Online Appendix for
"A Behavioral New Keynesian Model"

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This online appendix gives complements on the model (e.g. on the natural interest, on the numerical values chose, on variants with fully flexible prices, on variants to model the long run). It gives also additional proofs.

9 Complements

9.1 The "natural interest rate" in a behavioral economy

The natural interest rate is defined here as the interest rate that would prevail "if pricing frictions were removed", but keeping cognitive frictions (and before any deficits). Let us examine this in detail. Take the IS curve (29), coming back to the more basic notion of $\hat{c}_t := \ln c_t - \ln \bar{c}$:

$$\hat{c}_t = M\mathbb{E}_t\left[\hat{c}_{t+1}\right] + b_d d_t - \sigma\left(r_t - \bar{r}\right) \text{ (IS curve)} \tag{90}$$

where $r_t = i_t - \mathbb{E}_t \pi_{t+1}$ is the real rate.

Consider also the case with productivity shocks, so that $c_t = e^{\zeta_t} N_t$, so that the optimum frictionless consumption (see the derivation in (95)) is

$$\hat{c}_t^n = \frac{1+\phi}{\gamma+\phi}\zeta_t$$

So, if we removed all pricing frictions, we'd have $\hat{c}_t = \hat{c}_t^n$, and if we were in an environment with no deficit, we'd have:

$$\hat{c}_t^n = M\mathbb{E}_t\left[\hat{c}_{t+1}^n\right] - \sigma\left(r_t^n - \bar{r}\right), \tag{91}$$

which gives us the value of the natural rate, $r_t^n = \bar{r} + \frac{M\mathbb{E}_t\left[\hat{c}_{t+1}^n\right] - \hat{c}_t^n}{\sigma}$. We define the output gap as $x_t := \hat{c}_t - \hat{c}_t^n$ (up to second order terms). So, taking (90) and subtracting (91) we have:

$$x_t = M\mathbb{E}_t\left[x_{t+1}\right] + b_d d_t - \sigma\left(r_t - r_t^n\right)$$

which is the formulation in the paper.

Note that we could have defined the "natural" rate as the rate that would prevail in an economy without pricing frictions, and given the actual deficits, i.e. defined it as the solution $\tilde{r}_t^n$ of:

$$\hat{c}_t^n = M\mathbb{E}_t\left[\hat{c}_{t+1}^n\right] + b_d d_t - \sigma\left(\tilde{r}_t^n - \bar{r}\right)$$

i.e. $r_t^n = \bar{r} + \frac{b_d d_t + M\mathbb{E}_t\left[\hat{c}_{t+1}^n\right] - \hat{c}_t^n}{\sigma}$. And then the IS curve would become:

$$x_t = M\mathbb{E}_t\left[x_{t+1}\right] - \sigma\left(r_t - r_t^n\right)$$

This would be mathematically equivalent, but the language would become more complicated. Then a policy change (via deficits) would change the natural rate. For instance, a temporary rise of the deficit would decrease the natural rate (as it makes people want to spend more). With that definition, the natural rate is not very "natural".

9.2 The ex ante benefits of the possibility of future fiscal policy

Now I explore how behavioral agent change a lot policy at the ZLB, and indeed use the model's ability to have non-trivial monetary and fiscal policy. To make the point, I suppose that we have a "crisis period" $I = (T_1, T_2)$, with $r_t^n < 0$ during that period, so that the ZLB binds. But $r_t^n > 0$ outside that period. However, with fiscal policy and behavioral agents, the first best can be restored.

Proposition 9.1 (Optimal mix of fiscal and monetary policy in a ZLB environment). *The following monetary and fiscal policies yield the first best ($x_t = \pi_t = 0$) at all dates.* During the crisis *($t \in (T_1, T_2)$), use fiscal policy*

$$d_t = -\frac{\sigma r_t^n}{b_d},$$

i.e. run a deficit with low interest rates, $i_t = 0$. After the crisis *($t \geq T_2$), pay back the accumulated debt by running a government fiscal surplus and keeping the economy afloat with low rates, e.g. $d_t = R^{-1}\left(B_{T_2} - B_0\right)\left(1 - \rho_d\right)\rho_d^{t-T_2} < 0$ for some $\rho_d \in (0,1)$, and adjust $i_t = r_t^n + \frac{b_d d_t}{\sigma}$ to ensure full macro stabilization, $x_t = \pi_t = 0$.* Before the crisis *($t < T_1$), there is no preventive action to do, so set $i_t = d_t = 0$.*

Proof. The proof is simply by examination of the basic equations of the NK model, (29)-(30). We adjust the instruments so that $x_t = \pi_t = 0$ at all dates. Note that there are multiple ways to soak up the debt after the crisis, so that $d_t = R^{-1}(B_{T_2} - B_0)(1-\rho_d)\rho_d^{t-T_2}$ is simply indicative. $\square$

The ex-ante preventive benefits of potential ex-post fiscal policy. Proposition 9.1 shows that "the possibility of fiscal policy as ex-post cure produces ex-ante benefits". Imagine that fiscal policy is not available. Then, the economy is depressed at the ZLB during (T_1, T_2). However, it is also depressed before: because the IS curve is forward looking, output threatens to be depressed before T_1, and that can put the economy to the ZLB at a time T_0 before T_1.[76] Hence, the threat of a ZLB-depression in (T_1, T_2) creates an earlier recession at (T_0, T_2) with $T_0 < T_1$. Intuitively, agents feel "if something happens, monetary policy will be impotent, so large dangers loom". However, if the government has fiscal policy in its arsenal, the agents feel "worse case, the government will use fiscal policy, so there is no real threat", and there is no recession in (T_0, T_1). Hence, there is a possibility of fiscal policy as an ex-post cure to produce ex-ante benefits.

In general, monetary and fiscal policies are substitutes (d_t and i_t enter symmetrically in (29)), so a great number of policies achieve the first best. However, fiscal policy d_t helps monetary policy if there is a constraint (e.g. at the ZLB), so the possibility of future fiscal policy is a complement to the monetary policy (as it relieves the ZLB).[77]

9.3 The economy with fully flexible prices

What happens if the economy has fully flexible prices? To study this, I revisit Galí (2015, Chapter 2.4), with behavioral agents.

