility of considering hypothetical special cases in which the lagged endogenous variables do not appear and using these case to infer--

as in the foregoing example--which of the solutions is relevant. That the procedure is in principle applicable in more general mojels, with a large number of endogenous variables, is demonstrated in the Appendix.

Referring again to the example provided by (1) and (13), it should be mentioned that the stipulation $|\rho_1| < 1.0$ serves to guarantee that the solution values in (18) will be real. For values of $\rho_1 > 1.0$, however, it would be possible to have $(\alpha-1)^2 + 4\alpha\rho_1 < 0$, so that the solution values to (18) would be complex. But a complex value for π_1 is inconsistent with our formulation, so the conclusion given by the suggested procedure is that in such a case no solution for p_t exists--the model's specification is in some sense internally inconsistent. In the case at hand the inconsistency apparently arises from the excessively explosive behavior of m_t and p_t implied by $\rho_1 > (\alpha-1)^2/(-4\alpha)$.

In describing the proposed solution procedure, undetermined coefficients with a minimal set of state variables, we have thus far considered only cases in which the system's disturbances are white noise. But the procedure is essentially the same for more general disturbance processes. Suppose, for example, that u_t is generated by a first-order autoregressive, first-order moving average process (often denoted ARMA(1,1)). In other words, assume that

$$u_t = \rho u_{t-1} + \varepsilon_t - \theta\varepsilon_{t-1} \quad ,$$

where $|\rho| < 1$. Then the latter should be regarded as one of the "structural" equations of the model, which makes u_t a dependent variable and u_{t-1} a predetermined variable. A solution in terms of predetermined variables and disturbances can then be obtained, with u_{t-1}, ε_t, and ε_{t-1} included in the (minimal) set of state variables. This sort of approach can evidently be extended to higher-order ARMA processes.

Another case that needs to be mentioned is that in which lagged expectations, such as $E_{t-j}x_t = E(x_t | \Phi_{t-j})$, appear. This case can be handled by including j-1 lagged values of all predetermined variables and disturbances. As an example, consider the model

$$x_t = \alpha_1 x_{t-1} + \alpha_2 E_{t-2} x_t + u_t$$

with $u_t = \rho u_{t-1} + \varepsilon_t$ and ε_t white noise. Then the minimal set of state variables would include x_{t-1}, u_{t-1}, ε_t, and ε_{t-1}. To solve for these coefficients in

$$x_t = \pi_1 x_{t-1} + \pi_2 u_{t-1} + \pi_3 \varepsilon_t + \pi_4 \varepsilon_{t-1}$$

we note that

$$E_{t-2} x_t = E_{t-2}(\pi_1 x_{t-1} + \pi_2 u_{t-1})$$

$$= \pi_1 (x_{t-1} - \pi_4 \varepsilon_{t-1}) + \pi_2 (u_{t-1} - \varepsilon_{t-1}).$$

Substitution into the original equation and recognition of identities then leads to

$$\pi_1 = \alpha_1/(1 - \alpha_2)$$

$$\pi_2 = \rho/(1 - \alpha_2)$$

$$\pi_3 = 1$$

$$\pi_4 = - \alpha_2(\rho_1 + \alpha_1)/(1 + \alpha_2)$$

More generally, lagged expectations of x_{t+i} could appear. If $i \geq 0$, the previous discussion applies directly. If $i < 0$, it may be possible to delete some of the lagged values of predetermined variables or disturbances.

The foregoing procedural rules agree with those described by Aoki and Canzoneri (1979, p. 64). They appear to provide sufficient scope for handling any system meeting Shiller's (1978) definition of "the general linear rational expectations model". Thus, it is possible, as well as potentially desirable, to single out solutions that do not contain bubble-type or bootstrap components.

IV. Taylor's Model

Let us now apply the procedure suggested above to the model discussed by Taylor (1977). This model is of special interest because it is one in which a potential solution multiplicity, described by Taylor, cannot be eliminated by assuming-- as some writers have done-- that the conditionally expected price path must be stable. The multiplicity can, nevertheless, be eliminated by means of the procedure described above.

As initially specified, Taylor's model includes IS- and LM-type functions and an aggregate supply function with a real-balance term. But because he considers a case with a constant money stock, Taylor is able to reduce the system to a single equation involving actual and expected values of the (log of the) price level. That equation is

$$(19) \quad E_{t-1}p_{t+1} = E_{t-1}p_t + \delta_1 p_t + \delta_0 + u_t \quad ,$$

where u_t is again a white noise disturbance. The (composite) parameter δ_1 is assumed to be non-zero but Taylor admits both positive and negative values. The value of the constant term, δ_0, is dependent upon the (constant) value of the money stock.

Let us then assume rational expectations and apply the procedure recommended above, undetermined coefficients with a minimal set of state variables. In this model, the only necessary state variables are 1 and u_t. Thus the solution to (19) will be of the form[11/]

$$(20) \quad p_t = \pi_0 + \pi_1 u_t \quad ,$$

if appropriate values are chosen for π_0 and π_1. The expectational variables will then be

$$(21) \quad E_{t-1}p_t = \pi_0 + \pi_1 E_{t-1}u_t = \pi_0$$

and

$$(22) \quad E_{t-1}p_{t+1} = \pi_0 + \pi_1 E_{t-1}u_{t+1} = \pi_0 \quad .$$

Substitution of (20), (21), and (22) into (19) gives

$$\pi_0 = \pi_0 + \delta_1(\pi_0 + \pi_1 u_t) + \delta_0 + u_t \quad ,$$

which implies that $\pi_0 = -\delta_0/\delta_1$ and $\pi_1 = -1/\delta_1$. Accordingly, the solution obtained by this procedure is

$$(23) \quad p_t = -(\delta_0/\delta_1) - (1/\delta_1)u_t \quad .$$

The foregoing solution is mentioned by Taylor as the only one with a finite variance when $\delta_1 > 0$. But, as stated above, he admits the possibility that $-2 < \delta_1 < 0$, in which case he finds that "even after imposition of the stability condition multiple equilibria remain" (p. 1382). How does Taylor obtain these additional equilibria? In terms of our approach, what he does is augment the set of state variables so as to include p_{t-1} and u_{t-1}, in addition to 1 and u_t. Thus he considers solutions of the form

$$(20') \quad p_t = \pi_0 + \pi_1 u_t + \pi_2 p_{t-1} + \pi_3 u_{t-1} \quad .$$

In this case, the expectational expressions are

$$(21') \quad E_{t-1}p_t = \pi_0 + \pi_2 p_{t-1} + \pi_3 u_{t-1}$$

and

$$(22') \quad E_{t-1}p_{t+1} = \pi_0 + \pi_2(\pi_0 + \pi_2 p_{t-1} + \pi_3 u_{t-1}) \quad .$$

