Online Supplementary Appendix to "Entry and Exit in OTC Derivatives Markets"

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In this Appendix we offer a complete analysis of the three type model. Section B.1 characterizes the equilibrium conditional on entry. Section B.2 shows that all entry equilibria and social optimal must be symmetric. Section B.3 studies equilibrium entry, and Section B.4 studies the social optimum with entry. Section B.5 characterizes the restrictions on market composition imposed by equilibrium and socially optimal entry. Section B.6 studies the model when the bargaining weight of customers and dealers are asymmetric. Section B.7 studies equilibrium exit, and Section B.8 studies the social optimum with exit.

B The three-type model

B.1 Equilibrium conditional on entry

We first cover, in Section B.1.1, the equilibrium conditional on entry when the distribution of traders is symmetric, as will arise in the symmetric entry or exit equilibrium we will study later. Next, in Section B.1.2, we offer a characterization of the equilibrium when the distribution of traders is asymmetric, which is needed to establish equilibrium uniqueness in the model with entry.

B.1.1 Symmetric distribution of traders

A simple case for which the equilibrium can be derived explicitly is when $\Omega = \{0, \frac{1}{2}, 1\}$ and the distribution of traders is symmetric (see section B.1.2 for a full treatment of the asymmetric case with three types). This case is very tractable because the distribution of traders can be parameterized by a single number, the fraction of traders in $\omega = \frac{1}{2}$ banks: $n(\frac{1}{2}) \equiv n$. Given symmetry, it must be the case that $n(0) = n(1) = \frac{1-n}{2}$.

Case 1: when $k(1+n) < 1$. Then, we guess and verify that risk sharing is imperfect. Thus, all traders sign k contracts whenever they meet a trader with $\tilde{\omega} \neq \omega$. This implies that:

$$g(0) = kn + k\frac{1-n}{2} = \frac{k(1+n)}{2}.$$

An $\omega = 0$ trader meets an $\omega = \frac{1}{2}$ trader with probability n, and an $\omega = 1$ trader with probability $\frac{1-n}{2}$. In both case, she sells k contracts. We also have $g(1) = 1 - g(0)$ and $g(\frac{1}{2}) = \frac{1}{2}$. For these exposures to be part of an equilibrium, we need to verify that risk-sharing is imperfect, that is, $g(0) < \frac{1}{2}$. Clearly, a necessary and sufficient condition is that $k(1+n) < 1$.

With this in mind, we can calculate the average gross exposure in the OTC market:

$$\mathcal{G}(k) = (1-n)k\frac{1+n}{2} + nk(1-n) = \frac{k(1-n)(1+3n)}{2}. \quad \text{(B.1)}$$

The first term is the total gross exposure of extreme-ω banks, and the second term is the total gross exposure of middle-ω banks. We can calculate, similarly, the absolute net exposure per capita. For extreme-ω banks, net and gross exposures are the same, and for middle-ω banks the net exposure is zero. The same calculation as above thus shows that the average net exposure is:

$$\mathcal{N}(k) = \frac{k(1-n)(1+n)}{2}. \tag{B.2}$$

The ratio of gross-to-net exposures is:

$$\mathcal{R}(k) \equiv \frac{\mathcal{G}(k)}{\mathcal{N}(k)} = \frac{1+3n}{1+n}. \tag{B.3}$$

The ratio of gross-to-net exposures, $\mathcal{R}(k)$, is independent of k, since both $\mathcal{G}(k)$ and $\mathcal{N}(k)$ grows linearly with k. When $k(1+n) = 1$, i.e., when full risk sharing is just achieved, we have that $\mathcal{N}(k) < \mathcal{G}(k)$, as shown in the general case by Proposition 3.

Case 2: $k(1+n) \geq 1$. In this case, full risk sharing must obtain in equilibrium. Suppose first that $k(1-n) < 1$. Then, $\mathcal{R}(k) > 1$, i.e. the gross to net the intermediation services of middle-ω are essential to achieve full risk-sharing. Indeed, if $\omega = 0$ traders only transacted with $\omega = 1$ traders and $\mathcal{R}(k) = 1$, then $g(0)$ would be equal to $k\frac{1-n}{2} < \frac{1}{2}$.

Lemma 14. *Suposse that $k(1+n) \geq 1$ and $k(1-n) < 1$. Then, the minimum average gross exposure is*

$$\mathcal{G}(k) = \frac{1-n}{2}\left[2 - k(1-n)\right].$$

It is achieved when the volume of customer-to-customer trades is maximized, $\frac{1-n}{2}\gamma(0,1) = \frac{k(1-n)}{2}$, and the volume of customer-to-dealer trades is minimized, $n\gamma\left(0, \frac{1}{2}\right) = \frac{1}{2} - \frac{k(1-n)}{2}$, subject to achieving full risk sharing, $g(0) = g\left(\frac{1}{2}\right) = g(1) = \frac{1}{2}$.

Since full risk sharing obtained, the average net exposure is $\mathcal{N} = \frac{1-n}{2}$, and so

$$\mathcal{R}(k) = 2 - k(1-n).$$

One sees that this ratio is strictly decreasing in k and reaches 1 when $k(1-n) = 1$. When $k(1-n) \geq 1$, the ratio of gross-to-net exposures stays equal to one. This lemma also makes it clear that, when k increases, agents are better able to direct their trade towards their best counterparties: the volume of direct customer-to-customer trade increases, and the volume of indirect customer-to-dealer trade decreases.

Taken together, we obtain that the ratio of gross-to-net exposures, $\mathcal{R}(k)$, is as shown in Figure 4.

B.1.2 Asymmetric distribution of traders

In this section we consider the model with three types, $\Omega = \{0, \frac{1}{2}, 1\}$, but when the distribution of traders is asymmetric, $n(0) \neq n(1)$. We characterize the equilibrium conditional on entry for all possible pairs $n(0)$ and $n(1)$, and all $k > 0$. With asymmetry, type $\omega = \frac{1}{2}$ banks are no longer pure intermediaries: they typically change their exposures while providing intermediation. This arises because the average exposure in the market is different from $\frac{1}{2}$, and because the probability

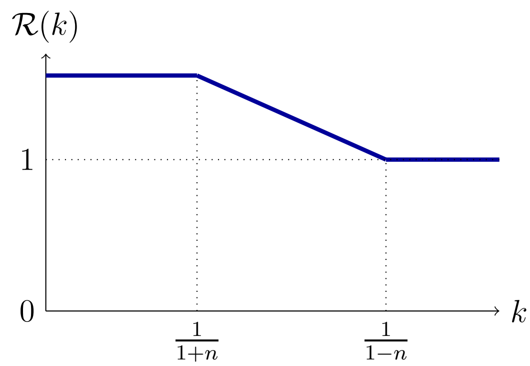

Figure 4: The ratio of gross-to-net exposures as a function of k, in the three-type model.

of pairing with an $\omega = 0$ trader is in general different from the probability of pairing with an $\omega = 1$ trader.

Down the road, we will use the results of this section to rule out asymmetric *entry*, when the cost of risk bearing is quadratic, and when the distribution of traders in the economy at large is symmetric, $\pi(0) = \pi(1)$.

With an asymmetric type distribution, there are four possible configurations for post-trade exposures, which we cover in order.

Case 1: $g(0) < g(\frac{1}{2}) < g(1)$. In this case, bilateral exposures are always equal to the trade size limit, k, which implies that:

$$\begin{aligned} g(0) &= k\left[n(\tfrac{1}{2}) + n(1)\right] = k\left[1 - n(0)\right] \\ g(\tfrac{1}{2}) &= \tfrac{1}{2} - kn(0) + kn(1) \\ g(1) &= 1 - k\left[1 - n(1)\right]. \end{aligned}$$

Moreover, using the above formula for post-trade exposures, we find that $g(0) < g(\frac{1}{2})$ if and only if $2k\left[1 - n(1)\right] < 1$ and that $g(\frac{1}{2}) < g(1)$ if and only if $2k\left[1 - n(0)\right] < 1$. Conversely, if these two inequalities are satisfied, then the above post-trade exposures are the basis of an equilibrium. Taken together, we obtain that:

Lemma 15. *In an equilibrium with three types, $g(0) < g(\frac{1}{2}) < g(1)$ if and only if*

$$2k\left[1 - n(0)\right] < 1 \tag{B.4}$$

$$2k\left[1 - n(1)\right] < 1. \tag{B.5}$$

Moreover, (B.4) *and* (B.5) *are jointly satisfied only if $k < 1$.*

The last statement of the Lemma obtains by adding up (B.4) and (B.5), which gives

$$2k\left[2 - n(0) - n(1)\right] < 2.$$

Since $1 - n(0) - n(1) \geq 0$, this implies that $k < 1$.

Case 2: $g(0) = g(\frac{1}{2}) < g(1)$. In this case, banks of type $\omega = 0$ and $\omega = \frac{1}{2}$ pool their risk fully, and their traders always go corner when they are paired with traders of type $\omega = 1$. This implies that $n(0) + n(\frac{1}{2}) > 0$ otherwise traders of type $\omega = 0$ and $\omega = \frac{1}{2}$ would never be paired and so $g(0) \neq g(\frac{1}{2})$. Therefore:

$$g(0) = g(\tfrac{1}{2}) = \frac{\frac{1}{2}n(\frac{1}{2})}{n(0) + n(\frac{1}{2})} + kn(1)$$
$$g(1) = 1 - k\,[1 - n(1)]$$

After some algebra, we obtain that $g(0) = g(\frac{1}{2}) < g(1)$ if and only if:

$$n(0) > 2k\,[1 - n(1)]\left[1 - \tfrac{1}{2k}\right].$$

Another necessary condition for such an equilibrium is that banks of type $\omega = 0$ and type $\omega = \frac{1}{2}$ banks can pool their risks with bilateral trades of size less than k. That is:

$$\gamma(0, \tfrac{1}{2})n(\tfrac{1}{2}) = \tfrac{1}{2} - \gamma(0, \tfrac{1}{2})n(0) \Rightarrow \gamma(0, \tfrac{1}{2}) \leq k.$$

Solving this equation for $\gamma(0, \frac{1}{2})$ we obtain the condition $2k\,[1 - n(1)] \geq 1$. Conversely, if all these conditions are satisfied, then the above post-trade exposures are the basis of an equilibrium. Taken together, we obtain that:

Lemma 16. *In an equilibrium with three types, $g(0) = g(\frac{1}{2}) < g(1)$ if and only if*

$$n(0) > 2k\,[1 - n(1)]\left[1 - \tfrac{1}{2k}\right] \tag{B.6}$$
$$2k\,[1 - n(1)] \geq 1. \tag{B.7}$$

Moreover, (B.6) *and* (B.7) *are jointly satisfied only if $k \in [\frac{1}{2}, 1)$, $n(1) < \frac{1}{2}$, and $n(0) > n(1)$.*

Let us now prove the last four statements of the lemma. First, we note that (B.7) implies that $2k \geq 1$, and so $k \geq \frac{1}{2}$. Second, adding $n(1)$ on both sides of (B.6), we obtain that $n(0) + n(1) > n(1) + [1 - n(1)]\,[2k - 1]$. Since $n(0) + n(1) \leq 1$ and since, by (B.7), $1 - n(1) > 0$, this implies that $2k - 1 < 1$, i.e., that $k < 1$. Third, (B.7) can be written as $n(1) \leq 1 - \frac{1}{2k}$. Since, as shown previously, $k < 1$, this implies that $n(1) < \frac{1}{2}$. Finally, substituting (B.7) into (B.6), we obtain that $n(0) > 1 - \frac{1}{2k}$. Given that (B.7) can be written as $n(1) \leq 1 - \frac{1}{2k}$, this implies that $n(0) > n(1)$.

Case 3. Symmetrically with Case 2, we obtain that:

Lemma 17. *In an equilibrium with three types, $g(0) < g(\frac{1}{2}) = g(1)$ if and only if*

$$n(1) > 2k\,[1 - n(0)]\left[1 - \tfrac{1}{2k}\right] \tag{B.8}$$
$$2k\,[1 - n(0)] \geq 1. \tag{B.9}$$

Moreover, (B.8) *and* (B.9) *are jointly satisfied only if $k \in [\frac{1}{2}, 1)$, $n(0) < \frac{1}{2}$, and $n(1) > n(0)$.*

Case 4. By construction, if none of the conditions of Lemma 15-17 are satisfied, then there is full risk sharing and $g(0) = g(\frac{1}{2}) = g(1)$.

Taking stock. We now use the results of Lemma 15-17 to characterize the equilibrium conditional on entry, for any pairs of $n(0)$ and $n(1)$, and any trade size limit, k.

- When trade size limits are small: $k \in (0, \frac{1}{2})$. Then, since $n(0)$ and $n(1)$ are positive, the two conditions of Lemma 15 are satisfied, and so $g(0) < g(\frac{1}{2}) < g(1)$. Note that, if $n(0) > n(1)$, then $g(\frac{1}{2}) < \frac{1}{2}$, and if $n(1) > n(0)$, $g(\frac{1}{2}) > \frac{1}{2}$. That is, middle-$\omega$ banks will find it optimal to change their exposures, because their traders meet trader of type $\omega = 0$ with higher probability than traders of type $\omega = 1$.

- When trade size limits are intermediate: $k \in [\frac{1}{2}, 1)$. With intermediate levels of trade size limits, the four possible configurations of post-trade exposures can arise in the equilibrium, depending on $n(0)$ and $n(1)$. This is illustrated graphically Figure 5.

  - The green area shows the combinations of $n(0)$ and $n(1)$ such that case 1 arises. Equation (B.4) is satisfied if and only if $n(0) > 1 - \frac{1}{2k}$, corresponding to the vertical frontier of the green area. Equation (B.5) is satisfied if and only if $n(1) > 1 - \frac{1}{2k}$, corresponding to the horizontal frontier of the green area.
  - The pink area shows the combinations of $n(0)$ and $n(1)$ such that case 2 arises. Equation (B.6) is satisfied if and only if $n(0) > 2k\,[1 - n(1)]\left[1 - \frac{1}{2k}\right]$, corresponding to the left and downward slopping frontier of the pink area. Equation (B.7) is satisfied if and only if $n(1) \leq 1 - \frac{1}{2k}$, corresponding to the upper horizontal frontier of the pink area.
  - The blue area shows the combinations of $n(0)$ and $n(1)$ such that case 3 arises. It is symmetric to the pink area.
  - The yellow area shows the combinations of $n(0)$ and $n(1)$ such that case 4 arises.

  Consider first the green area of the figure. In this case, both $n(0)$ and $n(1)$ are close to $\frac{1}{2}$ and $n(\frac{1}{2})$ is close to zero. Risk-sharing is more difficult and post-trade exposures are not equalized. Next, in the pink area of the figure, $n(0)$ is large, and both $n(1)$ and $n(\frac{1}{2})$ are small. Banks of type $\omega = 0$ and $\omega = \frac{1}{2}$ can equalize their exposure, but risk sharing is imperfect with banks of type $\omega = 1$. The blue area of the figure is the symmetric case. Finally, in the yellow area, $n(0)$ and $n(1)$ are small but the fraction of intermediaries, $n(\frac{1}{2})$ is large. This makes it easier to share risk and all banks equalize their exposures.

- When trade size limits are large, $k \geq 1$. As was shown in Lemma 15 through 17 a necessary condition for partial risk sharing is that $k < 1$. Therefore, for all $k \geq 1$, there is full risk sharing.

B.2 All equilibria and social optima are symmetric

We make the three-types model symmetric by assuming that $\Omega = \{0, \frac{1}{2}, 1\}$, $\pi(0) = \pi(1)$, and that the cost of risk bearing is quadratic. We show in this section that these symmetry assumptions imply that, in any equilibrium and any social optimum, participation is symmetric as well, i.e., $\mu(0) = \mu(1)$.

B.2.1 Equilibrium

The result that all equilibria must be symmetric is a consequence of the following proposition:

Proposition 13. *The extreme-ω banks which are on the short side of the OTC market enjoy larger surplus:*

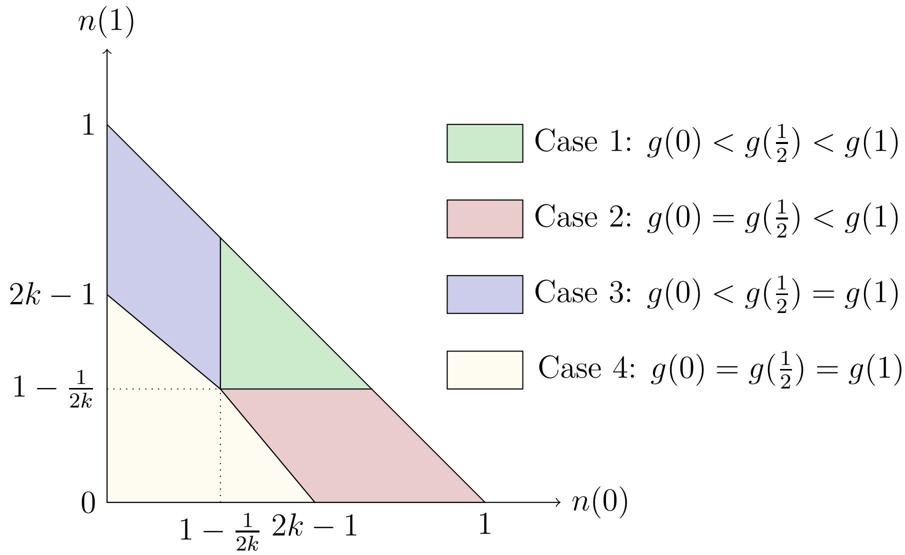

Figure 5: The four possible cases for post-trade exposures, as a function of the distribution of types, $n(0)$ and $n(1)$ such that $n(0) + n(1) \leq 1$.

- *if* $n(0) > n(1)$*, then* $K(0) \leq K(1)$ *and* $F(0) \leq F(1)$ *with at least one strict inequality;*
- *if* $n(1) > n(0)$*, then* $K(1) \leq K(0)$ *and* $F(1) \leq F(0)$ *with at least one strict inequality.*

The proposition relies on elementary properties of the competitive and frictional surplus, and so it applies both to the entry and the exit model. Combining this result with the equilibrium participation condition, we obtain:

Proposition 14. *Suppose that there are three types,* $\Omega = \{0, \frac{1}{2}, 1\}$*, that there is an equal measure of extreme types in the economy at large,* $\pi(0) = \pi(1)$*, and that the cost of risk bearing is quadratic. Then, all equilibria must be symmetric, i.e.,* $\mu(0) = \mu(1)$*.*

Towards a contradiction, assume that $\mu(0) > \mu(1)$ (the opposite case is symmetric). Then, by Proposition 13, we have that $\text{MPV}(0) < \text{MPV}(1)$. Now recall the equilibrium participation conditions:

$$\pi(0)\Phi\left[\text{MPV}(0)^-\right] \leq \mu(0) \leq \pi(0)\Phi\left[\text{MPV}(0)\right] \tag{B.10}$$

$$\pi(1)\Phi\left[\text{MPV}(1)^-\right] \leq \mu(1) \leq \pi(1)\Phi\left[\text{MPV}(1)\right]. \tag{B.11}$$

Given that $\text{MPV}(0) < \text{MPV}(1)$ and $\pi(0) = \pi(1)$, it follows that

$$\pi(0)\Phi\left[\text{MPV}(0)\right] \leq \pi(1)\Phi\left[\text{MPV}(1)^-\right].$$

From the equilibrium participation condition, we then obtain that $\mu(0) \leq \mu(1)$, which is a contradiction.