I say that the consumer perceives future inflation like the firms, i.e. use (50)

$$\mathbb{E}_t^{BR}[\pi_{t+1}] = \pi_t^d + m_\pi \mathbb{E}_t\left[\pi_{t+1} - \pi_t^d\right]$$

with $m_\pi \in [0,1]$, π_t^d is the default inflation, as in the main paper:

$$\pi_{t+1}^d = (1-\eta)\pi_t^d + \eta\left(\zeta\pi_t^{CB} + (1-\zeta)\pi_t\right)$$

If that channel is shut down, then simply $\pi_t^d = 0$, i.e. $\pi_t^{CB} = 0$, $\zeta = 1$.

[76] Future negative output gaps will creates a low output gap at times 0, 1, say, and so low that a central bank would need negative rates to fight those gaps.

[77] This "second instrument" could be very useful even in normal times, in a richer model with capital. Suppose that consumers get too optimistic about the future: the central bank should raise the interest rate. But then, that depresses investment. We do not get the first best any more, without a second instrument.

I suppose that the central bank follows a Taylor rule

$$i_t = j_t + \phi_\pi \pi_t$$

with $\phi_\pi \geq 0$.

In a model with flexible prices and no capital, the output gap is always 0. The behavioral IS curve still imposes:

$$r_t = i_t - \mathbb{E}_t^{BR}\left[\pi_{t+1}\right] - r_t^n$$

Take for simplicity an economy with constant $r_t^n = j_t$. Then, we have:

$$\phi_\pi \pi_t = \pi_t^d + m_\pi \mathbb{E}_t\left[\pi_{t+1} - \pi_t^d\right] \tag{92}$$

When is the equilibrium determinate?

Proposition 9.2 Determinacy in the flexible price economy) *Take the flexible price economy, in the simplest case with 0 default inflation ($\pi_t^d \equiv 0$, $\zeta = 1$). We have determinacy iff*

$$\phi_\pi > m_\pi \tag{93}$$

When there is non-zero default inflation ($\zeta < 1$), we have determinacy iff:

$$\phi_\pi > m_\pi + (1-\zeta)(1-m_\pi) \tag{94}$$

Proof. For the simple case, this is just because $\phi_\pi \pi_t = m_\pi \mathbb{E}_t\left[\pi_{t+1}\right]$, and we have determinacy iff $\phi_\pi > m_\pi$.

For the case $\zeta < 1$, we employ the same technique as for the proof of Proposition 5.3. □

Hence, we see a similar weakening of the Taylor criterion, from bounded rationality.

9.4 More general derivation for the mark-up

I give a more general derivation of (84). Recall that ζ_t is log productivity. The labor supply is still (60), $N_t^\phi = \omega_t C_t^{-\gamma}$, and as the resource constraint is $C_t = e^{\zeta_t} N_t$, $\omega_t = e^{-\phi\zeta_t} C_t^{(\gamma+\phi)}$. The real marginal cost is then $\Psi_t / P_t = \frac{\omega_t}{e^{\zeta_t}} = e^{-(1+\phi)\zeta_t} C_t^{(\gamma+\phi)}$. Then, recall the definition $\mu_t = p_t - \psi_t$, we obtain

$$\mu_t = (1+\phi)\zeta_t - (\gamma+\phi)c_t$$

Next, if the pricing frictions disappeared, the markup would be 0 (recall that the government has a subsidy to ensure that), i.e. consumption would be at c_t^n s.t.

$$0 = (1+\phi)\,\zeta_t - (\gamma+\phi)\,c_t^n$$

which gives the efficient level of consumption:

$$c_t^n = \frac{1+\phi}{\gamma+\phi}\zeta_t \tag{95}$$

So, the output gap is $x_t := c_t - c_t^n$ satisfies:

$$\mu_t = -(\gamma+\phi)\,x_t \tag{96}$$

9.5 Complements to the 2-period Model

This section gives complements to the 2-period model of Section 7

Discounted Euler equation in the 2-period model Also, this consumer satisfies a discounted Euler equation. Call $R = 1/\beta$ the steady state interest rate, so that $R_0 = R + \hat{r}_0$ and the perceived interest rate is: $R_0 = R + m_r\hat{r}_0$. Rewrite (62) as

$$c_0 = b\left(c_0 + \frac{c_1^d + m\hat{c}_1}{R + m_r\hat{r}_0}\right)$$

where $c_0^d = b\left(c_0^d + \frac{c_1^d}{R}\right)$. Then, we have:

$$\hat{c}_0 = b\left(\hat{c}_0 + \frac{m\hat{c}_1 - \frac{m_r}{R}\hat{r}_0}{R}\right)$$

i.e.

$$\hat{c}_0 = \frac{b}{1-b}\frac{1}{R}\left(m\hat{c}_1 - \frac{m_r}{R}\hat{r}_0\right).$$

In the rational model, we have $c_0 = \frac{b}{1-b}\frac{1}{R}c_1$ and $c_0 = c_1 = 1$. Hence, $\frac{b}{1-b}\frac{1}{R} = 1$. We obtain:

$$\hat{c}_0 = m\mathbb{E}_0\left[\hat{c}_1\right] - \frac{m_r}{R}\hat{r}_0. \tag{97}$$

This is a "discounted Euler equation" (with discount factor m), i.e. instead of the rational Euler equation, $\hat{c}_0 = \mathbb{E}[\hat{c}_1] - \hat{r}_0$. The same factor m gives power to fiscal policy, and yields a discounted Euler equation.

Derivation of (65). Call k_1 the wealth at the beginning of period 1 (before receiving labor income and profit), and $\mathcal{T}_1$ the transfer received from the government, and I_1 the profit income from the oligopolistic firms (so that $\omega_1 N_1 + I_1 = c_1$ when aggregating). The rational value function at time 1 is:

$$V^r(k_1, \mathcal{T}_1) = \max_{c_1, N_1} u(c_1, N_1) \text{ s.t. } c_1 \leq \omega_1 N_1 + I_1 + k_1 + \mathcal{T}_1.$$

The decision at time 0 is

$$\operatorname*{smax}_{c_0, N_0; \bar{m}} u(c_0, N_0) + \beta V^r \left(R_0 \left(\omega_0 N_0 + I_0 + \mathcal{T}_0\right), \bar{m}\mathcal{T}_1\right)$$

where $\bar{m}$ is optimized upon in the sparse max. Taking here provisionally the $\bar{m}$ as given, then the decision is simply:

$$\max_{c_0, N_0} u(c_0, N_0) + \beta V^r \left(R_0 \left(\omega_0 N_0 + I_0 + \mathcal{T}_0 - c_0\right), \bar{m}\mathcal{T}_1\right).$$

