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BLISS POINTS IN MEAN-VARIANCE PORTFOLIO MODELS

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ABSTRACT

When all financial assets have risky returns, the mean-variance portfolio model is potentially subject to two types of bliss points. One bliss point arises when a von Neumann-Morgenstern utility function displays negative marginal utility for sufficiently large end-of-period wealth, such as in quadratic utility. The second type of bliss point involves satiation in terms of beginning-of-period wealth and afflicts many commonly used mean-variance preference functions. This paper shows that the two types of bliss points are logically independent of one another and that the latter places the effective constraint on an investor's welfare. The paper also uses Samuelson's Fundamental Approximation Theorem to motivate a particular mean-variance portfolio choice model which is not affected by either type of bliss point.

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BLISS POINTS IN MEAN-VARIANCE PORTFOLIO MODELS

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Mean-variance portfolio models have played a major role in the development of financial economics. In monetary theory, for example, Tobin's [31] theory of liquidity preference formally justified the speculative demand for money in terms of diversified portfolios including both money and a risky security. In the theory of finance, the Sharpe [30]-Lintner [16] capital-asset-pricing model demonstrated the importance of risk in the equilibrium pricing of securities. Two important assumptions are uniformly employed in these pathbreaking studies. First, it is assumed that investors' preferences may be represented exactly by mean-variance (or mean-standard deviation) preference orderings. Second, although Markowitz [18] originally examines portfolio selection for the case in which all assets are risky, these early applications and extensions of the mean-variance approach proceed under the assumption that a riskless financial asset exists.

In the current economic environment of persistent and uncertain price inflation, and with the absence of an indexed security with zero default risk, the assumption that an asset with a riskless real return exists is clearly not applicable.[1/] By dropping the riskless-asset assumption, however, mean-variance analysis is subject to <u>two</u> potential bliss points. One bliss point is widely acknowledged and follows from Borch [5], who proves that the only von Neumann-Morgenstern [34] utility function which induces mean-variance preferences for <u>all</u> probability distributions of end-of-period is the quadratic utility function, which has a finite maximum with a corresponding satiation level of <u>end-of-period</u> wealth. The possible existence of a different bliss point is presented by Bierwag and Grove [4], who show that a bliss point exists in mean-variance portfolio models if a riskless asset is not available and indifference curves are convex in variance-mean space. The untenable implication of this second bliss point is that a satiation level of beginning-of-period wealth exists. In other words, there exist levels of initial wealth such that an investor maximizes utility by disposing of some of his wealth before selecting his portfolio.

The existence and implications of initial wealth satiation have been misinterpreted frequently. For example, both Borch [5] and Hakansson [10] apparently interpret initial wealth satiation as being the same as end-of-period wealth satiation in quadratic utility.[2/] However, as is shown in this paper, these bliss points are distinct. Indeed, in the quadratic utility case, initial wealth satiation occurs at a lower level of expected utility than end-of-period wealth satiation. Thus, those researchers who place importance on restricting the range of application of quadratic utility because of end-of-period wealth satiation—e.g., Hakansson [10]—should logically restrict its application further because of initial wealth satiation. Moreover, initial wealth satiation not only limits the usefulness of quadratic utility, but also many other common mean-variance utility functions.

The purpose of this paper is to investigate the conditions that generate initial wealth satiation in mean-variance portfolio models, and to clarify its implications. Following this introductory section, the mean-variance model is briefly reviewed and the notation used throughout the remainder of the paper is introduced. In the second section, the Bierwag and Grove [4] result is generalized, and the initial wealth satiation property is examined under alternative assumptions about the existence of a riskless asset and the specification of the mean-variance preference ordering. Examples of specific mean-variance preference orderings that do and do not display the initial wealth satiation property are presented in the third section. In the fourth section, Samuelson's [29] Fundamental Approximation Theorem is employed to construct a mean-variance model that does not display initial wealth satiation when viewed as a rigorous approximation to a well-behaved general expected utility model. The main conclusions of the paper are summarized in the final section.

I. The Mean-Variance Portfolio Choice Problem

The mean-variance portfolio choice model assumes that an investor's preference ordering over probability distributions of end-of-period wealth is represented by a function

$$U[E(W_1), V(W_1)] \tag{1}$$

where W_1 = end-of-period wealth ($W_1 = \underline{A}'\underline{R}$)

$\underline{A}$ = N × 1 vector of asset holdings in dollar amounts ($\underline{A}'\underline{1} = W_0$), with typical element A_i ($i = 1,2,\ldots,N$)

$\underline{R}$ = N × 1 vector of gross rates of return equal to $\underline{1} + \underline{r}$ with typical element R_i ($i = 1,2,\ldots,N$)

W_0 = beginning-of-period (or initial) wealth

$\underline{r}$ = N × 1 vector of net after-tax yields with typical element r_i ($i = 1,2,\ldots,N$)

$\underline{1}$ = N × 1 vector with each element equal to unity.

The investor's subjective assessment of the N × 1 mean vector and the N × N variance-covariance matrix of asset rates of return may be represented as [3/]

$$E(\underline{R}) = \underline{\mu}$$

$$V(\underline{R}) = E[(\underline{R}-\underline{\mu})(\underline{R}-\underline{\mu})'] = \Sigma$$

implying that the mean and variance of end-of-period wealth are

$$E(W_1) = \underline{A}'\underline{\mu}$$

$$V(W_1) = \underline{A}'\Sigma\underline{A}.$$

In the derivations below, all assets are assumed to be risky unless otherwise specified, and the variance-covariance matrix is taken as positive definite implying that stochastic domination does not exist. In addition, the mean-variance function (1) is assumed to be continuous and twice differentiable, and all investors are assumed to prefer less variance and more mean:

$$\partial U/\partial E \equiv U_E > 0$$

$$\partial U/\partial V \equiv U_V < 0$$

implying risk-averse behavior. (See Tobin [31].)

The Borch [5]-Feldstein [6]-Tobin [32] exchange indicates that sufficient conditions for an exact mean-variance preference ordering to be consistent with expected utility maximization are either that the utility function is quadratic, asset rates of return are distributed joint normally, or end-of-period wealth is distributed lognormally and the utility function is logarithmic.[4/] Alternatively, the mean-variance portfolio model may be viewed as an entity separate from the expected utility paradigm and stand on its own merits as an alternative criterion of portfolio selection. (See Markowitz [19].)

Under either of these interpretations, the Lagrangian expression associated with the constrained maximization of (1) is

$$L = U[\underline{A}'\underline{\mu}, \underline{A}'\Sigma\underline{A}] - \lambda(\underline{1}'\underline{A} - W_0). \qquad (2)$$

Differentiating this expression with respect to $\underline{A}$ yields the first-order-conditions for a constrained optimum:

$$U_E \cdot \underline{\mu} + 2 \cdot U_V \cdot \Sigma \underline{A} - \underline{1}\lambda = \underline{0} \qquad (3a)$$

$$\underline{1}'\underline{A} - W_0 \leqq 0 \qquad (3b)$$

$$\lambda \cdot (\underline{1}'\underline{A} - W_0) = 0 \qquad (3c)$$

$$\lambda \geqq 0. \qquad (3d)$$

For future reference, note that the Lagrangian multiplier λ solved from (3a) - (3d) may be interpreted as marginal mean-variance utility with respect to an increment of initial wealth W_0. In the remainder of this section, all initial wealth is assumed to be invested so that $\underline{1}'\underline{A} = W_0$. This assumption will be relaxed shortly.