Substitution into (19) now results in

$$\pi_0 + \pi_2\pi_0 + \pi_2^2 p_{t-1} + \pi_2\pi_3 u_{t-1} = \pi_0 + \pi_2 p_{t-1} + \pi_3 u_{t-1}$$

$$+ \delta_1(\pi_0 + \pi_1 u_t + \pi_2 p_{t-1} + \pi_3 u_{t-1}) + \delta_0 + u_t \quad .$$

The implied identities are then

$$(24) \quad \begin{aligned} \pi_2\pi_0 &= \delta_1\pi_0 + \delta_1 \\ 0 &= \delta_1\pi_1 + 1 \\ \pi_2^2 &= \pi_2 + \delta_1\pi_2 \\ \pi_2\pi_3 &= \pi_3 + \delta_1\pi_3 \quad . \end{aligned}$$

As before we have $\pi_1 = -1/\delta_1$, but equations (24) do not uniquely define the other parameters. The third equation implies that either $\pi_2 = 1 + \delta_1$ or $\pi_2 = 0$. In effect, Taylor's response was to opt for the former solution, $\pi_2 = 1 + \delta_1$.[12/] Given that choice, one is not able to pin down π_3 from the fourth equation. So by choosing $\pi_2 = 1 + \delta_1$ instead of $\pi_2 = 0$, Taylor was led to the conclusion that π_3 can take on an infinite number of values.

But of course the procedure recommended here would treat p_{t-1} as an extraneous variable and therefore lead to the $\pi_2 = 0$ solution. In this case the fourth of equations (24) would imply $\pi_3 = 0$, so a single solution would be indicated.

V. Fischer Black's Model

Next, let us turn our attention to another famous non-uniqueness example, that of Fischer Black (1974). Black's "basic model" is simply a special case of the system (1)-(2) above, the special case in which m is constant (so that $\mu_0 = m$, $\mu_1 = 0$, and $e_t \equiv 0$), $\gamma = 0$, and $u_t \equiv 0$. Of course the last of these conditions makes the system non-stochastic so that rational expectations collapses to "perfect foresight." In some respects perfect foresight systems may be easier to analyze that ones with non-degenerate rational expectations, but in the present context the stochastic system is conceptually clearer because it distinguishes a period's price level from its expectation. That is, the stochastic framework lets one know whether he is referring to p_t or $E_{t-j}p_t$. Consequently, let us adopt Black's assumptions $\mu_0 = m$, $\mu_1 = 0$, $e_t \equiv 0$, and $\gamma = 0$, but continue to view u_t as a non-degenerate, white-noise, random variable.

In this case, the system (1)-(2) becomes

$$(25) \quad m - p_t = \alpha(E_t p_{t+1} - p_t) + u_t \quad .$$

As in the Taylor problem, we seek a solution of the form

$$(26) \quad p_t = \pi_0 + \pi_1 u_t \quad .$$

It is easy to verify that $\pi_0 = m$ and $\pi_1 = 1/(\alpha-1)$ so that the solution is

$$(27) \quad p_t = m + \frac{1}{\alpha-1} u_t \quad .$$

Thus, p_t fluctuates randomly about m while the inflation rate, $\Delta p_t = p_t - p_{t-1}$, follows an ARMA (0,1) process with MA parameter equal to -1.

In the non-stochastic case considered by Black, with $u_t \equiv 0$, we would have $p_t = m$ and $\Delta p_t = 0$ for all t.

These last conclusions differ, of course, from Black's, which are that p_t is not defined while Δp_t approaches $+\infty$ or $-\infty$ as time passes, depending on whether the "initial" value of the inflation rate is positive or negative. Again, according to the view here proposed, Black's conclusion results from his inclusion of an extraneous state variable. Instead of (26) he in effect considers (as in the Taylor model) solutions of the form

$$(26') \quad p_t = \pi_0 + \pi_1 u_t + \pi_2 p_{t-1} + \pi_3 u_{t-1} \quad .$$

In this case, $E_t p_{t+1} = \pi_0 + \pi_2(\pi_0 + \pi_1 u_t + \pi_2 p_{t-1} + \pi_3 u_{t-1}) + \pi_3 u_t$ so the implied identities in 1, u_t, p_{t-1}, and u_{t-1} are as follows:

$$(28) \quad m = \alpha\pi_0 + \alpha\pi_0\pi_2 + (1-\alpha)\pi_0 = \pi_0(1+\alpha\pi_2)$$

$$0 = \alpha\pi_2\pi_1 + (1-\alpha)\pi_1 + \alpha\pi_3 + 1$$

$$0 = \alpha\pi_2^2 + (1-\alpha)\pi_2$$

$$0 = \alpha\pi_2\pi_3 + (1-\alpha)\pi_3 \quad .$$

The third of these has two roots, $\pi_2 = 0$ and $\pi_2 = (\alpha-1)/\alpha$. Our solution procedure would select $\pi_2 = 0$, giving (27) as the solution for p_t, but Black implicitly chooses $\pi_2 = (\alpha-1)/\alpha$. This choice implies $\pi_0 = m/\alpha$ and $\pi_3 = -1/\alpha$, but leaves π_1 undetermined (as is π_3 in Taylor's model). Let us nevertheless consider Black's non-stochastic case with $u_t \equiv 0$. Then his solution for p_t is