The first order conditions are:

$$u_{c_0} = \beta R_0 V_{k_1}$$
$$u_{N_0} = -\omega_0 \beta R_0 V_{k_1}$$

so that the intra-period labor supply condition $\omega_0 u_{c_0} + u_{N_0} = 0$ holds. Given that $V_{k_1} = u_{c_1}$, we obtain

$$u_{c_0}(c_0, N_0) = \beta R_0 u_{c_1}(c_1, N_1).$$

Now, we have $V^r_{k_1} = u'(c_1) = u'(k_1 + y_1)$ with $y_1 = \omega_1 N_1 + I_1 + m\mathcal{T}_1$, so

$$\frac{1}{c_0} = \frac{\beta R_0}{c_1}$$

with $c_1 = y_1 + R(y_0 - c_0)$ i.e. $c_0 + \frac{c_1}{R} = y_0 + \frac{y_1}{R}$, and with the Euler equation $c_1 = \beta R_0 c_0$:

$$c_0 = \frac{1}{1+\beta}\left(y_0 + \frac{y_1^s}{R}\right) = b\left(y_0 + \frac{y_1 + m\hat{y}_1}{R}\right).$$

9.6 Derivation of the Phillips curve (52) in continuous time

Here I show the derivation of the Phillips curve in continuous time. In exploring variants of the NK model, I found it much quicker to use this continuous-time derivation than the discrete time version (the 2-period model is also useful for basic conceptual issues).

If a firm can reset its price at time 0, it sets it to (using $\delta := \lambda + r$):

$$p_0^* - p_0 = \mathbb{E}^{BR}\left[\int_0^\infty \delta e^{-\delta t}\left(\mu x_t + p_t - p_0\right) dt\right] = \mathbb{E}^{BR}\left[\int_0^\infty \delta e^{-\delta t}\left(\mu x_t + \int_0^t \pi_s ds\right) dt\right]$$

and using

$$\int_{t=0}^\infty \delta e^{-\delta t}\left(\int_{s=0}^t \pi_s ds\right) dt = \int_{s=0}^\infty \pi_s ds\left(\int_{t=s}^\infty \delta e^{-\delta t} dt\right) = \int_{s=0}^\infty e^{-\delta s}\pi_s ds$$

we have

$$p_0^* - p_0 = \mathbb{E}^{BR}\left[\int_0^\infty e^{-\delta t}\left(\delta\mu x_t + \pi_t\right) dt\right]. \tag{98}$$

Now, we aggregate over all firms. Inflation at time 0 is $\pi_0 = \dot{p}_0 = \lambda\left(p_0^* - p_0\right)$. Hence, we have:

$$\pi_0 = \lambda\mathbb{E}^{BR}\left[\int_0^\infty e^{-\delta t}\left(\delta\mu x_t + \pi_t\right) dt\right]. \tag{99}$$

The discrete-time $\bar{m}^t$ becomes $e^{-\xi t}$, where $\xi \geq 0$ is the amoung of cognitive discounting (rationality corresponds to $\xi = 0$).

We assume the following for the perceived inflation process:

$$\begin{aligned}
\mathbb{E}_0^{BR}\left[\pi_t\right] &= h\pi_0^d + m_{f,\pi}e^{-\xi t}\mathbb{E}_0\left[\pi_t - h\pi_0^d\right] \\
\mathbb{E}_0^{BR}\left[x_t\right] &= m_{f,x}e^{-\xi t}\mathbb{E}_0\left[x_t\right]
\end{aligned}$$

with $h = 0$ in the basic model, and $h = 1$ in the extended model. So:

$$\pi_0 = \lambda\mathbb{E}\left[\int_0^\infty e^{-(\delta+\xi)t}\left(m_{f,x}\delta\mu x_t + m_{f,\pi}\pi_t\right) dt\right] + h\pi_0^d\lambda\left[\int_0^\infty e^{-\delta t}\left(1 - m_{f,\pi}e^{-\xi t}\right) dt\right]\pi_0^d.$$

To solve this, it is useful to use the differentiation operator, $D = \frac{d}{dt}$. With this notation, for a function f (sufficiently regular), the Taylor expansion formula can be written as:

$$f(t+\tau) = \sum_{k=0}^\infty f^{(k)}(t)\frac{\tau^k}{k!} = \sum_{k=0}^\infty\left(D^k\frac{\tau^k}{k!}\right) f = e^{\tau D} f$$

i.e.

$$f(t+\tau) = e^{\tau D} f(t). \tag{100}$$

Hence, we have (formally at least):

$$\int_0^\infty e^{-\rho\tau} f(\tau)\, d\tau = \int_0^\infty e^{-\rho t} e^{\tau D} f(0)\, d\tau = \frac{1}{\rho - D} f(0). \tag{101}$$

Hence (99) becomes (dropping the expectations for ease of notation):

$$\pi_t = \frac{\lambda}{\delta + \xi - D} \left(m_{f,x} \delta \mu x_t + \pi_t \right) + h\lambda \left(\frac{1}{\delta} - m_{f,\pi} \frac{1}{\delta + \xi} \right) \pi_t^d \tag{102}$$

and multiplying by $\delta + \xi - D$,

$$(\delta + \xi - D)\, \pi_t = \lambda \left(m_{f,x} \delta \mu x_t + \pi_t + h \left(\frac{1}{\delta} - m_{f,\pi} \frac{1}{\delta + \xi} \right) (\delta + \xi - D)\, \pi_t^d \right)$$

i.e. with $\kappa = m_{f,x} \lambda \delta \mu$

$$(r + \xi - D)\, \pi = \kappa x_t + h \left(\frac{1}{\delta} - m_{f,\pi} \frac{1}{\delta + \xi} \right) (\delta + \xi - D)\, \pi_t^d \tag{103}$$

i.e.

$$(r + \xi)\, \pi_t - \dot{\pi}_t = \kappa x_t + h \frac{\xi}{\delta} \left(\pi_t^d - \frac{\dot{\pi}_t^d}{\delta + \xi} \right)$$

This gives the continuous-time version of the Phillips curve in the basic model (equation 30, $h = 0$) and the extended model (equation 52, $h = 1$).

