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SOME BASIC IDEAS IN STOCHASTIC STABILITY

BY H. J. KUSHNER*

This paper provides several definitions of stability for stochastic systems. It also suggests techniques for ascertaining the stability of a stochastic system.

1. INTRODUCTION

Control theory and systems analysis is deeply concerned with problems of a stability nature, yet it is not a well-defined subject. It is concerned with a broad family of qualitative properties of stochastic and deterministic dynamical systems; asymptotic properties of the paths and their dependence on parameters; and, whether certain properties hold under perturbation of parameters, input, initial conditions, or structure of the system itself. Some of the basic ideas for the discrete time stochastic problem will be briefly discussed in this paper. The methods yield much information of value, but constitute only one tool or point of view among many which must be brought to bear in the analysis of stochastic systems.

Suppose that the discrete system

$$p_{n+1} = f(p_n, \xi_n) \tag{1}$$

represents a price adjustment mechanism, where $\{\xi_n\}$ is some sequence of random variables. Suppose that, if $\xi_n \equiv 0$, then $p_n \to p$, some stable price, independent of p_0. What information would we like to have when $\xi_n \not\equiv 0$? When can $p_n \to p$? Even if $x_n \to p$ in some sense, there are many statistical senses in which it can occur (with probability one, in probability, in r-th moment, etc.). Furthermore the random perturbation ξ_n (or its effects) would have to be proportional in some sense to $(p_n - p)$; i.e., the random effects would have to degenerate as p is approached. This would not be a very common situation. However, among the above choices for convergence the "with probability one" (w.p.1) convergence is the easiest to treat, and probably also yields the most information. The w.p.1 convergence is a property of the *path*, the others are properties of the distributions. As will soon be seen there are very many stability properties of (1) (or senses in which p_n converges) which we could try to investigate, and which would frequently be more appropriate than w.p.1 (or similar) convergence. Yet, the current status of the pertinent techniques and results is unsatisfactory.

Consider the following specific example.

Let a_s, a_d, b_s, b_d denote positive constants with $b_d > b_s$, $a_d > a_s$. Let supply be given by $s_n = d_s + b_s p_{n-1} - \xi_n$, demand by $d_n = a_d - b_d p_n$, subject to all $p_n \geq 0$. Under the "clearing" assumption, $s_n = d_n$ and

$$p_n = \max\left[0, \frac{(a_d - a_s)}{b_d} - \frac{b_s}{b_d} p_{n-1} + \frac{\xi_n}{b_d}\right]. \tag{2}$$

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With $b_d > b_s$, there is a stable price p. It is conceivable, but not likely, that ξ_n would be (in some sense) proportional to $(p_{n-1} - p)$. If the ξ_n were due to random variations in manufacturing efficiency, rejects, effects of other demands, etc., then one would expect that the paths $\{p_n\}$ would never settle down.

What notions of stability are then appropriate? We mentioned only a few of many possibilities.

(a) Bounded paths in probability;

$$\sup_n P\{|p_n| \geq N\} \to 0, \qquad N \to \infty.$$

(b) Bounded paths;

$$P\{\sup_n |p_n| \geq N\} \to 0, \qquad N \to \infty.$$

(c) Recurrence; the path (almost always) repeatedly returns to some bounded set.

(d) No finite escape time.

(e) Boundedness or convergence of some moment.

(f) Existence and uniqueness of an invariant measure u, and $u_n \to u$, for all u_0, where $u_n(A) = P_{u_0}\{p_n \in A\}$ = probability (given initial distribution u_0) that $p_n \in A$. u is invariant if $P_u\{p_1 \in A\} = u(A)$; the distribution maintains itself.

In (2), if $\{\xi_n\}$ is a sequence of independent identically distributed random variable with bounded variance, then all (a), (c)–(f) hold. In general, one would expect that ξ_n would depend on p_{n-1}. (a) and (b) speak for themselves. (c) is a type of stability—there is some bounded set so that wherever the paths go, they always ultimately return to that set. The property is often not hard to verify, and is required for (e)–(f). Property (f) is one of the more interesting, but is difficult to treat, and, even if verified, may not yield enough information, unless good estimates of other properties (moments, correlations with respect to u, etc.) are also obtained.

Next, in order to motivate some of the techniques, the deterministic case will be dealt with briefly, then some of the stochastic results will be presented for w.p.1 convergence, and some related properties. Then an example will be given, and, finally, we make some remarks concerning recurrence and invariance. A more thorough, but still elementary, treatment appears in [1], and more sophisticated treatments appear in [2], [3].