$$(29) \quad p_t = \frac{m}{\alpha} + \frac{\alpha-1}{\alpha} p_{t-1} \quad ,$$

and for the inflation rate we have

$$(30) \quad \Delta p_t = \frac{\alpha-1}{\alpha} \Delta p_{t-1} \quad .$$

Black uses the latter-- which is the same as his (6)-- to write

$$(31)\quad \Delta p_t = [(\alpha-1)/\alpha]^t \Delta p_0$$

and concludes that "if the initial value of $[\Delta p_0]$ is positive, the rate of inflation increases exponentially over time" while "if the initial value of $[\Delta p_0]$ is negative the rate of deflation increases exponentially over time" so that "there is a price level path consistent with every possible choice of the initial price level" (p. 57) despite the constancy of m. Thus, he suggests, holding m constant has no implications for inflation.

What Black neglects to mention, given his approach, is the possibility $\Delta p_t = 0$ for all t. Emphasis on the difference equation aspect of (30) leads one to ignore the "trivial" solution $\Delta p_t = 0$. But in this problem it is, trivial or not, an entirely sensible solution-- one in which each period's price level is independent of past events. This was overlooked by Black because, I suspect, the non-stochastic version of the model leads to a difference equation in p_t that obscures the solution in which each period's "equilibrium" value for p_t is independent of the past.

In Black's second model (1974, pp. 58-60), monetary growth in each period depends on the previous period's inflation rate:

$$(32)\quad \Delta m_t = k\Delta p_{t-1} \quad , \qquad k > 1.$$

First differencing of the money demand equation gives

$$(33)\quad \Delta m_t - \Delta p_t = \alpha(E_t\Delta p_{t+1} - E_{t-1}\Delta p_t) + \xi_t$$

where $\xi_t \equiv u_t - u_{t-1}$. For simplicity, let us assume that ξ_t is white noise (i.e., that $\{u_t\}$ is a random walk).[13/] Then our procedure suggests a solution of the form

(34) $\Delta p_t = \pi_1 \Delta p_{t-1} + \pi_2 \xi_t$.

in which case $E_t \Delta p_{t+1} = \pi_1 \Delta p_t$ and $E_{t-1} \Delta p_t = \pi_1 \Delta p_{t-1}$. Substitution into (33) then yields

(35) $k\Delta p_{t-1} = (1+\alpha\pi_1)[\pi_1 \Delta p_{t-1} + \pi_2 \xi_t] - \alpha\pi_1 \Delta p_{t-1} + \xi_t$,

so the implied identities are

(36) $k = (1+\alpha\pi_1)\pi_1 - \alpha\pi_1$

$0 = (1+\alpha\pi_1)\pi_2 + 1$.

The roots for the first of these are-- just as in (18)--

(37) $\pi_1 = \dfrac{(\alpha-1) \pm \sqrt{(\alpha-1)^2 + 4\alpha k}}{2\alpha}$,

and consideration of the case $k = 0$ indicates that the positive square root is relevant. So we have

(38) $\Delta p_t = \pi_1 \Delta p_{t-1} - \dfrac{1}{1+\alpha\pi_1} \xi_t$

with $\pi_1 = [(\alpha-1) + \sqrt{(\alpha-1)^2 + 4\alpha k}\,]/2\alpha$. If $k \leqq -(\alpha-1)^2/4\alpha$, π_1 will be real. If in addition $\alpha < -1$, π_1 will be less than 1.0 in absolute value. Thus in this case the behavior of the inflation rate Δp_t is stable. But π_1 will be greater than 1.0, and Δp_t will be explosive, if $|\alpha|$ is small enough.[14] In either case, the inflation rate will be uniquely defined if an initial value is given for Δp_0, so as to "start up" (32). And if p_0 and p_{-1} are given separately, the price level will also be uniquely defined.

Black's conclusions are different. The reason is that, as before, he includes an extraneous term in his solution. In particular, he obtains a solution of the form

(39) $\Delta p_t = a_1\lambda_1^t + a_2\lambda_2^t$,

where λ_1 and λ_2 are the roots in (37). Then, since two initial conditions are needed to determine a_1 and a_2, the path of the inflation rate is not defined if only a single value of Δp is given, even though that is enough to start up the money supply process. But use of my recommended procedure would in effect set $a_2 = 0$ in (39), in which case the nonuniqueness would disappear-- as we have seen in the previous paragraph.

It remains to consider the possibility $k > -(\alpha-1)^2/4\alpha$, in which case the roots in (37) are complex.[15] In terms of my solution procedure, this suggests that the model has no economically sensible solution. The problem, I believe, is that the posited money supply behavior lacks "process consistency," in the language of Flood and Garber (1980a). With large k, expected money supply growth is too rapid for agents to be willing to hold money. This conclusion contrasts sharply with Black's suggestion that the inflation rate will, as in the case with $k \leqq -(\alpha-1)^2/4\alpha$, approach zero as time passes when (as he assumes) $\alpha < -1$.

VI. Money Growth Model

Next we consider a two-asset growth model of the type introduced by Tobin (1965), the stability of which has been questioned by Sidrauski (1967), Nagatani (1970), Burmeister (1980a), and many others. Temporarily letting y, k, m, and s denote per-capita magnitudes of income, capital, money, and saving, with r and p the real interest rate and the price level, the model can be written as follows:

(40) $y = f(k)$ [production function]

(41) $r = f'(k)$ [marginal product condition]

(42) $s = \sigma[y + \frac{m}{p}(\frac{\dot{m}}{m} - n - \frac{\dot{p}}{p})]$ [saving]

(43) $\dot{k} = s - \frac{m}{p}(\frac{\dot{m}}{m} - n - \frac{\dot{p}}{p}) - nk$ [capital accumulation]

(44) $\frac{m}{p} = \ell(y, k, r + \dot{p}/p)$ [portfolio balance]

(45) $\dot{m}/m = \mu$ [money growth].