B.2.2 Social optimum

We have a similar proposition for social optima:

Proposition 15. *Suppose that there are three types, $\Omega = \{0, \frac{1}{2}, 1\}$, that there is an equal measure of extreme types in the economy at large, $\pi(0) = \pi(1)$, and that the cost of risk bearing is quadratic. Then, all solutions of the planning problem must be symmetric, i.e., $\mu(0) = \mu(1)$.*

The proof is identical to the one of Proposition (14) , with MSV replacing MPV.

B.3 Equilibrium entry with three types

As explained in the main body of the paper to study entry, we assume that the distribution of bank sizes is Pareto with parameter $1+\theta > 0$, for some $\theta > 0$, over the support $[\underline{S}, \infty)$.[9] This implies that the measure of traders in banks with size greater than some $S \geq \underline{S}$ is $(S/\underline{S})^{-\theta}$. The associated distribution of per-capita entry cost has support $[0, \bar{z}]$, $\bar{z} \equiv c/\underline{S}$, and CDF $\Phi(z) = (z/\bar{z})^{\theta}$. We have shown in Proposition 14 that, under these assumptions, entry must be symmetric in any equilibrium, i.e., $\mu(0) = \mu(1)$. We will thus look for an equilibrium in which $n(0) = n(1)$, and we will simplify notations by denoting $n(\frac{1}{2}) \equiv n$.

Marginal private values. After some algebra, gathered in Section B.9.3, we find that the marginal private value for extreme-ω banks is

$$\text{MPV}(0) = \text{MPV}(1) = \begin{cases} \frac{\Gamma'' k}{4}\left[1 - \frac{k\left(1-n^2\right)}{2}\right] & \text{if } k(1+n) < 1 \\ \frac{\Gamma''}{8} & \text{if } k(1+n) \geq 1. \end{cases}$$

For middle-ω banks, it is

$$\text{MPV}\left(\tfrac{1}{2}\right) = \begin{cases} \frac{\Gamma'' k}{4}\left(1-n\right)\left[1 - k\left(1+n\right)\right] & \text{if } k(1+n) < 1 \\ 0 & \text{if } k(1+n) \geq 1. \end{cases}$$

In the above, $\Gamma'' = \Gamma''[g]$ is a constant because the cost of risk bearing, $\Gamma[g]$, is quadratic. Another advantage of a quadratic cost of risk bearing is that, when the distribution of banks in the market is symmetric, entry incentives are symmetric as well: the marginal private value is the same for $\omega = 0$ as for $\omega = 1$ banks. One can verify that $\text{MPV}(0) > \text{MPV}\left(\frac{1}{2}\right)$, i.e., extreme-$\omega$ banks have stronger incentives to enter than middle-ω banks. This is in line with Proposition 4 from the general model.

Given that the cost distribution is continuous, the fixed point correspondence is actually a function and so the equilibrium conditions becomes $\mu(\omega) = \frac{1}{3}\Phi\left[\text{MPV}(\omega)\right]$ for $\omega \in \{0, \frac{1}{2}, 1\}$. In an equilibrium with positive participation, $\sum_{\tilde{\omega}} \mu(\tilde{\omega}) > 0$, and so we obtain:

$$n(\omega) = \frac{\Phi\left[\text{MPV}\left(\omega\right)\right]}{\sum_{\tilde{\omega}} \Phi\left[\text{MPV}\left(\tilde{\omega}\right)\right]},$$

for all $\omega \in \{0, \frac{1}{2}, 1\}$. Since entry incentives are symmetric for $\omega = 0$ and $\omega = 1$, and since $\sum_{\tilde{\omega}} n(\tilde{\omega}) = 1$, we can reduce this system of three equations to a one-equation-in-one-unknown problem, for the fraction of middle-ω banks, $n\left(\frac{1}{2}\right) = n$. Using the parametric expression for the

[9]Precisely, the fraction of banks with size larger than $S \geq \underline{S}$ is equal to $(S/\underline{S})^{-(1+\theta)}$. To ensure that there is a measure one of traders, we also need to assume that the total measure of bank establishments in the economy at large is $\frac{\theta}{1+\theta}\frac{1}{\underline{S}}$.

per-trade distribution of costs, $\Phi(z)$, we obtain:

$$n = \frac{F(n)^\theta}{2 + F(n)^\theta} \text{ where } F(n) \equiv \frac{\min\left\{\frac{S}{c}\text{MPV}\left(\frac{1}{2}\right), 1\right\}}{\min\left\{\frac{S}{c}\text{MPV}\left(0\right), 1\right\}}. \tag{B.12}$$

We obtain:

Proposition 16. *Equation (B.12) has a unique solution, n. It is equal to zero if $k \geq 1$, and it is strictly positive and strictly less than $\frac{1}{k} - 1$ if $k < 1$.*

When $k \geq 1$ then $k(1+n) \geq 1$: thus, there is full risk-sharing in any equilibrium, $\text{MPV}\left(\frac{1}{2}\right) = 0$, and so $n = 0$. Conversely, $n = 0$ is indeed an equilibrium. When $k < 1$, the solution of the fixed point equation cannot be such that $k(1+n) \geq 1$. Indeed, when $k < 1$ we need sufficiently positive entry of middle-ω banks to sustain full risk sharing, but if there is full risk sharing middle-ω banks have no incentive to enter. The proposition also shows that the symmetric equilibrium is unique. The force that underlies uniqueness is that the entry decisions of middle-ω banks are strategic substitutes. This is because, when more middle-ω banks enter, risk-sharing improves and intermediation profits are eroded.

Our first set of comparative statics is with respect to the fixed cost of entry, c:

Proposition 17. *The equilibrium fraction of middle-ω traders is continuous in c. Holding all other parameters the same, there are two cost thresholds $\underline{c} \geq 0$ and $\overline{c} > \underline{c}$ such that:*

- *If $c < \underline{c}$, all banks enter and $n = \frac{1}{3}$.*
- *If $c \in (\underline{c}, \overline{c})$, all extreme-$\omega$ banks enter, and only some of the middle-ω banks enter. When c increases, the fraction of middle-ω traders decreases, and the total measure of traders in the market decreases.*
- *If $c \geq \overline{c}$, only some of the extreme-ω and middle-ω banks enter. When c increases, the fraction of middle-ω traders, n, does not not change, but the total measure of traders in the market decreases.*

Moreover, $\underline{c} = 0$ if and only if $k \geq \frac{3}{4}$. In this case, $\lim_{c \to 0^+} n = \frac{1}{k} - 1$.

When c is small enough, $c < \underline{c}$, then all banks enter regardless of their size. When c is in the intermediate range $[\underline{c}, \overline{c}]$, then all extreme-$\omega$ banks and only large enough middle-ω banks enter. Indeed, extreme-ω banks have greater incentives to participate in the market. As c increases, fewer middle-ω banks enter, and the size of the market, as measured by the total measure of active traders, decreases. When c is large enough, then we have partial entry at all ω's. An increase in c reduces the size of the market but does not change its composition. This follows from the Pareto size distribution, which implies that, when c increases, the measures of middle- and extreme-ω traders are scaled down by the same constant.

Interestingly, for some parameters, some middle-ω banks may choose to stay out even in the limit $c \to 0$. This is because a decline in c creates two effects on entry incentives, going in opposite directions. On the one hand, when c declines, middle-ω banks find it less costly to enter. On the other hand, additional entry of middle-ω banks improves risk sharing and erodes their intermediation profits. Thus, the marginal private value, $\text{MPV}\left(\frac{1}{2}\right)$, declines, which reduces entry. If k is large enough, then risk sharing improves so much that the decline in cost is almost fully offset by the decline in $\text{MPV}\left(\frac{1}{2}\right)$, and some middle-$\omega$ banks do not enter even in the limit $c \to 0$.

Next, we turn to comparative statics with respect to k:

Proposition 18 (Changes in k)**.** *Changes in the trade size limit, k, have non-monotonic effects on intermediation activity:*

- *The measure of middle-ω traders, $\Phi\left[MPV\left(\frac{1}{2}\right)\right]$, is a non-monotonic function of k. It increases with k when k is close to zero, and it goes to zero as $k \to 1$.*
- *The fraction of middle-ω traders, n, is positive when $k = 0$ and equal to zero when $k = 1$. It decreases with k for $k \simeq 0$ and $\simeq 1$, but can increase with k otherwise.*

An increase in k has two opposite effects on middle-ω banks' entry incentives. On the one hand, there is a positive partial equilibrium effect: when k is larger, each trader in a given bank can increase the size of its position and thus earn larger profits. But, on the other hand, there is a general equilibrium effect: risk sharing improves, which reduces intermediation profits. The first effect dominates when $k \simeq 0$, increasing the measure of middle-ω traders. But the second effect dominates when $k \simeq 1$, decreasing the measure of middle-ω traders. To understand the effects on the *fraction* of middle-ω traders, note that, when $k \simeq 0$, an increase in k causes both middle-ω and extreme-ω traders to enter. But extreme-ω traders enter more, resulting in a decrease in n. When $k \simeq 1$, the risk-sharing is almost perfect and so n decreases towards zero.

These comparative statics translate into predictions about the evolution of gross exposures and net exposures as frictions decrease.

Corollary 6 (Exposures)**.** *A reduction in frictions has the following effects on exposures:*

- *When c decreases, both the average gross exposure, $\mathcal{G}$, and the ratio of gross-to-net exposures, $\mathcal{R}$, increase.*
- *When k increases, $k \simeq 0$ or $k \simeq 1$, the average gross exposure increases, but the ratio of gross to net exposure, $\mathcal{R}$, decreases.*

One sees that, in both cases, reducing frictions increases trading volume, in the sense of increasing gross notionals outstanding per capita. When c decreases, the increase in trading volume is due to an increase in intermediation. Middle-ω banks enter more, and the ratio of gross-to-net notional increases. When k increases, by contrast, the increase in trading volume comes about because of larger customer-to-customer trades, and less intermediation. In this case, the ratio of gross-to-net notional decreases. Therefore, according to the model, the evolution of the gross-to-net-notional ratio can help in telling apart a decrease in frictions due to an improvement in entry costs versus risk-management technologies.

B.4 Socially optimal entry with three types

We consider the planning problem when there are three types $\Omega = \{0, \frac{1}{2}, 1\}$, when the distribution of traders in the economy at large is uniform, $\pi(\omega) = \frac{1}{3}$, and when the cost of risk bearing is quadratic. As before in Section 6.1, we assume that the CDF of bank sizes is Pareto with coefficient $1+\theta > 0$, implying that the distribution of per-capita entry costs has support $[0, \bar{z}]$, $\bar{z} \equiv c/\underline{S}$, and CDF $\Phi(z) = (z/\bar{z})^{\theta}$.

B.4.1 Statement of the planning problem

We have shown in Proposition 15 that the planner always finds it optimal to choose a symmetric distribution of traders. Thus, the planner chooses only two variables: the size M of the market, and the fraction n of middle-ω traders. The fraction of extreme-ω traders is $1-n$, divided equally between $\omega=0$ and $\omega=1$ traders. With a symmetric distribution, we have $g(\frac{1}{2})=\frac{1}{2}$. Thus, the benefit associated with a given choice (M,n) can be written:

$$M\frac{1-n}{2}\bigg(\Gamma\left[0\right]-\Gamma\left[g(0)\right]+\Gamma\left[1\right]-\Gamma\left[g(1)\right]\bigg).$$

We have that:

$$\Gamma\left[0\right]-\Gamma\left[g(0)\right]+\Gamma\left[1\right]-\Gamma\left[g(1)\right]=\Gamma''\times g(0)\left[g(1)-g(0)\right]+\Gamma''\times g(0)^2,$$

where we used that $\Gamma'\left[g(1)\right]-\Gamma'\left[g(0)\right]=\Gamma''\times\left[g(1)-g(0)\right]$ since $\Gamma'\left[g\right]$ is linear. Since the post-trade exposures are symmetric, we have that $g(0)=1-g(1)$. Taken together, we obtain that the benefit is equal to:

$$Mf(n),\ \text{where}\ f(n)\equiv\frac{\Gamma''}{2}(1-n)g(0)\left[1-g(0)\right],\ \text{and}\ g(0)=\min\left\{\frac{1}{2},\frac{k(1+n)}{2}\right\}. \tag{B.13}$$

Using the previous analysis, the cost is equal to:

$$M^{1+1/\theta}h(n),\ \text{where}\ h(n)\equiv\frac{c}{\underline{S}}\frac{\theta}{1+\theta}3^{1/\theta}\left[2^{-1/\theta}(1-n)^{1+1/\theta}+n^{1+1/\theta}\right]. \tag{B.14}$$

The planning problem is thus to maximize:

$$Mf(n)-M^{1+1/\theta}h(n), \tag{B.15}$$

with respect to $(M,n)\in\mathbb{R}_+$ and subject to:

$$M(1-n)\leq\frac{2}{3}, Mn\leq\frac{1}{3}\ \text{and}\ n\in[0,1].$$

B.4.2 Preliminary results

The first immediate result is that

Lemma 18. *The planner finds it optimal to choose $M>0$.*

Indeed, for any $n\in[0,1]$ we have that $f(n)>0$. The result follows since the planner's marginal cost of entry is zero when $M=0$. □

Next, we show:

Lemma 19. *The function $f(n)$ is strictly decreasing in $n\in[0,1]$, and the function $h(n)$ is strictly convex and attains its minimum at $n=\frac{1}{3}$.*

From this Lemma it follows that:

Lemma 20. *The planner's solution satisfies $n\in\left[0,\min\left\{\frac{1}{3},\frac{1}{k}-1\right\}\right]$.*

Indeed, if $n>\frac{1}{3}$ then $M(1-n)<\frac{2}{3}$ since $M\leq 1$. Then, by decreasing n and keeping M the same, one increases the benefit $Mf(n)$, since it is decreasing in n, and reduce the cost $M^{1+1/\theta}h(n)$, since it achieves its minimum at $n=1/3$. Likewise, if $k(1+n)>1$, then one can lower n and M at the same time while keeping the measure of extreme-ω bank, $M(1-n)$, the same. This leaves the benefit $Mf(n)$ the same, but this lowers the cost of entry $M^{1+1/\theta}h(n)$ because there are now less middle-ω banks. $\square$

Next we establish useful identities about $f(n)$:

Lemma 21. *The function $f(n)$ satisfies:*

$$f(n) = (1-n)MPV(0) + nMPV(\tfrac{1}{2}) = (1-n)MSV(0) + nMSV(\tfrac{1}{2}) \tag{B.16}$$

$$MSV(0) = MPV(0) + \frac{1}{2}\left[F(0) - \bar{F}\right] = f(n) - nf'(n) \tag{B.17}$$

$$MSV(\tfrac{1}{2}) = MPV(\tfrac{1}{2}) + \frac{1}{2}\left(F(\tfrac{1}{2}) - \bar{F}\right) = f(n) + (1-n)f'(n). \tag{B.18}$$

Moreover, $MSV(0)$ is an increasing function and $MSV(\frac{1}{2})$ is a decreasing function of $n \in \left[0, \min\left\{\frac{1}{3}, \frac{1}{k} - 1\right\}\right]$. Finally:

$$MSV(0) \geq MPV(0) > MPV(\tfrac{1}{2}) \geq MSV(\tfrac{1}{2}), \tag{B.19}$$

Moreover $MSV(0) > MPV(0)$ and $MPV(\frac{1}{2}) > MSV(\frac{1}{2})$ if and only if $n \in \left(0, \frac{1}{k} - 1\right)$.

The first equality in equation (B.16) follows because payments of CDS premia are transfers which must net out to zero in the aggregate. Therefore, the average of private values must be equal to the social value, $f(n)$. The second equality in equation (B.16) follows from the Euler's formula for homogenous functions since, in the planner's problem, the benefit is homogenous of degree one in the measures of traders of different types. Equation (B.17) and (B.18), as well as the monotonicity results, follow from calculation gathered at the end of this section. We already know that MPV(0) $>$ MPV($\frac{1}{2}$). The inequality MSV(0) $\geq$ MPV(0) follows because, for an extreme-ω bank, the frictional surplus is higher than the average frictional surplus, $F(0) \geq \bar{F}$. The inequality MPV($\frac{1}{2}$) $\geq$ MSV($\frac{1}{2}$) follows because, for a middle-ω bank, the frictional surplus is lower than the average frictional surplus. As shown by the explicit formula gathered at the end of the section, the inequalities are strict whenever $n > 0$ and $n < \frac{1}{k} - 1$. $\square$

B.4.3 A first-order characterization of the planner's solution

Having established that the planner finds it optimal to choose strictly positive participation, $M > 0$, and given that we are working with a continuous distribution of cost, we know from Theorem 3 that the first-order conditions of the planning problem can be written:

$$\mu(\omega) = \Phi\left[\text{MSV}(\omega)\right] = \left(\min\left\{\frac{\underline{S}}{c}\text{MSV}(\omega), 1\right\}\right)^{\theta},$$

keeping in mind that $\bar{z} = c/\underline{S}$. Thus:

$$n = \frac{\frac{1}{3}\min\left\{1, \frac{\underline{S}}{c}\text{MSV}(\frac{1}{2})\right\}^{\theta}}{\frac{1}{3}\min\left\{1, \frac{\underline{S}}{c}\text{MSV}(\frac{1}{2})\right\}^{\theta} + \frac{2}{3}\min\left\{1, \frac{\underline{S}}{c}\text{MSV}(0)\right\}^{\theta}} = \frac{G(n)^{\theta}}{2 + G(n)^{\theta}}, \tag{B.20}$$

where

$$G(n) \equiv \frac{\min\left\{1, \frac{S}{c}\text{MSV}(\frac{1}{2})\right\}}{\min\left\{1, \frac{S}{c}\text{MSV}(0)\right\}}.$$

Direct calculations show that $G(0) > 0$, so the left-hand side of (B.20) is strictly less than the right-hand side when $n = 0$. If $\min\left\{\frac{1}{3}, \frac{1}{k} - 1\right\} = \frac{1}{k} - 1$, then at $n = \frac{1}{k} - 1$ we have that $G(1/k - 1) = 0$ so the left-hand side is strictly greater than the right-hand side. If $\min\left\{\frac{1}{3}, \frac{1}{k} - 1\right\} = \frac{1}{3}$, then at $n = \frac{1}{3}$ left-hand side is clearly greater than the right-hand side too. Therefore, equation (B.20) has a least one solution.

To show that this solution is unique, note that, by Lemma 21, $G(n)$ is strictly decreasing whenever it is strictly less than one, and constant when it is equal to one. Thus, if there is a solution $n < \frac{1}{3}$, then at this solution $G(n) < 1$ and so is strictly decreasing, implying that this solution is unique. Otherwise, the solution must be equal to $\frac{1}{3}$, and is therefore unique as well.