2. Deterministic (Discrete Parameter) Stability

Let

$$x_{n+1} = f(x_n) \tag{3}$$

represent an autonomous system, and $V(x)$ a non-negative function which tends to ∞ as $|x| \to \infty$. Suppose that

$$V(f(x)) - V(x) \equiv -k(x) \leq 0 \tag{4}$$

for some function $k(x) \geq 0$, and all x (such $V(x)$ are referred to as Liapunov functions). Then

(a) There is some $v \geq 0$ so that $V(x_n) \downarrow v$, $n \to \infty$.

(b) $0 \le V(x_n) = V(x_0) - \sum_{i=0}^{n-1} k(x_i)$ implies that $k(x_n) \to 0$ as $n \to \infty$.

(c) If $k(x)$ is continuous and $\{x_n\}$ bounded, then $x_n \to \{x : k(x) = 0\}$, $n \to \infty$.

Note that the purely *local* calculation gave us the *global* results (a)–(c). The dynamical property (3) was crucial to the local global implication.

Suppose that (4) only holds locally, say in a set including $Q_\lambda \equiv \{x : V(x) < \lambda\}$. Then (a)–(c) hold if $x_0 \in Q_\lambda$. Suppose that (4) holds for $x \notin Q_\lambda$, and $k(\cdot)$ is continuous. Then $x_n \to Q_\lambda$ as $n \to \infty$. Here Q_λ is an attracting set, and we have boundedness of the paths. We can draw no implication concerning the behavior of the paths in Q_λ, except that once in Q_λ they never leave Q_λ.

There are interesting stochastic analogs to all of these techniques and results.

3. Stochastic (Liapunov) Stability

Let $\{x_n\}$ denote a random process. To produce a stochastic analog to Section 2, we require that $\{x_n\}$ have a dynamical property; we suppose that it is a Markov process. Of course, it may be of interest to study the stability of only some components of x_n. Let E_x denote expectation with initial condition $x_0 = x$, and suppose that $\{x_n\}$ is a homogeneous (for convenience) Markov process.

Suppose that (analogous to (4))

$$E_x V(x_1) - V(x) \equiv -k(x) \le 0. \tag{5}$$

Then also $E_{x_n} V(x_{n+1}) - V(x_n) = -k(x_n) \le 0$. Define $V_n \equiv V(x_n)$. Many conclusions of interest can be drawn. We have (conditional expectation)

$$E[V_{n+1} | V_0, V_1, \ldots, V_n] \le V_n. \tag{6}$$

Such a sequence is called a supermartingale. It can be considered to represent the sequence of fortunes in an unfair game: Given the past history $V_0, \ldots, V_n$, the average value of the next fortune V_{n+1} is no greater than the current fortune V_n. Such processes are quite important in probability theory, and have been extensively studied (see e.g. [4]). We can assert

(a) (Martingale convergence theorem). There is a random variable $v \ge 0$ such that $V(x_n) \to v$ w.p.1.

(b) $0 \le E_x V(x_n) = V(x) - E_x \sum_0^{n-1} k(x_i)$ implies that $k(x_n) \to 0$ w.p.1, $n \to \infty$.

(c) If $k(x)$ is continuous and $\{x_n\}$ bounded, then $x_n \to \{x : k(x) = 0\}$, $n \to \infty$.

(d) $P_x\{\sup_{\infty > n \ge 0} V(x_n) \ge \lambda\} \le V(x)/\lambda$, for any $\lambda > 0$.

Thus, in the stochastic case also, the *local* estimate (5) yields *global* results. (b) is a consequence of (5) and the Borel–Cantelli lemma. (d) is a consequence of the fact that V_n is a non-negative supermartingale. Note that it gives us a bound on the path excursions. Suppose that (5) holds for $x \in Q_\lambda$. Then we can localize the result and obtain (a), (b) ($k(x_n) \to 0$), (c) for (almost) all paths which never leave Q_λ. (d) is still valid, and, hence, the paths never leave Q_λ with a probability at least $1 - V(x)/\lambda$. (a)–(c) are analogous to the results in the deterministic case. (d) is fundamentally stochastic: some paths may leave Q_λ, as opposed to the deterministic case. Indeed, if the right side of (d) were zero for all x, λ such that $V(x) < \lambda$, there would be no noise in the problem.