Here σ $(0 < \sigma < 1)$ is the fraction of disposable income saved and n is the rate of population growth. Assume that f and ℓ have the usual properties. By combining the first four equations and using $\dot{m}/m = \mu$ we can obtain

(46) $\dot{k} = \sigma f(k) - (1-\sigma)(m/p)(\mu-n-\dot{p}/p) - nk$.

And by substituting (40) and (41) into (44) we get

(47) $m/p = \phi(k, \dot{p}/p)$.

Given the (exogenous) behavior of m, (46) and (47) then determine the evolution of k and p. The usual analysis views (46), (47) as a system of two first order differential equations. With the usual properties assumed for f and ℓ, the equilibrium for (k,p) is a saddlepoint so the system will be stable only if the "initial value" of p is "correct," given $k(0)$ -- see, e.g., Burmeister (1980a).

In order to see how our solution procedure would treat this model, we need to adopt discrete-time, linear versions of (45)-(47). These will seem to be more plausible if k, m, and p are now thought of as logarithms of the corresponding variables. As before, the system will be made stochastic, in which case one is led to recognize that $\dot{p}/p$ in both (46) and (47) is appropriately interpreted as the expected rate of inflation. Thus our version of the model is as follows:

$$(48)\quad k_{t+1} - k_t = \alpha_0 + \alpha_1 k_t + \alpha_2(E_t p_{t+1} - p_t) + \alpha_3(m_t - p_t) + u_t \quad , -1 \leqq \alpha_1 < 0, \alpha_2 > 0, \alpha_3 > 0.$$

$$(49)\quad m_{t+1} - E_t p_{t+1} = \beta_0 + \beta_1 k_t + \beta_2(E_t p_{t+1} - p_t) + \beta_3(m_t - p_t) + v_t \quad , \quad \beta_1 > 0,\ \beta_2 < 0,\ \beta_3 > 0.$$

In this form the model is obviously quite similar to that of Lucas (1975, p. 1117). There are some differences, however: the relevant real return on capital is a function of k_t rather than k_{t+1}, the real-balance terms which Lucas deletes are here included, and m_{t+1} is deflated by $E_t p_{t+1}$.[16/]

A minimal set of state variables for (48) (49), given the policy equation

$$(50)\quad m_t - m_{t-1} = \mu \quad ,$$

is $1, k_t, m_t, u_t, v_t$. So solution equations will be of the form

(51) $k_{t+1} = \pi_{10} + \pi_{11}k_t + \pi_{12}m_t + \pi_{13}u_t + \pi_{14}v_t$,

(52) $p_t = \pi_{20} + \pi_{21}k_t + \pi_{22}m_t + \pi_{23}u_t + \pi_{24}v_t$.

The expectational representation is then

(53) $E_t p_{t+1} = \pi_{20} + \pi_{21}(\pi_{10} + \pi_{11}k_t + \pi_{12}m_t + \pi_{13}u_t + \pi_{14}v_t) + \pi_{22}(m_t + \mu)$.

Substituting (50)-(53) into (48) yields

(54) $$\pi_{10} + \pi_{11}k_t + \pi_{12}m_t + \pi_{13}u_t + \pi_{14}v_t = \alpha_0 + (\alpha_1 + 1)k_t$$

$$+ \alpha_2[\pi_{20} + \pi_{21}(\pi_{10} + \pi_{11}k_t + \pi_{12}m_t + \pi_{13}u_t + \pi_{14}v_t) + \pi_{22}(m_t + \mu)]$$

$$- (\alpha_2 + \alpha_3)[\pi_{20} + \pi_{21}k_t + \pi_{22}m_t + \pi_{23}u_t + \pi_{24}v_t] + \alpha_3 m_t + u_t \quad .$$

Let us record only the identities implied by equating coefficients of k_t and m_t . They are

(55) $$\pi_{11} = (\alpha_1 + 1) + \alpha_2\pi_{21}\pi_{11} - (\alpha_2 + \alpha_3)\pi_{21} \quad ,$$

$$\pi_{12} = \alpha_2\pi_{21}\pi_{12} + \alpha_3(1 - \pi_{22}) \quad .$$

Analogously, substitution into (49) yields

(56) $$(1 + \mu - \beta_3)m_t = -(\beta_3 + \beta_2)[\pi_{20} + \pi_{21}k_t + \pi_{22}m_t + \pi_{23}u_t + \pi_{24}v_t] + \beta_0 + \beta_1 k_t$$

$$+ (1 + \beta_2)[\pi_{20} + \pi_{21}(\pi_{10} + \pi_{11}k_t + \pi_{12}m_t + \pi_{13}u_t + \pi_{14}v_t) + \pi_{22}(m_t + \mu)] + v_t \quad ,$$

and, therefore, the identities

(57) $$0 = -(\beta_3 + \beta_2)\pi_{21} + \beta_1 + (1 + \beta_2)\pi_{21}\pi_{11} \quad ,$$

$$1 = \beta_3 - (\beta_3 + \beta_2)\pi_{22} + (1 + \beta_2)(\pi_{21}\pi_{12} + \pi_{22}) - \mu$$

Using the first of equations (55) and (57) yields the following quadratic for π_{11} :

$$(58)\quad \pi_{11} = \frac{-q \pm \sqrt{q^2-4(1+\beta_2)[(1+\alpha_1)(\beta_3+\beta_2)-\beta_1(\alpha_2+\alpha_3)]}}{2(1+\beta_2)}$$

where $q \equiv -(\beta_3 + \beta_2) + \alpha_2\beta_1 - (1 + \beta_2)(1 + \alpha_1)$. To determine which root should be used, note from (48) and (51) that $\pi_{11} = 0$ should be obtained (i.e., k_t deleted) if $1 + \alpha_1 = \beta_1 = 0$. It can then be found from (58) that the positive (negative) square root is relevant if $\beta_2 + \beta_3 < 0$ ($\beta_2 + \beta_3 > 0$). Given this value of π_{11}, π_{21} can readily be found from the first of either (55) or (57). Then π_{12} and π_{22} can be found uniquely as they enter linearly in the second of equations (55) and (57).

