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A MULTIPLICATIVE MODEL OF INVESTMENT IN HUMAN CAPITAL

Yoram Weiss

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National Bureau of Economic Research, Inc.
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I. Introduction

In recent years we have seen the rise of a bold and fruitful approach which attempts to explain the development of individual earnings as if they result from a continuous choice process. A basic part of this approach is the on-the-job training hypothesis (See Becker [1964], Mincer [1962, 1974], Ben Porath (1967], Rosen [1972, 1973]) whereby individuals face at each point in their lifetime, a set of options which involve the trading of current earnings in exchange for higher future earning capacity. Given these options the individual chooses an optimal strategy which is then reflected in his observed earnings profile.

The basic qualitative result of this approach is that investment is decreasing throughout life, and therefore observed earnings should increase as long as net investment is positive. There are, however, many additional aspects of lifetime earnings which can be analyzed within the investment framework. The purpose of this paper is to analyze the effects of changes in exogenous parameters such as the interest rate, the length of the working period and initial endowments on the shape of the observed earnings profile. Though this problem can be treated in general, we shall restrict ourselves to the following "inverse optimal" problem: find a form of the trade-off function between current and future earnings which leads to a logarithmic earnings function. Since such an earning function is most frequently used in econometric research (most notably by Mincer [1974]), it is natural to inquire what restrictions on it are implied by an optimal accumulation of human capital.

Limiting the earnings function form to the logarithmic class leads us to adopt a particular multiplicative specification for the trade-off between current and future earnings. Under this specification jobs are ranked according to the rate of growth in earning capacity which they offer. In this formulation the trade-off is described by a relation whereby a higher growth rate is associated with a sacrifice of a higher proportion of current earnings capacity. This specification which has been used by Blinder and Weiss [1976] and Rosen [1975], should be distinguished from the alternative additive specification: where jobs are ranked on the basis of the absolute growth which they provide, and costs are defined in absolute dollar terms rather than as a proportion of earning capacity. This special case of the Ben Porath [1967] model was analyzed in detail by Rosen [1973] Haley [1973], Lillard [1973], Wallace and Ihnen [1975], Brown [1976], and Heckman [1975]. The additive form which constrains the absolute growth in earnings does not place direct restrictions on the behavior of log earnings over the life time.

In the paper we demonstrate that logarithmic earning functions can be derived from optimal behavior. Specifically, the simple case which we analyze leads to piece wise linear log earnings functions. In contrast to "approximations" which derive logarithmic earnings functions by superimposing an arbitrary investment profile (see Mincer [1974, pp. 80-89] and Johnson [1970]). Such a derivation has the advantage that the effects on earnings of exogenous factors can be consistently analyzed. The model is sufficiently simple to allow a clear exposition of the basic elements which govern earnings in a static world. The same elements appear in the more complicated derivations currently available in the literature but it is more difficult to trace their impact. Finally, the multiplicative model provides additional information on the robustness of the results previously derived from the Ben-Porath specification. This is particularly important since the "production function" for human capital is not directly observable and alternative specification can only be compared in terms of their implications with respect to observed earnings.

II. The Model

We consider an individual who operates in a static world under perfect certainty. He is facing an investment opportunity frontier which can be generally described as:

$$(1) \qquad Y = F(K,\dot{K}) \quad , \quad F_1 > 0 \text{ and } F_2 < 0 \quad ,$$

where Y denotes current earnings, K is the unobserved stock of human capital (measured in efficiency units) and $\dot{K}$ is its derivative with respect to age. The positive partial derivative with respect to K and the negative partial derivative with respect to $\dot{K}$, reflect the trade-off between current and future earnings which is implicit in an equilibrium wage structure.

An <u>additive</u> specification of the trade-off function is:

$$(2) \qquad Y = RK - c(\dot{K})$$

where R is the "rental rate" on human capital. (Without loss of generality we shall subsequently assume R=1.) This form arises, for instance in the Ben-Porath model when the depreciation rate on human capital is assumed to be zero. The simplifying feature of this specification is that dollar investment costs are independent of the earnings capacity of the individual.

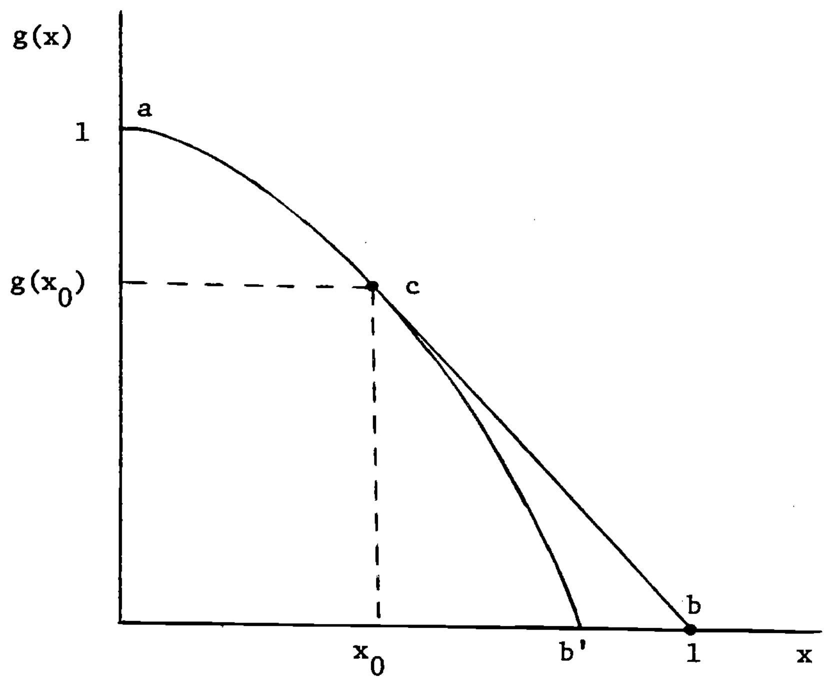
The multiplicative specification is:

$$(3) \qquad Y = KG\left(\frac{\dot{K}}{K}\right)$$

where K is earnings capacity, $\frac{\dot{K}}{K}$ is its rate of change, and $G(\frac{\dot{K}}{K})$ is the proportion of earnings capacity used to generate current earnings $1-G(\frac{\dot{K}}{K})$ is the proportion sacrificed), associated with each rate of growth. Note that we ignore the direct costs of training and assume that all costs are opportunity costs. The simplifying aspect of this specification is that "time" costs, i.e. the proportion of earning capacity which is sacrificed, depend only on the rate of growth in human capital and not on the level of accumulated stock.