The second-order conditions of the constrained optimum are satisfied if the indifference curves associated with the mean-variance utility function (1) are convex with respect to the origin in variance-mean space. Following Tobin [31], this convexity assumption implies that investors are risk-averse diversifiers.[5/] To facilitate the discussion of bliss points, it will prove useful to represent the solution to the mean-variance problem graphically, which is done immediately below.

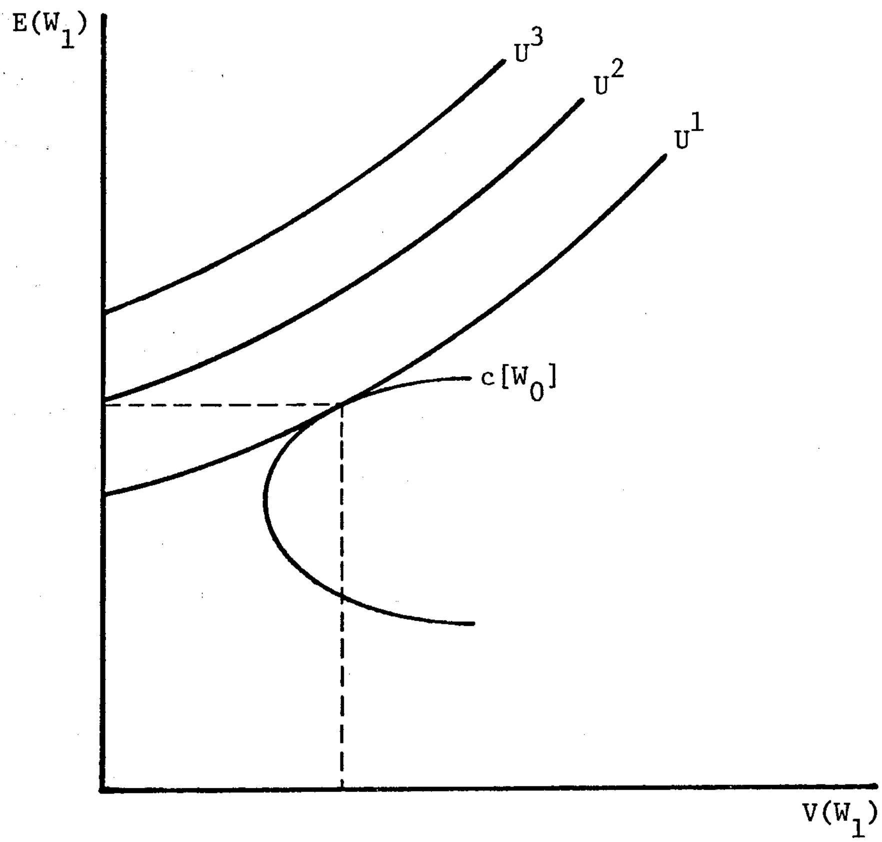
<u>Geometric Solution</u>. The solution to the mean-variance portfolio choice problem (2) is displayed in Figure 1.[6/] A family of three indifference curves is represented as U^1, U^2, and U^3, where $U^1 < U^2 < U^3$. Previous assumptions about the marginal expected utilities of the mean and variance of end-of-period wealth guarantee that the slope of a given indifference curve is positive—i.e.,

$$\left.\frac{dE}{dV}\right|_{dU=0} = -(U_V/U_E) > 0. \qquad (4)$$

The strict convexity of the indifference curves in Figure 1 further implies

FIGURE 1

SOLUTION TO THE MEAN-VARIANCE PORTFOLIO CHOICE PROBLEM WHEN ALL ASSETS ARE RISKY



$$\left.\frac{d^2E}{dV^2}\right|_{dU=0} = [-(U_E^2 \cdot U_{VV} + U_V^2 \cdot U_{EE}) + 2U_E \cdot U_V \cdot U_{EV}]/(U_E^3) > 0. \quad (5)$$

The optimal portfolio $\underline{A}^*$ is associated with the point of tangency of the efficiency locus $c[W_0]$ with the indifference curve U^1, which represents the point of highest attainable mean-variance utility. Following Markowitz [18], the efficiency locus is the boundary of feasible portfolios which have the smallest variance for a given mean value of end-of-period wealth. Analytically, the efficiency locus may be derived from the problem

$$\underset{\underline{A}}{\text{minimize}} \quad \underline{A}'\Sigma\underline{A} \text{ subject to } \underline{A}'\underline{\mu} = E \text{ and } \underline{A}'\underline{1} = W_0. \quad (6)$$

The solution to (6) is a parabola in variance-mean space dependent on the level of initial wealth and the parameters of the joint-probability distribution of asset rates of return:

$$V = \underline{A}'\Sigma\underline{A} = \frac{W_0^2(\underline{\mu}'\Sigma^{-1}\underline{\mu}) - 2W_0(\underline{\mu}'\Sigma^{-1}\underline{1})E + (\underline{1}'\Sigma^{-1}\underline{1})E^2}{(\underline{1}'\Sigma^{-1}\underline{1})(\underline{\mu}'\Sigma^{-1}\underline{\mu}) - (\underline{\mu}'\Sigma^{-1}\underline{1})^2}. \quad (7)$$

From the parabolic curvature of the efficiency locus the well-known result that investors with convex indifference curves will always select efficient portfolios is readily apparent.

II. Conditions for the Existence of Initial Wealth Satiation

Bierwag and Grove [4] were the first to demonstrate that mean-variance preference functions having strictly convex indifference curves display initial wealth satiation. This result is generalized below to include some preference orderings represented by concave indifference curves. In addition, the possible existence of initial wealth satiation is considered for different permutations of the mean-variance problem.

Before considering the conditions that give rise to initial wealth satiation, it is useful to define this concept in more precise terms.

Definition: Initial wealth satiation is attained at initial wealth W_0^* if all levels of initial wealth $W_0 < W_0^*$ yield lower mean-variance utility, and if $W_0 > W_0^*$, an investor will maximize utility by disposing an amount of initial wealth equal to $W_0 - W_0^*$.

Notice that this bliss point is similar to the wealth-satiation property inherent in quadratic utility as discussed by Borch [5]. However, this bliss point is defined in terms of initial wealth W_0, while wealth satiation in quadratic utility is more naturally defined in terms of end-of-period wealth W_1. Furthermore, as is shown in section III, the level of initial wealth at the unconstrained global maximum of expected quadratic utility is actually greater than the level of initial wealth at the bliss point defined above.

Despite the differences, the implications of initial wealth satiation are nevertheless similar to those of end-of-period wealth satiation in the quadratic utility case. In particular, for levels of initial wealth above that consistent with initial wealth satiation, an investor maximizes expected utility by disposing an amount of initial wealth equal to the difference. In other words, at sufficiently high levels of initial wealth, marginal mean-variance utility is negative with respect to increments of initial wealth. The implication of this bliss point is therefore highly untenable if limited-liability assets exist. In such a world, the axiom that more end-of-period wealth is always preferred to less also implies that more initial wealth is always preferred to less. To the extent that a mean-variance preference ordering is inconsistent with this latter proposition it is also inconsistent with a generally accepted norm of rational behavior.