The dynamic behavior of the system depends crucially, of course, on π_{11}. If we add to our sign conditions the weak restrictions $1 + \beta_2 < 0$ and $\beta_3 < |\beta_2|$, then π_{11} will be real and will have absolute value less than 1.0 so the system will be stable.

But the main point of the example is that application of the suggested procedure suggests that p_{t-1} does not enter the system (51) (52). Given the behavior of the exogenous variables, m_t, u_t, and v_t, the evolution of p_τ and k_τ is fully determined for $\tau = 1,2,\ldots$ once the single initial condition k_0 is assigned. There is no need to select a "correct" value-- indeed, any value-- for p_0.

Of course it would be possible, as in previous examples, to add an "extraneous" state variable to equations like (51) (52). In this case, one obvious contender would be p_{t-1}. But, as a result of the previous examples, one might conjecture that entering p_{t-1} could lead to identities that would have multiple solutions, one of which is zero. To see that this conjecture is correct, consider again the special case in which $1 + \alpha_1 = \beta_1 = 0$. Then the system determining p_t is (49)-(50) with k_t exogenous. Now add p_{t-1} and v_{t-1} to equation (52), obtaining

$$(52') \quad p_t = \pi_{20} + \pi_{21}k_t + \pi_{22}m_t + \pi_{23}u_t + \pi_{24}v_t + \pi_{25}p_{t-1} + \pi_{26}v_{t-1} \quad .$$

Next verify that

$$E_t p_{t+1} = \pi_{20} + \pi_{21}(\pi_{10} + u_t) + \pi_{22}(m_t + \mu)$$
$$+ \pi_{25}(\pi_{20} + \pi_{21}k_t + \pi_{22}m_t + \pi_{23}u_t + \pi_{24}v_t + \pi_{25}p_{t-1} + \pi_{26}v_{t-1})$$
$$+ \pi_{26}v_t \quad .$$

Substitution into (49) then gives a long expression that yields the following identities for p_{t-1} and v_{t-1}:

$$0 = (1 + \beta_2)\pi_{25}^2 - (\beta_2 + \beta_3)\pi_{25}$$

$$0 = (1 + \beta_2)\pi_{25}\pi_{26} - (\beta_2 + \beta_3)\pi_{26}$$

Clearly the first of these implies that either $\pi_{25} = 0$, or $\pi_{26} = (\beta_2 + \beta_3)/(1 + \beta_2)$. If the latter solution were chosen, the second of the identities would fail to determine π_{26}. But, just as in the Taylor example of Section IV, this results from the the inclusion of extraneous state variables.

VII. Additional Properties of the Solution Procedure

The purpose of this section is to make two points regarding our proposed solution procedure. The first point is that adoption of this procedure--undetermined coefficients with a minimal set of state variables-- is not equivalent to assuming stability for the model at hand. In order to demonstrate this, let us consider an example suggested by Burmeister (1980a, 1980b). In this example, money demand is given by the Cagan function, but price adjustments are presumed to be "sluggish." In particular, we adopt the following specification:

$$(59) \quad m_t^d = p_t + \gamma + \alpha(E_t p_{t+1} - p_t) \qquad \alpha < 0$$

$$(60) \quad p_t - p_{t-1} = \beta(m_{t-1} - m_{t-1}^d) + u_t \qquad \beta > 0$$

$$(61) \quad m_t = m \quad .$$

Thus prices rise when the money supply (assumed constant for simplicity) exceeds the quantity demanded.[17/] Now (59) and (60) can be combined to yield

$$(62) \quad p_t - p_{t-1} = \beta[m - p_{t-1} - \gamma - \alpha(E_{t-1}p_t - p_{t-1})] + u_t \quad .$$

The procedure suggests a solution for p_t of the form

$$(63) \quad p_t = \pi_0 + \pi_1 p_{t-1} + \pi_2 u_t \quad ,$$

so we have

$$(64) \quad E_{t-1}p_t = \pi_0 + \pi_1 p_{t-1} \quad ,$$

and substitution into (62) results in the implied identities:

$$(65)\quad \pi_0 = \beta(m - \gamma) - \alpha\beta\pi_0$$

$$\pi_1 = 1 - \beta + \alpha\beta - \alpha\beta\pi_1$$

$$\pi_2 = 1 \quad .$$

From the second of these we obtain

$$(66)\quad \pi_1 = 1 - \frac{\beta}{1+\alpha\beta} \quad .$$

Thus, for stability we must have

$$(67)\quad 1 + \alpha\beta > \beta/2 \quad .$$

But since $\alpha < 0$, this requirement is rather demanding. With $\alpha = -5$, for example, it implies $\beta < 0.182$. Accordingly, for even moderately large values of β, the solution defined by (63) and (65) will imply price level instability. Thus adoption of the undetermined coefficients solution procedure does not require that the model be stable, as some analysts have suggested.

The second point involves admissable parameter values. To illustrate the relevant issue let us refer to the model considered in Section II, which we repeat with $\gamma = \mu_0 = 0$ for convenience:

$$(1')\quad m_t - p_t = \alpha(E_t p_{t+1} - p_t) + u_t \qquad \alpha < 0$$

$$(2')\quad m_t = \mu_1 m_{t-1} + e_t \quad .$$

The solution provided by our procedure is, as above,

$$(68)\quad p_t = \frac{1}{1-\alpha(1-\mu_1)} m_t - \frac{1}{1-\alpha} u_t \quad .$$