The trade-off function $G(\frac{\dot{K}}{K})$ is best described within the framework of activity analysis, (See Rosen [1972]). One option which is open to the individual is full-time schooling. Let us denote the rate of growth which is obtained in this case by $a-\delta$ where δ is the depreciation rate. For this option the individual has to give up all his earnings, i.e. $G(a-\delta) = 0$. The job market also offers training opportunities. It is convenient to use an index x to rank the growth options associated with the various jobs as compared to the growth which can be obtained at schools. Thus, x runs between 0 and 1, x = 1 characterizes the school activity, x = 0 corresponds to the job in which $\frac{\dot{K}}{K} = -\delta$, that is no investment is performed. It is natural to assume that $G(-\delta) = 1$, i.e., no sacrifice of current earnings is necessary in this case. In Figure 1, we present the various options, assuming no mixtures of activities. The line ab' describes the options in the job market. Its negative slope indicates that in an equilibrium wage structure, jobs with better growth options are not provided freely. The point b describes the schooling option. If it is feasible to purchase linear mixtures of jobs or of schooling and jobs by an appropriate allocation of time, then the efficient frontier is the line acb. We shall denote this frontier by g(x). The point c is determined by the condition

Figure 1. The Earning Investment Frontier



$$-f'(x_0) = \frac{f(x_0)}{1-x_0} \tag{4}$$

where f(x) is the trade-off in the job market.

Notice our implicit assumption in describing the set of training options. We assume that a higher rate of growth can be achieved at school than in the job market if one is willing to give up all his current earning. On the other hand, pure on-the-job training is more efficient than full-time or part-time schooling if low levels of growth are desired, i.e. $x < x_0$. This specification is designed to capture, among other things, the discontinuity in investment in human capital which seems to occur upon leaving school, See Mincer [1974, p 94]).

We assume a perfect capital market and ignore the choice of leisure. The objective of each individual is to maximize the present value of his life time earnings. The maximization problem is thus:

$$\max_{\{x\}} \int_0^T e^{-r\tau} Kg(x)d\tau \tag{5}$$

$$\text{s.t } \frac{\dot{K}}{K} = ax - \delta \qquad 0 \leqslant x \leqslant 1 \ , \quad K(0) = K_0$$

The length of life is denoted by T, and r is the exogenously given rate of interest. Using the Hamiltonian function, the above can be transformed into the following maximization problem:

$$\max_{0 \leqslant x \leqslant 1} \quad e^{-r\tau} K[g(x) + \psi \ (ax - \delta)] \tag{6}$$

$$\text{with } \dot{\psi} = r\psi - g(x) - \psi(ax - \delta) \quad , \ \psi(T) = 0$$

This maximization problem is easy to interpret. The returns from human capital, (in the form of "full" wages per unit of capital) depend on the amount of investment. The "full" wage consists of the observed current wages, $Kg(x)$, and on the returns from investment $K\psi(ax - \delta)$, where:

$$\psi(\tau) = \int_{\tau}^{T} e^{-r(\xi - \tau)}[g(x) + \psi(ax - \delta)]\, d\xi \tag{7}$$

denotes the marginal returns of the investment activity. Note that at each point of time these benefits are equal to the present value of future optimal "full" rates of returns. The optimal path is such that for any given shadow price, ψ, the individual chooses the level of investment which maximizes the full wage.

The optimal path of investment can be presented graphically as a movement along the investment frontier $g(x)$, which is associated with the changing shadow price for investment. As long as $a\psi > -g'(x_0)$ the individual will specialize in schooling $(x = 1)$. If $a\psi = -g'(x_0)$, the individual will be indifferent among the various allocations of time between school and work at the job x_0. For $-g'(0) < a\psi < -g'(x_0)$ the individual will choose a tangency point in which $a\psi = -g'(x)$. Finally, if $a\psi < -g'(0)$ there will be no investment and the job with maximal current earning will be chosen.

Since we are interested, in this paper, in a model which is solvable in a closed form, we proceed by specifying a functional form for the investment frontier.

Suppose that the opportunity set for pure on-the-job training, $f(x)$, is given by:

$$(8) \qquad \frac{Y}{K} = [1 - \frac{1}{\beta}(\frac{\dot{K}}{K} + \delta)]^{\alpha} \qquad a > \beta > 0 \qquad 0 < \alpha < 1$$
$$\alpha < \beta/a$$

The parameter β can be interpreted as the efficiency of producing human capital on the job. Higher values of β imply that for a given growth rate a higher proportion of earning capacity is retained. The assumption that $\beta < a$ means that even if all earning capacity is given up, the rate of growth which is attained by pure on-the-job training will be less than that which can be achieved in school. In the same vein 'a' can be interpreted as the efficiency of producing human capital in school. Higher values of 'a' mean that upon giving up all earning capacity and choosing the schooling activity higher growth is attained. Finally, α is a parameter which governs the concavity of the opportunity set; we assume that $0 < \alpha < 1$. The condition $\alpha < \beta/a$ guarantees that for small levels of investment on-the-job training is more efficient.

Using the definition $x = \frac{1}{a}(\frac{\dot{K}}{K} + \delta)$ we obtain the following specification for $g(x)$.

$$(9) \qquad g(x) = \begin{cases} (1 - \frac{a}{\beta} x)^{\alpha} & \text{for } x \leqslant x_0 \\ (1 - \frac{a}{\beta} x_0)^{\alpha} - \alpha \frac{a}{\beta} (1 - \frac{a}{\beta} x_0)^{\alpha-1}(x-x_0) & \text{for } 1 \geqslant x \geqslant x_0 \end{cases}$$

$$\text{where } x_0 = \frac{\frac{\beta}{a} - \alpha}{1-\alpha} \quad , \; a > \beta \text{ and } \alpha < \frac{\beta}{a}$$

This particular form leads to an extremely simple optimal pattern for the observed net earnings. The rate of growth of earnings is piece-wise constant. Productive life is thus divided into three phases: a schooling phase in which no earnings are observed, an investment period in which observed earnings are positive and grow at a constant rate, and a non-investment period in which earnings decline at a constant rate. The length of each phase, as well as the slope of the earnings and investment profiles in each phase can be related to the basic parameters which the individual faces.