To find the conditions that lead to initial wealth satiation in mean-variance portfolio models, it is easiest to proceed in a manner parallel to the previous section. Recall that in solving an investor's portfolio selection problem, initial wealth was assumed to be fully invested. This assumption is now relaxed, and the level of invested wealth will be allowed to vary. By varying the level of invested wealth, a family of efficiency loci representing sets of feasible portfolios is obtained, each parameterized by the same joint-probability distribution of asset rates of return and by different levels of invested wealth. Two efficiency loci corresponding to invested wealth levels W_0^* and $W_0^\dagger$ are displayed as $c[W_0^*]$ and $c[W_0^\dagger]$ in Figure 2a. The boundary of the set of all feasible portfolios is given by the envelope of the efficiency loci—such as $c^*[W_0]$ in Figure 2a. Analytically, this envelope may be expressed as [7/]

$$V = E^2(\underline{\mu}'\Sigma^{-1}\underline{\mu})^{-1}. \tag{8}$$

Each point on the boundary of this envelope corresponds to the unique level of invested wealth given by

$$W = (V/E)(\underline{\mu}\Sigma^{-1}\underline{1}). \tag{9}$$

As is apparent from Figure 2a, initial wealth satiation exists if and only if mean-variance utility has a maximum along the parabola representing the envelope of all efficiency loci. This yields the following theorem:

<u>Theorem 1</u>: Initial wealth satiation exists if and only if the problem

$$\underset{W}{\text{maximize}} \quad U[E,V] \text{ subject to (9)} \tag{10}$$

has a finite solution W_0^*.

Geometrically, Theorem 1 merely states that a necessary and sufficient condition for initial wealth satiation is the existence of a level of invested

FIGURE 2a

THE EXISTENCE OF INITIAL WEALTH SATIATION
IN A MEAN-VARIANCE PORTFOLIO CHOICE PROBLEM

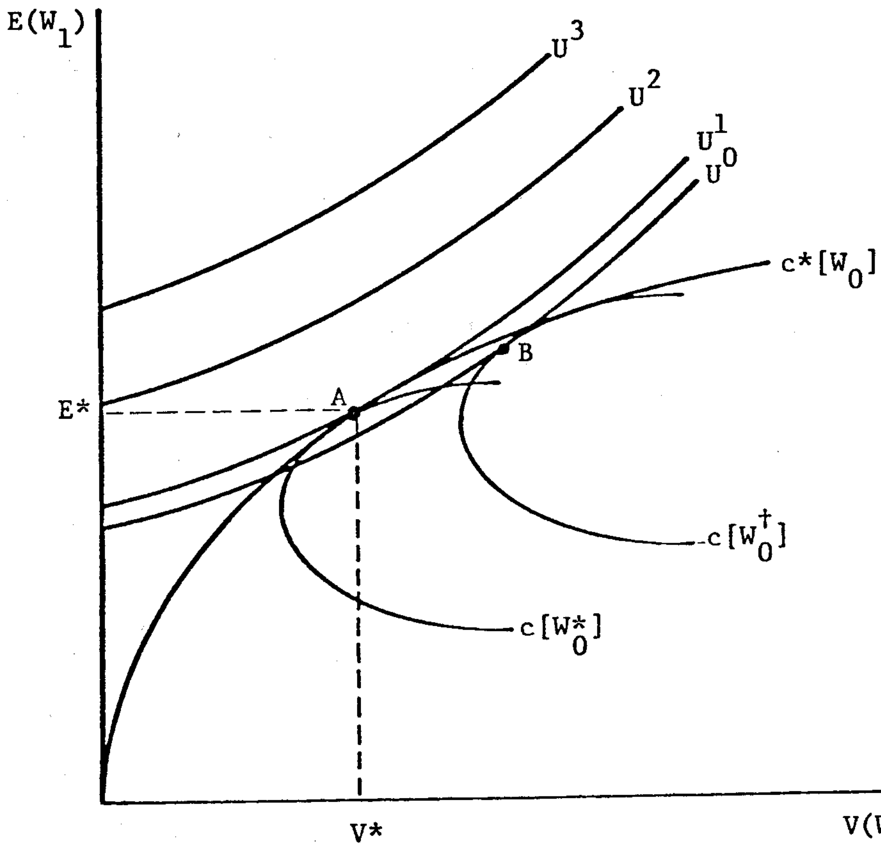
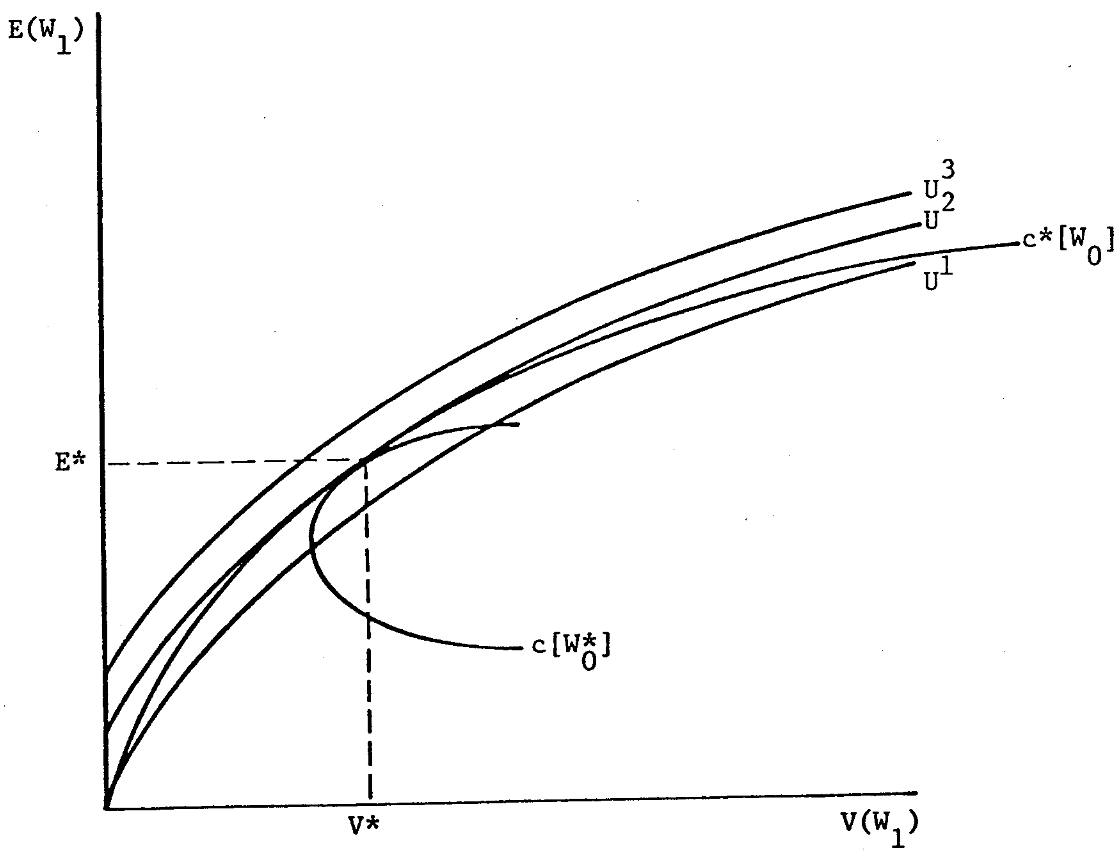


FIGURE 2b



wealth W_0^* such that an indifference curve is tangent to the envelope of the efficiency loci. In Figure 2a, this bliss level of initial wealth corresponds to point A. If an investor has initial wealth equal to $W_0^\dagger$, for example, and invests the entire amount, the optimal portfolio decision would be the portfolio represented by point B. Clearly, the level of utility could be made higher in this case by disposing an amount of initial wealth equal to $W_0^\dagger - W_0^*$, and moving to point A. It is easy to show that if initial wealth exceeds the satiation level, then marginal mean-variance utility of wealth (λ) is negative when the investor is fully invested, thus implying that some initial wealth should divested.