9.7 Stability criterion in the extended model: Auxiliary Routh-Hurwitz conditions

Our state vector is $\boldsymbol{z}_t := (x_t, \pi_t, \pi_t^d)'$. To study stability, we dispense with the forcing term $\alpha \pi_t^{CB}$, and are left with $\mathbb{E}_t \boldsymbol{z}_{t+1} = \boldsymbol{B} \boldsymbol{z}_t$ for

$$\boldsymbol{B} = \begin{pmatrix} \frac{\sigma \phi_x \beta^f + \beta^f + \kappa \sigma}{M \beta^f} & \frac{\sigma(\beta \phi_\pi - \alpha^f \eta \rho \chi - 1)}{M \beta^f} & \frac{\alpha^f ((\eta - 1)\rho + 1)\sigma}{M \beta^f} \\ -\frac{\kappa}{\beta^f} & \frac{\alpha^f \eta \rho \chi + 1}{\beta^f} & \frac{\alpha^f (-\eta \rho + \rho - 1)}{\beta^f} \\ 0 & \eta \chi & 1 - \eta \end{pmatrix}$$

with $\chi := 1 - \zeta$.

Consider the characteristic polynomial of B, $\Phi(\Lambda) := \det(\Lambda \boldsymbol{I} - \boldsymbol{B})$ (where $\boldsymbol{I}$ is the identity matrix):

$$\Phi(\Lambda) = \sum_{i=0}^{3} a_i \Lambda^i = \Pi_{i=1}^{3} (\Lambda - \Lambda_i)$$

where $\{\Lambda_i\}$ are (potentially complex) eigenvalues of B. We have

$$a_0 = -\frac{(1-\eta)\kappa\sigma}{M\beta^f}\phi_\pi - \frac{(1-\eta+\eta\alpha\chi))(\sigma\phi_x+1)}{M\beta^f} < 0$$

$$a_1 = \frac{\kappa\sigma}{M\beta^f}\phi_\pi + \frac{\sigma(1+\beta(1-\eta)+\eta\alpha\rho\chi))}{M\beta^f}\phi_x + \frac{(1-\eta)[\kappa\sigma+1+\beta+M]+\eta[1+\alpha\chi(M+\rho)]}{M\beta^f} > 0$$

$$a_2 = -\frac{\sigma}{M}\phi_x - \frac{(1-\eta)M\beta^f + \eta\alpha\rho\chi M + M + \beta + \kappa\sigma}{M\beta^f} < 0$$

$$a_3 = 1$$

When $\alpha \neq 0$, inflation π^d is a predetermined variable, not a jump variable. Hence, for determinacy, we must have one eigenvalue less than 1 in modulus (corresponding to the predetermined variable π_t^d) and the other two eigenvalues greater than 1 in modulus (corresponding to the 2 jump variables x_t and π_t).

This implies that a necessary condition is $\Phi(1) > 0$. We can calculate this term:

$$\frac{M\beta^f}{\eta\kappa\sigma}\Phi(1) = \phi_\pi - 1 + \frac{[1-\beta^f-\alpha\chi(1-\rho)](1-M+\sigma\phi_x)}{\kappa\sigma}$$

which is equivalent to (55). This, however, is not sufficient.

To derive sufficiency condition, consider a Mobius transformation of the characteristic polynomial:

$$\Psi(\lambda) := (\lambda-1)^3 \Phi\left(\frac{\lambda+1}{\lambda-1}\right) \tag{104}$$

There is a one-to-one mapping from any (non-unitary) root of $\Psi(\cdot)$ to a root of $\Phi(\cdot)$ by construction: $\lambda \mapsto \Lambda(\lambda) = \frac{\lambda+1}{\lambda-1}$. It is easy to show that $Re(\lambda) < 0$ if and only if $|\Lambda(\lambda)| < 1$. Thus, the conditions for B to have exactly two eigenvalues Λ outside the unit circle is the same as the conditions for $\Psi(\cdot)$ to have exactly two roots λ with non-negative real parts. We next use the Routh-Hurwitz theory, which has been developed to handle that case.

We can rewrite $\Psi(\lambda)$ as

$$\Psi(\lambda) = \sum_{i=0}^{3} b_i \lambda^i$$

where

$$\begin{aligned} b_3 &= 1 + a_2 + a_1 + a_0 \\ b_2 &= 3 + a_2 - a_1 - 3a_0 \\ b_1 &= 3 - a_2 - a_1 + 3a_0 \\ b_0 &= 1 - a_2 + a_1 - a_0 \end{aligned}$$

The criterion $b_3 > 0$ is exactly the Taylor criterion in the text. Also, by inspection, $b_0 > 0$ (it's linear in γ, and it's positive by inspection when $\gamma = 0$ and $\gamma = 1$). We assume that ϕ_π, ϕ_x are nonnegative.

Applying the Routh-Hurwitz stability criterion for polynomial, $\Psi(\lambda)$ has exactly two roots with non-negative real parts if and only if when going through the sequence

$$b_3 \to b_2 \to b_1'' := \frac{b_2 b_1 - b_3 b_0}{b_2} \to b_0$$

signs change exactly twice (see for example Meinsma (1995)). This is possible if and only if (b_2, b_1'') are not both positive, i.e. iff b_2 and $b_1' := b_2 b_1 - b_3 b_0$ are not both positive. Thus we have proven the following.

Proposition 9.3 (Equilibrium determinacy with behavioral agents – with backward looking terms) *Assume that ϕ_π, ϕ_x are nonnegative. A necessary and sufficient condition for equilibrium determinacy is that the Taylor criterion (55) in the text holds, and that the following "auxiliary Routh-Hurwiz condition" holds:*

$$b_2 \text{ and } b_1' := b_2 b_1 - b_3 b_0 \text{ are not both positive.} \tag{105}$$

The expression for b_1' is detailed below.

I conducted some numerical explorations, making sure that the main Taylor criterion was verified. Then, the auxiliary Routh-Hurwitz condition (105) was always verified. Without claiming that it is actually always verified, it seems that the "hard" economic essence is in the Taylor criterion of the main text, while auxiliary Routh-Hurwitz condition (105) is a much less demanding condition.