These solutions are: (See appendix for derivations)

$$\tau_1 = T + \frac{1}{r+\delta} \ln(1 - (r+\delta)\frac{\alpha}{\beta}) \tag{10}$$

$$\tau_0 = \tau_1 - \frac{1-\alpha}{\beta-r-\delta} \ln\left[\frac{\beta(a-r-\delta)(1-\alpha)}{(\beta-\alpha(r+\delta))(a-\beta)}\right] \tag{11}$$

$$\frac{\dot{Y}}{Y} = \begin{cases} \beta - \delta + \frac{\alpha}{1-\alpha}(\beta-r-\delta) & \text{for } \tau_0 \leq \tau < \tau_1 \\ -\delta & \text{for } \tau_1 \leq \tau \leq T \end{cases} \tag{12}$$

$$\frac{\dot{y}}{y} = \begin{cases} \alpha \quad \frac{\beta-r-\delta}{1-\alpha} + \beta y^{1/\alpha} & \text{for } \tau_0 \leq \tau < \tau_1 \\ 0 & \text{for } \tau_1 \leq \tau < T \end{cases} \tag{13}$$

where Y denotes observed earnings and $y = \frac{Y}{K} = g(x)$ is the proportion of earnings capacity used in the "production" of earnings and 1-y is the proportion invested. Even though these activities are performed jointly on the job one may think of y as the "proportion of time" spent in producing goods, and 1-y as the proportion of time spent in producing new knowledge. (see Mincer [1974, p. 19])

The boundary conditions for this system are:

$$(14) \qquad y(\tau_1) = 1, \qquad y(\tau_0) = g(x_0) = \left[\frac{\alpha}{1-\alpha} \left(\frac{a}{\beta} - 1\right) \right]^{\alpha}$$

$$(15) \qquad Y(\tau_0) = K_0 e^{(a-\delta)\tau_0} y(\tau_0)$$

where $Y(\tau_0)$ can be interpreted as observed starting salaries and K_0 is the exogenously given initial level of human capital.

As seen from the above set of equations there are some restrictions on the parameters which are implicit in a life time earnings profile which includes all three phases. The basic condition is:

$$(16) \qquad a > r+\delta \qquad \text{which implies} \qquad \beta > (r+\delta)\alpha$$

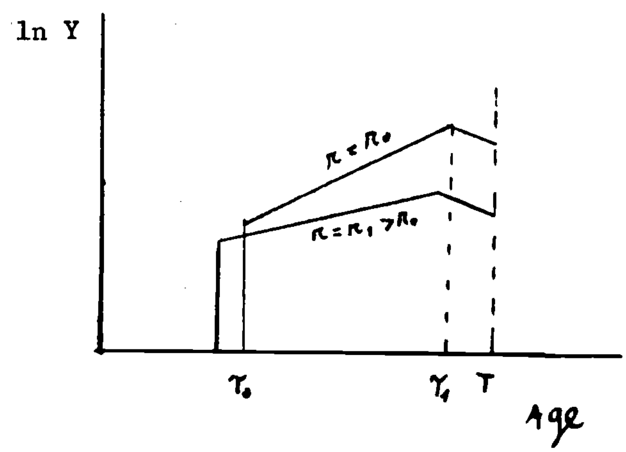
The interpretation of these two conditions is quite transparent; for positive investment to exist, it is necessary that the returns from investment exceed the costs associated with the postponement of earnings.

As suggested by Becker [1964, pp. 14-15] and Ben-Porath [1967], one may explain the general shape of the earnings profile in an investment framework. In particular the positive slope during the on-the-job investment period reflects positive and decreasing investment on the job.1/ The concavity of the log earnings profile depends, however, on the specific trade-off function $g(x)$. The specific form (9) which we adopted has the property that $\frac{\dot{y}}{y}$ increases with age as y increases. The degree of convexity in y (concavity in investment time) is just sufficient to offset the reduction in $\frac{\dot{K}}{K}$ as investment decreases.

The comparative statics of the model are also extremely simple. Consider first a change in the interest rate. An increase in the interest rate will tend to reduce the <u>slope</u> of the log earnings profile. (See figure 2.) This is directly evident from equation (12). It can be seen from equation (10) that the length of the no investment period, $T - \tau_1$, will increase; that is, the peak in earnings will be attained at an earlier age. Since $y(\tau_0)$ and $y(\tau_1)$ are both independent of r and since $\dot{y}$ is decreasing with r for every y, the individual will stay a longer period in the region of on-the-job investment. It follows that τ_0 must decrease, i.e., the individual will invest less in schooling. A similar result can be derived for the additive formulation (2). The only difference is that the increase in the interest rate reduces the absolute growth in earnings rather than its rate of growth.

Consider next the issue of differences in ability. One measure of increased ability is an increase in earning capacity which is uniform and independent of the (endogenous) level of skill. Differences in the initial stock of human capital K_0 will induce parallel shifts in logarithmic earnings function without any further effect on the length of the various investment periods. (This result is in contrast to that of the additive Ben-Porath model where increase in K_0 leads to a shorter time span in school. See Haley [1973].) An alternative specification is to associate increase in ability with an increase in the efficiency of "producing" human capital as represented by the parameters a and β. If a person is a better student at school (higher a) the effect will be higher y_0, while $\frac{\dot{Y}}{Y}$ and τ_1 remain the same. It is easy to show that τ_0 must go up. In other words, there will be a longer period in school with a lower investment on the job once out of school. The log earnings profile will shift in a parallel fashion with the peak remaining unchanged.