The Bierwag and Grove [4] result immediately follows as a straightforward application of Theorem 1:

> Corollary 1: Initial wealth satiation exists for all mean-variance utility functions exhibiting convex indifference curves in variance-mean space.

Again, this case conforms precisely to the family of indifference curves in Figure 2a. Notice, however, that this corollary describes a sufficient but not a necessary condition. In particular, if initial wealth satiation exists at W_0^*, then the second-order condition associated with the problem in Theorem 1 may be expressed as

$$\left.\frac{d^2E}{dV^2}\right|_{\substack{dU=0 \\ W_0=W_0^*}} > -\frac{1}{E}\left[\frac{dE}{dV}\right]^2_{\substack{dU=0 \\ W_0=W_0^*}} \tag{11}$$

where the left-hand side is the curvature of the indifference curve representing highest attainable mean-variance utility, evaluated at its tangency to the envelope of the efficiency loci, such as Point A in Figure 2a. From this condition it is apparent that some concave mean-variance indifference curve mappings also imply initial wealth satiation. Such a case is illustrated in Figure 2b. This latter fact has interesting implications for mean-variance models formulated in standard deviation-mean space, which are discussed below.

Permutations of Two-Moment Portfolio Selection Problems. Instead of representing an investor's objective function in terms of the mean and variance of end-of-period wealth, some researchers specify mean-standard deviation preference orderings. (See, for example, Tobin [31] and Sharpe [30].) In standard deviation-mean space, the sufficient conditions for a satiation level of initial wealth are slightly different. (See Jones [12].) In this case, the analogue of the envelope of the efficiency loci (7) is the straight line

$$S \;=\; E(\underline{\mu}'\Sigma^{-1}\underline{\mu})^{-1/2} \qquad (8')$$

where $S = V^{1/2}$. Convex indifference curves in standard deviation-mean space are not sufficient to ensure a satiation level of initial wealth in this model. Instead, a sufficient condition for such a satiation point to exist is that the indifference curves are convex and at least one indifference curve cuts the efficient portfolio line (8´) at two points. This latter condition holds if the marginal rate of substitution of variance for mean tends to infinity as the level of variance increases without bound. Geometrically, this is equivalent to the proposition that the indifference curves become vertical for large values of the standard deviation.

The results of the previous subsection may also be interpreted in standard deviation-mean (S-E) space. As a preliminary step, it is useful to indicate the relationship between indifference-curve mappings of the same preference ordering in S-E and V-E spaces. In terms of slope and curvature, this relationship is as follows:

$$\left.\frac{dE}{dS}\right|_{dU=0} = 2V^{1/2}\left[\frac{dE}{dV}\right]_{dU=0}$$

$$\left.\frac{d^2E}{dS^2}\right|_{dU=0} = 4V \cdot \left[\frac{d^2E}{dV^2}\right]_{dU=0} + 2\left[\frac{dE}{dV}\right]_{dU=0} .$$

One implication of the above is that convex indifference curves in S-E space, implying risk-averse diversifying behavior, do not necessarily imply convex indifference curves in V-E space. In particular, following Tobin [31], an investor is a risk-averse diversifier if

$$\left.\frac{d^2E}{dS^2}\right|_{dU=0} > 0$$

which implies

$$\left.\frac{d^2E}{dV^2}\right|_{dU=0} > -\frac{1}{2V}\left[\frac{dE}{dV}\right]_{dU=0} .$$

Combining these results with the second-order condition (11) leads to the following:

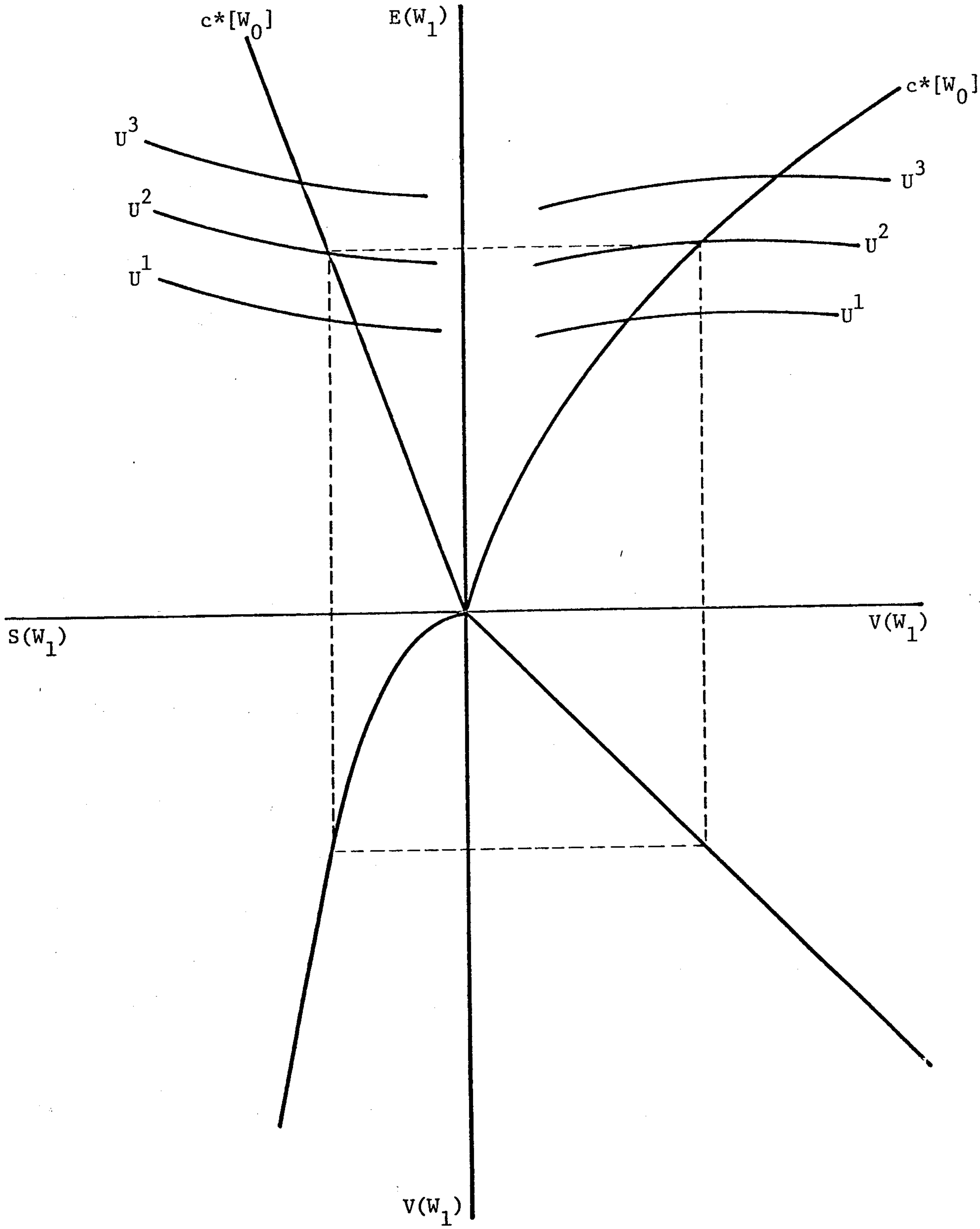
<u>Corollary 2</u>: Not all preference orderings consistent with risk-averse diversifying behavior imply initial wealth satiation.