Next suppose that there is an $\varepsilon > 0$ so that $E_x V(x_1) - V(x) = -k(x) \le -\varepsilon$ for $x \notin Q_\lambda$. Then a modification of the above result yields that x_n always returns to Q_λ.

Indeed, the average return time is $\leq V(x)/\varepsilon$. Q_λ is a "recurrent" set. The recurrence property gives us a type of stability analogous to (loosely) boundedness of paths.

In a sense the results are the best possible. (d) can be an equality if all we know is that (5) holds. Good Liapunov functions are tailored to the system as much as possible.

4. An Elementary Example

The following example is taken from a sampled data problem in control theory, and I do not know what its analog in Economics would be. Yet it does illustrate some of the basic features of the stochastic Liapunov function approach. We have a first order system $\dot{x} = -ax - K\varepsilon$, where ε is a 'feedback' quantity. The output is sampled at moments $t_n, n = 0, 1, \ldots$, where $t_n = \sum_0^{n-1} \Delta_i$ where Δ_i are independent random variables; for $t_n \leq t < t_{n+1}$, let $\varepsilon(t) = x(t_n)$. Define $x_n \equiv x(t_n)$. Such systems occur frequently in automatic control, and its stability will be analyzed. We have

$$x_{n+1} = A_n x_n,$$
$$A_n = [(1 + K/a)\, e^{-a\Delta_n} - K/a].$$

Stability problems often arise in such systems owing to the delayed information that is used as an input. Let $V(x) = |x|^s$ for some $s > 0$. Then

$$E_{x_n} V(x_{n+1}) - V(x_n) = (E|A_n|^s - 1)|x_n|^s. \tag{7}$$

If $\sup_n E|A_n|^s < 1$ for some $s > 0$, then $x_n \to 0$ w.p.1. The larger is s, the better the bound (d), since

$$P_x\{\sup_{\infty > n \geq 0} |x_n| \geq \lambda\} = P_x\{\sup_{\infty > n \geq 0} |x_n|^s \geq \lambda^s\} \leq |x|^s/\lambda^s. \tag{8}$$

Eventually, as s increases, we usually (though not always) have that $E|A_n|^s > 1$. Thus powers of Liapunov functions are not necessarily Liapunov functions—this is related to the difficulty of obtaining w.p.1 *instability* results for stochastic problems. In a sense, above, the fastest growing Liapunov function gives the best path estimate (8). The $\{A_n\}$ or $\{\Delta_n\}$ need not be identically distributed. Also Δ_n can depend on x_n; say, smaller errors x_n at t_n giving a longer wait Δ_n on the average, and conversely for large errors (whose sampling takes place more frequently). Suppose the distribution of Δ_n depends on x_n (but not on n otherwise) and that $E_x|A_1|^s \geq 1$ for small x ($x \in$ some Q_λ), and $E_x|A_1|^s < 1$ for $x \notin Q_\lambda$. Then we have recurrence—a natural situation in many applications—but not asymptotic stability w.p.1.

A variety of related situations can be investigated, where control (feedback policies) or parameters vary.

5. An Ergodic Result

For some $b > 0$, $k(x) \geq 0$, let

$$E_x V(x_1) - V(x) = -k(x) + b,$$

where $k(x) \geq b + \varepsilon$ (for some $\varepsilon > 0$) outside of some set, say Q_λ. Then, of course, we have recurrence. Also (we can also use $\underline{\lim}$ or $\overline{\lim}$ for lim, if appropriate)

$$\lim_n \frac{1}{n} E_x \sum_{i=0}^{n-1} k(x_i) \leq b - \lim_n \left(\frac{E_x V(x_n)}{n} \right) \leq b. \tag{8}$$

If $E_x V(x_n)/n \to 0$, then the limit exists and is simply b, and we have a type of moment estimate. Unfortunately, to show that (if true) $E_x k(x_n) \to b$ is considerably more difficult. This question is connected with the subject of invariant measures.

6. Invariant Measures

If $\{x_n\}$ is a Markov chain with transition matrix P and $u_n(i) = P_{u_0}(x_n = i)$, then ($u_n$ is a row vector) $u_{n+1} = u_n P$ and the problem of when $u_n \to u$, such that $u = uP$, is completely solved. The situation is far more difficult if x_n can take values in some Euclidean space.

Problem 1

When is there are least one invariant measure? For practical purposes there is a rather complete solution, e.g., there is one if [5]

(a) For a function $g(x) \to \infty$ as $|x| \to \infty$, $E_{u_0} g(x_n)$ is uniformly bounded for *some* u_0.