Figure 2. Effects of a Change in the Interest Rate



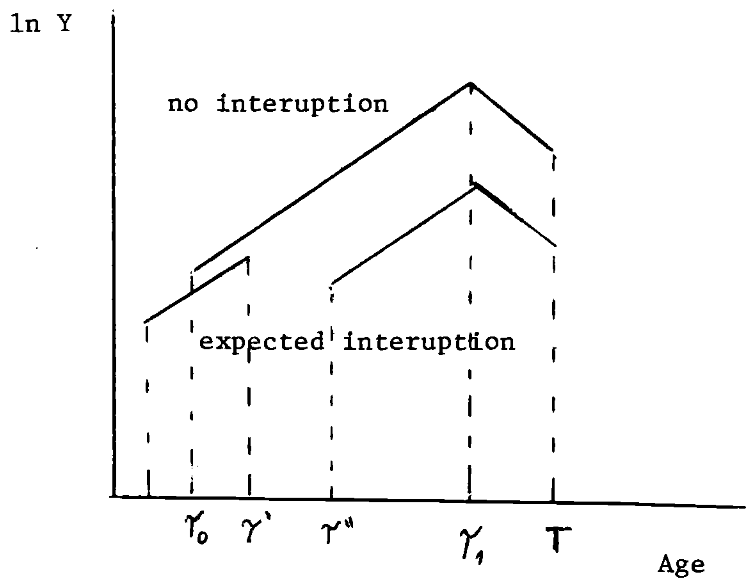
If a person is a better on-the-job student (higher β) $\frac{\dot{Y}}{Y}$ will increase while y_0 will decrease. The effect on τ_1 is positive and on τ_0 negative. In other words, this individual will invest less in schooling and more in on-the-job training. The log earnings profile will have a higher slope, and will peak at a later age.

The most realistic case seems to be that in which both a and β increase. The effect on the length of the schooling period is ambiguous in this case. An interesting special case is that in which the optimal level of schooling, τ_0 , remains the same. The implication of higher ability will be a higher log earnings profile with a higher slope and a later peak. Another special case is that in which a and β grow at the same rate so that y_0, the initial investment in on-the-job training, remains the same. In this case, higher ability will lead to more schooling, and the log earnings profile will have a higher slope and a later peak.

An important empirical phenomenon is the existence of considerable variation in the age at which a given level of schooling is obtained. To a large extent, postponement may result from many factors not incorporated in the present analysis, such as imperfections in the capital market, differences in preferences towards leisure, and uncertainty with respect to one's own abilities and preferences. Within our simple model, we can, however, deal with the effects of exogenous changes in the age of entry, due to, say, military service. A person who is a late starter (See Johnson and Stafford [1974]) will have a shorter horizon, and naturally will tend to invest less. If there is a positive period of specialization, the reduction in investment will take the form of a shorter schooling period. The log earning profile will be lower but its slope will remain the same. The age of entry and peak in earning will be unaffected. This is a somewhat unrealistic result which follows from the assumption that the age of retirement is exogenous. A perhaps more realistic assumption is that for brief postponements the length of the <u>working</u> period is constant (See Mincer [1974, p. 10-11]).

Finally, consider the effects of a disruption in the accumulation of experience, that is, exogenous changes in participation. If a woman, say, plans to stay out of the labor force for some interval $[\tau',\tau'']$ her profile will be as depicted in Figure 3. Note that upon returning to the labor force her earnings are somewhat lower reflecting the effects of depreciation. If the disruption is expected, the woman will also plan under a shorter horizon and therefore her specialization period will be shorter and the level of earning will be lower for every level of work experience. This horizon effect will be absent if the break is unexpected, but otherwise the results will be identical. This is a direct consequence of the independence between the investment rate and the level of human capital in our model. Notice that when log earning is plotted against experience, rather than age, the outcome is a flatter log earnings profile.

Figure 3. Effects of an Expected Break in Participation



III. Empirical Implications

The simple model just described has several important implications for empirical research.

1. Since in a static world schooling and investment on the job are governed by the same basic exogenous factors, individual differences in schooling are associated with corresponding differences in the slope of the earnings function. For instance, if the main source of individual variation is due to differences in the interest rate, we would observe a positive interaction between the level of schooling and the growth of earnings. (For some evidence on this point see Weiss and Lillard [1976].)

2. Log earnings profiles when estimated from either cross section (see Mincer [1974]) or longitudinal data (see Weiss and Lillard [1976]) tend to be concave in age (or experience). That is, the rate of growth in earnings appears to decline smoothly with age. As our simple model illustrates, this is <u>not</u> a general property of earnings profiles which arise from optimal accumulation of human capital. It is quite easy, however, to introduce concavity into the multiplicative model. One may either add age effect explicitly (see Weiss and Lillard [1967]) or choose an alternative formulation of the trade-off function[2]. (See Rosen [1975], Weiss [1974].)

3. The observed earnings of an individual at different points in time are systematically related through his choice of investment program. In fact, in the absence of exogenous shocks we can predict the earnings of an individual from a sample of his past earnings without having any additional information

about him. All the relevant information on his schooling, his ability and his access to the capital market is already incorporated in the past level and growth of his earnings. The particular form of the autoregressive scheme which emerges depends critically on the form of the trade off function (1). If one assumes that simple multiplicative form (3) then annual observations on this continuous process should approximately satisfy:

$$(17) \qquad \ln Y_t = \ln Y_{t-1} + \text{constant} .$$

Analogous formulations can be derived for the additive case, if it is further assumed that $C(\dot{K})$ is quadratic. Such a specification implies a second order linear differential equation in earnings (see Rosen [1973]) and discrete observations would approximately satisfy the autoregressive scheme:

$$(18) \qquad Y_t = aY_{t-1} + bY_{t-2} + \text{constant} .$$

Higher order linear schemes arise if one allows the trade-off function $F(\quad)$ to be a general quadratic. An interesting aspect of these schemes is the alternatings signs of the coefficients on past earnings (see Weiss [1974]). Autoregressive schemes can be applied to longitudinal data and yield estimates for some of the parameters. The rate of interest, for instance, can be directly estimated from (18). Furthermore they can be used to differentiate between alternative specifications of the trade-off function. A sharper discrimination can sometimes be made when the models are compared in terms of the autoregressive schemes which they imply rather than the explicit age profiles (which frequently become complicated nonlinear functions).