Taken together, we obtain:

Proposition 19. *The planner's problem has a unique solution. The fraction of middle-ω traders, n, is the unique solution of (B.20). It is strictly greater than zero and strictly less than $\frac{1}{k} - 1$. The size of the market, M, is the unique maximizer of (B.15) given n.*

Proceeding with exactly the same proof as in the case of equilibrium, but with private values being replaced by social values, we obtain:

Proposition 20. *Holding all other parameters the same, there are two cost thresholds $0 \leq \underline{c}_P < \overline{c}_P$ such that:*

- *If $c < \underline{c}_P$, all banks enter and $n_P = \frac{1}{3}$.*
- *If $c \in [\underline{c}_P, \overline{c}_P]$, all extreme-$\omega$ banks enter, and only some of the middle-ω banks enter. When c increases, the fraction of middle-ω traders decreases, and the total measure of traders in the market decreases.*
- *If $c > \overline{c}_P$, only some of the extreme-ω and middle-ω banks enter. When c increases, the fraction of middle-ω traders, n, does not not change, but the total measure of traders in the market decreases.*

Moreover, $\underline{c} = 0$ if and only if $\frac{1}{k} - 1 \leq \frac{1}{3}$. In this case, $\lim_{c \to 0^+} n = \frac{1}{k} - 1$.

B.4.4 Comparing the equilibrium and the planner's solution

For this section denote the equilibrium by (M_E, n_E) and the planner's solution by (M_P, n_P). We also recall that n_E solves:

$$n = \frac{F(n)^\theta}{2 + F(n)^\theta}, \text{ where } F(n) = \frac{\min\left\{1, \frac{S}{c}\text{MPV}(\frac{1}{2})\right\}}{\min\left\{1, \frac{S}{c}\text{MPV}(0)\right\}}.$$

Likewise, n_P solves:

$$n = \frac{G(n)^\theta}{2 + G(n)^\theta}, \text{ where } G(n) = \frac{\min\left\{1, \frac{S}{c}\text{MSV}(\frac{1}{2})\right\}}{\min\left\{1, \frac{S}{c}\text{MSV}(0)\right\}}.$$

Moreover, we know that both n_P and n_E are strictly greater than zero and strictly less than $\frac{1}{k} - 1$. In the interval $\left(0, \frac{1}{k} - 1\right)$, it follows from equation (B.19) in Lemma 21 that $G(n) \leq F(n)$ with an equality if and only if $G(n) = 1$. We thus obtain:

Proposition 21. *The fraction of middle-ω traders is smaller in the social optimum than in the equilibrium:*

$$n_P \leq n_E,$$

with an equality if and only if $n_P = \frac{1}{3}$.

Next we derive an equation for the market size, M, as a function of n. Since this equation holds both for the social optimum and for the equilibrium, it will allow us to compare M_P and M_E:

Lemma 22. *The socially optimal market size is $M_P = \mathcal{M}(n_P)$ and the equilibrium market size is $M_E = \mathcal{M}(n_E)$ where:*

$$\mathcal{M}(n) = \min \left\{ \frac{2}{3(1-n)}, \left[\frac{\theta}{1+\theta} \frac{f(n)}{h(n)} \right]^{\theta} \right\}. \tag{B.21}$$

Based on this Lemma, we obtain:

Proposition 22. *The measure of extreme-ω banks is greater in the social optimum than in the equilibrium:*

$$M_P(1-n_P) \geq M_E(1-n_E),$$

with an equality if and only if $n_P = \frac{1}{3}$.

If $\theta \leq 1$, then market size is smaller in the social optimum than in the equilibrium:

$$M_P \leq M_E,$$

with an equality if and only if $n_P = \frac{1}{3}$.

Finally, we can compare the cost thresholds:

Proposition 23. *The entry cost thresholds satisfy:*

$$\underline{c}_P \leq \underline{c}_E \leq \overline{c}_E \leq \overline{c}_P.$$

This ranking reflects the fact that the planner wants more extreme-ω customer banks to enter, and less middle-ω dealer banks. As c declines from $+\infty$ to zero, the planner sends all extreme-ω banks into the OTC market "sooner" than in equilibrium, i.e., for larger entry costs, and all middle-ω banks "later" than in equilibrium, i.e., for lower entry costs.

The proof is straightforward. The left-hand side inequality, $\underline{c}_P \leq \underline{c}_E$ follows directly from Proposition 21: whenever c is low enough so that $n_P = \frac{1}{3}$, $c \leq \underline{c}_P$, then it must be the case that $n_E = \frac{1}{3}$, i.e., that $c \leq \underline{c}_E$. The right-hand side inequality, $\overline{c}_E \leq \overline{c}_P$ follows from noting that, when $c = \overline{c}_E$, the measure of extreme-ω customer banks is maximal, $M_E\left(1 - n_E\right) = \frac{2}{3}$. By Proposition 22, it must be equal to $M_P\left(1 - n_P\right)$, implying in turn that $c \leq \overline{c}_P$.

B.5 The set of n arising in equilibrium or socially optimal entry

In this section we study the restriction on n implied by equilibrium and socially optimal entry. Namely, let $\mathcal{N}_E(c,\theta,k)$ and $\mathcal{N}_P(c,\theta,k)$ denote, respectively, the equilibrium and socially optimal fraction of middle-ω traders, n, when the exogenous parameters are $c \geq 0$, $\theta > 0$, and $k \in (0,1)$. Define $\mathcal{N}_E(0,\theta,k) \equiv \lim_{c\to 0^+} \mathcal{N}_E(c,\theta,k)$, and similarly for $\mathcal{N}_P(0,\theta,k)$.[10]

Lemma 23.

$$\bigcup_{c\geq 0,\theta>0} \mathcal{N}_E(c,\theta,k) = \bigcup_{c\geq 0,\theta>0} \mathcal{N}_P(c,\theta,k) = \left(0, \min\left\{\frac{1}{3}, \frac{1}{k}-1\right\}\right].$$

This shows that, given k, one can choose (c,θ) such that the equilibrium or the socially optimal n takes on any value in the interval $\left(0, \min\left\{\frac{1}{3}, \frac{1}{k}-1\right\}\right]$.

We establish this result for $\mathcal{N}_E$ – the case of $\mathcal{N}_P$ is identical. Let $\hat{n}_E(\theta) > 0$ denote the solution of

$$n = \frac{\text{MPV}(\frac{1}{2})^\theta}{2\text{MPV}(0)^\theta + \text{MPV}(\frac{1}{2})^\theta}.$$

From Proposition 17, we know that

$$\bigcup_{c\geq 0} \mathcal{N}(c,\theta,k) = \left[\hat{n}_E(\theta), \min\left\{\frac{1}{3}, \frac{1}{k}-1\right\}\right].$$

Next, we note that, since the right-hand side of the fixed point equation for $\hat{n}_E(\theta)$ is decreasing in n, we have that:

$$\hat{n}_E(\theta) \leq \frac{\left(\text{MPV}(\frac{1}{2})\big|_{n=0}\right)^\theta}{2\left(\text{MPV}(0)\big|_{n=0}\right)^\theta + \left(\text{MPV}(\frac{1}{2})\big|_{n=0}\right)^\theta}.$$

Noting that $\text{MPV}(\frac{1}{2})\big|_{n=0} < \text{MPV}(0)\big|_{n=0}$, this implies that $\lim_{\theta\to 0} n_E(\theta) = 0$. □

B.6 Three symmetric types with asymmetric bargaining

In this section we study the three-type model when bargaining power is asymmetric. We study conditions for the surplus sharing rule in the OTC market to implement the solution of the social planning problem. We show that such a rule exists. However, the share of the surplus that dealers should receive to implement the social optimum depends on all parameters of the model. In general, we show that there can be either over entry or under entry in equilibrium.

In what follows, we depart from our maintained assumption that traders share the surplus equally in all bilateral meetings. Instead, we assume that middle-ω traders can have a different bargaining power than extreme-ω traders. Namely, we assume that in a bilateral meeting between middle-ω trader and extreme-ω traders, a middle-ω trader has bargaining power $\beta \in [0,1]$, while traders of extreme-ω banks have bargaining power $1-\beta$. When traders from two extreme-ω banks

[10] If $\frac{1}{k}-1 < \frac{1}{3}$ and $c = 0$, then there is a continuum of equilibria with $n \in \left[\frac{1}{k}-1, \frac{1}{3}\right]$ in which since middle-ω bank are indifferent between entering or not. However, all the equilibria with $n > \frac{1}{k}-1$ are not robust to adding a small entry cost: they cannot be obtained as the limit of a sequence of equilibria with positive entry cost, as the entry cost goes to zero.

meet, then we assume as before that they share the surplus equally. Finally, to consider entry, we take the distribution of banks' size to be Pareto, as before.

B.6.1 Equilibrium entry

We start with a derivation of the fixed-point equation for an equilibrium with positive entry. First, we note that asymmetric bargaining will not change equilibrium post-trade exposures conditional on entry, $g(\omega)$: indeed, these post-trade exposures are entirely pinned down by the bilateral optimality condition (8) and do not depend on the surplus sharing rule between traders in the OTC market. The only equilibrium object that differs with asymmetric bargaining is $R(\omega, \tilde{\omega})$, the bilateral price.

Second, going through the same calculations as in Proposition 13, one can show that, if $n(0) > n(1)$, then $\text{MPV}(0) < \text{MPV}(1)$, and vice versa. This implies, as in Proposition 14, that all equilibria must be symmetric, i.e., $\mu(0) = \mu(1)$.

Third, we can calculate marginal private values with asymmetric bargaining. For an extreme-ω bank:

$$\text{MPV}(0) = K(0) + \Gamma'' k \left[(1-\beta) \times n |g(\tfrac{1}{2}) - g(0)| + \frac{1}{2} \times \frac{1-n}{2} |g(1) - g(0)| \right].$$

Note that, in the formula, the bargaining power of an $\omega = 0$ trader is $1 - \beta$ with dealers, but $\frac{1}{2}$ with customers. Collecting and simplifying terms, we obtain:

$$\text{MPV}(0) = \begin{cases} \frac{\Gamma''}{2} \left[\frac{k(1+n)}{2} \right]^2 + \frac{\Gamma'' k}{2} \left(\frac{1+n}{2} - \beta n \right) [1 - k(1+n)] & \text{if } k(1+n) < 1 \\ \frac{\Gamma''}{8} & \text{if } k(1+n) \geq 1. \end{cases}$$

Traders of middle-ω banks only meet customers, so that they have a bargaining power equal to β in all of their bilateral meetings. It thus follows that $\text{MPV}(\frac{1}{2}) = \beta F(\frac{1}{2})$. After some quick calculations, we obtain:

$$\text{MPV}(\tfrac{1}{2}) = \begin{cases} \beta \frac{\Gamma'' k}{2} (1-n) [1 - k(1+n)] & \text{if } k(1+n) < 1 \\ 0 & \text{if } k(1+n) \geq 1, \end{cases}$$

Up to these different formulas for marginal private values, the fixed-point equation for an equilibrium with positive entry is the same as before:

$$n = \frac{F(n)^\theta}{2 + F(n)^\theta} \text{ where } F(n) = \frac{\min\left\{ \frac{S}{c} \text{MPV}(\frac{1}{2}), 1 \right\}}{\min\left\{ \frac{S}{c} \text{MPV}(0), 1 \right\}}. \tag{B.22}$$

A straightforward application of the intermediate value theorem shows that an equilibrium with positive entry exists.

B.6.2 A simple condition for implementation

Next, we derive a simple condition for an equilibrium with asymmetric bargaining to implement the social planning solution. As we know, all that is needed for implementation is that $\text{MPV}(\omega) = \text{MSV}(\omega)$ for all $\omega \in \Omega$. With three types, and given that both the equilibrium and the social planning solution are symmetric, we only need to verify this condition for $\omega = \frac{1}{2}$. Indeed, since the planner's objective (before cost) is homogenous of degree one:

$$(1-n)\text{MPV}(0) + n\text{MPV}(\tfrac{1}{2}) = (1-n)\text{MSV}(0) + n\text{MSV}(\tfrac{1}{2}) \tag{B.23}$$

for any n, as already shown in Lemma 21. Therefore, if $\text{MPV}(\frac{1}{2}) = \text{MSV}(\frac{1}{2})$, then it follows that $\text{MPV}(0) = \text{MSV}(0)$. With this in mind, we calculate:

$$\begin{aligned}\text{MSV}(\tfrac{1}{2}) &= F(\tfrac{1}{2}) - \frac{1}{2}\bar{F} \\ &= \begin{cases} \frac{(1-n)}{2} \times (1-n)\frac{\Gamma'' k}{2}\left[1 - k\,(1+n)\right] & \text{if } k(1+n) < 1 \\ 0 & \text{if } k(1+n) \geq 1 \end{cases}\end{aligned}$$

after algebra which we collect in Section B.9.10. Direct comparison with $\text{MPV}(\frac{1}{2})$ shows that:

Proposition 24. *Let n_P denote the fraction of middle-ω traders in the planning solution. Then, a sufficient condition for an equilibrium with asymmetric bargaining to implement the social planning solution is $\beta = \frac{1-n_P}{2} > 0$. If $n_P \in \left(0, \min\left\{\frac{1}{k} - 1, \frac{1}{3}\right\}\right)$, then this condition is also necessary.*

The expression for the bargaining power β is simple and intuitive. First, the bargaining power should be less than $\frac{1}{2}$. This is an expected result, given that we already know that $\beta = \frac{1}{2}$ leads to $n_E > n_P$. Second, β is decreasing in n_P. This reflects the fact that, when many dealers enter in the planning solution, the social value of the *marginal* dealer bank is smaller, and so the surplus sharing rule should leave more surplus to customers. Finally, note that since $n < \frac{1}{3}$, then $\beta > 0$: the planner should leave some surplus to dealers.

When the solution of the planning problem is at a corner, then the Proposition only delivers a sufficient condition. Indeed, if $n_P = 0$, which occurs for instance when $k \geq 1$, then $\text{MSV}(\frac{1}{2}) = 0$, and $\text{MPV}(\frac{1}{2}) = 0$ for any β, and so implementation obtains regardless of the surplus sharing rule between dealers and customers. This confirms a result of Theorem 3. If $n_P = \frac{1}{3}$, then the planner wants all banks to enter, and so $\frac{S}{c}\text{MSV}(0) > \frac{S}{c}\text{MSV}(\frac{1}{2}) \geq 1$. Clearly, these inequalities can also hold with marginal private values, even tough they are not equal to marginal social values, so that the equilibrium can coincide with the planning solution.

B.6.3 Equilibrium vs. social planning

Whenever the solution of the planning problem is interior we obtain a sharp comparison between any equilibrium and the planning solution:

Proposition 25. *Suppose $n_P \in \left(0, \min\left\{\frac{1}{k} - 1, \frac{1}{3}\right\}\right)$. Then, in any equilibrium, $n_E > n_P$ if $\beta > \frac{1-n_P}{2}$, and $n_E < n_P$ if $\beta < \frac{1-n_P}{2}$.*

The Proposition reveals that, depending on parameters, dealers may enter too much or too little in equilibrium relative to the social optimum. To see this, first recall Lemma 23. Given any $n_P \in (0, \bar{n}_P)$, where $\bar{n}_P = \min\left\{\frac{1}{3}, \frac{1}{k} - 1\right\}$, then as long as $k \in (0,1)$ there are some cost and Pareto parameters, c and θ, such that this n_P solves the planning problem. Conversely, if $n_P > \bar{n}_P$, then there do not exit any cost and Pareto parameters such that n_P solves the planning problem. Taken together, this implies that, as long as $k < 1$ and $n_P > 0$:

- If $\beta \geq \frac{1}{2}$, then dealers always enter too much in equilibrium, i.e., $n_E > n_P$.
- If $\beta \in \left(\frac{1-\bar{n}_P}{2}, \frac{1}{2}\right)$, then there exists some cost and Pareto parameters such that dealers enter too much, and others such that they enter too little.

- If $\beta \in \left[0, \frac{1-\bar{n}_P}{2}\right)$, then dealers always enter too little.

B.6.4 An explicit bargaining model implying $\beta < \frac{1}{2}$

One may wonder why it is natural to assume that intermediaries have bargaining power less than $\frac{1}{2}$. In this subsection, we follow a suggestion of an anonymous referee: we show that $\beta < \frac{1}{2}$ arises naturally when customers understand that intermediaries make profit via round-trip trades, and when the matching protocol allows them to appropriate some of that profit.

The matching and bargaining protocol. Assume that, when two customers meet ($\omega = 0$ meets $\omega = 1$), they split the surplus equally. But, when a customer bargains with a dealer ($\omega \in \{0, 1\}$ meets $\omega = \frac{1}{2}$), she understands that the dealer is engaged in a "round trip" trade. Namely, assume that, for all intermediated trades, bargaining occurs amongst four traders in a quadrilateral match, with two traders from the same $\omega = \frac{1}{2}$ bank, and two traders from different extreme-ω bank, one of type $\omega = 0$ and one of type $\omega = \frac{1}{2}$. The bargaining protocol is as follows. Half of the times, the dealer (traders of type $\omega = \frac{1}{2}$) first bargains with the seller of protection ($\omega = 0$) over k contract, and then it bargains with a buyer of protection ($\omega = 1$) over k contracts. The other half of the times, it is the other way around: the dealer first bargains with the buyer of protection and then with the seller of protection.

We assume that, at each stage of this matching and bargaining protocol, the surplus is split equally. We will show however that, when the two stages of matching and bargaining protocol are taken together, the dealer appropriate a fraction $\beta < \frac{1}{2}$ of the round-trip trade surplus.

Terms of trade when the sequence of transaction is $0 \to \frac{1}{2} \to 1$. To solve the model, we go backward. We assume that the two $\omega = \frac{1}{2}$ traders have already purchased k contracts from the $\omega = 1$ trader, and that they are now seeking to re-sell these contracts to a buyer of protection. The total trading surplus for this second trade is $k\left\{\Gamma'\left[g(1)\right] - \Gamma'\left[g(\frac{1}{2})\right]\right\}$. Indeed, the threat point of the two traders of type $\omega = \frac{1}{2}$ is to hold on to the contract they have already purchased. The price at which the two traders of type $\omega = \frac{1}{2}$ will re-sell the contract, the "ask", is:

$$A = \frac{1}{2}\left\{\Gamma'\left[g(1)\right] + \Gamma'\left[g(\tfrac{1}{2})\right]\right\}$$

Now consider the first trade, between the two traders of type $\omega = \frac{1}{2}$ and the trader of type $\omega = 0$. The total surplus is now $A - \Gamma'\left[g(0)\right]$: the trader of type $\omega = 0$ understands that the two $\omega = \frac{1}{2}$ traders will resell the contracts at a price that is higher than their reservation value, $A \geq \Gamma'\left[g(\frac{1}{2})\right]$. Therefore, the price at which the two traders of type $\omega = \frac{1}{2}$ will purchase the contract, the "bid", is:

$$B = \frac{1}{2}\left\{A + \Gamma'\left[g(0)\right]\right\}$$

Hence, the profit of the two traders of type $\omega = \frac{1}{2}$ over the round-trip trade is:

$$A - B = \frac{k}{2}\left\{\frac{\Gamma'\left[g(1)\right] + \Gamma'\left[g(\frac{1}{2})\right]}{2} - \Gamma'\left[g(0)\right]\right\} \leq \frac{k}{2}\left\{\Gamma'\left[g(1)\right] - \Gamma'\left[g(0)\right]\right\}.$$

One sees that this bargaining protocol erodes the profit of the traders of type $\omega = \frac{1}{2}$. The reason is that, in the first leg of the transaction, the counterparty of type $\omega = 0$ is able to extract some of the profit that the two traders of type $\omega = \frac{1}{2}$ make by re-selling the contract to a $\omega = 1$ trader.