I next record the values of the $b's$:

$$b_0 := \frac{[(2-\eta)(1+\beta^f)+\eta\alpha\chi(1+\rho)](1+M+\sigma\phi_x)+(2-\eta)\kappa\sigma(1+\phi_\pi)}{M\beta^f}$$

$$b_1 := \frac{1}{M\beta^f}[(-4+(3+\beta^f)\eta-(3+\rho)\alpha\eta\chi)(1-M+\sigma\phi_x)+\kappa\sigma(\eta+(-4+3\eta)\phi_\pi)+4M(-1+\beta^f+\eta(1-\alpha\chi))]$$

$$b_2 = \frac{1}{M\beta^f}[(2-2\beta+\eta\beta-3\eta+(3-\rho)\alpha\eta\chi)(1-M+\sigma\phi_x)+\kappa\sigma(-2+\eta+(2-3\eta)\phi_\pi)+2\eta M(-1+\beta^f+\alpha\chi(1-\rho))]$$

$$b_3 = \frac{\eta}{M\beta^f}[(1-\beta^f-\alpha\chi(1-\rho))(1-M+\sigma\phi_x)+\kappa\sigma(-1+\phi_\pi)]$$

9.8 Details of the Numerical Examples

Here are the values used for the numerical illustrations in the paper.

Table 1 summarizes the main inputs, which are sufficient statistics for the output of the model, summarized in Figures 1-6 (with extra parameters in Table 3 for the extended model). These sufficient statistics can in turn be rationalized in terms of "ancillary" parameters shown in Table 2. We call these parameters "ancillary" because they matter only via their impact on the aforementioned sufficient statistics listed in Table 1. For instance, the value of κ can come from many combinations of $\kappa = \left(\frac{1}{\theta}-1\right)(1-\beta\theta)(\gamma+\phi)\, m_x^f$ can come from many combinations of θ, γ, ϕ etc. Table 2 shows one such combination.

The values are broadly consistent with those of the New Keynesian literature. The inattention parameters are drawn to be close to the myopia found in Galí and Gertler (1999). The inattention to the output gap, m_x^f, is there to match a low slope of the Phillips curve, κ.

Table 1:
KEY PARAMETER INPUTS

Cognitive discounting by consumers and firms	$M = M^f = 0.85$
Sensitivity to interest rates	$\sigma = 0.20$
Slope of the Phillips curve	$\kappa = 0.053$
Rate of time preference	$\beta = 0.99$
Deviation from Ricardian equivalence	$b^d = 0.0096$
Relative welfare weight on output	$\vartheta = 0.05$

Notes. This table reports the coefficients used in the model. Units are quarterly.

Table 2:
ANCILLARY PARAMETERS

Coefficient of risk aversion	$\gamma = 1$
Inverse of Frisch elasticity	$\phi = 1$
Survival rates of prices	$\theta = 0.7$
Demand elasticity	$\varepsilon = 5.3$
Attention parameters	
Cognitive discounting (slope of attention)	$\bar{m} = 0.85$
Consumer's attention to interest rates and income	$m_r = 0.2$, $m_y = 1$
Firms' attention to inflation and output gap	$m_\pi^f = 1$, $m_x^f = 0.2$

Notes. This table reports the coefficients used in the model to generate the parameters of Table 1. Units are quarterly.

Table 3:
ENRICHED MODEL: NEW KEYNESIAN PARAMETER INPUTS

Backward looking coefficient	$a = 0.35$
Updating speed in default inflation	$\eta = 0.05$
Weight on central bank guidance	$\zeta = 0.7$

Notes. Units are quarterly.

10 Further proofs

Proof of Lemma 4.1 The proof mimics the ones in Woodford (2003) and Galí (2015). We have

$$W=-\frac{1}{2}u_c c\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left[(\gamma+\phi)x_t^2+\epsilon var_i(p_t(i))\right]$$

where $var_i(p_t(i))$ is the dispersion of prices at time t. As in Woodford (2003, Chapt. 6),

$$\begin{aligned}\sum_{t=0}^{\infty}\beta^t var_i(p_t(i)) &= \frac{\theta}{(1-\theta)(1-\beta\theta)}\sum_{t=0}^{\infty}\beta^t\pi_t^2+\frac{1}{1-\beta\theta}v_{-1}\\ &= \frac{\gamma+\phi}{\bar{\kappa}}\sum_{t=0}^{\infty}\beta^t\pi_t^2+\frac{1}{1-\beta\theta}v_{-1}\end{aligned}$$

using (86), and calling $v_{-1}:=var_i(p_{-1}(i))$.

Hence,

$$\begin{aligned}W &= -\frac{1}{2}u_c c\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left[(\gamma+\phi)x_t^2+\epsilon\frac{\gamma+\phi}{\bar{\kappa}}\pi_t^2\right]-\frac{1}{2}u_c c\epsilon\frac{1}{1-\beta\theta}v_{-1}\\ &= -\frac{1}{2}u_c c(\gamma+\phi)\frac{\epsilon}{\bar{\kappa}}\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left(\pi_t^2+\frac{\bar{\kappa}}{\epsilon}x_t^2\right)+W_-\\ &= -\frac{1}{2}K\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left[\pi_t^2+\vartheta x_t^2\right]+W_-\end{aligned}$$

with

$$\begin{aligned}K &:= u_c c(\gamma+\phi)\frac{\epsilon}{\bar{\kappa}}=u_c c(\gamma+\phi)\frac{\epsilon}{\kappa}m^f\\ \vartheta &:= \frac{\bar{\kappa}}{\epsilon}=\frac{\kappa}{m^f\varepsilon}\\ W_- &:= -\frac{1}{2}u_c c\epsilon\frac{1}{1-\beta\theta}var_i(p_{-1}(i))\end{aligned}\tag{106}$$

I used $\kappa=\bar{\kappa}m^f$ from equation (85). Note that K and ϑ are independent of behavioral factors, when expressed in terms of primitives including the components of $\bar{\kappa}$, ε. However, when they're expressed in terms of κ, the behavioral term m^f intervenes.