Now in Section II it was assumed that $|\mu_1| < 1$. But suppose that $\mu_1 > 1$. Unless $\mu_1 = (\alpha-1)/\alpha$, the expressions in (68) will nevertheless continue to provide an apparent solution. In fact, (68) gives a finite value for p_t even if $\mu_1 > (\alpha-1)/\alpha$, i.e., even in the "process inconsistency" case in which

$$(69) \quad \sum_{j=0}^{\infty} E_t m_{t+j} [\alpha/(\alpha-1)]^j = m_t \sum_{j=0}^{\infty} [\mu_1 \alpha/(\alpha-1)]^j$$

fails to converge. But convergence of this series is required for (68) to be obtainable by the alternative procedure of solving

$$(70) \quad p_t = (\frac{1}{1-\alpha})[m_t - u_t] + (\frac{\alpha}{\alpha-1})E_t p_{t+1}$$

by repeated substitution into the (expectational) future, as in

$$(71) \quad p_t = -\frac{u_t}{1-\alpha} + \frac{1}{1-\alpha} E_t[m_t + am_{t+1} + a^2 m_{t+2} + \cdots] + \lim_{j\to\infty} a^j E_t p_{t+j}$$

$$= \frac{-u_t}{1-\alpha} + \frac{m_t}{1-\alpha} [1 + a\mu_1 + a^2\mu_1^2 + \cdots] + \lim_{j\to\infty} a^j E_t p_{t+j}$$

with $a = \alpha/(\alpha-1)$. Consequently, Flood and Garber (1980a) have contended that p_t would not be finite in this case.

Now the Flood-Garber contention is attractive, but it is nevertheless true that (68) satisfies equations (1') and (2'). What, then, should one make of this case? In my opinion, it points out the desirability of specifying admissible parameter values as an integral part of the model. The crucial fact is that the coefficient on m_t in (68), $\pi_1 = 1 / [1 - \alpha(1 - \mu_1)]$, approaches $+\infty$ as $\mu_1 \to (\alpha - 1) / \alpha$ from below and approaches $-\infty$ as $\mu_1 \to (\alpha - 1) / \alpha$ from above.

Thus, there is an infinite discontinuity in π_1 at $\mu_1 = (\alpha - 1) / \alpha$. It seems to be necessary to require that values on only one side of the discontinuity are admissible. Such a requirement is to some extent analogous to the familiar condition in static models that the Jacobian relevant for comparative-static experiments be non-zero -- for otherwise "multiplier" values will not give good predictions of the comparative-static effects in question. In the present case, the effects of e_t realizations on p_t will be entirely different depending on whether $\mu_1 < (\alpha - 1) / \alpha$. A sensible specification would include the last inequality as part of the model.

VIII. Conclusions

The argument of this paper does not, it should be said clearly, claim that there is only one solution path consistent with rational expectations in general or in the models considered. What it does claim is that one path can be singled out for special attention and that, relative to that path, others which satisfy the model occur because unnecessary or "extraneous" state variables are permitted to influence expected (and therefore actual) values of endogenous variables. In a sense, then, the argument proposes a condition or criterion which, if adopted, would lead to a unique solution.[18/]

The question then naturally arises, should this criterion be adopted? As it has no obvious choice-theoretic (utility maximizing) rationale, there seems to be no basis for suggesting that it should invariably be adopted, i.e., whatever the purpose of the analysis. But that is hardly surprising. Since its adoption directly rules out extraneous elements or "bubbles," adoption would clearly be out of place in any analysis the object of which is to determine whether bubbles exist.[19/] But of course most analyses do not have such an objective, and for many of these other, more mundane problems, the solution provided by the proposed criterion would seem to be well-suited.[20/]

In any event, there is no reason-- according to our argument-- to abandon rational expectations in favor of other currently-available expectational hypotheses. One cannot escape arbitrariness by simply rejecting the hypothesis-- that expectations are consistent with the model at hand-- in favor of its negation.

And while the proposed criterion cannot be used in cosmic analyses designed to determine whether extraneous variables-- or explosive instability resulting from the same-- are features of a market economy, it would also be entirely inappropriate in such analyses to presume[21] that such extraneous variable will necessarily be of importance. For it has been shown that, in several cases, the proposed criterion leads to cogent, economically plausible solutions. These have been ignored or deemphasized in the literature, apparently because the more general solution descriptions have obscured the special status of these paths.

Appendix

The object here is to show that the procedure described in Section III will yield unique solutions in models with several expectational variables. We first consider the case in which lagged endogenous variables are absent and then turn to the more difficult case in which the minimal set of state variables includes lagged endogenous variables.

Let y_t be the mxl vector of endogenous variables in the model and let z_t be the kxl vector of exogenous variables. Consider the psuedo-reduced form of the system

$$(A-1) \qquad y_t = AE_t y_{t+1} + Bz_t + u_t$$

where A and B are mxm and mxk parameter matrices and u_t is a mxl white noise vector. There will be adequate generality if we assume that z_t is generated by a first-order autoregressive scheme, viz.,

$$(A-2) \qquad z_t = Rz_t + e_t,$$

where R is a kxk parameter matrix and e_t is a kxl white noise vector.

In this system the minimal set of state variables includes only z_t and u_t so the solution equation will be of the form

$$(A-3) \qquad y_t = \Pi z_t + \Gamma u_t$$

where Π and Γ are mxk and mxm matrices of undetermined coefficients. The conditional expectation in (A-1) is then

(A-4) $$E_t y_{t+1} = \Pi E_t z_{t+1} + \Gamma E_t u_{t+1} = \Pi R z_t.$$

Substitution into (A-1) then yields

(A-5) $$\Pi z_t + \Gamma u_t = A\Pi R z_t + B z_t + u_t$$

so the identities in question are $\Gamma = I$ and

(A-6) $$\Pi = A\Pi R + B.$$

To see that Π is determined uniquely by (A-6) we transform A into the Jordan canonical form

(A-7) $$A = G^{-1}JG$$

where the eigenvalues λ_i of A appear on the diagonal of J. Premultiplying (A-6) by G then gives

(A-8) $$Q = JQR + S$$

where $Q = G\Pi$ and $S = GB$. If Q is uniquely determined by (A-8), $\Pi = G^{-1}Q$ will be unique.