Surplus sharing under symmetry. Anticipating a symmetric equilibrium, we have $\Gamma'[g(1)] - \Gamma'\left[g(\frac{1}{2})\right] = \Gamma'\left[g(\frac{1}{2})\right] - \Gamma'[g(0)]$. This implies that the profit of the two traders of type $\omega = \frac{1}{2}$ is the same for both possible sequences of transactions. After some algebra, it is easy to see that the profit of the two traders of type $\omega = \frac{1}{2}$ is:

$$\frac{3}{8}\left\{\Gamma'[g(1)] - \Gamma'\left[g(\tfrac{1}{2})\right]\right\} + \frac{3}{8}\left\{\Gamma'\left[g(\tfrac{1}{2})\right] - \Gamma'[g(0)]\right\}.$$

Hence, the bargaining protocol proposed here is equivalent to assuming that traders of type $\omega = \frac{1}{2}$ traders have a bargaining power of $\frac{3}{8} < \frac{1}{2}$ when they bargain with traders of type $\omega \in \{0, 1\}$.

B.7 Equilibrium exit with three types

We now study equilibrium exit in our simple parametric example with three-types. We assume that the distribution of traders' types in the market arises from the entry model studied above, in Section 6.1. As we have shown in Section B.5, this implies the restriction $\pi(\frac{1}{2}) \leq \min\{\frac{1}{3}, \frac{1}{k} - 1\}$. We consider a simple discrete distribution of exit costs: all banks have to pay the same cost, z, per trader capita, in order to resume trading in the OTC market. We now characterize all equilibria, for all values of z.

We already know from Proposition 14 that all exit equilibria must be symmetric, i.e., that $\mu(0) = \mu(1)$. As long as a positive measure of traders stay in the market, the fraction of middle-ω traders is equal to:

$$n = \frac{\mu\left(\frac{1}{2}\right)}{\mu\left(\frac{1}{2}\right) + 2\mu(0)},$$

and the matching probability after exit is:

$$\alpha = \rho + (1-\rho)\left[2\mu(0) + \mu\left(\tfrac{1}{2}\right)\right].$$

The marginal private values for extreme-ω (customer) and middle-ω (dealer) banks are:

$$\text{MPV}(0) = \text{MPV}(1) = \frac{\Gamma''}{4}\alpha k\left[1 - \frac{\alpha k\left(1-n^2\right)}{2}\right]$$

$$\text{MPV}(\tfrac{1}{2}) = \frac{\Gamma''}{4}\alpha k(1-n)\left[1 - \alpha k(1+n)\right].$$

We will abstract from the coordination failure that can result in multiple equilibria: whenever there is more than one equilibrium with positive participation, we will consider the equilibrium in which featuring the highest level of participation. We offer two main propositions about this equilibrium. First, it has an attractive property for welfare analysis:

Proposition 26. *Given any cost shock z, the equilibrium with highest participation maximizes utilitarian welfare amongst all possible equilibria.*

Second, we can characterize fully banks' exit behavior as a function of the cost shock:

Proposition 27. *There are three cost thresholds, $z_{1E} < z_{2E} < z_{3E}$, such that, in the equilibrium with highest participation:*

- *$z \in [0, z_{1E}]$, all banks stay:*

$$2\mu(0) = 1 - \pi \text{ and } \mu\left(\tfrac{1}{2}\right) = \pi;$$

- *$z \in (z_{1E}, z_{2E})$, extreme-ω banks stay and middle-ω banks exit partially:*

$$2\mu(0) = 1 - \pi \text{ and } 0 < \mu\left(\tfrac{1}{2}\right) < \pi;$$

- *$z \in [z_{2E}, z_{3E}]$, extreme-ω banks stay and middle-ω exit fully:*

$$2\mu(0) = 1 - \pi \text{ and } \mu\left(\tfrac{1}{2}\right) = 0;$$

- *$z > z_{3E}$: all banks exit fully:*

$$2\mu(0) = 0 \text{ and } \mu\left(\tfrac{1}{2}\right) = 0.$$

Finally, the measures of extreme-ω and middle-ω traders in the market are continuous and decreasing in z except at the threshold z_3, where the measure of extreme-ω traders has a downward jump.

The thresholds appearing in the proposition have intuitive interpretations. For example, the first threshold, z_{1E}, is the lowest cost that makes a middle-ω bank indifferent between staying or not, when all other banks stay:

$$z_{1E} = \text{MPV}\left(\tfrac{1}{2}\right)\Big|_{2\mu(0)=1-\pi \text{ and } \mu(1/2)=\pi}.$$

When $z = z_{1E}$, $\text{MPV}(0) > \text{MPV}\left(\frac{1}{2}\right) = z$, and so all extreme-$\omega$ banks find it optimal to stay.

The Proposition shows that middle-ω banks are the most vulnerable to shocks: for any z, a middle-ω bank is more likely to exit than an extreme-ω bank. The reason is, as before, that intermediation is a small profit margin activity: it provides no fundamental gains from trade and so there is less to gain by staying in the market.

One other thing we learn from the Proposition is that the measure of customer banks can be discontinuous in z at $z = z_{3E}$. Such a discontinuity is expected since the model has multiple equilibria, and so selections of the equilibrium map typically have discontinuities. What is perhaps more interesting is that this discontinuity occurs for relatively large shocks, that it goes downward so it represents a liquidity dry up and not a boom, and that it is characterized by a withdrawal of customer banks and not of dealer banks.

B.8 Socially optimal exit with three types

We now consider the planning problem with exit when there are three types $\Omega = \{0, \frac{1}{2}, 1\}$, and when the cost of risk bearing is quadratic. As for the equilibrium, we assume that the exit cost is equal to z for all banks. Therefore, the CDF of exit costs is a step function: $\Phi(\tilde{z}) = 0$ for $\tilde{z} < z$, and $\Phi(\tilde{z}) = 1$ for $\tilde{z} \geq z$. We know from Proposition 15 that all solutions of the planning problem must be symmetric. We thus directly look for solutions such that $\mu(0) = \mu(1)$, and we denote $n(\frac{1}{2})$ by n.

The planner's objective. Ignoring constant terms and using that $\mu(0) = \mu(1)$, the planner's objective is

$$\mu(0)\Big\{\Gamma[0] - \Gamma[g(0)] + \Gamma[1] - \Gamma[g(1)]\Big\} - 2\mu(0)z - \mu(\tfrac{1}{2})z.$$

Given that $\Gamma[g]$ is quadratic,

$$\Gamma[\omega] = \Gamma[g(\omega)] + \Gamma'[g(\omega)][\omega - g(\omega)] + \frac{\Gamma''}{2}[\omega - g(\omega)]^2.$$

Keeping in mind that $g(0) - 0 = g(0) = 1 - g(1)$, this implies that:

$$\begin{aligned}\Gamma[0] - \Gamma[g(0)] + \Gamma[1] - \Gamma[g(1)] &= \left(\Gamma'[g(1)] - \Gamma'[g(0)]\right)g(0) + \Gamma''g(0)^2 \\ &= \Gamma''[g(1) - g(0)]\,g(0) + \Gamma''g(0)^2 = \Gamma''g(0)[1 - g(0)],\end{aligned}$$

where the second line follows because $\Gamma'[g] = \Gamma'[0] + \Gamma''g$ and because $g(0) = 1 - g(1)$.

The optimization problem. Taken together, we obtain that the planner's problem is to choose $\mu(0) = \mu(1)$ and $\mu(\frac{1}{2})$ in order to maximize:

$$\mu(0)\Gamma''g(0)[1 - g(0)] - 2\mu(0)z - \mu(\tfrac{1}{2})z,$$

subject to

$$0 \leq 2\mu(0) \leq 1 - \pi \text{ and } 0 \leq \mu(\tfrac{1}{2}) \leq \pi,$$

where $\pi \equiv \pi(\frac{1}{2})$, while $g(0)$, α, and n solve:

$$g(0) = \min\left\{\frac{1}{2}, \frac{k\alpha(1+n)}{2}\right\},\ \alpha = \rho + (1-\rho)\left[2\mu(0) + \mu(\tfrac{1}{2})\right],\ \text{ and } n = \frac{\mu(\frac{1}{2})}{2\mu(0) + \mu(\frac{1}{2})}.$$

Note that we can replace the constraint $g(0) = \min\left\{\frac{1}{2}, \frac{k\alpha(1+n)}{2}\right\}$ by $g(0) = \frac{k\alpha(1+n)}{2}$. This is because, under both constraints, the planner will find it optimal to choose $\mu(0)$ and $\mu(\frac{1}{2})$ such that $\frac{k\alpha(1+n)}{2} \leq \frac{1}{2}$. Indeed, if $\mu(\frac{1}{2})$ were such that $\frac{k\alpha(1+n)}{2} > \frac{1}{2}$, then $\mu(\frac{1}{2}) > 0$ and the planner's objective could be increased by reducing $\mu(\frac{1}{2})$.

B.8.1 Calculation of surpluses

Given symmetry, the competitive surpluses are:

$$\begin{aligned}K(0) = K(1) &= \frac{\Gamma''}{2}g(0)^2 \\ K(\tfrac{1}{2}) &= 0.\end{aligned}$$

Note that, in a symmetric equilibrium, $g(0) > 0$ because a type $\omega = 0$ trader finds counterparties of type $\omega \in \{\frac{1}{2}, 1\}$ with probability $\frac{1+n}{2} > 0$. Therefore, $K(0) = K(1) > K(\frac{1}{2})$.

The frictional surplus for extreme-ω banks is

$$F(0) = F(1) = \Gamma'' k \left\{ n\left[\tfrac{1}{2} - g(0)\right] + \frac{1-n}{2}\left[g(1) - g(0)\right] \right\} = \Gamma'' k \left\{\tfrac{1}{2} - g(0)\right\},$$

after noting that $g(1) = 1 - g(0)$. Finally, the frictional surplus for middle-ω banks is

$$F(\tfrac{1}{2}) = \Gamma'' k(1-n)\left\{\tfrac{1}{2} - g(0)\right\}$$

Clearly, $F(0) = F(1) \geq F(\frac{1}{2})$. With these results in mind, we obtain:

Lemma 24. *For any n, $MSV(0) = MSV(1) > MSV(\frac{1}{2})$. Therefore, if the planner finds it optimal to choose $\mu(\frac{1}{2}) > 0$, he finds it optimal to choose $\mu(0) = \pi(0) = \mu(1) = \pi(1)$.*

For the second part of the Lemma we note that if $\mu(\frac{1}{2}) > 0$, then there is strictly positive participation. The first-order conditions of the planning problem are the same as in the model of entry (see Theorem 3). These imply that $\mu(\frac{1}{2}) \leq \pi(\frac{1}{2})\Phi\left[\text{MSV}(\frac{1}{2})\right]$. Given that $\mu(\frac{1}{2}) > 0$ and that $\Phi(\tilde{z}) = 0$ for all $\tilde{z} < z$, this implies that $\text{MSV}(\frac{1}{2}) \geq z$. Since $\text{MSV}(0) = \text{MSV}(1) > \text{MSV}(\frac{1}{2}) \geq z$, we obtain that $\Phi\left[\text{MSV}(0)^-\right] = \Phi\left[\text{MSV}(1)^-\right] = 1$, and the result follows.

This lemma allows us to break down the planning problem into two simpler, one-dimensional component planning problems. The first component problem is to maximize social welfare with respect to the measure of middle-ω traders, $\mu(\frac{1}{2})$, holding $\mu(0) = \pi(0) = \mu(1) = \pi(1)$. The second component problem is to maximize social welfare with respect to the measure of extreme-ω traders, $\mu(0) = \mu(1)$, holding $\mu(\frac{1}{2}) = 0$. The solution of the planner's problem corresponds to the component problem with largest value.

B.8.2 The first component planning problem

Suppose that $\mu(0) = \mu(1) = \pi(0) = \pi(1)$. The first component planning problem is:

$$\hat{W}(\tfrac{1}{2}) = \max \pi(0)\Gamma'' g(0)\left[1 - g(0)\right] - 2\pi(0)z - \mu(\tfrac{1}{2})z,$$

with respect to $0 \leq \mu(\frac{1}{2}) \leq \pi$ and subject to:

$$g(0) = \frac{k\alpha(1+n)}{2},\ \alpha = \rho + (1-\rho)\left[1 - \pi + \mu(\tfrac{1}{2})\right],\ \text{ and } n = \frac{\mu(\frac{1}{2})}{1 - \pi + \mu(\frac{1}{2})}.$$

Manipulating the above constraints we find that:

$$\mu(\tfrac{1}{2}) = \frac{n(1-\pi)}{1-n} \text{ and } \alpha = \rho + \frac{(1-\rho)(1-\pi)}{1-n}$$

Plugging this into the formula for $\text{MSV}(\frac{1}{2})$ we obtain after algebraic manipulations:

Lemma 25. *When $2\mu(0) = 1 - \pi$ and $\mu(\frac{1}{2}) \in [0, \pi]$:*

$$\frac{\partial \hat{W}}{\partial \mu(\frac{1}{2})} = MSV(\tfrac{1}{2}) - z = \frac{\Gamma'' k}{4}\left[1 - k\alpha(1+n)\right]\left[1-n\right]\left[2\alpha - \rho(1+n)\right] - z$$
$$= \frac{\Gamma'' k}{4}\left[1 - k\alpha(1+n)\right]\left[\rho(1-n)^2 + 2(1-\rho)(1-\pi)\right] - z,$$

is a strictly decreasing function of n, and therefore a strictly decreasing function of $\mu(\frac{1}{2})$.

This shows that the first component planning problem is strictly concave in $\mu(\frac{1}{2})$, implying that its solution is fully characterized via first-order conditions.

Proposition 28. *The first component planning problem has a unique solution, such that:*

- *if* $z < MSV\left(\frac{1}{2}\right)|_{2\mu(0)=1-\pi \text{ and } \mu(1/2)=\pi}$, *then* $\mu(\frac{1}{2}) = \pi$;
- *if* $z > MSV\left(\frac{1}{2}\right)|_{2\mu(0)=1-\pi \text{ and } \mu(1/2)=0}$, *then* $\mu(\frac{1}{2}) = 0$;
- *otherwise,* $\mu(\frac{1}{2})$ *is the unique solution of* $MSV\left(\frac{1}{2}\right)|_{2\mu(0)=1-\pi \text{ and } \mu(1/2)} = z$.

B.8.3 The second component planning problem

The second component planning problem is to choose an optimal $\mu(0)$, holding $\mu(\frac{1}{2}) = 0$. In this case, we have that $n = 0$, and so $g(0) = \frac{k\alpha}{2}$. The planner's problem simplifies to:

$$\hat{W}(0) = \max 2\mu(0)\left\{\Gamma''\frac{k\alpha}{4}\left[1 - \frac{k\alpha}{2}\right] - z\right\},$$

with respect to $2\mu(0) \in [1, 1-\pi]$ and subject to $\alpha = \rho + (1-\rho)2\mu(0)$. The given that $k\alpha < 1$, the term in the curly bracket is strictly increasing in $\mu(0)$. Therefore, the optimum is to choose $2\mu(0) = 1 - \pi$ if it makes the term in curly bracket positive, and $\mu(0) = 0$ otherwise.

Proposition 29. *The second component planning problem is solved either by* $\mu(0) = 0$*, or by* $2\mu(0) = 1 - \pi$.

B.8.4 The full planning problem

Here, we first note that the value attained by the second component planner problem when he chooses $2\mu(0) = 1 - \pi$, is achieved by the first component planner when he chooses $\mu(\frac{1}{2}) = 0$. Therefore, if there is positive entry in the full planning problem, the value of the first component planning problem must be at least as large as that of the second component planning problem. If there is no entry, the value of the second component planning problem must be negative. Therefore, we obtain that:

Proposition 30. *The value of the planning problem is* $\max\{\hat{W}(\frac{1}{2}), 0\}$. *Moreover, there are three thresholds* $z_{1P} \leq z_{2P} \leq z_{3P}$ *such that, in a social optimum:*

- $z \in [0, z_{1P})$, *all banks stay:*

  $$2\mu(0) = 1 - \pi \text{ and } \mu\left(\tfrac{1}{2}\right) = \pi;$$

- $z \in (z_{1P}, z_{2P})$, *extreme-*ω *banks stay and middle-*ω *banks exit partially:*

  $$2\mu(0) = 1 - \pi \text{ and } 0 < \mu\left(\tfrac{1}{2}\right) < \pi;$$

- $z \in (z_{2P}, z_{3P})$, *extreme-*ω *banks stay and middle-*ω *banks exit fully:*

  $$2\mu(0) = 1 - \pi \text{ and } \mu\left(\tfrac{1}{2}\right) = 0;$$

- $z > z_{3P}$: *all banks exit fully:*

  $$2\mu(0) = 0 \text{ and } \mu\left(\tfrac{1}{2}\right) = 0.$$

Finally, the measures extreme-ω and middle-ω traders in the market are continuous and decreasing in z except perhaps at one point, where it has a downward jump.

Finally, we compare socially optimal and equilibrium exit. In what follows, if the social optimum has multiple solutions, which may occur at the thresholds, we pick the one with the highest level of participation.

Proposition 31. *Assume that $\rho < 1$ and let $\mu_E(\frac{1}{2})$ and $\mu_P(\frac{1}{2})$ denote the measure of middle-ω traders who stay in the OTC market, in equilibrium and in the planner's problem. Then, there are three thresholds $z_1 \leq z_2 < z_3$ such that:*

- *if $z < z_1$, $\mu_E(\frac{1}{2}) = \mu_P(\frac{1}{2}) = \pi(\frac{1}{2})$;*
- *if $z \in (z_1, z_2)$, $\mu_E(\frac{1}{2}) > \mu_P(\frac{1}{2})$;*
- *if $z \in (z_2, z_3)$, $\mu_E(\frac{1}{2}) < \mu_P(\frac{1}{2})$;*
- *if $z > z_3$, $\mu_E(\frac{1}{2}) = \mu_P(\frac{1}{2}) = 0$.*

Moreover, $z_1 = z_2$ if $\rho\left[1 + \pi(\frac{1}{2})\right] \leq 1$.

B.9 Omitted proofs in the analysis of the three-type model

B.9.1 Proof of Lemma 14

Since gross exposures have to be larger than net exposures, we have, for an $\omega = 0$ bank, that $G^+(0) \geq g(0) = \frac{1}{2}$ and $G^-(0) \geq 0$. Moreover:

$$\frac{1}{2} = g(0) = n\gamma\left(0, \tfrac{1}{2}\right) + \frac{1-n}{2}\gamma(0,1) \Rightarrow n\gamma\left(0, \tfrac{1}{2}\right) = \frac{1}{2} - \frac{1-n}{2}\gamma(0,1) \geq \frac{1}{2} - \frac{1-n}{2}k > 0,$$

since $\gamma(0,1) \leq k$ and since $k(1-n) < 1$. Symmetrically, for a $\omega = 1$ bank, $G^+(1) \geq 0$, $G^-(1) \geq 1 - g(1) = \frac{1}{2}$, and $n\gamma\left(1, \frac{1}{2}\right) \leq \frac{1-n}{2}k - \frac{1}{2} < 0$. Putting all these inequalities together, we obtain that:

$$\begin{aligned}\mathcal{G}(k) &\geq \frac{1-n}{2}G^+(0) + \frac{1-n}{2}G^-(1) + n\frac{1-n}{2}\left(|\gamma(0,\tfrac{1}{2})| + |\gamma(\tfrac{1}{2},1)|\right) \\ &\geq \frac{1-n}{2} + 2 \times \frac{1-n}{2}\left(\frac{1}{2} - \frac{1-n}{2}k\right) = \frac{1-n}{2}\left[2 - k(1-n)\right].\end{aligned}$$

One can easily verify that this lower bound is achieved by the proposed CDS contracts.