Complement to the Proof of Proposition 4.4 Here is the derivation of i_t. Substitute (49) into the Phillips curve:

$$\pi_t = \beta M^f \mathbb{E}_t \pi_{t+1} + \kappa \left(-\frac{\kappa}{\vartheta}\right) \pi_t + \nu_t \Rightarrow \pi_t = \frac{\beta M^f \vartheta}{\vartheta + \kappa^2} \mathbb{E}_t \pi_{t+1} + \frac{\vartheta}{\vartheta + \kappa^2} \nu_t$$

Iterating forward:

$$\begin{aligned}\pi_t &= \sum_{\tau=t}^{\infty} \left(\frac{\beta M^f \vartheta}{\vartheta + \kappa^2}\right)^{\tau - t} \frac{\vartheta}{\vartheta + \kappa^2} \mathbb{E}_t \nu_\tau = \sum_{\tau=t}^{\infty} \left(\frac{\beta M^f \vartheta \rho}{\vartheta + \kappa^2}\right)^{\tau - t} \frac{\vartheta}{\vartheta + \kappa^2} \nu_t = \frac{\vartheta}{\vartheta + \kappa^2} \frac{1}{1 - \frac{\beta M^f \vartheta \rho}{\vartheta + \kappa^2}} \nu_t \\ &= \frac{\vartheta}{\vartheta + \kappa^2 - \beta M^f \vartheta \rho_\nu} \nu_t = \vartheta \Phi \nu_t\end{aligned}$$

for $\Phi := (\vartheta + \kappa^2 - \beta M^f \vartheta \rho_\nu)^{-1}$. It quickly follows that $x_t = -\kappa \Phi \nu_t$.

Plug these expressions for x_t and π_t into the Behavioral IS curve, we can solve for the nominal interest rate:[78]

$$i_t = \frac{x_t - M \mathbb{E}_t x_{t+1}}{-\sigma} + \mathbb{E}_t \pi_{t+1} = \frac{-\kappa \Phi \nu_t + M \kappa \Phi \mathbb{E}_t \nu_{t+1}}{-\sigma} + \vartheta \Phi \mathbb{E}_t \nu_{t+1}$$

Again, $\mathbb{E}_t \nu_{t+1} = \rho_\nu \nu_t$. Simplifying the expression gives us

$$i_t = (\kappa \sigma^{-1} (1 - M \rho_\nu) + \vartheta \rho_\nu) \Phi \nu_t.$$

11 Another Approach to Long-Run Changes

Here, I record another way of handling long run changes, alternative to the one discussed in Section5. [79]In behavioral models, agent's actions and thoughts are anchored at a "default".[80] The "default" corresponds to: if the agent does not think, what kind of inflation does he expect? So far, I have assumed a constant default at 0 – this streamlines the analysis, at little cost in most situations. However, let us explore how to have a richer default, and what the consequences are.

[78]Take for simplicity $r_t^n = 0$.

[79]Here, there is common knowledge that in the long run, frictions are unimportant – something that may be behaviorally a bit bold. Hence, I downgrade this section to the appendix.

[80]In Bayesian models, the "default" is basically called the "prior" – which is a complex probability distribution, whereas the default is typically a point estimate.

11.1 Modelling the impact of long run policy expectation

For clarity, it is useful to be somewhat general and abstract. Suppose that we have a system

$$\boldsymbol{z}_t = \boldsymbol{A}(m)\,\mathbb{E}_t[\boldsymbol{z}_{t+1}] + b(m)\,a_t \tag{107}$$

where a_t is some exogenous "action" by the external world (e.g. the central bank), and $\boldsymbol{z}_t$ by endogenous variables. We assume, for the subjective model m of the agents, $\boldsymbol{A}(m)$ has eigenvalues that are less than 1 in modulus. Also, $\boldsymbol{A}^r = \boldsymbol{A}(\iota)$ and $b^r = b(\iota)$ are the responses that would happen where agents are rational (with ι a vector of ones, representing full rationality); but $\boldsymbol{A}^r$ could have unstable roots (with eigenvalues greater than 1 in modulus). For instance, in our NK setup in (37) with passive policy ($\phi_x = \phi_\pi = 0$), the behavioral response is $\boldsymbol{A}(m) = \begin{pmatrix} M & \sigma \\ \kappa M & \beta M^f + \kappa\sigma \end{pmatrix}$, and the rational response is $\boldsymbol{A}^r = \boldsymbol{A}(\iota) = \begin{pmatrix} 1 & \psi \\ \kappa & \beta + \kappa\psi \end{pmatrix}$, where (from 19.3), I define

$$\psi := \frac{1}{\gamma R} \tag{108}$$

which is basically the rational IES in the continuous time limit.

Suppose that we have a constant long run action: $a_t = a$ for all t. Then, (36) gives that the rational response z should satisfy:[81]

$$z = \boldsymbol{A}^r z + b^r a,$$

hence $z = H^r a$, with

$$H^r := (1 - \boldsymbol{A}^r)^{-1}\, b^r. \tag{109}$$

Now consider an agent who forms, at time t, some view of the long run action, e.g.

$$a_t^{LR} = \lim_{\tau\to\infty} \mathbb{E}_t[a_{t+\tau}], \tag{110}$$

but we will shortly consider a smoother version of this concept. Given the long run action a_t^{LR}, the rational action is $\boldsymbol{z}_t^{LR} = b^r a_t^{LR}$.

Next, I posit that agents reason about the economy in the "deviation from the long run", e.g.

[81] Here I make an (arguably mild) equilibrium selection: I assume that a constant impulse a generates a constant response z.

they think about a world:[82]

$$\hat{z}_{\tau|t} = \boldsymbol{A}(m)\,\hat{z}_{\tau+1|t} + b(m)\,\hat{a}_{\tau|t} \tag{111}$$

where $\hat{z}_{\tau|t}$ and $\hat{a}_{\tau|t}$ are the deviations from the time-t default:

$$\hat{Z}_{\tau|t} := Z_\tau - m_{LR} Z_t^{LR} \text{ for } Z = z, a.$$

Here, again $m_{LR} \in [0,1]$ is the weight on the LR as an anchor. In the formulation so far, we had $m_{LR} = 0$. If $m_{LR} = 1$, they think of economic outcomes as a deviation from the long run. I assume here that in their simulation at time t, they set an anchor for the whole future path $\hat{z}_{\tau|t}$ at times τ after t. I assume that agents do have access to this notion of "normatively correct long run response", $H(\iota)\,a_t^{LR}$. It is a bit of a strong assumption, though less strong than that of the traditional model. In future drafts, I plan to reexamine this assumption, and perhaps change it.

Calling $H(m) = \sum_{i\geq 0} \boldsymbol{A}(m)^i\, b(m) = (1 - \boldsymbol{A}(m))^{-1}\, b(m)$, so that $H(\iota) = H^r$.