An example of this situation is illustrated in Figure 3, where indifference curves consistent with risk-averse diversifying behavior in S-E space, but not consistent with initial wealth satiation, are transformed into V-E space. Note that by Corollary 1, convex indifference curves in S-E space that do not imply initial wealth satiation must correspond to concave indifference curves in V-E space.8/

Another possible specification of the mean-variance utility problem is

FIGURE 3

RISK-AVERSE DIVERSIFYING PORTFOLIO BEHAVIOR AND THE ABSENCE OF INITIAL WEALTH SATIATION



that with portfolio rate of return—instead of end-of-period wealth—as the argument of the mean-variance objective function. In this case, the mean-variance portfolio-choice problem is

$$\underset{\underline{h}}{\text{maximize}} \quad U[E(R_p),\ V(R_p)] \text{ subject to } 0 \leq \underline{1}'\underline{h} \leq 1 \qquad (2')$$

where

$\underline{h}$ = N × 1 vector of proportional holdings of assets ($\underline{h} = (1/W_0)\cdot\underline{A}$), with typical element h_i (i = 1,2,...,N)

R_p = portfolio rate of return ($R_p = W_1/W_0$).

This specification differs from the end-of-period wealth case analyzed above in only one respect: the efficiency loci are conditional on the fraction of the portfolio invested ($0 \leq \underline{1}'\underline{h} \leq 1$) rather than on W_0. Moreover, if $\underline{1}'\underline{h} = 1$ at the optimum of (2′), any increases in initial wealth will not result in wealth satiation because portfolio composition is selected independently from the level of initial wealth. (See Roley [26].) Conversely, if maximum mean-variance utility occurs for a less-than-fully invested portfolio (h<1), wealth satiation analogous to the end-of-period wealth case arises, although in this instance it is in terms of a proportion of initial wealth and not the level.

Mean-Variance Analysis with a Riskless Asset. With the existence of a riskless asset, the mean-variance portfolio-choice problem does not generally have an initial wealth satiation point. Specifically, the conditions that imply the absence of initial wealth satiation may be summarized as follows:

Theorem 2: Initial wealth satiation does not exist if a riskless asset with positive rate of return R_F is available.

Proof: This may be shown by respecifying (2) for the one-riskless-asset case. By suitably partitioning the variance-covariance matrix Σ and the mean vector of asset rates of return $\underline{\mu}$ to conform with the availability of a riskless asset, the first-order conditions of the constrained optimum may be solved for the Lagrangian multiplier

$$\lambda = U_E \cdot R_F.$$

Again, λ may be interpreted as the marginal mean-variance utility of initial wealth, and for initial wealth satiation to exist it must admit negative values. However, with the assumed signs of U_E and R_F, λ is always positive.

Geometrically, if a riskless asset exists, then any point in variance-mean space is obtainable with a sufficiently large level of initial wealth. Hence, in this case a boundary analogous to the envelope of the efficiency loci (8) does not exist and the problem analogous to that in Theorem 1 does not have a finite solution.

III. Specific Mean-Variance Expected-Utility Models and Initial Wealth Satiation

In this section the possible existence of initial wealth satiation is examined in three common expected-utility models which reduce to exact mean-preference orderings. These three models are quadratic utility, negative exponential utility with joint-normally distributed asset rates of return, and logarithmic utility with lognormally distributed end-of-period wealth. It is shown below that the first two expected-utility models imply initial wealth satiation while the latter is not.

Quadratic Utility. Perhaps the most interesting example of initial wealth satiation involves the case of quadratic utility

$$U[W_1] = W_1 - bW_1^2, \; b > 0. \tag{12}$$

While this utility function has been severely criticized by Hicks [11], Arrow [1], and Samuelson [28] for displaying increasing absolute risk aversion, it nevertheless is the only von Neumann-Morgenstern utility function that reduces to an exact mean-variance preference ordering for all probability distributions of end-of-period wealth. (See Borch [5].) The quadratic utility function also possesses a global maximum at

$$W_1^* = 1/2b$$

implying the existence of a satiation level of end-of-period wealth.

Expected quadratic utility immediately follows from (12) and may be written as

$$E(U[W_1]) = E(W_1) - b\cdot E(W_1)^2 - b\cdot V(W_1). \qquad (12')$$

To demonstrate that expected quadratic utility has a point of initial wealth satiation Theorem 1 may be applied directly. The first- and second-order conditions associated with problem (10) are

$$1 - 2b\cdot(1+(\underline{\mu}'\Sigma^{-1}\underline{\mu})^{-1})\cdot E(W_1) = 0$$

$$- 2b\cdot(1+(\underline{\mu}'\Sigma^{-1}\underline{\mu}^{-1}) < 0.$$

These conditions are jointly satisfied for the unique level of invested wealth

$$W_0^* = (1/2b)\cdot\underline{1}'(\Sigma+\underline{\mu}\ \underline{\mu}')\underline{\mu}.$$

Consequently, a satiation level of initial wealth exists at W_0^*, and all initial wealth above this level will be divested.

The existence of initial wealth satiation in the quadratic utility case is illustrated in Figure 4, where the indifference curves, the efficiency locus for W_0^*, and the envelope of the efficiency loci are labeled as before. By setting W_1 equal to W_1^* in (12′), the maximum possible level of expected utility is

$$U^* = 1/4b.$$

The level of expected utility associated with initial wealth W_0^* is

$$U^{**} = (\underline{\mu}'\Sigma^{-1}\underline{\mu})\cdot[4b\cdot(1+\underline{\mu}'\Sigma^{-1}\underline{\mu})]^{-1}$$

which is always less than that of the unconstrained maximum ($U^*>U^{**}$). Thus, contrary to Borch [5] and Hakansson [10] among others, it is initial wealth satiation, not end-of-period wealth satiation, that effectively places the upper limit on the level of expected quadratic utility. Moreover, those researchers who place restrictions on the subjective probability distribution of rates of return and initial wealth to ensure that $W_1<W_1^*$ misdirect their focus. Restrictions ensuring that $W_1<W_1^*$ do not necessarily preclude $W_0>W_0^*$. Hence, initial wealth should be restricted to be less than W_0^* in order to circumvent the effective bliss point problem in the quadratic utility model.

<u>Negative Exponential Utility with Joint-Normally Distributed Asset Rates of Return</u>. An expected-utility model that also enjoys widespread use is derived from the combined assumptions of negative exponential utility and joint-normally distributed asset rates of return

$$\begin{aligned} U[W_1] &= -\exp(-bW_1) \\ \underline{R} &\sim N(\underline{\mu},\Sigma). \end{aligned} \tag{13}$$

One of the attractive features of this specification is that absolute risk aversion is nondecreasing. This model does, however, exhibit increasing relative risk aversion.[9/] Another unattractive feature of this model is that it is inconsistent with the limited liability status of many financial assets because the range of the normal variate is unbounded below.

The expected-utility model consistent with the above assumptions can be shown to be maximized if the following is maximized:

$$U[E(W_1),\ V(W_1)] = E(W_1) - (b/2)\cdot V(W_1). \tag{13'}$$

To obtain the satiation level of initial wealth, equation (13′) may be substituted into Theorem 1, with the constrained optimum yielding the following first- and second-order conditions:

$$1 - b\cdot(\underline{\mu}'\Sigma^{-1}\underline{\mu})^{-1}E(W_1) = 0$$

$$- b\cdot(\underline{\mu}'\Sigma^{-1}\underline{\mu}) < 0$$

which are satisfied for the unique level of initial wealth

$$W_0^* = (1/b)\cdot(\underline{1}'\Sigma^{-1}\underline{\mu}).$$

The initial wealth satiation point inherent in this expected-utility model is illustrated in Figure 5. This example serves to highlight the important fact that initial wealth satiation is an issue completely unrelated to whether the von Neumann-Morgenstern utility function possesses an unconstrained maximum, since $U[W_1]$ in (13) is monotonically increasing in W_1.