B.9.2 Proof of Proposition 13

First, let us note that, since $K(0) = \frac{\Gamma''}{2}g(0)^2$ and $K(1) = \frac{\Gamma''}{2}\left[1 - g(1)\right]^2$, it follows that $K(0) < K(1)$ if and only if $g(0) < 1 - g(1)$. Second, let us consider the four possible cases described in Section B.1.2.

Case 1: $g(0) < g(\frac{1}{2}) < g(1)$. Since $g(0) = k\left[1 - n(0)\right]$ and $g(1) = 1 - k\left[1 - n(1)\right]$, we obtain that $K(0) < K(1)$ if and only if $n(0) > n(1)$, as claimed. Turning to the frictional surplus, we have

$$\begin{aligned}F(0) &= \Gamma''k\left[n(\tfrac{1}{2})\left\{\tfrac{1}{2} - k\left[1 - n(1)\right]\right\} + n(1)\left\{1 - k\left[1 - n(0)\right] - k\left[1 - n(1)\right]\right\}\right] \\ F(1) &= \Gamma''k\left[n(\tfrac{1}{2})\left\{\tfrac{1}{2} - k\left[1 - n(0)\right]\right\} + n(0)\left\{1 - k\left[1 - n(0)\right] - k\left[1 - n(1)\right]\right\}\right].\end{aligned}$$

Therefore:

$$F(1) - F(0) = \Gamma'' k \left[n(0) - n(1)\right] \left[n(\tfrac{1}{2}) + \{1 - k\left[1 - n(0)\right] - k\left[1 - n(1)\right]\}\right].$$

Keeping in mind that, in case 1, $2k\left[1 - n(0)\right] < 1$ and $2k\left[1 - n(1)\right] < 1$, we find that $F(1) > F(0)$ if and only if $n(0) > n(1)$.

Case 2: $g(0) = g(\frac{1}{2}) < g(1)$**.** Since, as shown in Lemma 16, $n(0) > n(1)$, we only need to show that $K(0) \leq K(1)$ and $F(0) \leq F(1)$ with at least one strict inequality. First, we show that $K(0) < K(1)$. As noted above, it is equivalent to show that $g(0) < 1 - g(1)$. Now recall that, by (B.7), $1 - g(1) = k\left[1 - n(1)\right] \geq \frac{1}{2}$. Therefore, a sufficient condition for $g(0) < 1 - g(1)$ is that $g(0) < \frac{1}{2}$, which can be written as:

$$\frac{\frac{1}{2}\left[1 - n(0) - n(1)\right]}{1 - n(1)} + kn(1) < \tfrac{1}{2} \Leftrightarrow -\frac{\frac{1}{2}n(0)}{1 - n(1)} + kn(1) < 0 \Leftrightarrow n(0) > 2kn(1)\left[1 - n(1)\right].$$

But (B.7) shows that $n(1) \leq 1 - \frac{1}{2k}$. Therefore, a sufficient condition for the last inequality to hold is that $n(0) > 2k\left[1 - \frac{1}{2k}\right]\left[1 - n(1)\right]$, which is (B.6). Next, consider the frictional surplus:

$$F(0) = \Gamma'' kn(1)\left[g(1) - g(0)\right]$$
$$F(1) = \Gamma'' k\left[1 - n(1)\right]\left[g(1) - g(0)\right].$$

Clearly, $F(0) < F(1)$ if and only if $n(1) < \frac{1}{2}$, which follows from Lemma 16.

Case 3: $g(0) < g(\frac{1}{2}) = g(1)$**.** This is symmetric to case 2.

Case 4: $g(0) = g(\frac{1}{2}) = g(1)$**.** In this case the frictional surplus is zero, and so we only need to study the competitive surplus. Since there is full risk sharing, we have

$$g(1) = g(0) = \tfrac{1}{2}n(\tfrac{1}{2}) + n(1) = \tfrac{1}{2}\left[1 - n(0) - n(1)\right] + n(1).$$

As shown before $K(0) < K(1)$ if and only if $g(0) < 1 - g(1)$. By the expression above, it follows immediately that $g(0) < 1 - g(1)$ if and only if $n(0) > n(1)$.

B.9.3 Marginal private values in the three types model

With a quadratic cost of risk bearing, $\Gamma''\left[g\right]$ is constant, so that a second-order Taylor series approximation implies:

$$\Gamma\left[\omega\right] = \Gamma\left[g(\omega)\right] + \Gamma'\left[g(\omega)\right]\left[\omega - g(\omega)\right] + \frac{\Gamma''}{2}\left[g(\omega) - \omega\right]^2 \Rightarrow K(\omega) = \frac{\Gamma''}{2}\left[g(\omega) - \omega\right]^2.$$

Moreover, since $\Gamma'\left[g\right] = \Gamma'\left[0\right] + \Gamma'' \times g$, the frictional surplus is:

$$F(\omega) = \sum_{\tilde{\omega}} k\Gamma''\big|g(\omega) - g(\tilde{\omega})\big|n(\tilde{\omega})$$

Now consider a symmetric distribution of traders, $n\left(\frac{1}{2}\right) \equiv n$ and $n(0) = n(1) = \frac{1-n}{2}$. When $k(1+n) < 1$, there is partial risk sharing so that $g(0) = 1 - g(1) = k\frac{1+n}{2}$. Plugging this back into the competitive and frictional surplus, we obtain that:

$$\begin{aligned}\text{MPV}(0) &= \frac{\Gamma''}{2}\left[g(0)^2 + k\left\{n\left(\frac{1}{2} - g(0)\right) + \frac{1-n}{2}\left(g(1) - g(0)\right)\right\}\right] \\ &= \frac{\Gamma''}{2}\left[\left(\frac{k(1+n)}{2}\right)^2 + k\left(\frac{1}{2} - \frac{k(1+n)}{2}\right)\right] = \frac{\Gamma'' k}{4}\left[1 - \frac{k\left(1-n^2\right)}{2}\right]\end{aligned}$$

When $k(1+n) > 1$, MPV(0) is constant equal to $\frac{\Gamma''}{8}$. Similar calculations lead to:

$$\begin{aligned}\text{MPV}\left(\tfrac{1}{2}\right) &= \frac{\Gamma'' k}{2}(1-n)\left[\frac{1}{2} - g(0)\right] = \frac{\Gamma'' k}{2}(1-n)\left(\frac{1}{2} - \frac{k(1+n)}{2}\right) \\ &= \frac{\Gamma'' k}{4}(1-n)\left[1 - k(1+n)\right].\end{aligned}$$

When $k(1+n) > 1$, MPV $\left(\frac{1}{2}\right)$ is constant equal to zero.

B.9.4 Proof of Proposition 16

The right-hand side of (B.12) is continuous, is positive at $n = 0$ and equal to 0 at $n = 1/k - 1$. Thus, by the intermediate value theorem, a solution exists. To establish uniqueness it is enough to show that the right-hand side is a decreasing function of n. We thus distinguish three cases. If both $\text{MPV}(0) > \text{MPV}\left(\frac{1}{2}\right) \geq \frac{c}{\underline{S}}$, then all banks enter, $F(n) = 1$, and so the property holds. If $\text{MPV}(0) \geq \frac{c}{\underline{S}}$ but $\text{MPV}\left(\frac{1}{2}\right) < \frac{c}{\underline{S}}$, then $F(n) = \frac{\underline{S}\text{MPV}\left(\frac{1}{2}\right)}{c}$, which is clearly decreasing in $n \in (0, 1/k - 1)$. If $\text{MPV}\left(\frac{1}{2}\right) < \text{MPV}(0) < \frac{c}{\underline{S}}$, then

$$F(n) = \frac{(1-n)(1-k(1+n))}{1 - k/2(1-n^2)} \Longrightarrow F'(n) = \frac{-1 + k/2 + k^2n^2/2 + kn}{\left(1 - k/2(1-n^2)\right)^2}.$$

Given that $n \in (0, 1/k - 1)$, we have $k < 1/(1+n)$ and so the numerator in the expression of $F'(n)$ is less than:

$$-1 + \frac{1}{2(1+n)} + \frac{n^2}{2(1+n)^2} + \frac{n}{1+n} = \frac{1}{2}\left(-\frac{1}{1+n} + \frac{n^2}{(1+n)^2}\right) < 0$$

where the last inequality follows because $n^2 < 1$ and $1/(1+n) > 1/(1+n)^2$. This establishes the claim. □

B.9.5 Proof of Proposition 17

Continuity is obvious given that the equilibrium is unique and that the equilibrium fixed point equation is continuous. Now, given that $\text{MPV}(0) > \text{MPV}\left(\frac{1}{2}\right)$, there are three cases to consider.

When MPV $\left(\frac{1}{2}\right) \geq \frac{c}{\underline{S}}$. We show in this paragraph that there exists $\underline{c} \geq 0$ such that $\text{MPV}\left(\frac{1}{2}\right) \geq \frac{c}{\underline{S}}$ if and only if $c \leq \bar{c}$.

Suppose that $\text{MPV}\left(\frac{1}{2}\right) \geq \frac{c}{\underline{S}}$. Then all banks enter, implying that $n = \frac{1}{3}$. In particular, given that $n = \frac{1}{3}$, we have:

$$\text{MPV}\left(\tfrac{1}{2}\right) \geq \frac{c}{\underline{S}} \Leftrightarrow c \leq \frac{\Gamma''\underline{S}}{4}\frac{2k}{3}\left(1 - \frac{4k}{3}\right).$$

Since $c \geq 0$, this condition can be written:

$$c \leq \underline{c} \equiv \min \left\{ \frac{\Gamma'' \underline{S}}{4} \frac{2k}{3} \left(1 - \frac{4k}{3}\right), 0 \right\}.$$

Conversely, if the condition $c \leq \bar{c}$ is satisfied, $n = \frac{1}{3}$ solves the equilibrium equation (B.12) and all banks enter.

Clearly, when $c \leq \underline{c}$, all banks enter and so the measure of traders in the economy is constant. Finally, one sees that $\underline{c} = 0$ if and only if $k \geq \frac{3}{4}$ which is equivalent to $\frac{1}{k} - 1 \leq \frac{1}{3}$.

When MPV$(0) < \frac{c}{\underline{S}}$. We show in this paragraph that there is some $\bar{c} > 0$ such that there is partial entry of extreme- and middle-ω banks if and only if $c > \bar{c}$.

If $\mathrm{MPV}(0) < \frac{c}{\underline{S}}$, the fixed point equation, and thus n, does not depend on c. Let $\underline{n}$ denote the solution of the fixed point equation in this case and let

$$\overline{c} \equiv \underline{S}\mathrm{MPV}\left(0\right) = \frac{\Gamma'' k}{4} \left[1 - \frac{k(1-\underline{n}^2)}{2}\right] \underline{S}.$$

In words, $\overline{c}$ is the cost threshold such that $\mathrm{MPV}\left(0\right) = \frac{c}{\underline{S}}$ when $n = \underline{n}$. By construction, if $\mathrm{MPV}(0) < \frac{c}{\underline{S}}$, we have that $c > \overline{c}$. Conversely, if $c > \overline{c}$, then $n = \underline{n}$ solves the equilibrium fixed point equation, and there is partial entry of extreme- and middle-ω banks.

One sees that, when $c > \overline{c}$, both $\frac{c\mathrm{MPV}(0)}{\underline{S}}$ and $\frac{c\mathrm{MPV}\left(\frac{1}{2}\right)}{\underline{S}}$ are increasing in c, implying that the total measure of traders in the economy is decreasing in c.

When MPV$\left(0\right) \geq \frac{c}{\underline{S}}$ but MPV$\left(\frac{1}{2}\right) < \frac{c}{\underline{S}}$. From the previous two paragraphs, this occurs if and only if $c \in (\underline{c}, \overline{c}]$. Because $\mathrm{MPV}\left(0\right) \geq \frac{c}{\underline{S}}$, all extreme-$\omega$ traders enter: the measure of extreme-ω traders is thus equal to $2/3$ and does not change with c. Together with the fact that the *fraction* of middle-ω traders, n, is decreasing in c, this implies that the *measure* of middle-ω traders is decreasing in c. Thus, the total measure of traders is decreasing in c.

Finally to see that $\underline{c} < \overline{c}$, we consider two cases. First, if $\underline{c} = 0$, then by definition we have that $\overline{c} > 0 = \underline{c}$. If $\underline{c} > 0$, then we note that at $c = \overline{c}$, we have that $\frac{\underline{S}\mathrm{MPV}(0)}{\overline{c}} = 1$. But $\mathrm{MPV}\left(\frac{1}{2}\right) < \mathrm{MPV}\left(0\right)$ for all $n \in [0,1]$, implying that $\mathrm{MPV}\left(\frac{1}{2}\right) < \frac{\overline{c}}{\underline{S}}$ and so middle-ω banks enter partially. But we know from the first paragraph that middle-ω banks enter fully if and only if $c \geq \underline{c}$. By contrapositive, this implies that $\overline{c} > \underline{c}$.

The limit of n as $c \to 0^+$. If $\frac{1}{3} < \frac{1}{k} - 1$, then $\underline{c} > 0$ and so $n = \frac{1}{3}$ for all $c \in (0, \underline{c})$. Therefore, the limit of n as $c \to 0$ is $n = \frac{1}{3}$. Now suppose that $\frac{1}{k} - 1 \leq \frac{1}{3}$ and suppose that $\lim_{c \to 0^+} n < \frac{1}{k} - 1$ (note that this limit exists since n is decreases with c). Then, both $\mathrm{MPV}\left(0\right)$ and $\mathrm{MPV}\left(\frac{1}{2}\right)$ are bounded away from zero when as $c \to 0^+$. This implies that, when c is sufficiently close to zero, $F(n) = 1$ and $n = \frac{1}{3}$, a contradiction. □

B.9.6 Proof of Proposition 18

We first establish a number of preliminary results, and then turn to the results stated in the proposition.

Lemma 26. *When all banks enter partially, $MPV(0) < \frac{c}{\underline{S}}$, the fraction of middle-ω traders decreases with k.*

When $\text{MPV}(0) < \frac{c}{\underline{S}}$:

$$F(n,k) = \frac{(1-n)\,[1-k(1+n)]}{1-k(1-n^2)/2} \Longrightarrow \frac{\partial F}{\partial k} = -\frac{1-n}{[1-k(1-n^2)/2]^2}\left\{\frac{1}{2}+n+\frac{n^2}{2}\right\} < 0.$$

But we already know that $\partial F/\partial n < 0$, from the proof of Proposition 16, in Section B.9.4. Thus, the equilibrium fixed-point equation, (B.12), is increasing in both n and k, establishing the claim.

Lemma 27. *When middle-ω banks enter partially and extreme-ω banks enter fully, $MPV\left(\frac{1}{2}\right) < \frac{c}{\underline{S}} < MPV(0)$, the fraction of middle-ω traders, n, increases in k if $k(1+n) < \frac{1}{2}$, and decreases in k if $k(1+n) > \frac{1}{2}$.*

When $\text{MPV}\left(\frac{1}{2}\right) < \frac{c}{\underline{S}} < \text{MPV}\,(0)$:

$$F(n,k) = \frac{\underline{S}\Gamma''}{4c}\,[1-n]\times k\times[1-k(1+n)]\,.$$

It thus follows that $F(n,k)$ increases with k when $k(1+n) < \frac{1}{2}$ and decreases with k when $k(1+n) > \frac{1}{2}$. But we already know from the proof of Proposition 16, in Section B.9.4, that $\partial F/\partial n < 0$, establishing the claim.[11]

Lemma 28. *When $k \simeq 0$, the fraction of middle-ω traders, n, decreases with k, and entry incentives for all banks, $MPV(0)$ and $MPV\left(\frac{1}{2}\right)$, both increase with k.*

Note that $\text{MPV}(0)$ and $\text{MPV}\left(\frac{1}{2}\right)$ are less than $\Gamma''k/4$, which goes to zero when k goes to zero. Therefore, for k sufficiently small, we know that both $\text{MPV}(0)$ and $\text{MPV}\left(\frac{1}{2}\right)$ must be smaller than $\frac{c}{\underline{S}}$, and the first part of the result follows from Lemma 26 that n is a decreasing function of k. Let $n(0)$ denote the limit of n as $k \to 0$. By continuity, $n(0)$ must solve the fixed point equation

$$n - \frac{F(n,0)^\theta}{2+F(n,0)^\theta} = 0 \Leftrightarrow 2n - (1-n)^{\theta+1} = 0, \text{ where } F(n,k) = \frac{(1-n)(1-k(1+n))}{1-k(1-n^2)/2}.$$

In particular, $n(0) \in (0,1)$. Given that $\partial F(n,k)/\partial n \leq 0$, the partial derivative of the above equation with respect to n must be larger than one. We can thus apply the implicit function theorem and find that n is continuously differentiable with respect to k in a neighborhood of $k = 0$, with bounded derivatives. The second part of the result then follows by differentiating $\text{MPV}\,(0)$ and $\text{MPV}\left(\frac{1}{2}\right)$ with respect to k.

Lemma 29. *When $k \simeq 1$, the fraction of middle-ω traders can be written as $n(k) = \max\{n_1(k), n_2(k)\}$, where both $n_1(k)$ and $n_2(k)$ are continuously differentiable and satisfy $n_1(1) = n_2(1) = n_1'(1) = n_2'(1) = 0$.*

For this Lemma we start by noting that since $n \in (0, 1-1/k)$, we must have that $n \to 0$ as $k \to 1$. It then follows that when $k(1+n) \to 1$, $\frac{\underline{S}\text{MPV}\left(\frac{1}{2}\right)}{c} \to 0$, and that n must solve either one

[11] One can easily find parameter values such that, in equilibrium, either $k(1+n) < \frac{1}{2}$ or $k(1+n) > \frac{1}{2}$. To find parameter values such that $k(1+n) < \frac{1}{2}$, one first picks some k such that $k(1+\frac{1}{3}) < \frac{1}{2}$. Then, given this k, one picks $c \in (\underline{c}, \overline{c})$, as constructed in the proof of Proposition 2, so that $\text{MPV}\left(\frac{1}{2}\right) < \frac{c}{\underline{S}} < \text{MPV}(0)$. But then we know that the equilibrium n has to be less than $\frac{1}{3}$, and so it follows from $k(1+\frac{1}{3}) < \frac{1}{2}$ that $k(1+n) < \frac{1}{2}$ as well. Likewise, if $k > \frac{1}{2}$ and $c \in (\underline{c}, \overline{c})$, then $k(1+n) > \frac{1}{2}$ and n decreases with k.

of the following two equations:

$$n - \frac{F_1(n,k)^\theta}{2 + F_1(n,k)^\theta} = 0, \quad \text{where } F_1(n,k) = \frac{\underline{S}\text{MPV}\left(\frac{1}{2}\right)}{c} = \frac{\underline{S}}{c}\frac{\Gamma'' k}{4}\left[1-n\right]\left[1-k(1+n)\right] \tag{B.24}$$

$$n - \frac{F_2(n,k)^\theta}{2 + F_2(n,k)^\theta} = 0, \quad \text{where } F_2(n,k) = \frac{\frac{\underline{S}\text{MPV}\left(\frac{1}{2}\right)}{c}}{\frac{\underline{S}\text{MPV}(0)}{c}} = \frac{\left[1-n\right]\left[1-k(1+n)\right]}{1-\frac{k}{2}(1-n^2)}. \tag{B.25}$$

The same calculations as in the proof of Proposition 16, in Section B.9.4, show that each of these equations has a unique solution in $(0, 1-1/k)$, which we denote by $n_1(k)$ and $n_2(k)$ respectively. Clearly, both $n_1(k)$ and $n_2(k)$ go to zero as k goes to one. Moreover the implicit function theorem applies to both equations at $(n,k) = (0,1)$ since both are continuously differentiable in (n,k), with a partial derivative with respect to n that is greater than one. Moreover, since $F_1(0,1) = F_2(0,1) = 0$ and $\theta > 1$, it follows that the partial derivative with respect to k is zero at $(n,k) = (0,1)$. Therefore, $n_1'(1) = n_2'(1) = 0$.

