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A TEST FOR SYSTEMATIC VARIATION IN REGRESSION COEFFICIENTS

BY DAVID A. BESLEY*

This paper offers a statistical test of the constancy of the parameters of a linear regression. The $F$ test is based on transformed residuals which result from OLS applied to the given equation under the null hypothesis of constancy.

SOME NOTATION

We consider the model

\[ y(t) = x'(t)\beta(t) + \epsilon(t) \]
\[ \beta(t) = \Gamma z(t) + \delta(t) \]

where

$x(t)$, $z(t)$ $K$ and $R$ vectors, respectively.

$\epsilon(t)$ spherically distributed with $E\epsilon\epsilon' = \sigma^2I$.

$\delta(t)$ independent over time with $E\delta\delta' = \sigma^2\Omega$.

(See preceding article for motivation.

In what follows we consider the special case $\sigma_1^2 = 0$, i.e., variation in $\beta(t)$ is systematic and non random. Hence, we may write

\[ y(t) = x(t)\Gamma z(t) + \epsilon(t) \]
\[ = [x(t) \otimes z(t)]\Lambda + \epsilon(t) \]

where

\[ \Lambda = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_p \end{bmatrix} \]

Let

\[ Y = [y(t)] \]
\[ X = \begin{bmatrix} x(1) \\ \vdots \\ x(T) \end{bmatrix} \]
\[ Z = \begin{bmatrix} z(1) \\ \vdots \\ z(T) \end{bmatrix} \]
\[ D = \begin{bmatrix} x(1) \otimes z(1) \\ \vdots \\ x(T) \otimes z(T) \end{bmatrix} \]

Then (2) becomes

\[ Y = D\Lambda + \epsilon \]

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and we note that we may write

\[ D = [I \quad I \quad \cdots \quad I] X \otimes I, \]

where \( I_i = \text{diag} Z_i \) and \( Z_i \) is the \( i \)th column of \( Z \).

Thus, (3) becomes

\[ Y = \sum_{i=1}^{n} I_i X_i + \epsilon. \]

**REMARKS**

Our purpose here is to determine a test of the null hypothesis that \( \beta(t) = \beta \)

i.e., is constant, for all \( t \). Clearly a regression could be run on (3) directly if the \( z \)'s were known, but alternative modeling tests would be cumbersome given the size of \( D'D^{-1} \) even for moderate \( K \) and \( R \).

In what follows a two-step test is determined that looks to be efficient and does not require inversion of \( D'D \). Alternative \( Z \) matrices may be compared with a minimum of computation. The first step is OLS of \( Y \) on \( X \) without regard to \( Z \). The second step consists of regressing a transformed set of residuals from step one on the similarly transformed \( z \)'s. \( H_0 \) may be tested with the results of the second regression.

**STEP ONE: OLS \( Y \) on \( X \)**

First regress \( Y \) on \( X \) to get

\[ b = (X'X)^{-1}X'Y \]

\[ = (X'X)^{-1}X'Z \alpha + (X'X)^{-1}X'e \]

\[ = (X'X)^{-1}X' \sum_{r} J_r Z_r + (X'X)^{-1}X'e \]

and

\[ c = Y - Xb = HY \]

\[ = H(\alpha + \epsilon) \]

\[ = [H J_1 X \cdots H J_n X] \alpha + He \]

\[ \equiv [T_1 \ldots T_n] \alpha + He \]

\[ \equiv \sum_{i=1}^{n} V_i' \alpha + He \]

where the \( V_i \) are the residual matrices from an auxiliary regression of \( J_r X \) on \( X \).
This regression need not be run in practice. The relevance of \( V_1 \) is seen from

\[ H \mathcal{J} X = \mathcal{J} X - X(X'X)^{-1}X'X = \mathcal{J} X - \lambda B_1 \equiv V_1, \]

where \( B_1 \) is the set of regression coefficients from \( \mathcal{J} X = XB_1 + V_1. \)

Thus we have

\[ \mathcal{C} = \Sigma V_1^2 + \mathcal{H} \mathcal{C}. \]

We recall that \( H \) is idempotent, has rank \( T - K \), and hence there exists an orthogonal \( C \) such that 

\[ C'HPC = \left[ \begin{array}{cc} I & 0 \\ 0 & 0 \end{array} \right] \equiv G. \]

Further we note \( HV_1 = V_1 \), \( r = 1 \ldots R \) and \( H \mathcal{C} = \mathcal{C} \). Hence, we may write

\[ C'HPC\mathcal{C} = CHPC\Sigma V_1^2 + CHPC\mathcal{C} \]

or

\[ GC\mathcal{C} = GC\Sigma V_1^2 + GC\mathcal{C}. \]

and, partitioning \( C = [C_1 C_2] \) so that the first \( T - K \) rows of (9) become

\[ C_1'HPC = C_1'HPC\Sigma V_1^2 + C_1'HPC\mathcal{C} \]

This last inequality comes from noting that \( V_1 = H \mathcal{J} X \), and hence \( C_1V_1 = C_1'HPC\Sigma V_1^2 + C_1'HPC\mathcal{C} \) and \( C_1V_1 = C_1'HPC\mathcal{C} \). We have also let \( C_1 = \eta \). We also note that \( \eta \) is spherically distributed, since \( E\eta = 0, V\eta = V^2 = E\Sigma C_1 \Sigma C_1 = \sigma^2 \Sigma^2 + \sigma^2 I_{T-K} \), due to the orthogonality of \( C \).