First consider the subcase in which J is diagonal. Then (A-8) can be written as

(A-9) $$q_i = \lambda_i q_i R + s_i \qquad i = 1,\dots,m,$$

where q_i is the ith row of Q and s_i the ith row of S. Since λ_i is a scalar,

$$q_i[I - \lambda_i R] = s_i \qquad i = 1,\ldots,m, \tag{A-10}$$

and the q_i are uniquely determined if $[I - \lambda_i R]^{-1}$ exists for $i = 1,\ldots,m$. But $[I - \lambda_i R]^{-1}$ will exist unless λ_i^{-1} is an eigenvalue of R so, with A and R representing independent parameters, Π will be unique for almost all values of A and R.

In the subcase in which A is not diagonalizable, equations (A-9) and (A-10) must be modified. But the same conclusions result because J is triangular and again has the eigenvalues of A on its diagonal.

The same sort of result holds if the expectational variable is $E_t y_{t+j}$ for $j = 2,3,\ldots$. Then we have $E_t y_{t+j} = \Pi R^j z_t$ and the discussion proceeds as before but with R^j instead of R in (A-10).

Now we turn to the case in which lagged endogenous variables appear in the system so that (A-1) is replaced with

$$y_t = AE_t y_{t+1} + Bz_t + Cy_{t-1} + u_t, \tag{A-11}$$

where C is mxm, and instead of (A-3) we have

$$y_t = \Pi z_t + \Omega y_{t-1} + \Gamma u_t \tag{A-12}$$

with Π, Ω, and Γ to be determined. In this case the expectational representation is

$$E_t y_{t+1} = \Pi R z_t + \Omega y_t = \Pi R z_t + \Omega(\Pi z_t + \Omega y_{t-1} + \Gamma u_t). \tag{A-13}$$

The counterpart of (A-5) is then

$$\Pi z_t + \Omega y_{t-1} + \Gamma u_t = A\Pi R z_t + A\Omega(\Pi z_t + \Omega y_{t-1} + \Gamma u_t) + B z_t + C y_{t-1} + u_t. \tag{A-14}$$

so the relevant identities are

$$\begin{aligned} \Pi &= A\Pi R + A\Omega\Pi + B \\ \Omega &= A\Omega\Omega + C \\ \Gamma &= A\Omega\Gamma + I. \end{aligned} \tag{A-15}$$

The first and third of these would be uniquely solvable for Π and Γ if Ω were given, but clearly the second is not linear in the elements of Ω. This is the multivariable counterpart of the nonlinear identities that are discussed in Section III. In the present case, the number of solutions -- not all real, most probably -- is 2m.

A unique solution may nevertheless be obtained by a generalization of the approach proposed in Section III. Let us demonstrate this in the special case in which B = 0 in (A-11), which implies $\Pi=0$ but retains the troublesome non-linear equation involving Ω. In this case the model is

$$y_t = AE_t y_{t+1} + C y_{t-1} + u_t \tag{A-16}$$

and the solution equation is

(A-17) $$y_t = \Omega y_{t-1} + \Gamma u_t.$$

There will be no loss of generality if we take A to be invertible, as we shall.

The first step is to put the system in the form used by Blanchard and Kahn (1980). Thus we define

(A-18) $$x_t \equiv y_{t-1}$$

and write

(A-19) $$\begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A^{-1}C & A^{-1} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} 0 \\ -A^{-1} \end{bmatrix} u_t.$$

Then we have

(A-20) $$y_t = \Omega x_t + \Gamma u_t$$

(A-21) $$E_t y_{t+1} = \Omega(\Omega x_t + \Gamma u_t)$$

so (A-19) becomes

(A-22) $$\begin{bmatrix} \Omega x_t + \Gamma u_t \\ \Omega^2 x_t + \Omega\Gamma u_t \end{bmatrix} =$$

$$\begin{bmatrix} 0 & I \\ -A^{-1}C & A^{-1} \end{bmatrix} \begin{bmatrix} x_t \\ \Omega x_t + \Gamma u_t \end{bmatrix} + \begin{bmatrix} 0 \\ -A^{-1} \end{bmatrix} u_t.$$

The portion of this system corresponding to x_t then implies

$$\text{(A-23)} \qquad \begin{bmatrix} \Omega \\ \Omega^2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A^{-1}C & A^{-1} \end{bmatrix} \begin{bmatrix} I \\ \Omega \end{bmatrix}$$

Let the square matrix on the right be denoted M; it is 2m x 2m in dimension. Let Λ be the matrix with eigenvalues of M on its diagonal and let P^{-1} be the matrix of eigenvectors. Then $M = P^{-1}\Lambda P$ so we can premultiply (A-23) by P and obtain

$$\text{(A-24)} \qquad \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \Omega \\ \Omega^2 \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} I \\ \Omega \end{bmatrix}$$

For our purposes, the assignment of eigenvalues to Λ_1 and Λ_2 is both crucial and unusual. Note that each eigenvalue λ satisfies

$$\text{(A-25)} \qquad \det[M - \lambda I] = \det \begin{bmatrix} -\lambda I & I \\ -A^{-1}C & A^{-1}-\lambda I \end{bmatrix}$$

$$= \det[A^{-1} - \lambda I] \det[-\lambda I + I(A^{-1} - \lambda I)^{-1}A^{-1}C] = 0.$$

Thus when C=0, half of the eigenvalues of M equal zero while the other half equal the eigenvalues of A^{-1}, which are all non-zero. We then take Λ_1 to include the eigenvalues that approach 0 as $C \to 0$ and Λ_2 to include the others (which approach the eigenvalues of A^{-1}). That this assignment can be implemented follows from the fact that each eigenvalue is a continuous function of the elements of M.