The last thing to show is that $n(k) = \max\{n_1(k), n_2(k)\}$. To see this, note that if $n = n_1(k)$, then it must be the case that $\underline{S}\text{MPV}(0)/c \geq 1$, from which it follows that $F_1(n_1(k),k) \geq F_2(n_1(k),k)$, and so that $n_1(k) \geq n_2(k)$. Similarly, when $n = n_2(k)$, we have that $\underline{S}\text{MPV}(0)/c \leq 1$, that $F_2(n_2(k),k) \geq F_1(n_2(k),k)$, and so $n_2(k) \geq n_1(k)$.

Proof of the proposition. For the first statement of the first bullet point, recall Lemma 28: when k is close to zero, $\text{MPV}(\frac{1}{2})$ is an increasing function of k and $\text{MPV}\left(\frac{1}{2}\right) < \frac{c}{\underline{S}}$. It thus follows that $\Phi\left[\text{MPV}\left(\frac{1}{2}\right)\right]$ is increasing in k. For the second statement of the first bullet point, recall Lemma 29: when $k \to 1$, n goes to zero. At the same time, the total measure of traders in the market cannot exceed one, and so $\Phi\left[\text{MPV}\left(\frac{1}{2}\right)\right] \leq n$. The second bullet point follows directly from Lemma 27 and Lemma 26.

B.9.7 Proof of Corollary 6

The average gross exposures, $\mathcal{G}(k,n)$, and the gross-to-net notional ratio, $\mathcal{R}(k,n)$, are both increasing in $n \in [0, \frac{1}{3}]$ (we make the dependence of these exposures on n explicit to facilitate the exposition). Since n is non-increasing in c, the first bullet point follows.

For the second bullet point, consider first gross exposures. The result when $k \simeq 0$ follows by applying the same argument as in the proof of Lemma 28. When $k \simeq 1$, the argument follows by first noting that, since $\mathcal{G}(k,n)$ is increasing in $n \in (0, \frac{1}{3})$, we can write $\mathcal{G}(k, n(k)) = \max\{\mathcal{G}(k, n_1(k)), \mathcal{G}(k, n_2(k))\}$. Using that the $n_i(k)$'s are continuously differentiable with $n_i(1) = n_i'(1) = 0$, one obtains that $\mathcal{G}(k, n_i(k))$ is increasing for $k \simeq 1$, and so is $\mathcal{G}(k, n(k))$. The result in the second bullet point about the gross-to-net notional ratio, $\mathcal{R}(k, n(k))$, follows because, on the one hand, $\mathcal{R}(k,n)$ is an increasing function of n and, from Proposition 18, $n(k)$ is decreasing in k for $k \simeq 0$ and $k \simeq 1$.

B.9.8 Proof of Lemma B.9.9

The planner's first order condition can be written:

$$\mu(0) \leq \pi(0)\Phi\left[\text{MSV}(0)\right] \text{ with "} = \text{" if } \mu(0) < \pi(0)$$
$$\mu(\tfrac{1}{2}) \leq \pi(\tfrac{1}{2})\Phi\left[\text{MSV}(\tfrac{1}{2})\right] \text{ with "} = \text{" if } \mu(\tfrac{1}{2}) < \pi(\tfrac{1}{2})$$

Now recall that $\pi(0) = \pi(\frac{1}{2}) = 1/3$, that $\mu(0) = M\frac{1-n}{2}$, $\mu(\frac{1}{2}) = Mn$, and that $\Phi(z) = (\min\{z/\bar{z}, 1\})^\theta$, with $\bar{z} = c/\underline{S}$. Thus, we can rewrite the first-order conditions as:

$$
\begin{aligned}
-\frac{c}{\underline{S}} 3^{1/\theta} M^{1/\theta} \left(\frac{1-n}{2}\right)^{1/\theta} + \mathrm{MSV}(0) \geq 0, \text{ with `` = '' if } M\frac{1-n}{2} < \frac{1}{3} = \pi(0) \\
-\frac{c}{\underline{S}} 3^{1/\theta} M^{1/\theta} n^{1/\theta} + \mathrm{MSV}(\tfrac{1}{2}) \geq 0, \text{ with `` = '' if if } Mn < \frac{1}{3} = \pi(\tfrac{1}{2}).
\end{aligned}
$$

Now multiply the first equation by $1-n$, the second equation by n, and add them up. Using expression (B.14) for $h(n)$, as well as identity (B.16), while keeping in mind that $\mu(0) < \pi(0)$ implies $\mu(\frac{1}{2}) < \pi(\frac{1}{2})$, we obtain:

$$
\begin{aligned}
& - M^{1/\theta} \frac{1+\theta}{\theta} h(n) + f(n) \geq 0, \text{ with `` = '' if } \mu(0) < \pi(0) \\
\Leftrightarrow & M \leq \left[\frac{\theta}{1+\theta} \frac{f(n)}{h(n)}\right]^\theta, \text{ with `` = '' if } \mu(0) < \pi(0).
\end{aligned}
$$

Moreover, we know that, if $\mu(0) = \pi(0)$, then:

$$
\frac{M(1-n)}{2} = \frac{1}{3}
$$

Finally note that, given identity (B.16), these last two relationships also hold for the equilibrium. The result follows. □

B.9.9 Proof of Proposition 22

Proof of the first part. Note that if $n_P = \frac{1}{3}$, then we know from Proposition 21 that $n_E = \frac{1}{3}$, so both $M_E = M_P = 1$ and the result follows. If $n_P < \frac{1}{3}$, then from Proposition 21 we know that $n_E > n_P$. Consider, then, the function:

$$
(1-n)\mathcal{M}(n) = \min\left\{\frac{2}{3}; (1-n)\left[\frac{\theta}{1+\theta}\frac{f(n)}{h(n)}\right]^\theta\right\},
$$

for $n \in [n_P, n_E]$. Let us take the derivative of the second argument of the minimum:

$$
\begin{aligned}
\frac{d}{dn}\left((1-n)\left[\frac{f(n)}{h(n)}\right]^\theta\right) &\propto - f(n)h(n) + \theta(1-n)f'(n)h(n) - \theta(1-n)f(n)h'(n) \\
&\propto - f(n)\left[2^{-1/\theta}(1-n)^{1+1/\theta} + n^{1+1/\theta}\right] \\
&\quad + \theta(1-n)f'(n)\left[2^{-1/\theta}(1-n)^{1+1/\theta} + n^{1+1/\theta}\right] \\
&\quad - (1+\theta)(1-n)f(n)\left[-2^{-1/\theta}(1-n)^{1/\theta} + n^{1/\theta}\right],
\end{aligned}
$$

where the last line follows from calculating the derivative of $h(n)$. Factoring $(1-n)^{1+1/\theta}$ and $n^{1/\theta}$ we obtain that the above is equal to the sum of two terms:

$$\begin{aligned}\text{Term (I)} &= 2^{-1/\theta}(1-n)^{1+1/\theta}\left[-f(n)+\theta(1-n)f'(n)+(1+\theta)f(n)\right]\\ &= 2^{-1/\theta}\theta(1-n)^{1+1/\theta}\left[f(n)+(1-n)f'(n)\right]\\ &= 2^{-1/\theta}\theta(1-n)^{1+1/\theta}\text{MSV}(\tfrac{1}{2}).\end{aligned}$$

The second term is:

$$\begin{aligned}\text{Term (II)} &= n^{1/\theta}\left[-nf(n)+\theta(1-n)nf'(n)-(1+\theta)(1-n)f(n)\right]\\ &= n^{1/\theta}\left[-f(n)-\theta(1-n)\left\{f(n)-nf'(n)\right\}\right]\\ &= n^{1/\theta}\left[-f(n)-\theta(1-n)\text{MSV}(0)\right].\end{aligned}$$

Collecting the two we obtain that:

$$\text{Term (I)} + \text{Term (II)} = -n^{1/\theta}f(n)+\theta\,(1-n)\left[2^{-1/\theta}(1-n)^{1/\theta}\text{MSV}(\tfrac{1}{2})-n^{1/\theta}\text{MSV}(0)\right].$$

The first term on the right-hand side is strictly negative. The second term is also negative for all $n\in[n_P,n_E]$, because it is a decreasing function of n and it is negative when $n=n_P$. To see that the second term is indeed negative when $n=n_P$, we note first that since $n_P<\frac{1}{3}$, we have:

$$G(n)=\frac{\frac{S}{c}\text{MSV}(\frac{1}{2})}{\min\left\{1,\frac{S}{c}\text{MSV}(0)\right\}}\geq\frac{\text{MSV}(\frac{1}{2})}{\text{MSV}(0)}.$$

Plugging this into the fixed point equation for n_P, we obtain:

$$n_P\geq\frac{\left(\frac{\text{MSV}(1/2)}{\text{MSV}(0)}\right)^{\theta}}{2+\left(\frac{\text{MSV}(1/2)}{\text{MSV}(0)}\right)^{\theta}}.$$

Rearranging we obtain that:

$$2^{-1/\theta}(1-n_P)^{1/\theta}\text{MSV}(\tfrac{1}{2})-n_P^{1/\theta}\text{MSV}(0)\leq 0, \tag{B.26}$$

and the result follows. □

Proof of the second part. When $n_P=\frac{1}{3}$ then $n_E=\frac{1}{3}$ and $M_E=M_P$, so the result follows. When $n_P<\frac{1}{3}$, then we proceed as follows. First, we prove below that:

Lemma 30. *The function $\mathcal{M}(n)$ is either single peaked or strictly increasing over $\left[0,\min\left\{\frac{1}{3},\frac{1}{k}-1\right\}\right]$.*

Let n_M denote the argument maximum of $\mathcal{M}(n)$. By the above Lemma, we have that $\mathcal{M}(n)$ is strictly increasing over $[0,n_M]$. Then, we show below that:

Lemma 31. *Both n_P and n_E are less than n_M.*

Since $n_P<n_E$ by Proposition 21, and since, by Lemma , the equilibrium size of the market is $\mathcal{M}(n_E)$ and the socially optimal size of the market is $\mathcal{M}(n_P)$, the result follows.

A preliminary characterization of n_P. As shown above in (B.26), the social planner's fixed point equation can be written:

$$2^{-1/\theta}(1-n)^{1/\theta}\text{MSV}(\tfrac{1}{2}) - n^{1/\theta}\text{MSV}(0) \leq 0 \Leftrightarrow H(n) \leq 0 \text{ with “=” if and only if } \mu(0) < \pi(0) = \frac{1}{3},$$

and where

$$H(n) \equiv f'(n)h(n) - \frac{\theta}{1+\theta} f(n)h'(n).$$

The formula for $H(n)$ follows by replacing $\text{MSV}(\frac{1}{2})$ and $\text{MSV}(0)$ by their expression in term of $f(n)$ and $f'(n)$, in Lemma 21, and rearranging. Since $\text{MSV}(\frac{1}{2})$ is decreasing and $\text{MSV}(0)$ is increasing, the function $H(n)$ is decreasing. It is strictly negative at $n = \min\left\{\frac{1}{3}, \frac{1}{k} - 1\right\}$. So there exists a unique $\hat{n}_P \in \left[0, \min\left\{\frac{1}{3}, \frac{1}{k} - 1\right\}\right]$ such that $H(\hat{n}_P) = 0$ and:

$$n_P \geq \hat{n}_P, \text{ with “=” if and only if } \mu(0) < \pi(0) = \frac{1}{3}.$$

Proof of Lemma 30. Define first

$$n_M \equiv \arg \max_{n \in \left[0, \min\left\{\frac{1}{3}, \frac{1}{k}-1\right\}\right]} \mathcal{M}(n)$$

$$\hat{n}_M \equiv \arg \max_{n \in \left[0, \min\left\{\frac{1}{3}, \frac{1}{k}-1\right\}\right]} \frac{f(n)}{h(n)}.$$

Taking the derivative, we obtain that:

$$\frac{d}{dn}\left[\frac{f(n)}{h(n)}\right] \propto f'(n)h(n) - f(n)h'(n) = H(n) - \frac{1}{1+\theta} f(n)h'(n). \tag{B.27}$$

Note that

$$\frac{d}{dn} f(n)h'(n) = f'(n)h'(n) + f(n)h''(n) > 0,$$

for $n \in \left[0, \frac{1}{3}\right]$. Indeed, $f'(n) < 0$ for $n \in [0, 1]$, $h'(n) < 0$ for $n \in \left[0, \frac{1}{3}\right]$ since it achieves its minimum at $n = \frac{1}{3}$, and $h''(n) > 0$ since it is convex. Since we have already shown that $H(n)$ is decreasing, we thus conclude that $\frac{d}{dn}\left[\frac{f(n)}{h(n)}\right]$ is decreasing over $\left[0, \min\left\{\frac{1}{3}, \frac{1}{k} - 1\right\}\right]$. Moreover, it is strictly positive at $n = 0$, since $H(0) > 0$ and $h'(0) < 0$. Therefore, $\frac{f(n)}{h(n)}$ is either strictly increasing or single peaked on $\left[0, \min\left\{\frac{1}{3}, \frac{1}{k} - 1\right\}\right]$. Now let

$$\hat{n}_S = \sup\left\{n \in \left[0, \min\left\{\frac{1}{3}, \frac{1}{k} - 1\right\}\right] \; : \; \frac{2}{3(1-n)} < \left[\frac{\theta}{1+\theta}\frac{f(n)}{h(n)}\right]^{\theta}\right\}$$

with the convention that $\hat{n}_S = 0$ if this set is empty. Over $n \in [0, \hat{n}_S]$, $\mathcal{M}(n) \leq \frac{2}{3(1-n)} \leq \frac{2}{3(1-\hat{n}_S)} = \mathcal{M}(\hat{n}_S)$, since $\frac{2}{3(1-n)}$ is increasing. Therefore:

$$\hat{n}_S = \arg \max_{n \in [0, \hat{n}_S]} \mathcal{M}(n).$$

Note that $\mathcal{M}(n) = \left[\frac{\theta}{1+\theta}\frac{f(n)}{h(n)}\right]^\theta$ for $n \geq \hat{n}_S$, which is single peaked at $\hat{n}_M$. Therefore if $\hat{n}_S \leq \hat{n}_M$, then $n_M = \hat{n}_M$ and if $n_S > \hat{n}_M$, then $n_M = \hat{n}_S$. Taken together, we obtain:

$$n_M = \max\{\hat{n}_S, \hat{n}_M\}.$$

Finally, we verify that $\mathcal{M}(n)$ is increasing for $n \leq n_M$, and decreasing for $n \geq n_M$. First we note that $\mathcal{M}(n)$ must be increasing over $[0, \hat{n}_M]$ because it is the minimum of two increasing functions. If $n_S \leq \hat{n}_M$, then $\mathcal{M}(n) = \left[\frac{\theta}{1+\theta}\frac{f(n)}{h(n)}\right]^\theta$ for $n \geq \hat{n}_M$ and so is decreasing. If $\hat{n}_S > \hat{n}_M$ then for $n \in [\hat{n}_M, \hat{n}_S]$, since $\frac{f(n)}{h(n)}$ is decreasing and $\frac{2}{3(1-n)}$ is increasing, we have:

$$\frac{2}{3(1-n)} \leq \frac{2}{3(1-\hat{n}_S)} = \mathcal{M}(\hat{n}_S) \leq \left[\frac{\theta}{1+\theta}\frac{f(\hat{n}_S)}{h(\hat{n}_S)}\right]^\theta \leq \left[\frac{\theta}{1+\theta}\frac{f(n)}{h(n)}\right]^\theta.$$

Therefore, $\mathcal{M}(n) = \frac{2}{3(1-n)}$ and so is increasing. Finally, for $n > \hat{n}_S > \hat{n}_M$, we have $\mathcal{M}(n) = \left[\frac{\theta}{1+\theta}\frac{f(n)}{h(n)}\right]^\theta$ which is decreasing.

Proof of $n_P \leq n_M$ in Lemma 31. To see this consider first the case $n_P = \hat{n}_P$. Then $H(n_P) = 0$ and, from (B.27),

$$\frac{d}{dn}\left[\frac{f(n)}{h(n)}\right] = -\frac{1}{1+\theta}f(n_P)h'(n_P) > 0,$$

since $n_P < \frac{1}{3}$. This implies that $n_P < \hat{n}_M$ and, since $n_M = \max\{\hat{n}_M, \hat{n}_S\}$, that $n_P < n_M$. The other case is when $n_P > \hat{n}_P$. Then, $\mu(0) = \pi(0) = \frac{1}{3}$ and so $\mathcal{M}(n_P) = \frac{2}{3(1-n_P)} \leq \left[\frac{\theta}{1+\theta}\frac{f(n_P)}{h(n_P)}\right]^\theta$. Therefore $n_P \leq \hat{n}_S$ and the result follows because $n_M = \max\{\hat{n}_M, \hat{n}_S\}$.