Logarithmic Utility with Lognormally-Distributed End-of-Period Wealth. Feldstein [6] proposed the logarithmic utility function with lognormally distributed end-of-period wealth

$$\begin{aligned} U[W_1] &= \log(W_1) \\ \log(W_1) &\sim N(E[\log(W_1)],\ V[\log(W_1)]) \end{aligned} \tag{14}$$

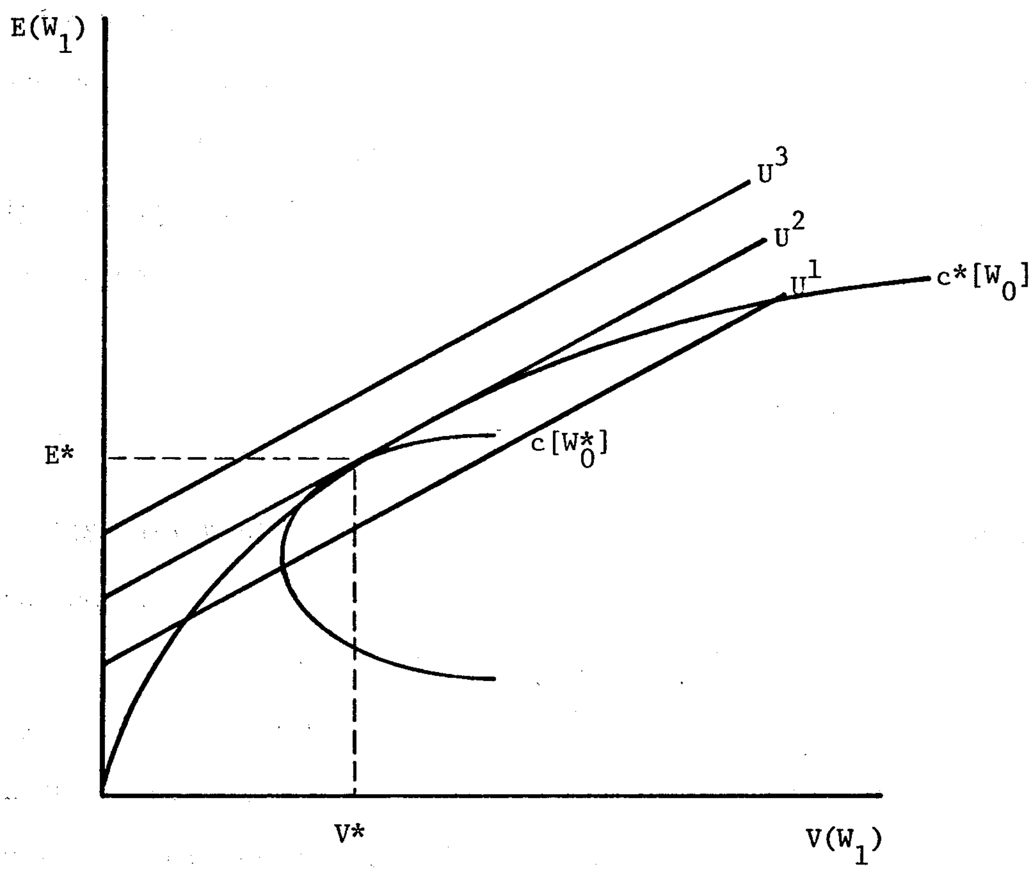
as an alternative to the quadratic utility function used by Tobin [31] in the analysis of liquidity preference. Combining the above two assumptions and taking the expected value of the utility function yields

$$E(U[W_1]) = \log[E(W_1)] - (1/2)\cdot\log[(V(W_1)/E(W_1)^2)+1]. \qquad (14')$$

This mean-variance preference ordering embodies decreasing abolute risk aversion and constant relative risk aversion, which implies investor behavior at least as plausible as that in the other two cases considered. (See, for example, Friend and Blume [8], Lintner [17], and Rubinstein [27].) However, the model is also based on the assumption of lognormally-distributed end-of-period wealth, and it is well known that a linear combination of lognormally-distributed asset rates of return—such as that representing end-of-period

FIGURE 5

NEGATIVE EXPONENTIAL UTILITY WITH JOINT-NORMALLY DISTRIBUTED ASSET RATES OF RETURN



wealth—is not distributed lognormally.

The possible existence of initial wealth satiation in this specification may again be investigated by applying Theorem 1, which yields the following necessary first- and second-order conditions:

$$1/E(W_1) = 0$$
$$-1/E(W_1)^2 < 0.$$

In this case the first-order condition is not satisfied for finite values of expected end-of-period wealth, and the expected-utility maximization problem does not have a bliss point. Geometrically, the mean-variance indifference curves implied by (14′) are strictly concave, as illustrated in Figure 6.[10/] Moreover, the curvature of each indifference curve is sufficiently flat so that the conditions implied by Theorem 1 are not satisfied. In contrast to the other two models considered, therefore, this model does not have either an unconstrained maximum or a point of initial wealth satiation.

IV. The Fundamental Approximation Theorem and Initial Wealth Satiation

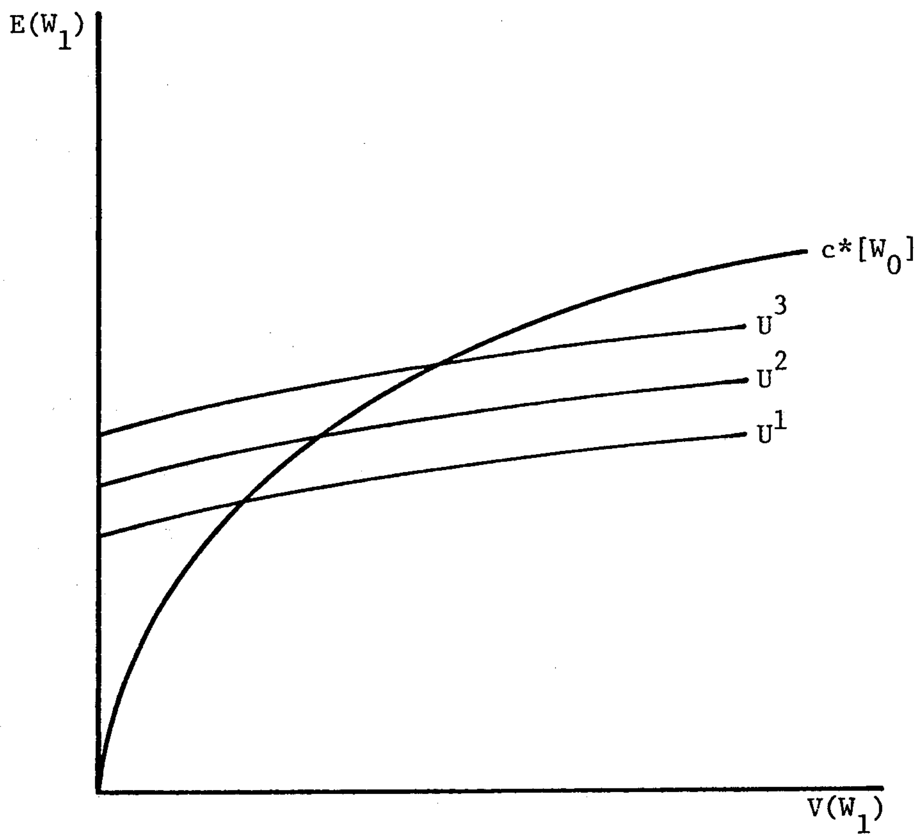
In this section, Samuelson's [29] Fundamental Approximation Theorem is used to show that if the conditions which enable a particular mean-variance model to closely approximate an arbitrary expected-utility model are satisfied, then the initial wealth satiation level inherent in the model should be ignored and the optimal portfolio should be calculated assuming the investor remains fully invested. The restrictiveness of the assumptions used to obtain this result are discussed in turn following the basic derivation.