(b) $\{u_n\}$ are weakly compact for some $u = u_0$; e.g., if $P_{u_0}\{|x_n| \geq N\} \to 0$ as $N \to \infty$, uniformly in n, for some u_0.

(c) There is a compact set A such that for

$$\frac{1}{n} \sum_1^n u_i \equiv u^n, \overline{\lim_n}\, u^n(A) > 0.$$

((c) is implied by our recurrence criterion in Section 6). The proofs are rather involved.

Problem 2

Uniqueness? [6]. The basic criterion is that the state space not contain any proper self-contained subset. A set B is self-contained if $P_x(x_1 \in B) = 1$ for all $x \in B$. However, the criterion is not always easy to verify.

Problem 3

Assuming existence and uniqueness of an invariant measure u, when does (and how) $u_n \to u$ for any initial measure u_0? There is a fairly general criterion due to Doob [7]. Let m and u denote probability measures. m is said to be absolutely continuous with respect to u if $m(A) > 0$ implies $u(A) > 0$. m is said to be singular with respect to u if $m(A) > 0$ implies $u(A) = 0$. Here m is concentrated on a u-null set. For any m and u, there is a unique decomposition $m = m^s + m^c$, where m^s and m^c are singular and continuous, resp., with respect to u.

Let u denote an invariant measure, and decompose u_n into $u_n^s + u_n^c$ with respect to u. If $u_n^s(A) \to 0$, each A, as $n \to \infty$, then $u_n \to u$. All one need do is verify that $u_n^s \to 0$, often not easily done.

There are, however, several cases where it can be readily verified (see also Doob [4]). Suppose that there is a transition density $p(x, n, y)$ so that for some n_0, $p(x, n, y) > 0$ each x, y, for $n \geq n_0$. Then

$$u(A) = \int_A dy \left[\int p(x, n, y) u(dx) \right]$$

and u has a density which is *nowhere zero*. Similarly u_n $(n > 0)$ has a density and $u_n^s \equiv 0$. One must still prove that such a density $p(x, n, y)$ exists.

The requirement that such a density exists is restrictive. One can refine the ideas somewhat, but (as for nonlinear controllability, and for some similar reasons), much work needs to be done before a satisfactory understanding will be available.

Even if one can prove the desired convergence, the information will often be of limited value. Suppose that $x_{n+1} = f(x_n, \xi_n)$ represents a price adjustment mechanism where $\{\xi_n\}$ are independent and identically distributed. With $\xi_n \equiv 0$, let $\{x_n\}$ be stable in the large in the sense that there is a bounded set which, asymptotically, contains all paths but otherwise we let the system have any behavior at all; e.g., there can be many limit cycles (stable and unstable), etc. Suppose that, with ξ_n reput back into the dynamics, there is a transition density $p(x, n, y)$ of the type above, and the process is recurrent. Then there is a unique invariant measure and $u_n \to u$ for any u_0. Thus the noise has wiped out all the detail of the deterministic behavior. The convergence result gives us little information on the path behavior, correlation of functions, etc., unless both u and $p(x, 1, y)$ are available. So even establishing this type of convergence is only a first step in the analysis of the process. Indeed, important as the above mentioned stability concepts are, it is only one approach to the analysis of stochastic systems. One can, and no doubt should, investigate criteria for various types of stochastic stability. Yet, in doing so, especially in applications where invariant measures are involved, it is important not to lose sight of the important questions concerning the path behavior which arise as soon as the stability question is settled.

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References

[1] H. J. Kushner, *Introduction to Stochastic Control Theory*, Holt, Rinehart and Winston, New York, 1971.

[2] H. J. Kushner, *Stochastic Stability and Control*, Academic Press, New York, 1967.

[3] H. J. Kushner, "Stochastic Stability," in *Stability of Stochastic Dynamical Systems*, vol. 294, Lecture Notes in Math., Springer, New York, 1972.

[4] J. L. Doob, *Stochastic Processes*, John Wiley & Sons, New York, 1953.

[5] V. F. Benes, "Finite Regular Invariant Measures for Feller Processes," *J. Appl. Prob.*, 5 (April), 1968.

[6] S. R. Foguel, "Existence of Invariant Measures for Markov Processes II," *Proc. Amer. Math. Soc.*, 17 (1966), pp. 387–389.

[7] J. L. Doob, "Asymptotic Properties of Markoff Transition Probabilities," *Trans. Amer. Math. Soc.*, 63 (1948), pp. 393–421.