Proof of $n_E \leq n_M$ in Lemma 31. If $\mathcal{M}(n_E) = \frac{2}{3(1-n_E)}$, then $n_E \leq \hat{n}_S$ and so the result follows as before because $n_M = \max\{\hat{n}_S, \hat{n}_M\}$. If $\mathcal{M}(n_E) = \left[\frac{\theta}{1+\theta}\frac{f(n_E)}{h(n_E)}\right]^\theta$, then $\mu(0) < \pi(0) = \frac{1}{3}$ and $\frac{S}{c}\text{MPV}(0) < 1$. Since $\text{MPV}(\frac{1}{2}) < \text{MPV}(0)$, then we also have that $\frac{S}{c}\text{MPV}(\frac{1}{2}) < 1$. Therefore, n_E is the unique solution of

$$n_E = \frac{\left[\frac{\text{MPV}(1/2)}{\text{MPV}(0)}\right]^\theta}{2 + \left[\frac{\text{MPV}(1/2)}{\text{MPV}(0)}\right]^\theta},$$

which, after rearranging, can be written as:

$$2^{-1/\theta}(1-n)^{1/\theta}\text{MPV}(\tfrac{1}{2}) - n^{1/\theta}\text{MPV}(0) = 0. \tag{B.28}$$

Note that the left-hand side is a decreasing function of n. Also one can verify that:

$$2^{-1/\theta}(1-n)^{1/\theta} \propto h(n) - \frac{\theta}{1+\theta}nh'(n)$$
$$n^{1/\theta} \propto h(n) + \frac{\theta}{1+\theta}(1-n)h'(n),$$

with the same constant of proportionality. Therefore, equation (B.28) can be rewriten as:

$$\begin{aligned}
&\mathrm{MPV}(\tfrac{1}{2})\left[h(n)-\frac{\theta}{1+\theta}nh'(n)\right]-\mathrm{MPV}(0)\left[h(n)+\frac{\theta}{1+\theta}(1-n)h'(n)\right]=0\\
\Leftrightarrow&h(n)\left[\mathrm{MPV}(\tfrac{1}{2})-\mathrm{MPV}(0)\right]-\frac{\theta}{1+\theta}h'(n)\left[n\mathrm{MPV}(\tfrac{1}{2})+(1-n)\mathrm{MPV}(0)\right]=0\\
&h(n)\left[\mathrm{MPV}(\tfrac{1}{2})-\mathrm{MPV}(0)\right]-\frac{\theta}{1+\theta}h'(n)f(n)=0.
\end{aligned}$$

Now, at $n=\hat{n}_M$, we have that $f'(n)h(n)=f(n)h'(n)$. Therefore, at $n=\hat{n}_M$, the above equilibrium equation is proportional to

$$\mathrm{MPV}(\tfrac{1}{2})-\mathrm{MPV}(0)-\frac{\theta}{1+\theta}f'(n),$$

Plugging in the formulas for $\mathrm{MPV}(\frac{1}{2})$, $\mathrm{MPV}(0)$, and $f'(n)$ we obtain, after some rearranging that, at $n=\hat{n}_M$, the equilibrium equation is proportional to

$$-k\frac{1-n^2}{2}-n\left[1-\theta+\theta k(1+n)\right]<0$$

if $\theta\leq 1$. It follows from this that $n_E<\hat{n}_M$ and thus $n_E<n_M$. □

B.9.10 Proof of Proposition 24

First, we need to calculate $\mathrm{MSV}(\frac{1}{2})$ when $k(1+n)\leq 1$,

$$\begin{aligned}
\mathrm{MSV}(\tfrac{1}{2})&=F(\tfrac{1}{2})-\frac{1}{2}\bar{F}\\
&=\frac{\Gamma''k}{2}(1-n)\left[1-k(1+n)\right]-\frac{1}{2}\left\{n\frac{\Gamma''k}{2}(1-n)\left[1-k(1+n)\right]+(1-n)\frac{\Gamma''k}{2}\left[1-k(1+n)\right]\right\}\\
&=\left(1-n-\frac{n(1-n)+(1-n)}{2}\right)\frac{\Gamma''k}{2}\left[1-k(1+n)\right]\\
&=\frac{(1-n)}{2}\times(1-n)\frac{\Gamma''k}{2}\left[1-k(1+n)\right].
\end{aligned}$$

The sufficiency of the condition is obvious. Necessity is implied by Proposition 25, which is proved below.

B.9.11 Proof of Proposition 25

Uniqueness when $\beta<\frac{1}{2}$. We start with a preliminary result:

Lemma 32. *If $\beta\leq\frac{1}{2}$, then there is a unique equilibrium with positive entry.*

To prove this Lemma, note that:

$$\frac{\partial\mathrm{MPV}(0)}{\partial n}=\frac{\Gamma''k}{2}\left(\frac{1}{2}-\beta\right)\left[1-k(1+n)\right]+\frac{\Gamma''k^2}{\beta}n\geq 0,$$

when $\beta\leq\frac{1}{2}$. Given that $\mathrm{MPV}(\frac{1}{2})$ is decreasing in n, this implies that the right-hand side of the equilibrium equation is decreasing in n, and the result follows.

Proof of the main proposition. Consider any equilibrium n_E. It solves

$$n = \frac{F(n)^\theta}{2 + F(n)^\theta} \text{ where } F(n) = \frac{\min\left\{\frac{S}{c}\text{MPV}(\frac{1}{2}), 1\right\}}{\min\left\{\frac{S}{c}\text{MPV}(0), 1\right\}}.$$

Also, recall that n_P is the unique solution of the fixed point equation:

$$n = \frac{G(n)^\theta}{2 + G(n)^\theta} \text{ where } G(n) = \frac{\min\left\{\frac{S}{c}\text{MSV}(\frac{1}{2}), 1\right\}}{\min\left\{\frac{S}{c}\text{MSV}(0), 1\right\}}.$$

This is the same as the equilibrium equation, but with MSV's replacing MPV's. Now suppose that n_P is interior, that is $n_P \in \left(0, \min\left\{\frac{1}{3}, \frac{1}{k} - 1\right\}\right)$. This implies that $\frac{1}{k} - 1 > 0$, $k < 1$, and so that that $k(1 + n_E) < 1$ and $\text{MPV}(\frac{1}{2}) > 0$. We then have two cases to consider:

- If $\beta > \frac{1}{2}$ then, if $n_E = \frac{1}{3}$, the result is obvious because of our maintained assumption that $n_P < \frac{1}{3}$. Otherwise, suppose that $n_E < \frac{1}{3}$. Then, we note that, at $n = n_E$, $\text{MPV}(\frac{1}{2}) > \text{MSV}(\frac{1}{2})$ and $\text{MPV}(0) < \text{MSV}(0)$. Given that n_E solves the equilibrium equation and $n_E < \frac{1}{3}$, this implies that:

  $$n_E > \frac{G(n_E)^\theta}{2 + G(n_E)^\theta}.$$

  Then, we obtain that $n_E > n_P$ because the right-hand side is decreasing and n_P is the unique solution of corresponding fixed point equation.

- If $\beta \leq \frac{1}{2}$, then we start from the fixed-point equation:

  $$n_P = \frac{G(n_P)^\theta}{2 + G(n_P)^\theta}$$

  If $\beta > \frac{1-n_P}{2}$ then we have that $\text{MPV}(\frac{1}{2}) > \text{MSV}(\frac{1}{2})$ and $\text{MPV}(0) < \text{MSV}(0)$. Given that $n_P < \frac{1}{3}$, we have $\frac{S}{c}\text{MSV}(\frac{1}{2}) < 1$ and so this implies that:

  $$n_P < \frac{F(n_P)^\theta}{2 + F(n_P)^\theta}.$$

  The result that $n_E > n_P$ then follows because we have shown in the above Lemma that the equilibrium is unique. The case $\beta < \frac{1-n_P}{2}$ is symmmetric.

B.9.12 Proof of Propositions 26 and 27

We start with Proposition 10 and consider all possible equilibrium configurations for $\mu(0)$ and $\mu\left(\frac{1}{2}\right)$.

Type-a equilibrium: all banks stay. For this to be an equilibrium, we need that $\text{MPV}(0) \geq z$ and $\text{MPV}\left(\frac{1}{2}\right) \geq z$, when $2\mu(0) = 1 - \pi$, $\mu\left(\frac{1}{2}\right) = \pi$, and $\alpha = 1$. Since $\text{MPV}(0) > \text{MPV}\left(\frac{1}{2}\right)$, it is necessary and sufficient that $\text{MPV}\left(\frac{1}{2}\right) \geq z$. Therefore, if we define

$$z_{1E} \equiv \text{MPV}\left(\tfrac{1}{2}\right)\Big|_{2\mu(0)=1-\pi,\mu\left(\frac{1}{2}\right)=\pi \text{ and } \alpha=1} = \frac{\Gamma''}{4}k(1-\pi)\left[1-k(1+\pi)\right], \tag{B.29}$$

we obtain:

Lemma 33 (Type-a equilibrium)**.** *There exists an equilibrium in which all bank stay if and only if $z \leq z_{1E}$.*

Clearly, since the measures of traders, $\mu(0)$ and $\mu(\frac{1}{2})$ are constant, they are continuous in z.

Type-b equilibrium: extreme-ω banks stay, and middle-ω banks partially exit. For this to be an equilibrium, we need that $\text{MPV}(0) \geq z$ and $\text{MPV}\left(\frac{1}{2}\right) = z$ when $2\mu(0) = 1-\pi$, $0 < \mu\left(\frac{1}{2}\right) < \pi$, and $\alpha = \rho + (1-\rho)\left[1-\pi+\mu\left(\frac{1}{2}\right)\right]$. Since $\text{MPV}(0) > \text{MPV}\left(\frac{1}{2}\right)$, a necessary and sufficient condition for a type-b equilibrium is that there exists a solution $0 < \mu\left(\frac{1}{2}\right) < \pi$ to the equation:

$$\text{MPV}\left(\tfrac{1}{2}\right) = \frac{\Gamma''}{4}k\alpha(1-n)\left[1-k\alpha(1+n)\right] = z \tag{B.30}$$

where

$$n = \frac{\mu\left(\frac{1}{2}\right)}{1-\pi+\mu\left(\frac{1}{2}\right)} \text{ and } \alpha = \rho + (1-\rho)\left[1-\pi+\mu\left(\tfrac{1}{2}\right)\right].$$

Solving the last two equations for α as a function of n gives $\alpha = \rho + (1-\rho)\frac{1-\pi}{1-n}$. Substituting back into the formula for $\text{MPV}\left(\frac{1}{2}\right)$ delivers:

$$\text{MPV}\left(\tfrac{1}{2}\right) = \frac{\Gamma''}{4}k\left[\rho(1-n)+(1-\rho)(1-\pi)\right]\left[1-k\left(\rho+(1-\rho)\frac{1-\pi}{1-n}\right)(1+n)\right],$$

One sees that $\text{MPV}\left(\frac{1}{2}\right)$ is a strictly decreasing function of n. Since n is itself a strictly increasing function of $\mu\left(\frac{1}{2}\right)$, this means that equation (B.30) has a solution $0 < \mu\left(\frac{1}{2}\right) < \pi$ if and only if it is strictly greater than z when $\mu\left(\frac{1}{2}\right) = 0$, and strictly smaller than z when $\mu\left(\frac{1}{2}\right) = \pi$. Precisely, if we let

$$z_{2E} = \text{MPV}\left(\tfrac{1}{2}\right)\Big|_{2\mu(0)=1-\pi,\mu\left(\frac{1}{2}\right)=0 \text{ and } \alpha=1-(1-\rho)\pi} = \frac{\Gamma''}{4}k\left[1-(1-\rho)\pi\right]\left\{1-k\left[1-(1-\rho)\pi\right]\right\}, \tag{B.31}$$

we obtain:

Lemma 34 (Type-b equilibrium)**.** *There exists an equilibrium in which extreme-ω banks stay, $2\mu(0) = 1-\pi$, and middle-ω partially exit, $0 < \mu\left(\frac{1}{2}\right) < \pi$, if and only if $z \in (z_{1E}, z_{2E})$.*

The measure $\mu(0)$ is constant and is obviously continuous. The measure $\mu\left(\frac{1}{2}\right)$ is also continuous because it is bounded and is the unique solution of an equation which is continuous in $\mu\left(\frac{1}{2}\right)$ and z.

Type-c equilibrium: extreme-ω banks stay and middle-ω banks exit. For this to be an equilibrium, we need that $\text{MPV}(0) \geq z$ and $\text{MPV}\left(\frac{1}{2}\right) \leq z$ when $2\mu(0) = 1-\pi$, $\mu\left(\frac{1}{2}\right) = 0$, and $\alpha = 1-(1-\rho)\pi$. Define, then

$$z_{3E} \equiv \text{MPV}(0)\big|_{2\mu(0)=1-\pi,\mu\left(\frac{1}{2}\right)=0,\text{ and }\alpha=1-(1-\rho)\pi} = \frac{\Gamma''}{4}k\left[1-(1-\rho)\pi\right]\left\{1-\frac{k}{2}\left[1-(1-\rho)\pi\right]\right\}. \tag{B.32}$$

Note that, since $\text{MPV}(0) > \text{MPV}\left(\frac{1}{2}\right)$, we have that $z_{3E} > z_{2E}$. We thus obtain:

Lemma 35 (Type-c equilibrium)**.** *There exists an equilibrium in which extreme-ω banks stay, $2\mu(0) = 1-\pi$, and middle-ω exit, $\mu\left(\frac{1}{2}\right) = 0$, if and only if $z \in [z_{2E}, z_{3E}]$.*

In a type-c equilibrium, the measures of traders are constant and so are obviously continuous in z.

Type-d equilibrium: extreme-ω banks partially exit and middle-ω banks exit. For this to be an equilibrium, we need that $\text{MPV}(0) = z$ and $\text{MPV}\left(\frac{1}{2}\right) \leq z$ when $0 < 2\mu(0) < 1-\pi$, $\mu(\frac{1}{2}) = 0$, and $\alpha = \rho + (1-\rho)2\mu(0)$. To characterize the range of z such that a type-d equilibrium exists, we first note that in this case $n = 0$ and so

$$\text{MPV}(0) = \frac{\Gamma''}{4}k\alpha\left[1-\frac{k\alpha}{2}\right].$$

This function is strictly increasing in α, and $\alpha \in [\rho, 1-(1-\rho)\pi]$ as $\mu(0)$ increases from 0 to $1-\pi$. Thus, if we let

$$z_{4E} = \frac{\Gamma''}{4}k\rho\left[1-\frac{k\rho}{2}\right],$$

we obtain:

Lemma 36 (Type-d equilibrium)**.** *There exists an equilibrium in which extreme-ω banks partially exit, and middle-ω banks exit, $0 < 2\mu(0) < 1-\pi$ and $\mu\left(\frac{1}{2}\right) = 0$ if and only if $z \in [z_{4E}, z_{3E})$.*

One sees that when $z \in [z_{4E}, z_{3E})$, there are multiple equilibria with positive entry: there is a type-d equilibrium, and either a type-a, type-b, or type-c equilibrium.

Type-e equilibrium: extreme-ω and middle-ω banks exit. There is always such an equilibrium.

The equilibrium with highest participation. If $z < z_{4E}$ or if $z = z_{3E}$, then there is a unique equilibrium with positive participation. If $z \in (z_{4E}, z_{3E})$, then there are two equilibria with positive participation: a type-d equilibrium and some other equilibrium which can be of type-a, type-b, or type-c. In all cases, this other equilibrium features strictly more participation, since $2\mu(0) = 1-\pi$. If $z > z_{3E}$, then there is no equilibrium with positive participation.

Welfare in the equilibrium with highest participation. Suppose that $z \in [z_{4E}, z_{3E})$ and that a type-d equilibrium exists. Then, in this equilibrium, $\text{MPV}(0) \leq z$ and $\text{MPV}\left(\frac{1}{2}\right) < z$. At the same time, we have shown that there exists another equilibrium which is either of type-a, type-b, or type-c. Given $z \in [z_{4E}, z_{3E})$, we know that in this other equilibrium, $\text{MPV}(0) > z$. At the same time, since participation decisions are optimal, $\mu\left(\frac{1}{2}\right)\left[\text{MPV}\left(\frac{1}{2}\right) - z\right] \geq 0$. Taken together, this implies that, whenever a type-d equilibrium exists, it is welfare dominated by the other equilibrium with positive participation.

B.10 Proof of Lemma 25

Using results of Section B.9.3 above, we know that the marginal private values are:

$$\begin{aligned}\mathrm{MPV}(0) &= \frac{\Gamma''}{4}k\alpha\left[1-\frac{k\alpha}{2}(1-n^2)\right]\\ \mathrm{MPV}(\tfrac{1}{2}) &= \frac{\Gamma''}{4}k\alpha\,[1-n]\,[1-k\alpha(1+n)]\end{aligned} \tag{B.33}$$

We also know that

$$\mathrm{MSV}(\omega)=\mathrm{MPV}(\omega)+\frac{\alpha}{2}F(\omega)-\frac{\rho}{2}\bar{F}.$$

Using the formula for the frictional surpluses derived before Lemma 24, and using that $g(0)=\frac{k\alpha(1+n)}{2}$, we obtain:

$$\begin{aligned}F(0) &= \frac{\Gamma''k}{2}\,[1-k\alpha(1+n)]\\ F(\tfrac{1}{2}) &= \frac{\Gamma''k}{2}(1-n)\,[1-k\alpha(1+n)]=\frac{2}{\alpha}\mathrm{MPV}(\tfrac{1}{2})\\ \bar{F} &= \frac{\Gamma''k}{2}(1-n^2)\,[1-k\alpha(1+n)]\,.\end{aligned}$$

Plugging back into the expressions for MSV(0) and $\mathrm{MSV}(\frac{1}{2})$ above, we obtain after some algebra that:

$$\begin{aligned}\mathrm{MSV}(0) &= \frac{\Gamma''k}{4}\left\{\alpha\left[1-\frac{k\alpha}{2}(1-n^2)\right]+\alpha\,[1-k\alpha(1+n)]-\rho(1-n^2)\,[1-k\alpha(1+n)]\right\}\\ \mathrm{MSV}(\tfrac{1}{2}) &= \frac{\Gamma''k}{4}\,[1-k\alpha(1+n)]\,[1-n]\,[2\alpha-\rho(1+n)]\end{aligned} \tag{B.34}$$

B.11 Proof of Proposition 30

For this we let:

$$\begin{aligned}z_1^\star &= \mathrm{MSV}\left(\tfrac{1}{2}\right)\big|_{2\mu(0)=1-\pi\text{ and }\mu(1/2)=\pi}\\ z_2^\star &= \mathrm{MSV}\left(\tfrac{1}{2}\right)\big|_{2\mu(0)=1-\pi\text{ and }\mu(1/2)=0}\\ z^\star &= \inf\{z\geq 0\,:\,\hat{W}(\tfrac{1}{2})\leq 0\}.\end{aligned}$$

Note that $z^\star$ is uniquely defined and strictly positive, since $\hat{W}(\frac{1}{2})$ is continuous in z (by the Theorem of the Maximum), strictly decreasing in z, and strictly negative when $z\to\infty$. Then, if we let

$$z_{1P}=\min\{z_1^\star,z^\star\},z_{2P}=\min\{z_2^\star,z^\star\}\text{ and }z_{3P}=z^\star,$$

the result follows. Because the first-order condition of the first component planning problem is continuous in z and $\mu(\frac{1}{2})$, it follows that the solution of the planning problem is continuously decreasing in z as long as $z < z^\star$. All measures drop to zero when $z > z^\star$. In particular, $\mu(0)$ drops discontinuously from $\frac{1-\pi}{2}$ to zero.

B.12 Proof of Proposition 31

A preliminary result. We start with the following:

Lemma 37. *Suppose that* $2\mu(0) = 1 - \pi$. *Then:*

- *if* $\rho(1+\pi) < 1$, $MSV(\frac{1}{2}) > MPV(\frac{1}{2})$ *for all* $\mu(\frac{1}{2}) \in [0, \pi]$;
- *if* $\rho(1+\pi) = 1$, *then* $MSV(\frac{1}{2}) > MPV(\frac{1}{2})$ *for all* $\mu(\frac{1}{2}) \in [0, \pi)$, *and* $MSV(\frac{1}{2}) = MPV(\frac{1}{2})$ *for* $\mu(\frac{1}{2}) = \pi$;
- *if* $\rho(1+\pi) > 1$, *then there is some* $\mu^\star \in (0, \pi)$ *such that* $MSV(\frac{1}{2}) > MPV(\frac{1}{2})$ *for all* $\mu(\frac{1}{2}) \in [0, \mu^\star)$, *and* $MSV(\frac{1}{2}) < MPV(\frac{1}{2})$ *for all* $\mu(\frac{1}{2}) \in (\mu^\star, \pi]$.