Proposition 11.1 *In the model with a non-zero long-run a_t^{LR}, we have*

$$\boldsymbol{z}_t = \sum_{\tau \geq t} \mathbb{E}_t\left[\boldsymbol{A}(m)^{\tau - t}\, b(m)\, a_\tau\right] + m_{LR}\left(H(\iota) - H(m)\right) a_t^{LR}.$$

Proof. We have

$$\begin{aligned}
\boldsymbol{z}_t = \hat{z}_{t|t} + m_{LR}\boldsymbol{z}_t^{LR} &= \sum_{\tau\geq t} \mathbb{E}_t\left[\boldsymbol{A}(m)^{\tau-t}\, b(m)\left(a_\tau - m_{LR} a_t^{LR}\right)\right] + m_{LR}\boldsymbol{z}_t^{LR} \\
&= \sum_{\tau\geq t} \mathbb{E}_t\left[\boldsymbol{A}(m)^{\tau-t}\, b(m)\, a_\tau\right] - m_{LR}[\sum_{\tau\geq t} \boldsymbol{A}(m)^{\tau-t}\, b(m)] a_t^{LR} + m_{LR} H(\iota)\, a_t^{LR} \\
&= \sum_{\tau\geq t} \mathbb{E}_t\left[\boldsymbol{A}(m)^{\tau-t}\, b(m)\, a_\tau\right] + m_{LR}\left(H(\iota) - H(m)\right) a_t^{LR}.
\end{aligned}$$

□

This is what we had before, $\boldsymbol{z}_t = \sum_{\tau\geq t} \mathbb{E}_t\left[\boldsymbol{A}(m)^{\tau-t}\, b(m)\, a_t\right]$ (equation (40)) with a new term, $\left(H(\iota) - H(m)\right) a_t^{LR}$ for the adjustment to long run impact. When agents are rational, $m = \iota$ and this term is 0: there is no need for an extra adjustment term. When agents are less than rational, the anchor on the long term helps them be more rational.

[82] This is in the tradition of cognitive modelling, where thinking is anchored at a "default" and the agent considers partial adjustments from it (cf. Tversky and Kahneman (1974), Gabaix (2014, 2016)). Here, the default is the long run, which itself is influenced by the past, as we shall soon see.

To get clean expressions, I use the notations:

$$M = 1 - \xi,\ M_f = \frac{1-\rho}{\beta}, \beta = 1 - \rho + \chi \tag{112}$$

so that the rational case corresponds to $\xi = 0$ for consumers and $\chi = 0$ for firms; the expressions are similar in discrete and continuous time. Indeed, $\left(1 - \beta M^f\right)(1 - M) = \rho\xi$ then. Simple calculations show that we have:[83]

$$H(m) = \frac{1}{\rho\xi - \kappa\sigma}(-\rho, \kappa)\,\sigma, \qquad H(\iota) = \left(\frac{\rho - \chi}{\kappa}, 1\right)$$

and $b^{LR}(m) = H(\iota) - H(m)$ is equal to

$$b^{LR}(m) = \frac{\rho\xi}{\rho\xi - \kappa\sigma}\left(\frac{\rho}{\kappa}, 1\right)' - \left(\frac{\chi}{\kappa}, 0\right). \tag{113}$$

Here, I gather the model.

Proposition 11.2 (Behavioral New Keynesian model, with adjustment for the long run) *The generalization of the model with adjustment for the long run is as follows: Define $\widehat{x}_\tau$, $\widehat{\pi}_\tau$ to be the solutions of the model of Proposition 2.5:*

$$\widehat{x}_\tau = M\mathbb{E}_\tau\left[\widehat{x}_{\tau+1}\right] + b_d d_t - \sigma\left(i_\tau - \mathbb{E}_{\tau+1}\widehat{\pi}_\tau - r_\tau^n\right) \tag{114}$$

$$\widehat{\pi}_\tau = \beta M^f \mathbb{E}_\tau\left[\widehat{\pi}_{\tau+1}\right] + \kappa\widehat{x}_\tau. \tag{115}$$

The actual values of output and inflation are:

$$(x_t, \pi_t) = (\widehat{x}_t, \widehat{\pi}_t) + m_{LR} b^{LR} i_t^{LR}$$

where b^{LR} is given in (113), and i_t^{LR}, the perception of long run policy, is given by $i_t^{LR} = \lim_{\tau\to\infty}\mathbb{E}_t\left[i_{t+\tau}\right]$ in the "strict long run" case; or (117) applied to $a_t = i_t$ in the "smoothed long run" case.

If indeed the action is constant, then $a^{LR} = \bar{a}$, and the true long run is

$$\bar{z}^{LR} = \left[(1 - m_{LR}) H(m) + m_{LR} H(\iota)\right]\bar{a}.$$

[83] Actually, the value of κ is a bit different in the rational model. This little bug will be fixed in the next iteration of the paper.

Lemma 11.3 (In the long run, does inflation increase or decrease with interest rates?) *Suppose that the nominal interest rate (minus the RBC normative interest rate) is constant at $\bar{\imath}$ in the long run. Then, the steady state inflation is is:*

$$\bar{\pi} = \left[-\left(1 - m_{LR}\right) \frac{\kappa\sigma}{\rho\xi - \kappa\sigma} + m_{LR} \right] \bar{\imath}. \tag{116}$$

Hence, if $m_{LR} = 1$, long-run Fisher neutrality holds. More generally, if m_{LR} is close enough to 1, inflation increases with the interest rate in the long run.

Let me now detail the "smoothed long run" case.

11.2 A "Smoothed Long Run"

The simplest way to model the long run is $a_t^{LR} = \lim_{\tau\to\infty} \mathbb{E}_t \left[a_{t+\tau}\right]$ (equation (110)). But, this notion captures only "mathematical infinity" and will not capture policies that last for 80 years, rather than forever. In addition, expectations of BR agents may be slow to adjust. Hence, I use a smooth generalization of (110):

$$a_t^{LR} = \sum_{D,\tau\geq 0} \mathbb{E}_{t-D} \left[a_{t-D+\tau}\right] g\left(D\right) f\left(\tau\right). \tag{117}$$

Here, D represents a delay in the adjustment of the information set (as in Gabaix and Laibson 2002, Mankiw and Reis 2002), distributed according to $g\left(D\right) = \phi e^{-\phi D}$ (i.e. $g\left(D\right) = \phi\left(1-\phi\right)^D$ in discrete time). Also, $f\left(\tau\right)$ is the weight put on the future τ periods ahead. In practice, I take $f\left(\tau\right) = \zeta^2 e^{-\zeta\tau}\tau$, which puts more weight on the future than on the immediate present.