4. The breaks in participation which characterize the work career of many women lead to flatter log earning profiles, when viewed as a function of experience. (For empirical evidence see Johnson and Stafford [1974], Polachek and Mincer [1976]). It is important to note that this result is independent of possible discrimination against women in the labor force. If, for instance, the rental rate for human capital of females is half that of males their investment pattern will be unaffected. As long as opportunity costs in the <u>market</u> are the sole costs of training, such discrimination would affect the benefits and costs of training equally. In the present context discrimination can lead to flatter profiles for women only if it is increasing with the level of skill.

5. It is sometimes argued that short work horizons are likely to lead to flatter earnings profiles. This had been suggested by Johnson and Stafford [1974a and 1974b] as an explanation of flatter profiles of women and "late starters". As we have seen this need not be the case. In our simple model the reduction in investment of individuals with relatively short horizon takes the form of reduced schooling rather than a lower rate of investment in on-the-job training. Consequently, the level of earnings is affected but not its rate of growth. In fact, if the comparison holds schooling constant, those with shorter horizons probably face a lower rate of interest or are of higher learning ability, therefore their earning profile may well have a higher slope. It is worth noting, however, that if age effects are introduced explicitly (see Lillard and Weiss [1976]) late starters <u>do</u> tend to have a flatter profile, but this reflects the effects of age on the capacity to learn rather than the shorter horizon effect.

Footnotes

1. It is possible that there exists an automatic process of learning from experience which is to some extent independent of individual decisions (that is, g(x) approaches 1 at a positive $(\frac{\dot{K}}{K} + \delta)$. In such a context, the theory only explains <u>differences</u> in the slope of the earnings profiles in terms of differential investment. It is clearly <u>not</u> necessary to assume positive investment for the purpose of explaining a positive slope of the earnings profile.

2. The relation between the form of g(x) and the concavity of the log earning profile during the investment period is given by:

$$\ddot{z} = [\dot{x}]^2 F(x)$$

where $z = \ln Y$ and

$$F(x) = 2\frac{g''}{g} - \frac{g'''}{g''}\frac{g'}{g} - [\frac{g'}{g}]^2 .$$

When $g(x) = (1 - \frac{a}{\beta}x)^{\alpha}$, then $F(x) = 0$ for all x.

For any function g(x) such that $g > 0$, $g' < 0$, $g'' < 0$, a sufficient condition for $F(x) < 0$ and thus $\ddot{z} < 0$ is that $g''' \geq 0$.

For a detailed discussion of the case in which $g''' = 0$, see Rosen [1975]. Needless to say, under our specification $g''' < 0$.

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Appendix A

The purpose of this appendix is to derive equations 10 to 13 in the text and to prove some comparative statics results.

The problem which we solve is the maximization of (6) subject to (9) in the text.

The necessary conditions for optimum are:

$$(A1)\quad g'(x) + a\psi \geq 0 \text{ if } x_0 \geq x \leq 1$$
$$g'(x) + a\psi \leq 0 \text{ if } x = 0$$
$$g'(x) + a\psi = 0 \text{ if } 0 < x < x_0$$

and

$$(A2)\quad \dot{\psi} = (r+\delta)\psi - \psi ax - g(x) \qquad \psi(T) = 0 \ .$$

In the case of an interior solu.ion we can take the derivative of the first order condition with respect to age to obtain a differential equation for x.

$$(A3)\quad \dot{x} = \frac{g'(x)}{g''(x)}[r + \delta] - [\frac{-g(x) + xg'(x)}{g''(x)}]\ a$$

The rate of increase in observed earning is given by

$$(A4)\quad \frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} + \frac{g'(X)}{g(x)}\dot{x} = ax - \delta + \frac{g'(x)}{g(x)}\dot{x}$$

and substituting for $\dot{x}$ we obtain

$$\text{(A5)} \quad \frac{\dot{Y}}{Y} = ax - \delta + \frac{g'(x)}{g(x)}\frac{g'(x)}{g''(x)}[r+\delta]$$

$$- \frac{g'(x)}{g(x)} \frac{g'(x)}{g''(x)} \left[\frac{-g(x) + xg'(x)}{g'(x)}\right] a$$

Under the special functional form (9):

$$\text{(A6)} \quad \frac{[g'(x)]^2}{g(x)g''(X)} = \frac{\alpha}{\alpha-1} \text{ and } \frac{-g(x) + xg'(x)}{g'(x)} = \frac{\beta}{\alpha a} + \frac{x(\alpha-1)}{\alpha};$$

hence

$$\text{(A7)} \quad \frac{\dot{Y}}{Y} = \beta - \delta + \frac{\alpha}{1-\alpha}(\beta-r-\delta)$$

which is equation (12) in the text.

We can also determine the length of each of the phases in the individual investment program. During the last phase of zero investment we have:

$$\text{(A8)} \quad \dot{\psi} = (r+\delta)\psi-1 \qquad \text{and} \qquad \psi(\tau) = \frac{1}{r+\delta}\left[1 - e^{-(r+\delta)(T-\tau)}\right]$$

the age of the peak in earnings is determined by the condition:

$$\text{(A9)} \quad \psi(\tau_1) = \frac{-g'(0)}{a} = \frac{1}{r+\delta}(1 - e^{-(r+\delta)(T-\tau_1)})$$

or

$$\text{(A10)} \quad T-\tau_1 = \frac{-1}{r+\delta}\ln(1 - (r+\delta)\frac{\alpha}{\beta})$$

To determine the length of the investment on the job phase, we have to solve equation 13 in the text and use the boundary conditions in equation 14.

Define $q = y^{1/\alpha} = 1 - \frac{a}{\beta} x$, then equation 13 in the text can be rewritten as:

(A11) $\dot{q} = Aq + Bq^2$ where $A = \frac{\beta - r - \delta}{1-\alpha}$ and $B = \frac{\beta}{\alpha}$;

with the solution:

(A12) $\tau - \tau_0 = \frac{1}{A} \left[\ln \frac{q}{A+Bq} - \ln \frac{q_0}{A+Bq_0} \right]$;

using the boundary conditions we obtain:

(A13) $\tau_1 - \tau_0 = \frac{-1}{A} \left[\ln(A+B) + \ln(\frac{q_0}{A+Bq_0}) \right]$.

