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THE APPLICABILITY OF THE KALMAN FILTER IN THE DETERMINATION OF SYSTEMATIC PARAMETER VARIATION

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The basic optimality theorem for the Kalman filter is stated and generalized to account for a conditional mean varying systematically with respect to additional variates z. The relevance of the resulting "state" estimator is discussed in the context of determining systematic parameter variation in a linear regression model. The Kalman filter is seen to have essentially the same drawbacks as the moving-window technique discussed in an earlier study.

In the context of a regression model with time-varying parameters

(1) $$y(t) = x'(t)\beta(t) + \varepsilon(t),$$

an increasingly popular model for the parameter variation is the Kalman Filter which specifies that $\beta(t)$ changes state according to the dynamic transition equation

(2) $$\beta(t + 1) = \Phi\beta(t) + v(t).$$

THE KALMAN FILTER

The relevant theorem suggesting the use of the Kalman Filter is

Theorem 1:

Given (1) and (2) with

(3) $$\begin{aligned} Ev(t)v'(t) &= R_1 \\ Ev(t)\varepsilon'(t) &= 0 \\ E\varepsilon(t)\varepsilon'(t) &= R_2, \end{aligned}$$

v and ε multivariate normal,

then, the estimator of $\beta(t + 1)$ that minimizes

(4) $$Eg[a'(\beta(t + 1) - \hat{\beta}(t + 1))]$$

for any symmetric, nondecreasing function g, is given by the recursive system

$$\hat{\beta}(t + 1) = \Phi\hat{\beta}(t) + K(t)[y(t) - x'(t)\hat{\beta}(t)]$$

where

(5) $$K(t) = \Phi P(t)x'(t)[x(t)P(t)x'(t) + R_2]^{-1}$$

$$P(t + 1) = [\Phi - K(t)x(t)]P(t)\Phi' + R_1$$

and where Φ may be a function of time [1, Ch. 7].

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This model has been treated by Duncan [3], Sarris [10], [11], and Mehra [4], [5], [6], and even for Φ constant over time, it would seem to offer an efficient means of approximating complex parameter variation in many economic circumstances.

The Kalman Filter and Systematic Parameter Variation

In many other circumstances, however, it is reasonable to assume that at least some component of $\beta(t)$ varies systematically with respect to additional variates, z. Such systematic parameter variation is treated by Quandt [7], [8], [9] and by Belsley [2].

The above theorem may be generalized to take into account a systematic component of parameter variation in the following [1, p. 234]:

Theorem 2:

Given the system

$$y(t) = x'(t)\beta(t) + \varepsilon(t) \tag{6}$$

$$\beta(t+1) = \Phi\beta(t) + \Gamma w(t) + v(t)$$

$$Ev(t)v'(s) = \delta_{s,t}R_1 \tag{7}$$

$$Ev(t)\varepsilon'(s) = \delta_{s,t}R_{12}$$

$$E\varepsilon(t)\varepsilon'(s) = \delta_{s,t}R_2$$

where $\delta_{s,t}$ is the Kronecker delta, then the estimators of $\beta(t+1)$, optimal in the sense of (4), are determined from the following recursive system:

$$\hat{\beta}(t+1) = \Phi\hat{\beta}(t) + \Gamma w(t) + K(t)[y(t) - x'(t)\hat{\beta}(t)]$$

where

$$K(t) = [\Phi P(t)x'(t) + R_{12}][x(t)P(t)x'(t) + R_2]^{-1}$$

$$P(t+1) = \Phi P(t)\Phi' + R_1 - K(t)[R_2 + x(t)P(t)x'(t)]K'(t).$$

Linear Systematic Parameter Variation

It is of interest to note here that the preceding Theorem 2 covers the case of linear systematic parameter variation treated by Belsley in [2], but seems to offer no particular advantage over the moving-window technique suggested there.

In [2] Belsley treats model (1) with

$$\beta(t) = \Gamma z(t) + u(t) \tag{8}$$

where Γ is a matrix of parameters and the $z(t)$ are additional variates systematically determining $\beta(t)$.

The elements of Γ are estimated (and the relevance of the z's tested) by using an estimated time series of the $\beta(t)$'s obtained independently of the z's by a moving-window regression applied to (1) without regard to parameter variation. This technique is shown to have a bias that becomes smaller the more slowly the z variates move over time. [See 2, p. 491].

The Belsley model can be cast into the general Kalman framework of Theorem 2 if $u(t)$ is assumed to be distributed with independent increments, for then we may write from (8),

$$\beta(t+1) = \beta(t) + \Gamma\Delta z(t+1) + v(t). \tag{9}$$

where

$$v(t) = \Delta u(t+1)$$

$$Ev(s)v'(t) = \delta_{s,t}R_2.$$

The model (1) and (9) clearly fits into Theorem 2 with $\Phi = I$ and $w(t) = \Delta z(t+1)$.

It would seem that the general Kalman technique would offer an alternative means of estimating the $\beta(t)$ time series required in [2]. But the Kalman technique clearly requires knowledge of the z's from the outset, and hence is not appropriate for testing among alternative z's when the exact z variates are not known.

For slowly moving z series ($\Delta z(t)$ small), however, (9) is approximated by (2), and under these conditions the estimator of Theorem 1, which does not require knowledge of the z's, may offer a good means of determining a $\beta(t)$ series for testing alternative z's. This method should be compared to the moving-window method used in [2]. It is interesting to note that the "slowly moving z" requirement is exactly that needed to justify the moving-window technique as a good approximation.

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