Returning to (A-24) we write out the row equalities as follows:

(A-26) $$(P_{11} + P_{12}\Omega)\Omega = \Lambda_1(P_{11} + P_{12}\Omega)$$

(A-27) $$(P_{21} + P_{22}\Omega)\Omega = \Lambda_2(P_{21} + P_{22}\Omega).$$

These suggest the two solution expressions

(A-28) $$\Omega^{(1)} = - P_{12}^{-1}P_{11}$$

(A-29) $$\Omega^{(2)} = - P_{22}^{-1}P_{21}.$$

Each of these satisfies the model and gives a value for Ω that will, with the implied Γ, provide a solution to the model. But by our extended principle of minimal state variables, we require a solution expression that makes $\Omega=0$ in the case in which C=0. Now when C=0 we have $\Lambda_1=0$ by construction and (A-26) becomes

(A-30) $$(P_{11} + P_{12}\Omega)\Omega = 0.$$

But the latter will be satisfied by $\Omega^{(1)}$ whether or not it equals a matrix of zeros; in other words

$$(P_{11} + P_{12}\Omega^{(1)})\Omega^{(1)} = 0$$

does not imply $\Omega^{(1)}=0$. By contrast, (A-27) will be satisfied by $\Omega^{(2)}$ only if $\Omega^{(2)}=0$. Thus $\Omega^{(2)}$ is the only solution expression for Ω that is guaranteed to take on the value 0 when C=0, as our principle requires. Thus, $-P_{22}^{-1}P_{21}$ is the value of Ω in (A-17) that excludes bootstrap effects.

If one's selection criterion required stability, then $\Omega^{(2)}$ would be the indicated value of Ω if the eigenvectors of M were arranged so that each of the diagonal elements of Λ_1 has modulus less than 1.0. This is, however, a different criterion and can evidently lead to a different composition of Λ_1 and thus to a different solution. The continuity of the eigenvalues with respect to the elements of C suggests, nevertheless, that the composition of Λ_1 will be the same when the elements of C are small. Note finally that the criterion of Section III, unlike the stability criterion, provides a unique solution when more than m of the eigenvalues are real and have absolute values less than 1.0.

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Footnotes

[1/] Shiller, for example, says "The existence of so many solutions to the rational expectations model implies a fundamental indeterminacy for these models" (1978, p. 33). Blanchard states that "in models where anticipations of future endogenous variables influence current behavior, there exists an infinity of solutions under the assumption of rational expectations" (1979, p. 114). Burmeister (1980b) suggests that "one of the most crucial issues in rational expectations modelling ... concerns the dynamic properties of rational expectations paths and the manner in which the stability properties of these expectations serves to make determinate the stochastic properties of the actual variables" (1980b, pp. 800-801).

[2/] Where Barro says "equilibrium analysis," "rational expectations" would for present purposes be more accurate.

[3/] Here I use the term "Cagan model of inflation" to refer to any system that includes the money demand function (1) plus a money supply specification as behavioral equations which, with an assumption about expectations, determine p_t and m_t.

[4/] More generally, as Shiller (1978) and Flood and Garber (1980b) show, one can add $\pi_3(\psi^{t+1} + \zeta_{t+1})$, if ζ_{t+1} is any random variable satisfying $E(\zeta_{t+1}|\Phi_t) = 0$. The additional stochastic component is not of particular interest in the present context.

[5/] For what it is worth, it should be noted that in his original article Muth (1961, pp. 325-7) excluded such terms by his choice of solution procedure. As it is the same as Taylor's (1977), it does not rule out all types of solution multiplicity.

6/ Since first drafting this paper, I have learned that Wallace (1980) has made a similar argument in the context of a particular (overlapping generations) model. He does not, however, propose extensions such as those introduced below.

7/ See, for example, Flood and Garber (1980b).

8/ That there may be several minimal sets is not of importance, since they will imply (in the class of models considered) identical realizations of the endogenous variables. For example, in the model (1) (2) the set of 1, m_{t-1}, e_t, u_t is minimal but yields the same realizations for p_t as the minimal set 1, m_t, u_t.

9/ This statement implicitly assumes that the roots to the parameter identities are real. If instead they are complex, the appropriate conclusion, I believe, is that the model does not possess an economically sensible solution. (If some are real and the others complex, then the latter are irrelevant and the former can be considered as described in the text.)

10/ It can, for example, be used to rationalize the choice of solution values made by Lucas (1975, p. 1118). Examination of his (2.7) and (2.9) suggests that, in the special case with $\alpha_2 = 0$, π_{12} will equal zero in (2.11). Also, with $\alpha_2 = 0$ the solution for π_{11} will, by (2.14), be the one given in (2.19). Then $\pi_{12} = 0$ implies $\pi_{22} = 1$ by (2.18) and (2.19) consequently implies that π_{21} is as in Lucas's (2.20).

11/ It should be noted that the symbols π_0, π_1,... have different meanings here and in Taylor (1977).

12/ Taylor did not explicitly recognize the second possibility at this stage of the argument. But his proposed criterion for choosing among solutions-- minimum variance-- leads him at a later point to come back to the solution in (23).

13/ Thus we change the stochastic specification from the "basic model" case.

14/ Note that when $\alpha = 0$ the system reduces to $\Delta p_t = k\Delta p_{t-1} + \xi_t$, which is clearly unstable for $k > 1$.

15/ This sort of possibility is mentioned by King (1978) and Barro (1981).

16/ Equation (49) looks somewhat different than (47) but is, I believe, the discrete-time counterpart. In any case, the results obtained below also hold if (49) is replaced with $m_t - p_t = \beta_0 + \beta_1 k_t + \beta_2(E_t p_{t+1} - p_t) + v_t$.

17/ Burmeister (1980b) recognizes that the specification (60) is questionable.

18/ Other possible criteria are discussed by Blanchard (1979), who finds none satisfactory. Taylor (1977) also reviews earlier suggestions and proposes a criterion of his own-- that of minimizing the unconditional variance of the price level. In monetary models of the type considered above, Taylor's criterion leads to the same solution as the one proposed in this paper. It is not clear that such would be true in more complex models, however. In any event, their rationale is different.

19/ It seems unlikely that any theoretical analysis will be able to rule out the possibility of bubbles. If so, investigations of the type mentioned will need to be empirical. One such investigation has been carried out by Flood and Garber (1980b).

[20] For an example of an application of this type, see McCallum (1980).

[21] As some writers have come very close to doing. See, for example, Hahn (1969), and Shell and Stiglitz (1967).