To prove this Lemma, we note that, by combining equations (B.33) and (B.34), it follows that $\text{MSV}(\frac{1}{2}) > \text{MPV}(\frac{1}{2})$ if and only if

$$\alpha - \rho(1+n) > 0. \tag{B.35}$$

When $2\mu(0) = 1 - \pi$, we have $\alpha = \rho + (1-\rho)\left[1 - \pi + \mu(\frac{1}{2})\right]$, and $n = \frac{\mu(1/2)}{1-\pi+\mu(1/2)}$. This implies that

$$\alpha = \rho + \frac{(1-\rho)(1-\pi)}{1-n}.$$

Plugging this back into (B.35), we obtain that $\text{MSV}(\frac{1}{2}) > \text{MPV}(\frac{1}{2})$ if and only if

$$H(n) > 0, \text{ where } H(n) = \frac{(1-\rho)(1-\pi)}{1-n} - \rho n.$$

We note that $H(n)$ is a strictly convex function, equal to infinity both at $n = -\infty$ and at $n = 1$, and that $H(n)$ achieves its minimum over $(-\infty, 1)$ at $\underline{n} \in (-\infty, 1)$ solving:

$$\rho(1-\underline{n}) = \frac{(1-\rho)(1-\pi)}{1-\underline{n}} \Leftrightarrow H(\underline{n}) = \rho(1-2\underline{n}).$$

Now consider the first two bullet points of the Lemma, i.e., $\rho(1+\pi) \leq 1$. Then $H(\pi) \geq 0$, with an equality if and only if $\rho(1+\pi) = 1$. If $\underline{n} \leq \pi$, then since $n \leq \pi \leq \frac{1}{3}$, we have from the above that $H(\underline{n}) > 0$, implying that $H(n) > 0$ for all $n \in [0, \pi]$. If $\underline{n} > \pi$, then by strict convexity $H'(n) < 0$ for all $n \in [0, \pi]$. Together with the fact that $H(\pi) \geq 0$, this implies that $H(n) > 0$ for all $n \in [0, \pi)$, and $H(\pi) \geq 0$ with an equality if and only if $\rho(1+\pi) = 1$.

Next, consider the third bullet point of the Lemma, i.e., $\rho(1+\pi) > 1$. Then, given our maintained assumption that $\rho < 1$ and given that $\pi \leq \frac{1}{3}$, we have $H(0) > 0$. Given $\rho(1+\pi) > 1$, we have $H(\pi) < 0$. Since $H(n)$ is strictly convex and goes to infinity at $n = -\infty$ and $n = 1$, it follows that there exists a unique $n^\star \in (0, \pi)$ such that $H(n) > 0$ for $n \in [0, n^\star)$, and $H(n) < 0$ for $n \in (n^\star, \pi]$. The result follows.

Proof of the main result, when $\rho(1+\pi) \leq 1$. In that case, we let:

$$z_1 = z_2 = z_{1E}, \text{ and } z_3 = z_{2P}.$$

- First, by construction, $z_{1E} > 0$.

- Second, we argue that $z_{1E} < z_{2P}$. At $z = z_{1E}$, $\text{MPV}(\frac{1}{2}) = z_{1E}$ when evaluated at $2\mu(0) = 1-\pi$ and $\mu(\frac{1}{2}) = \pi$. Therefore, $\text{MPV}(0) > \text{MPV}(\frac{1}{2}) = z_{1E}$ when evaluated at $2\mu(0) = 1 - \pi$ and $\mu(\frac{1}{2}) = \pi$, implying that social welfare in equilibrium is strictly positive. By implication, social welfare obtained by maximizing over $\mu(\frac{1}{2})$, holding $2\mu(0)$ constant equal to $1 - \pi$ must also be strictly positive, that is, $\hat{W}(\frac{1}{2}) > 0$. This means that $z_{1E} < z^\star$, as defined in the proof of Proposition 30. Moreover, by Lemma 37, $\text{MSV}(\frac{1}{2}) \geq \text{MPV}(\frac{1}{2}) = z_{1E}$ when evaluated at $2\mu(0) = 1 - \pi$ and $\mu(\frac{1}{2}) = \pi$. This implies that $z_{1E} \leq z_1^\star < z_2^\star$, as defined in the proof of Proposition 30. Since $z_3 = \min\{z_2^\star, z^\star\}$, the result follows.

- Third, we argue that, when $z < z_1$, $\mu_E(\frac{1}{2}) = \mu_P(\frac{1}{2}) = \pi$. To see this we note that, when $z < z_1$, $\text{MPV}(0) > \text{MPV}(\frac{1}{2}) > z$ when evaluated at $2\mu(0) = 1 - \pi$ and $\mu(\frac{1}{2}) = \pi$. Therefore, by the same reasoning as in the previous bullet point, $\hat{W}(\frac{1}{2}) > 0$, so that the solution of the planning problem coincides with that of the first component planning problem. Since $\text{MSV}(\frac{1}{2}) \geq \text{MPV}(\frac{1}{2})$ when evaluated at $2\mu(0) = 1 - \pi$ and $\mu(\frac{1}{2}) = \pi$, it follows that the solution of this first component planning problem is to choose $\mu_P(\frac{1}{2}) = \pi$.

- Fourth, we argue that, when $z \in (z_1, z_3)$, then $\mu_P(\frac{1}{2}) > \mu_E(\frac{1}{2})$. If $\mu_E(\frac{1}{2}) = 0$, this is obvious because, since $z < z_{2P}$, we know that $\mu_P(\frac{1}{2}) > 0$. If $\mu_E(\frac{1}{2}) > 0$, then we have that $\text{MPV}(0) > \text{MPV}(\frac{1}{2}) = z$ when evaluated at $2\mu(0) = 1 - \pi$ and $\mu(\frac{1}{2}) = \mu_E(\frac{1}{2})$. Therefore, $\hat{W}(\frac{1}{2}) > 0$ and the planning problem coincide with the solution of the first component planning problem. Since $\text{MSV}(\frac{1}{2}) > \text{MPV}(\frac{1}{2})$, it thus follows that $\mu_P(\frac{1}{2}) > \mu_E(\frac{1}{2})$.

- Fifth, we argue that, when $z > z_3$, then $\mu_P(\frac{1}{2}) = \mu_E(\frac{1}{2}) = 0$. By our choice of z_3, we have that $\mu_P(\frac{1}{2}) = 0$. Suppose, towards a contradiction, that $\mu_E(\frac{1}{2}) > 0$. Then, by the same reasoning as before, we would have that $\hat{W}(\frac{1}{2}) > 0$, so that $z \in (z_2^\star, z^\star)$. But $z > z_2^\star$ implies that $\text{MSV}(\frac{1}{2}) < z$ when evaluated at $2\mu(0) = 1 - \pi$ and at any $\mu(\frac{1}{2}) \in [0, \pi]$. In particular, $\text{MPV}(\frac{1}{2}) \leq \text{MSV}(\frac{1}{2}) < z$ when evaluated at $2\mu(0) = 1 - \pi$ and $\mu(\frac{1}{2}) = \mu_E(\frac{1}{2})$, which is a contradiction.

Proof of the main result, when $\rho(1 + \pi) > 1$. In that case we let:

$$\begin{aligned}
z_1 &= z_{1P}, \\
z_2 &= \text{MPV}(\tfrac{1}{2})\big|_{2\mu(0)=1-\pi \text{ and } \mu(1/2)=\mu^\star} = \text{MSV}(\tfrac{1}{2})\big|_{2\mu(0)=1-\pi \text{ and } \mu(1/2)=\mu^\star} \\
z_3 &= z_{2P}.
\end{aligned}$$

- First, we argue that when $z < z_1$, $\mu_E(\frac{1}{2}) = \mu_P(\frac{1}{2}) = \pi$. By construction, $\text{MSV}(\frac{1}{2}) > z$ when evaluated at $2\mu_P(0) = 1 - \pi$ and $\mu_P(\frac{1}{2}) = \pi$. Since $\pi > \mu^\star$, it follows from Lemma 37 that $\text{MPV}(0) > \text{MPV}(\frac{1}{2}) > \text{MSV}(\frac{1}{2}) > z$ when evaluated at the same point. Thus, $2\mu_E(0) = 1-\pi$ and $\mu_E(\frac{1}{2}) = \pi$.

- Second, we argue that $z_1 < z_2$. This follows directly because $\text{MSV}(\frac{1}{2})$ is a strictly decreasing function of $\mu(\frac{1}{2})$ when $2\mu(0)$ is fixed at $1 - \pi$.

- Third, we argue that, when $z \in (z_1, z_2)$, we have $\mu_E(\frac{1}{2}) > \mu_P(\frac{1}{2})$. Since $z \leq z_2$, the equation

$$\text{MPV}(\tfrac{1}{2})\big|_{2\mu(0)=1-\pi, \mu(\frac{1}{2})} \geq z, \text{ with “=” if } \mu(1/2) < \pi,$$

has a unique solution $\mu(\frac{1}{2}) < \mu^\star$. This implies in particular that $\hat{W}(\frac{1}{2}) > 0$, so that $2\mu_P(0) = 1 - \pi$ and $z < z^\star$. By Lemma 37, it follows that $\text{MPV}(\frac{1}{2}) > \text{MSV}(\frac{1}{2})$ when evaluated at $2\mu(0) = 1 - \pi$ and $\mu(\frac{1}{2}) = \mu_E(\frac{1}{2})$. Together with the fact that $\mu_P(\frac{1}{2}) < \pi$, since $z_1 < z < z^\star$, this implies that $\mu_P(\frac{1}{2}) < \mu_E(\frac{1}{2})$.

- Fourth, we argue that $z_3 > z_2$. At $z = z_2$, $2\mu_E(0) = 1 - \pi$ and $\mu_E(\frac{1}{2}) = \mu^\star \in (0, \pi)$, which implies that $\hat{W}(\frac{1}{2}) > 0$, hence $z_2 < z^\star$. Since $\text{MSV}(\frac{1}{2}) = \text{MPV}(\frac{1}{2}) = z_2$ when evaluated at $2\mu(0) = 1 - \pi$ and $\mu(\frac{1}{2}) = \mu^\star$, this also implies that $z_2 < z_2^\star$. Thus, $z_2 < \min\{z^\star, z_2^\star\} = z_{2P} = z_3$.

- Fifth, we argue that, when $z \in (z_2, z_3)$, then $\mu_E(\frac{1}{2}) < \mu_P(\frac{1}{2})$. Indeed, in this case, the solution of the planning problem has $2\mu_P(0) = 1 - \pi$ and $0 < \mu_P(\frac{1}{2}) < \mu^\star$. Since $\mu_P(\frac{1}{2}) < \mu^\star$, it follows from Lemma 37 that $z = \text{MSV}(\frac{1}{2}) > \text{MPV}(\frac{1}{2})$ when evaluated at $2\mu(0) = 1 - \pi$ and $\mu(\frac{1}{2}) = \mu_P(\frac{1}{2})$. Therefore, in equilibrium, $\mu_E(\frac{1}{2}) < \mu_P(\frac{1}{2})$.

- Sixth, we argue that, when $z > z_3$, then $\mu_P(\frac{1}{2}) = \mu_E(\frac{1}{2}) = 0$. Since $z > z_3 = z_{2P}$, it follows that $\mu_P(\frac{1}{2}) = 0$. Towards a contradiction, assume that $\mu_E(\frac{1}{2}) > 0$. Then, since $z > z_2$, it must be the case that $\mu_E(\frac{1}{2}) < \mu^\star$. Moreover, since $\mu_E(\frac{1}{2}) > 0$, it follows that $\hat{W}(\frac{1}{2}) > 0$, so that $z < z^\star$ and $z > z_2^\star$. But this means that $\text{MSV}(\frac{1}{2}) \leq z$ when evaluated at the optimum $2\mu_P(0) = 1 - \pi$ and $\mu_P(\frac{1}{2}) = 0$. By implication, $\text{MSV}(\frac{1}{2}) < z$ when evaluated at $2\mu(0) = 1 - \pi$ and any $\mu(\frac{1}{2}) > 0$, in particular at $\mu(\frac{1}{2}) = \mu_E(\frac{1}{2})$. Since $\mu_E(\frac{1}{2}) < \mu^\star$, we obtain by an application of Lemma 37 that $\text{MPV}(\frac{1}{2}) < \text{MSV}(\frac{1}{2}) < z$ when evaluated at $2\mu_E(0) = 1 - \pi$ and $\mu_E(\frac{1}{2})$, which is a contradiction.

C Some auxiliary mathematical results

C.1 Some results about increasing functions

Lemma 38. *Consider an increasing and right-continuous function $\Phi(z)$. Given any two sequences y_k and z_k converging to y and z, respectively, if $y_k \leq \Phi(z_k)$ for all k, then $y \leq \Phi(z)$.*

Proof. Let $\bar{z}_k = \sup_{\ell \geq k} z_\ell$. Then $z_k \leq \bar{z}_k$, and since $\Phi(z)$ is increasing it follows that $y_k \leq \Phi(\bar{z}_k)$. Moreover, since $\bar{z}_k$ decreases and converges to z, and since $\Phi(z)$ is right-continuous, we can go to the limit and we obtain $y \leq \Phi(z)$. □

The same argument applies to left-continuous and increasing functions:

Lemma 39. *Consider an increasing and left-continuous function $\Psi(z)$. Given any two sequences y_k and z_k converging to y and z, respectively, if $y_k \geq \Psi(z_k)$ for all k, then $y \geq \Psi(z)$.*

Proof. Let $\underline{z}_k = \inf_{\ell \geq k} z_\ell$. Then $z_k \geq \underline{z}_k$, and since $\Psi(z)$ is increasing it follows that $y_k \geq \Psi(\underline{z}_k)$. Moreover, since $\underline{z}_k$ increases and and converges to z, and since $\Psi(z)$ is left-continuous, we can go to the limit to obtain $y \geq \Psi(z)$. □

Another perhaps obvious result is:

Lemma 40. *Suppose that $\Phi(z)$ is increasing and right-continuous. Then, the function $z \mapsto \Phi(z^-) \equiv \lim_{y \to z, y<z} \Phi(y)$ is increasing and left-continuous.*

Proof. Take $z_1 < z_2 < z_3$. Then $\Phi(z_1^-) \leq \Phi(z_1) \leq \Phi(z_2) \leq \Phi(z_3^-)$, and so it follows that $z \mapsto \Phi(z^-)$ is increasing. For left continuity, note that by definition of $\Phi(z^-)$, and keeping in mind that $\Phi(z)$ is increasing, we have that for all $\varepsilon > 0$, there exists $\eta > 0$ such that $z - \eta < z_1 < z_2 < z$ implies that $\Phi(z^-) - \varepsilon \leq \Phi(z_1) \leq \Phi(z_2) \leq \Phi(z^-)$. Taking the limit $z_1 \to z_2$, we obtain that $\Phi(z^-) - \varepsilon \leq \Phi(z_2^-) \leq \Phi(z^-)$ for all $z - \eta < z_2 < z$, and we are done. □

C.2 Some results about the quantile function

Given any cdf $\Phi(z)$ over the support $[\underline{z}, \bar{z}]$, we can define the quantile function:

$$\psi(q) = \inf\left\{z \in [\underline{z}, \bar{z}] \,:\, \Phi(z) \geq q\right\}.$$

Lemma 41. *The quantile function, $\psi(q)$, is increasing, left continuous, and it satisfies $\psi(0) = \underline{z}$ and $\psi(1) = \overline{z}$, $\psi \circ \Phi(z) \leq z$ and $\Phi \circ \psi(q) \geq q$. Moreover:*

$$\Phi(z) = \sup\left\{q \in [0,1] \,:\, \psi(q) \leq z\right\}.$$

Proof. Let $Z(q) = \{z \in [\underline{z}, \bar{z}] : \Phi(z) \geq q\}$. Then for $q' \geq q$, $Z(q') \subseteq Z(q)$, which implies after taking the infimum that $\psi(q') \geq \psi(q)$. To show left-continuity, assumed toward a contradiction that there exists a sequence $q_n < q$ such that $\lim_{n \to \infty} q_n = q$ and $\psi(q_n) \leq \psi(q) - \varepsilon$ for some $\varepsilon > 0$. Then, by definition of $\psi(q)$, for each q_n we can find some z_n such that $\Phi(z_n) \geq q_n$, $z_n \leq \psi(q_n) + \frac{\varepsilon}{2}$, which implies that $z_n \leq \psi(q) - \frac{\varepsilon}{2}$ by our maintained assumption. Now let $\bar{z}_n = \sup_{k \geq n} z_k$. Since $\bar{z}_n \geq z_n$ and since $\Phi(z)$ is increasing, we have that $\Phi(\bar{z}_n) \geq q_n$. Moreover, since $\bar{z}_n$ is decreasing and converges to $\limsup z_n$, and since $\Phi(z)$ is right-continuous, we have that $\lim \Phi(\bar{z}_n) = \Phi\left(\limsup z_n\right) \geq q$. At the same time, by our choice of z_n, we have that $\limsup z_n \leq \psi(q) - \frac{\varepsilon}{2}$, which is a contradiction.

The property that $\psi \circ \Phi(z) \leq z$ follows from the observation that, if $q = \Phi(z)$ then $z \in Z\,[\Phi(z)]$. To show that $\Phi \circ \psi(q) \geq q$, note that for all $z \geq \psi(q)$, $\Phi(z) \geq q$. Letting $z \to \psi(q)$ and using the right-continuity of $\Phi(z)$, we obtain that $\Phi \circ \psi(q) \geq q$.

For the last property, let us define, for $z \in [\underline{z}, \overline{z}]$:

$$\hat{\Phi}(z) \equiv \sup \{q \in [0,1] \,:\, \psi(q) \leq z\}\,.$$

Using symmetric arguments as above, one can show that $\hat{\Phi}(z)$ is increasing, right-continuous, satisfies $\hat{\Phi}(\underline{z}) = 0$ and $\hat{\Phi}(\overline{z}) = 1$. In addition, $\hat{\Phi} \circ \psi(q) \geq q$ and $\psi \circ \hat{\Phi}(z) \leq z$. Our objective now is to establish that $\hat{\Phi}(z) = \Phi(z)$.

First note that, since $\Phi(z)$ is increasing, $\psi \circ \hat{\Phi}(z) \leq z$ implies that $\Phi \circ \psi \circ \hat{\Phi}(z) \leq \Phi(z)$. But since we have shown that $\Phi \circ \psi(q) \geq q$, this implies that $\hat{\Phi}(z) \leq \Phi(z)$. Second, note that, since $\hat{\Phi}(z)$ is increasing, $\psi \circ \Phi(z) \leq z$ implies that $\hat{\Phi} \circ \psi \circ \Phi(z) \leq \hat{\Phi}(z)$. But we also have that $\hat{\Phi} \circ \psi(q) \geq q$, which implies in turn that $\Phi(z) \leq \hat{\Phi}(z)$. $\square$