It is the transformed residuals \( f = C_1V_1 \) that we make use of in step two. The transformation \( C_1 \) comes from finding an orthogonal set of eigenvectors of \( H = I - X(X'X)^{-1}X' \), and hence \( f \) depends only on knowledge of \( X \) and \( Y \) and does not require knowledge of \( Z \).

**Step Two**

It is clear from (10) that the residuals from step one depend in a very involved way on the interrelation of \( X \) and \( Z \) through the terms \( \mathcal{J} X \). However, under the null hypothesis \( H_0: \beta^0 = \beta \), these terms disappear, and a simpler test is available.

Consider a mechanical regression of \( f \) on \( Z \) transformed by \( C_1 \) (which depends only on \( X \)):

\[ f = C_1'Z\delta + \phi. \]

\[ \text{In passing we note from (6) that} \quad \delta = \Sigma YX + \text{weighted sum of the } \gamma, \quad \text{and } V\delta = \sum YX + \text{weighted sum of the } \gamma. \]

Hence, \( \delta = \Sigma YX \), a weighted sum of the \( \gamma \), and \( V\delta = \sum YX + \text{weighted sum of the } \gamma. \)

This latter sum goes from \( r = 2 \) to \( R \) since, if \( Z_i \) the first col. of \( Z \) is a column vector of all ones, then \( \mathcal{J}_i = 1 \) and hence \( V_1 = \mathcal{J} X = \mathcal{J}_i X = \mathcal{J}_i X \). These residuals must necessarily be zero, since \( \mathcal{J}_i = 1 \) does the trick of minimizing the sum of squares. Hence, \( C_1V_1 = 0 = C_1'\mathcal{J} X = C_1'X. \)
OLS gives

\[(12) \quad d = (ZC_1C_1Z)^{-1}ZC_1f\]

and from (10)

\[
= (ZC_1C_1Z)^{-1}ZC_1\sum_{r=1}^{R} \beta_r X_{ir} + (ZC_1C_1Z)^{-1}Z C_1\epsilon
\]

where \(Q \equiv C_1C_1\).

Under the null hypothesis \(H_0: \beta(t) = \beta; \beta_r = 0\) for \(r = 2 \ldots R\), and hence the first term of (12) is 0. That is, under \(H_0:\)

\[(13) \quad d = (ZQZ)^{-1}ZQ\epsilon = (ZQZ)^{-1}Z C_1f.\]

In addition, from (10) we have under \(H_0\) that

\[(14) \quad f = C_1\epsilon.\]

Further, we note for future reference that \(Q\) is idempotent since \(QQ = C_1C_1C_1C' = C_1C_1 = \epsilon - \text{of rank } T - K.\)

Now consider the residuals \(g\) of this second step; using (13) and (14),

\[(15) \quad g = f - C_1Zd = C_1\epsilon - C_1Z(ZQZ)^{-1}ZQ\epsilon = C_1[I - Z(ZQZ)^{-1}Z]Q\epsilon = N\epsilon \quad \text{where we let } N = C_1[I - Z(ZQZ)^{-1}Z].\]

Now

\[
g^2 = \epsilon N N\epsilon
\]

\[
= \epsilon[I - Q(ZQZ)^{-1}Z]C_1C_1[I - Z(ZQZ)^{-1}Z]Q\epsilon
\]

\[
= \epsilon[Q - Q(ZQZ)^{-1}Z]Q[I - Q(ZQZ)^{-1}Z]Q\epsilon
\]

\[(16) \quad = \epsilon MM\epsilon \quad \text{where } M = Q - Q(ZQZ)^{-1}ZQ\]

since \(M\) is seen to be idempotent with \(\text{tr } M = T - K - R.\) And hence,

\[(17) \quad g^2 = \sigma^2\epsilon^2X_{1-k-R}
\]

From (13) we have

\[(18) \quad d = (ZQZ)^{-1}ZQ\epsilon = B\epsilon\]

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and

\[ BM = (Z'QZ)^{-1} Z' Q (Q - Q Z (Z'Q Z)^{-1} Z' Q) \]

\[ = (Z'QZ)^{-1} Z' Q - (Z'QZ)^{-1} Z' Q = 0. \]

Hence, the linear form (18) is distributed independently of the quadratic form (17) and the usual tests of significance on \( d \) may take place. Under \( H_0: E_d = 0 \), and hence a \( t \) value for a specific \( d \) at \( T - K - R \) degrees of freedom in excess of the test level rejects the null hypothesis.

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