The Fundamental Approximation Theorem may be stated as

Fundamental Approximation Theorem: Assume that the random rate of return on each asset, R_i, for i = 1,2,...,N, belongs to a family of

FIGURE 6

LOGARITHMIC UTILITY WITH LOGNORMALLY DISTRIBUTED END-OF-PERIOD WEALTH



probability distributions $F_i[R;\sigma^2]$ parameterized by σ^2 such that as σ^2 tends to zero the distribution $F_i[R;\sigma^2]$ tends to the sure event $\text{Prob}\{R_i = m\} = 1$ in the sense that:

$$\lim_{\sigma^2 \to 0} E(R_i - m) = 0$$

$$\lim_{\sigma^2 \to 0} E(R_i - m)/\sigma^2 = C_{i,1} \neq 0$$

$$\lim_{\sigma^2 \to 0} E[(R_i - m^2]/\sigma^2 = C_{i,2} \neq 0$$

$$\lim_{\sigma^2 \to 0} E[(R_i - m)^n]/\sigma^2 = 0, \; n=3,4,\ldots$$

where $m > 0$. Furthermore, assume that the investor's von Neumann-Morgenstern utility function, $U[W_1]$, satisfies $U' > 0$, $U'' < 0$, and possesses either an exact Taylor series expansion about arbitrary levels of wealth or possesses a bounded derivative of order "k" for some $k > 2$. Suppose that $\underline{h}_{\sigma^2}$ solves the portfolio choice problem

$$\underset{\underline{h}}{\text{maximize}} \; E_{\sigma^2}\{U[W_0 \cdot \underline{h}'\underline{R}]\} \text{ subject to } \underline{1}'\underline{h} \leq 1 \qquad (15)$$

and $\underline{h}^*_{\sigma^2}$ solves the associated mean-variance problem

$$\underset{\underline{h}}{\text{maximize}} \; \underline{h}'\underline{\bar{r}}_{\sigma^2} - (1/2)\cdot(\rho/m)\cdot\underline{h}'\Sigma_{\sigma^2}\underline{h} \qquad (16)$$

$$\text{subject to } \underline{1}'\underline{h} = 1$$

where $\underline{\bar{r}}_{\sigma^2} = E_{\sigma^2}(\underline{R}) - \underline{1}m$

$$\Sigma_{\sigma^2} = E_{\sigma^2}[(\underline{R}-\underline{\mu})(\underline{R}-\underline{\mu})']$$

$$\rho = -m\cdot W_0 \cdot U''[m\cdot W_0]/U'[m\cdot W_0],$$

then $\lim_{\sigma^2 \to 0} |\underline{h}_{\sigma^2} - \underline{h}^*_{\sigma^2}| = 0$.

A proof of this theorem appears in Samuelson [29]. The essential economic implication of this theorem is the following: if portfolio risk is small, the solution to the expected-utility maximization problem (15) is well approximated by the solution to the associated mean-variance problem (16).

Several characteristics of the mean-variance problem (16) are noteworthy. First, the problem is essentially a quadratic approximation to (15).[11/] Second, the problem is expressed in terms of portfolio rate of return. Finally, the parameter ρ in (16) is nothing more than the investor's relative risk aversion evaluated at $m \cdot W_0$.[12/]

As discussed in the second section, the problem (15) technically has a satiation point in terms of portfolio shares if maximum mean-variance utility occurs for a less-than-fully invested portfolio ($\underline{1}'\underline{h}<1$), rather than for $\underline{1}'\underline{h} = 1$. However, by the Fundamental Approximation Theorem, problem (16) should be solved for the optimal portfolio share vector assuming the investor is fully invested. In a sense, initial wealth satiation should be ignored in this case. An intuitive explanation for this conclusion follows from the first-order conditions for a constrained optimum of (15)

$$E_{\sigma^2}(U' \cdot \underline{R}) - \underline{1}\lambda_{\sigma^2} = \underline{0} \qquad (17)$$

where λ_{σ^2} is the Lagrangian multiplier associated with the wealth constraint in (15). Taking the limit of (17) as σ^2 tends to zero implies

$$\lim_{\sigma^2 \to 0} \lambda_{\sigma^2} = m \cdot U'[m \cdot W_0].$$

Thus, for portfolio risk sufficiently small, the marginal expected utility of initial wealth $\lambda^*_{\sigma^2}$ is positive and initial wealth is fully invested. In other words, if the marginal utility with respect to wealth in (15) is positive, the optimal portfolio obtained from the mean-variance approximation (16) should be represented as one that is fully invested.

The usefulness of this result turns on the issue of whether portfolio risk is, in fact, typically small. This issue has been examined fairly extensively, and it has been shown that portfolio risk is effectively small if asset returns follow a Gaussian diffusion process and if either of the following conditions are present: (1) the investor's holding period is short, or (2) asset trading can take place continuously in time.[13/] Jones [12] also shows that the conditions of the Fundamental Approximation Theorem are effectively met in the lognormal securities market models of Lintner [15], Bawa and Chakrin [2], and Ohlson and Ziemba [25].

V. Summary of Conclusions

When all financial assets are assumed to be risky, arguably the most relevant case because of persistent and uncertain price inflation, a broad class of mean-variance utility models is consistent with initial wealth satiation. Bierwag and Grove [4] first noticed this property, but in subsequent studies involving mean-variance utility models, it has either been ignored or misinterpreted. In these latter instances, end-of-period wealth satiation inherent in quadratic utility has been mistaken for the Bierwag and Grove result which instead involves initial wealth satiation.

In this paper, the Bierwag and Grove result was generalized, and specific case applications involving quadratic utility, negative exponential utility with joint-normally distributed asset rates of return, and logarithmic utility with lognormally distributed end-of-period wealth were examined. In the quadratic utility case, it was shown that initial wealth satiation, not end-of-period wealth satiation, effectively places an upper bound on the model in terms of rational investor behavior. Moreover, in general applications including other utility functions, the existence of initial wealth satiation was found to be independent of the existence of end-of-period wealth satiation.

By appealing to Samuelson's [29] defense of mean-variance analysis, however, it was shown that if a particular mean-variance model is interpreted as an approximation to a well-behaved expected utility model, the possible existence of a satiation point in the mean-variance approximation is not relevant. In this case, the optimum of the mean-variance problem asymptotically approaches the solution to the expected utility model, and if the marginal utility with respect to wealth is positive, the optimal portfolio should be represented as one that is fully invested.

Footnotes

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1/ For studies involving the all-risky assets case, see, for example, Merton [22] and Landskronner [13].

2/ Borch [5] and Hakansson [10] interpret the Bierwag and Grove [4] result as implying that indifference curves in standard deviation-mean space are concentric circles with the point of highest utility represented by a single point at the center. This bliss point corresponds to end-of-period wealth satiation in quadratic utility. Bierwag and Grove [4], however, do not examine the case in which indifference curves in standard deviation-mean space have this representation. Instead, they assume convex indifference curves in variance-mean space, and show that this assumption implies a preference ordering in asset space represented by concentric circles. The center of these circles represents the point of initial wealth satiation.

3/ Throughout this paper all probability-distributions of end-of-period wealth are assumed to have finite first- and second-moments.

4/ However, the lognormality of end-of-period wealth does not follow from lognormally-distributed asset rates of return.