The schooling (or specialization period) is then found as a residual using the identity:

(A14) $\tau_0 = T - (T-\tau_1) - (\tau_1 - \tau_0)$.

Equation A14 can be also used to derive an explicit solution for the investment profile. This solution assumes the form:

(A15) $y^{1/\alpha} = \frac{Ae^{A(\tau-\tau_0)}}{C - Be^{A(\tau-\tau_0)}}$ where $C = \frac{A+Bq_0}{q_0}$.

We conclude with a brief discussion of comparative statics. The effect of a on the schooling level is:

$$\text{(A16)} \qquad \frac{\partial \tau_0}{\partial a} = -\frac{1-\alpha}{\beta-r-\delta}\left[\frac{1}{(a-r-\delta)} - \frac{1}{(a-\beta)}\right] = -\frac{1-\alpha}{\beta-r-\delta}\left[\frac{r+\delta-\beta}{(a-r-\delta)(a-\beta)}\right] > 0.$$

To determine the effect of β, let us rewrite equation (9) as:

$$\tau_0 = T - \frac{1}{A}\ln\left[\frac{A+Bq_0}{(A+B)q_0}\right] + \frac{1}{r+\delta}\ln\left(1-\frac{r+\delta}{B}\right)$$

where $A = \dfrac{\beta-r-\delta}{1-\alpha}$, $\quad B = \beta/\alpha$, $\quad q_0 = \dfrac{\alpha}{1-\alpha}\left(\dfrac{a}{\beta}-1\right)$.

Note that $A + Bq_0 = \dfrac{a-r-\delta}{1-\alpha}$ is independent of β. We thus have:

$$\text{(A17)} \qquad \frac{\partial \tau_0}{\partial \beta} = \frac{1}{A^2}\ln\left(\frac{A+Bq_0}{(A+B)q_0}\right) + \frac{1}{A}\left(\frac{1}{A+B}\left(\frac{dA}{d\beta}+\frac{dB}{d\beta}\right) + \frac{1}{q_0}\frac{dq_0}{d\beta} + \frac{1}{B^2-(r+\delta)B}\frac{dB}{d\beta}\right.$$

After some manipulations we arrive at:

$$\frac{\partial \tau_0}{\partial \beta} = \frac{1}{A^2(1-\alpha)}\left[\ln\left(\frac{A+Bq_0}{(A+B)q_0}\right) - \frac{A}{Bq_0}(1-q_0)\right].$$

Due to the concavity of the log function,

$$\ln\left(\frac{A+Bq_0}{(A+B)q_0}\right) < \frac{A+Bq_0}{A+B} - 1 = \frac{A(1-q_0)}{(A+B)q_0}.$$

It follows that:

$$\text{(A18)} \qquad \frac{\partial \tau_0}{\partial \beta} < \frac{1}{A^2(1-\alpha)}\left[\frac{A(1-q_0)}{(A+B)q_0} - \frac{A}{Bq_0}(1-q_0)\right] < 0.$$

The effect of late entry is analyzed by modifying the maximization problem to:

$$\text{(A19)} \qquad \max_{x} \int_{\tau_s}^{T} e^{-r(\tau-\tau_s)} K g(x) d\tau$$

$$\text{s.t.} \quad \frac{\dot{K}}{K} = ax - \delta \ , \ 0 \leq x \leq 1 \ , \ K(\tau_s) = K_s$$

Where τ_s is the age of entry. Upon a change of variables, $\mu = \tau - \tau_s$, the problem becomes identical to the discussed previously except that T is replaced by $T-\tau_s$. It is seen that the length of time spent in the last two phases is unaffected. It follows from (A14) that the period of time spent in school must decrease.

The effects of an expected interruption in career can be analyzed by defining an indicator $h(\tau)$ such that:

$$h(\tau) = \begin{cases} 0 & \text{for} \quad \tau \varepsilon [\tau', \tau''] \\ 1 & \text{otherwise} \end{cases}$$

and rewriting the maximization problem as:

$$\text{(A20)} \qquad \max_{\{x\}} \int_{0}^{T} e^{-r\tau} hKg(x) d\tau$$

$$\text{s.t.} \qquad \frac{\dot{K}}{K} = ahx - \delta \qquad 0 \leq x \leq 1 \ , \ K(0) = K_0$$

when $h(\tau) = 1$ the solution is identical to the one derived previously. On the other hand $\frac{\dot{K}}{K} = -\delta$ when $h(\tau) = 0$. Therefore during periods of participation the rate of growth in Y is the same, the only effect is on the initial Y at each phase.

Appendix B

The purpose of this appendix is to derive the autoregressive schemes in earnings for the linear and multiplicative specifications of the investment frontier. It reproduces the main results of my "Notes on Income Generating Functions," (Princeton 1974).

Consider first a model which leads to a linear earnings generating function of the form.

B1. $Y_t = \alpha_1 Y_{t-1} + \alpha_2 Y_{t-2} + \cdot \cdot \cdot \cdot \cdot$

where Y_t denotes annual observed earnings, or in continuous time formulation:

B2. $Y_i = \alpha_1 Y_{i-1} + \alpha_2 Y_{i-2} \cdot \cdot \cdot \cdot \cdot$

where Y_i denotes the i'th order derivative with respect to time Rosen [1973] has considered some special cases of this form.

Let us write the individual's maximization problem as

B3. $\underset{K_t}{\text{MAX}} \int_{to}^{T} e^{-rt} F(K,K_1)dt.$

where K is the amount of human capital and K_1 is its rate of change. We may assume $F_K > 0$, and $F_{K_1} < 0$.

The optimal accumulation path satisfies the Euler condition:

B4. $e^{-rt} F_K = \frac{d}{dt} e^{-rt} F_{K_1}$ (we assume interior solution)

or

4'. $F_K = -rF_{K_1} + F_{KK_1} K_1 + F_{K_1K_1} K_2$,

A necessary and sufficient condition for a linear autoregressive scheme for observed earnings is that the observed (net) earning function $Y = F(K,K_1)$ is quadratic in K and K_1. Equation B2 then becomes a second order linear equation.