When $\zeta \to 0$ and $\phi \to \infty$, a_t^{LR} converges to $\lim_{\tau\to\infty} \mathbb{E}_t \left[a_{t+\tau}\right]$. Let us evaluate its value for a typical case.

Lemma 11.4 *Suppose that a policy change $a_t = e^{-\alpha t} a_0$ is announced at time 0. Then, agents' perception of its long-run value is (in continuous time):*

$$a_t^{LR} = \frac{\zeta^2\phi}{\left(\zeta+\alpha\right)^2\left(\phi-\alpha\right)} \left(e^{-\alpha t} - e^{-\phi t}\right) a_0.$$

For instance, if this is a permanent change, $\alpha = 0$, then $a_t^{LR} = \left(1 - e^{-\phi t}\right) a_0$. There is a delayed adjustment captured by ϕ. When $\phi = \infty$ (no delay in expectations), then $a_t^{LR} = a_0$, expectations adjust immediately.

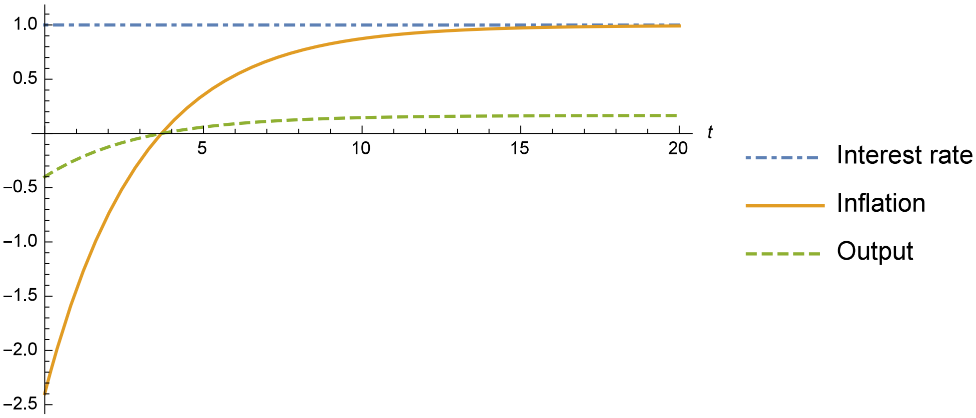

Figure 7: Impact of a permanent rise in the nominal interest rate. At time 0, the nominal interest rate is permanently increased by 1%. The Figure traces the impact on inflation and output. Units are percents.

When the policy change will mean-revert ($\alpha > 0$), then the "strict long run" is just 0: $\lim_{\tau \to \infty} \mathbb{E}_t [a_{t+\tau}] = 0$. However, the "smoothed long run" a_t^{LR} is not 0. Hence, a shock lasting, say, 50 years but not an infinite number of years is captured by the smoothed long run.

11.3 Impact of a Permanent Rise in the Nominal Interest Rate

We can now study the impact of a permanent rise in the nominal interest rate: i_t increase by $J = 1\%$.

The effect is shown in Figure 7. On impact, the rise in the rate lowers inflation and output – this is the conventional Keynesian effect. In the long run, however, the Fisherian prediction holds: the 1% rise in the interest rate is coupled with a 1% rise in inflation, so that the long term interest rate is unchanged.[84]

As in Section 5.3, this effect is very hard to obtain in a conventional New Keynesian model (Cochrane (2015)). However, here again, the bounded rationality of the agents overturns this result, with just one bounded equilibrium.

To see analytically why, it is worth examining two polar cases. First, take the full rationality case in Proposition 2.5, we have

$$x = x - \sigma (J - \pi), \tag{118}$$

$$\pi = \beta \pi + \kappa x_t. \tag{119}$$

[84]However, this neutrality is partial – as in the basic NK model, output does increase permanently if inflation is permanently higher. This effect, however, is quite small.

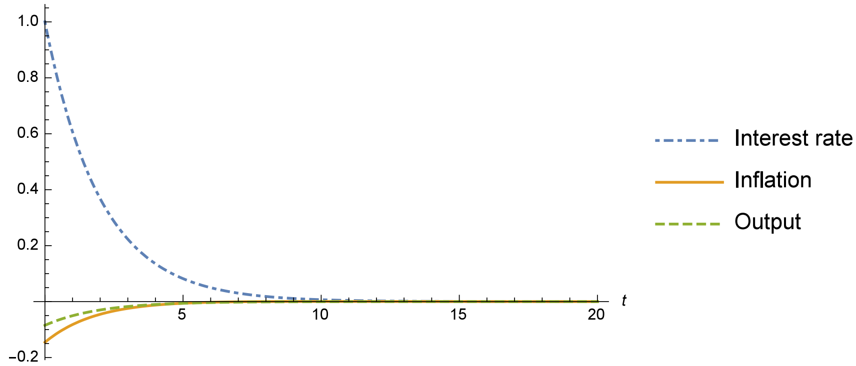

Figure 8: Impact of a permanent rise in the nominal interest rate. At time 0, the nominal interest rate is temporarily increased by 1%. The Figure traces the impact on inflation and output. Units are percents.

Then, the solution with a constant coefficient is : $\pi_t = J$, $x = \frac{(1-\beta)}{\kappa} J$. Hence, a higher interest rate leads to higher inflation, as in the Fisher neutrality.

However, take a completely myopic model, so that $M = M^f = 0$. Then, Proposition 2.5 reduces to:

$$x_t = -\sigma \left(i_t - \mathbb{E}_t \pi_{t+1} - r_t^n \right),$$
$$\pi_t = \kappa x_t \ ,$$

hence:

$$\pi_t = \kappa \sigma \mathbb{E}_t \pi_{t+1} - \sigma J$$

so that a higher interest rate leads to *lower* inflation.

The enriched model with long run expectations gives something in between those two polar models. Hence, it generates the first Keynesian dynamics, with a high interest rate lowering inflation, and the long run Fisher effect of a higher inflation to restore constant real rates.

11.4 Impact of a Temporary Rise in Interest Rates

Let us now consider a short term rise in the interest rate, Figure 8. We find indeed that a temporary rise in interest rates decreases inflation and output. This is a result that was hard to get in NK models (though again this depends on issues of equilibrium selection), see Cochrane (2015). Here we get it easily, with a determinate equilibrium.

References

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