5/ Tobin [31] relates this type of risk-averse behavior to indifference curves in standard deviation-mean space. In variance-mean space, convex indifference curves are a sufficient but not a necessary condition for risk-averse-diversifying behavior, as discussed in the next section.

6/ See Merton [22] for graphical presentations of both the all-risky assets and one-riskless asset cases.

7/ The envelope of the efficiency loci may be derived from the problem

$$\underset{\underline{A}}{\text{minimize}} \quad \underline{A}'\Sigma\underline{A} \text{ subject to } \underline{A}'\underline{\mu} = E.$$

8/ Analytically, a risk-averse diversifier does not have a point of initial wealth satiation if

$$\left.\frac{dE}{dS}\right|_{dU=0} < \frac{E}{S} .$$

Thus, the restrictions placed on the portfolio selection problem by Tsiang [33] guarantee that initial wealth satiation does not occur. In particular, he assumes that (1) $E/S > 1$, i.e., the efficiency envelope consists only of portfolios with greater mean than standard deviation and (2) $[dE/dS]_{dU=0} < 1$. For criticisms of this latter assumption, see Bierwag [3] and Levy [14].

9/ Nevertheless, Arrow [1] has argued on both theoretical and empirical grounds that relative risk aversion should be, and is, an increasing function of wealth.

10/ As demonstrated by Feldstein [6], in standard deviation-mean space the indifference curves are initially convex and then concave after the inflection point. In variance-mean space, however, it may be verified that the indifference curves are strictly concave.

11/ This will be the case if rates of return are generated by any continuous-time diffusion process.

12/ Tsiang [33] extends the Fundamental Approximation Theorem for utility functions exhibiting constant absolute and constant relative risk aversion. He claims that portfolio risk need not be small absolutely for mean-variance analysis to be a good approximation to expected utility maximization. Rather, all that is required is that portfolio risk be small relative to the size of initial wealth. Tsiang claims that this condition will be met for most individual investors provided that wealth is correctly defined to include both financial and nonfinancial assets—e.g., human capital.

13/ For examples using condition (1), see Merton and Samuelson [24], Friend, Landskronner, and Losq [9], Landskronner [13], and Friedman and Roley [7]. For examples using condition (2), see Merton [20, 21, 22].

References

1. Arrow, Kenneth J. Aspects of the Theory of Risk-Bearing. Helsinki: The Yrjö Jahsson Foundation, 1965.

2. Bawa, V. and Chakrin, L. "Optimal Portfolio Choice and Equilibrium in a Lognormal Securities Market." Mimeo, Bell Labs, 1977.

3. Bierwag, G. O. "The Rationale of the Mean-Standard Deviation Analysis: Comment." American Economic Review. LXIV (June, 1974), 431-3.

4. Bierwag, G. O., and Grove, M. A. "Indifference Curves in Asset Analysis." Economic Journal, LXXVI (June, 1966), 337-43.

5. Borch, Karl. "A Note on Uncertainty and Indifference Curves." Review of Economic Studies. XXXVI (January, 1969), 1-4.

6. Feldstein, Martin S. "Mean-Variance Analysis in the Theory of Liquidity Preference and Portfolio Selection." Review of Economic Studies. XXXVI (January, 1969), 5-12.

7. Friedman, Benjamin M., and Roley, V. Vance. "A Note on the Derivation of Linear Homogeneous Asset Demand Functions." Mimeo, Harvard University, 1979.

8. Friend, Irwin, and Blume, Marshall E. "The Demand for Risky Assets." American Economic Review. LXV (December, 1975), 900-22.

9. Friend, Irwin; Landskronner, Yoram; and Losq, Etienne. "The Demand for Risky Assets Under Uncertain Inflation." Journal of Finance. XXXI (December, 1976), 1287-97.

10. Hakansson, Nils H. "Mean-Variance Analysis in a Finite World." Journal of Financial and Quantitative Analysis. VII (September, 1972), 1873-80.

11. Hicks, John R. "Liquidity." Economic Journal. LXXII (December, 1962), 787-802.

12. Jones, David S. A Structural Econometric Model of the United States Equity Market. Ph.D. dissertation, Harvard University, 1979.

13. Landskronner, Yoram. "The Determinants of the Market Price of Risk in the Absence of a Riskless Asset." Friend and Bicksler (eds.), Risk and Return in Finance, Volume II. Cambridge, Ma.: Ballinger Publishing Company, 1977.

14. Levy, Haim. "The Rationale of the Mean-Standard Deviation Analysis: Comment." American Economic Review. LXIV (June, 1974), 434-41.

15. Lintner, John. "Equilibrium in a Random Walk and Log Normal Securities Market." Mimeo, Harvard University, 1972.

16. Lintner, John. "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets." Review of Economics and Statistics. XLVII (February, 1965), 13-37.

17. Lintner, John. "The Lognormality of Security Returns, Portfolio Selection and Market Equilibrium." Mimeo, Harvard University, 1975.

18. Markowitz, Harry. "Portfolio Selection." Journal of Finance. VII (March, 1952), 77-91.

19. Markowitz, Harry. Portfolio Selection. New Haven: Yale University Press, 1969.

20. Merton, Robert C. "Lifetime Portfolio Selection Under Uncertainty: The Continuous Time Case." Review of Economics and Statistics. LI (August, 1969), 247-57.

21. Merton, Robert C. "Optimum Consumption and Portfolio Rules in a Continuous Time Model." Journal of Economic Theory. III (December, 1971), 373-413.

22. Merton, Robert C. "An Analytic Derivation of the Efficient Portfolio Frontier." Journal of Financial and Quantitative Analysis. VII (September, 1972), 1851-72.

23. Merton, Robert C. "An Intertemporal Capital Asset Pricing Model." Econometrica, XLI (September, 1973), 867-87.

24. Merton, Robert C., and Samuelson, Paul A. "Fallacy of the Log-Normal Approximation to Optimal Portfolio Decision Making Over Many Periods." Journal of Financial Economics. I (May, 1974), 67-94.

25. Ohlson, J. A., and Ziemba, W. T. "Portfolio Selection in a Lognormal Market When the Investor Has a Power Utility Function." Journal of Financial and Quantitative Analysis. XI (March, 1976), 57-71.

26. Roley, V. Vance. A Structural Model of the U.S. Government Securities Market. Ph.D. dissertation, Harvard University, 1977.

27. Rubinstein, Mark. "The Strong Case for the Generalized Logarithmic Utility Model as the Premier Model of Financial Markets." Mimeo, University of California at Berkeley, 1976.

28. Samuelson, Paul A. "A General Proof that Diversification Pays." Journal of Financial and Quantitative Analysis. II (March, 1967), 1-13.

29. Samuelson, Paul A. "The Fundamental Approximation Theorem of Portfolio Analysis in Terms of Means, Variances, and Higher Moments." Review of Economic Studies. XXXVII (October, 1970), 537-42.

30. Sharpe, William F. "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk." Journal of Finance. XIX (September, 1964), 425-42.

31. Tobin, James. "Liquidity Preference as Behavior Toward Risk." Review of Economic Studies. XXV (February, 1958), 65-86.

32. Tobin, James. "Comment on Borch and Feldstein." Review of Economic Studies. XXXVI (January, 1969), 13-4.

33. Tsiang, S. C. "The Rationale of the Mean-Standard Deviation Analysis, Skewness Preference, and the Demand for Money." American Economic Review. LXII (June, 1972), 354-71.

34. von Neumann, John, and Morgenstern, Oskar. Theory of Games and Economic Behavior. New York: John Wiley & Sons, 1944.