B5. $K_2 = \lambda_0 + \lambda_1 K + \lambda_2 K_1$

We can now derive a differential equation in Y. Note that if F is quadratic then Y is a linear function in K, K^2, K_1, K_1^2, KK_1. Using condition (B5), we can write Y_1 as a function of the same variables similarly for Y_2, Y_3, etc. We thus have a linear system of equations

B6. $Z = AX$ where $Z = \begin{pmatrix} Y-b_1 \\ Y_1-b_2 \\ Y_2-b_3 \\ Y_3-b_3 \\ Y_4-b_4 \end{pmatrix}$ $\quad X = \begin{pmatrix} K \\ K^2 \\ K_1 \\ K_1^2 \\ KK_1 \end{pmatrix}$

The rows of A and the constant vector b are determined recursively, by the relation

$$\alpha_t' = \alpha_{t-1}' B$$

where α_t' is a five element row vector of the matrix A. α_1' is given from the objective function. B is the matrix.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ \lambda_1 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 2\lambda_0 & 2\lambda_2 & 2\lambda_1 \\ 0 & \lambda_1 & 0 & 1 & \lambda_2 \end{pmatrix}$$

Also $b_1 = 0$ and $b_t = \alpha_{t-1,3}\lambda_0$.

If A is of full rank, we solve $Y_5 - b_5 = \alpha_1' B^5 A^{-1} X$ and get a fifth order linear differential equation. However A is in general not of full rank and the order of the linear differential equation will be $\rho(A)$.

For example:

B7. If $Y = AK - (\dot{K})^2 \frac{a}{2}$

then $Y_2 = 2\lambda_2 Y_1 + 2\lambda_0 A$

where $\lambda_2 = r$. $\lambda_0 = -A/a$

B8. If $Y = AK - \frac{B}{2}K^2 - (\dot{K})^2 \frac{a}{2}$

then $Y_2 = 2\lambda_2 Y_1 + 4\lambda_1 Y + 2\lambda_0 A$

where $\lambda_2 = r \quad \lambda_1 = \frac{B}{a} \quad \lambda_0 = \frac{-A}{a}$

B9. If $Y = AK - (\dot{K} + \delta K)^2 \frac{a}{2}$

then $Y_3 + \delta Y_2 = 3\lambda^*(Y_2 + \delta Y_1) - 2(\lambda^*)^2(Y_1 + \delta Y) + \frac{2A^2}{a}(r + \frac{1}{2}\delta)$

where $\lambda^* = r + \delta$

(the optimality condition is rewritten in this case as $\dot{I} = -\frac{A}{a} + \lambda^* I$ where $I = \dot{K} + \delta K$).

B10. Finally, if $Y = AK - \frac{B}{2}K^2 - (\dot{K} + \delta K)^2 \frac{a}{2}$

then $Y_4 = 3\lambda_2 Y_3 + (5\lambda - 2\lambda_2^2)Y_2 - 6\lambda_2\lambda_1 Y_1 - 4\lambda_1 Y + \frac{A^2}{a}(2\lambda_1 + 2\lambda_2)$

where $\lambda_1 = \frac{B}{a} + \delta(r + \delta)$

$\lambda_2 = r$

It is possible that due to transaction costs adjustment is not smooth and decisions are revised only on a set of discrete points. To provide the discrete analogue of the problem let us subdivide the interval (o,T) into equal subintervals by introducing the points $t_0, t_1, t_2, \ldots t_n$ where $t_0 = 0$, $t_n = T$ and $t_i - t_{i-1} = \Delta t$ is the time interval. The function $K(t)$ is assumed to remain fixed in each such interval, and we have the correspondence $t = t_0, t_1, t_2 \ldots t_n$ and $K = K_0, K_1, K_2 \ldots K_n$. The objective function can be rewritten as

$$\max_{K_1, K_2 \ldots K_n} \quad V = \sum_{i=0}^{n} (1 + r\Delta t)^{-i} F(K_{i-1}, \frac{K_i - K_{i-1}}{\Delta t})\Delta t$$

The first order conditions are given by

$$\frac{\partial V}{\partial K_j} = 0 \quad \text{for } j = 1, 2 \ldots n \quad (K_0 \text{ is predetermined}).$$

or

$$(1+r\Delta t)F_{K'}(K_{i-1}, \frac{K_{i-1} - K_{i-1}}{\Delta t}) - F_{K'}(K_i, \frac{K_{i+1} - K_i}{\Delta t}) + F_K(K_i, \frac{K_{i+1} - K_i}{\Delta t})\Delta t = 0$$

Dividing through by Δt and allowing it to approach zero we get the Euler condition [B4'] as $\lim_{\Delta t \to 0} \frac{\partial V}{\partial K_j} \frac{1}{\Delta t}$. For finite differences, eg $\Delta t = 1$, we get due to the assumption that F is quadratic, that:

$Y_t = F(K_t, K_{t-1}) = \alpha_{11}K_t + \alpha_{12}K_t^2 + \alpha_{13}K_{t-1} + \alpha_{14}K_{t-1}^2 + \alpha_{15} K_t K_{t-1}$. In this analysis we shall restrict ourselves to the case $\alpha_{11}=0$.

The optimality condition assumes the form:

$$\text{B5)'} \quad K_t = \lambda_0 + \lambda_1 K_{t-1} + \lambda_2 K_{t-2}$$

where $\lambda_0 = - \dfrac{\alpha_{13}}{\alpha_{15}}$

$$\lambda_1 = 2[\frac{\alpha_{12}}{\alpha_{15}} \lambda_2 - \frac{\alpha_{14}}{\alpha_{15}}]$$

$$\lambda_2 = 1 + r$$

The analogues to equations 7, 9, 10, are

B7'. $$Y_t = (2 + 2r + r^2)Y_{t-1} - (1 + 2r + r^2)Y_{t-2} - \frac{A^2}{a}(2 + r)$$

B9'. $$Y_t = (1 - \delta + \frac{1+r}{1-\delta} + (\frac{1+r}{1-\delta})^2)Y_{t-1} - (\frac{(1+r)^2}{1-\delta} + 1+r + \frac{(1+r)^3}{(1-\delta)^3})Y_{t-2}$$

$$+ (1-\delta)(\frac{1+r}{1-\delta})^3 Y_{t-3} - \frac{A^2}{2a(1-\delta)^2}(1-\delta-r + \frac{1+r}{1-\delta} - 2\frac{(1+r)^2}{1-\delta})$$

B10'. $$Y_t = (\lambda_1 + \lambda_1^2 + 2\lambda_2)Y_{t-1} + (\lambda_2 - \lambda_2^2 - \lambda^3 - 2\lambda_1\lambda_2)Y_{t-2} - 2\lambda_2^2 Y_{t-3}$$

$$+ \lambda_2^3 Y_{t-4} + \text{constant.}$$

where $\lambda_1 = 2[\frac{\alpha_{12}}{\alpha_{15}}\lambda_2 - \frac{\alpha_{14}}{\alpha_{15}}]$ and $\lambda_2 = 1 + r$

The parameters α_{12}, α_{14}, α_{15} are the coefficients in the earnings function of K_t^2, K_{t-1}^2, and $K_t K_{t-1}$ respectively.

The sign pattern of the coefficients is somewhat surprising. In the simplest case 7', the partial effect of Y_{t-2} holding Y_{t-1} constant is negative. This is equivalent to a positive (explosive) relation between ΔY_{t+1} and ΔY_t. Indeed equation 7' can be rewritten as

$$\Delta Y_t = (1+r)^2 \Delta Y_{t-1} - \frac{A^2}{a}(2+r)$$

More generally we notice that at least some lagged values appear with a <u>negative</u> coefficient. One hesitates to interpret these as partial derivatives, since all income levels are determined endogenously and simultaneously.

Consider now the multiplicative model and let $F(K,K') = f(K)g(x)$ where $x = \frac{1}{a}(\frac{\dot{K}}{K} + \delta)$ and $f(K)$ is assumed to be of constant elasticity, i.e. $f(K) = AK^\beta \quad 0 \leq \beta \leq 1$.

The Euler condition assumes the form

B11. $$\dot{x} = \frac{g'(x)}{g''(x)}(r+\beta\delta) - \beta a \frac{x g'(x)}{g''(x)} + a\beta \frac{g(x)}{g''(x)}$$

Let us denote $Z = \log Y$, then

B12. $$\dot{Z} = \beta(ax - \delta) + \frac{g'(x)}{g(x)}\dot{x}$$

Using equation B11 the change in log earnings, $\dot{Z}$, can be written as a function of x, e.g. $\dot{Z} = R(x)$. If R is a monotone function of x we can solve $x = R^{-1}(\dot{Z})$. Finally $\ddot{Z} = R'(x)\dot{x}$ is also a function of x. Under the invertibility assumption it follows that a second order differential equation exists such that

B13. $\ddot{Z} = \sigma(\dot{Z})$

This again is an autoregressive scheme, but in terms of log earnings. The particular form of the function $\sigma(\dot{Z})$ will, of course, depend on the choice of $g(x)$.

We have already considered the degenerate case in which $g(x) = (1-x)^{\alpha}$, and $R(x)$ = constant. We have seen that this form leads to a linear log earnings profile during the phase of on-the-job investment.

A relatively simple form, which leads to a concave log earnings profile is $g(x) = (1-x)(1+x) = 1-x^2$. In this case

B14. $$\dot{x} = (r+\beta\delta)x - \frac{\beta a}{2}(1+x^2) < 0$$

B15. $$\dot{Z} = \frac{-\beta\delta + 2\beta ax - 2x^2(r+\frac{\beta\delta}{2})}{(1+x)(1-x)} \quad \begin{matrix} > 0 \text{ for large } x \\ < 0 \text{ for small } x \end{matrix}$$

B16. $$\ddot{Z} = -4\left[\frac{\beta a}{2}(1+x^2) - (r+\beta\delta)x\right]^2 < 0$$

(we assume that $\beta a > r + \beta\delta$, note that $\frac{1}{x} + x > 2$ for $x \in [0,1]$)

$\dot{Z}$ is monotone increasing in s, in the relevant range. We can solve for x in terms of $\dot{Z}$. The resulting relation between $\ddot{Z}$ and $\dot{Z}$ is, however, not particularly attractive.

A form of g(x) which is somewhat more tractable is a combination of the two previous cases in which g(x) is the positive portion of the unit circle. That is $g(x) = (1-x^2)^{\frac{1}{2}}$. In this case

B17. $\dot{x} = (1-x^2)(r+\beta\delta)x - \beta a) < 0$

B18. $\dot{Z} = 2\beta ax - \beta\delta - x^2(r+\beta\delta) \begin{array}{l} < 0 \text{ for small } x \\ > 0 \text{ for large } x \end{array}$

B19. $\ddot{Z} = -2(\beta a - r(\beta+\delta)x)^2(1-\frac{2}{x}) < 0$

Finally the autoregressive scheme assumes the form

B20. $$\ddot{Z} = -2\Big[(\beta a)^2 - \frac{2(\beta a)^4}{r+\beta\delta} + (\beta+\delta\dot{Z})(\frac{(\beta a)^2}{r+\beta\delta} + 2(\beta a)^2 - (r+\beta\delta))$$
$$- (\dot{Z}+\beta\delta)^2 - \frac{2\beta a}{r+\beta\delta}\ \{(a\beta)^2 - (r+\beta\delta)(\dot{Z}+\beta\delta))^{3/2}\Big]$$

The coefficient of $\beta+\delta\dot{Z}$ is negative, that of $(\dot{Z}+\beta\delta)^2$ is positive.

To summarize, it is easy to find simple g() function which yield an earnings profile which is concave in the logs. These forms, however, in general do not lead to a simple autoregressive scheme. In practice a quadratic approximation $\ddot{Z} = a + \beta\dot{Z} + C(\dot{Z})^2$ may be advisable.