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A TEST FOR SYSTEMATIC VARIATION IN REGRESSION COEFFICIENTS

BY DAVID A. BELSLEY*

This paper offers a statistical test of the constancy of the parameters of a linear regression. The F test is based on transformed residuals which result from OLS applied to the given equation under the null hypothesis of constancy.

SOME NOTATION

We consider the model

$$y(t) = x'(t)\beta(t) + \varepsilon(t) \tag{1}$$

$$\beta(t) = \Gamma z(t) + u(t)$$

where

$x(t), z(t)$ K and R vectors, respectively,
$\varepsilon(t)$ spherically distributed with $E\varepsilon\varepsilon' = \sigma^2 I$,
$u(t)$ independent over time with $Euu' = \sigma_u^2\Omega$.

(See preceding article for motivation.

In what follows we consider the special case $\sigma_u^2 = 0$, i.e., variation in $\beta(t)$ is systematic and non random. Hence, we may write

$$y(t) = x'(t)\Gamma z(t) + \varepsilon(t) \qquad \Gamma = [\gamma_1 \cdots \gamma_R] \tag{2}$$

$$= [x'(t) \otimes z'(t)]\Lambda + \varepsilon(t)$$

where

$$\Lambda = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_R \end{bmatrix}.$$

Let

$$Y = [y(t)], \qquad \underset{T \times K}{X = \begin{bmatrix} x'(1) \\ \vdots \\ x'(T) \end{bmatrix}}, \qquad \underset{T \times R}{Z = \begin{bmatrix} z'(1) \\ \vdots \\ z'(T) \end{bmatrix}} \qquad \underset{T \times KR}{D = \begin{bmatrix} x'(1) \otimes z'(1) \\ \vdots \\ x'(T) \otimes z'(T) \end{bmatrix}}.$$

Then (2) becomes

$$Y = D\Lambda + \varepsilon \tag{3}$$

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and we note that we may write

$$D = [\mathscr{Z}_1 \mathscr{Z}_2 \ldots \mathscr{Z}_R][X \otimes I], \tag{4}$$

where $\mathscr{Z}_r = \text{diag } Z_r$ and Z_r is the rth column of Z.

Thus, (3) becomes

$$Y = \sum_{r=1}^{R} \mathscr{Z}_r X \gamma_r + \varepsilon. \tag{5}$$

Remarks

Our purpose here is to determine a test of the null hypothesis that $\beta(t) = \beta$, i.e., is constant, for all t. Clearly a regression could be run on (3) directly if the z's were known, but alternative modeling tests would be cumbersome given the size of $(D'D)^{-1}$ even for moderate K and R.

In what follows a two-step test is determined that looks to be efficient and does not require inversion of $D'D$. Alternative Z matrices may be compared with a minimum of computation. The first step is OLS of Y on X without regard to Z. The second step consists of regressing a transformed set of residuals from step one on the similarly transformed z's. H_0 may be tested with the results of the second regression.

STEP ONE: OLS Y ON X

First regress Y on X to get

$$\begin{aligned} b &= (X'X)^{-1}X'Y \\ &= (X'X)^{-1}X'D\Lambda + (X'X)^{-1}X'\varepsilon \\ &= (X'X)^{-1}X' \sum_r \mathscr{Z}_r X \gamma_r + (X'X)^{-1}X'\varepsilon \end{aligned} \tag{6}$$

and

$$\begin{aligned} e &\equiv Y - Xb = HY \qquad (H = I - X(X'X)^{-1}X') \\ &= H(D\Lambda + \varepsilon) \\ &= [H\mathscr{Z}_1 X \ldots H\mathscr{Z}_R X]\Lambda + H\varepsilon \\ &\equiv [V_1 \ldots V_R]\Lambda + H\varepsilon \\ &= \sum_{r=1}^{R} V_r \gamma_r + H\varepsilon \end{aligned}$$

where the V_r are the residual matrices from an auxiliary regression of $\mathscr{Z}_r X$ on X.

This regression need not be run in practice. The relevance of V_r is seen from

$$H\mathscr{Z}_r X_r = \mathscr{Z}_r X - X(X'X)^{-1}X'\mathscr{Z}_r X = \mathscr{Z}_r X - XB_r \equiv V_r,$$

where B_r is the set of regression coefficients from $\mathscr{Z}_r X = XB_r + V_r$.[1]

Thus we have

$$e = \Sigma V_r \gamma_r + H\varepsilon. \tag{8}$$

We recall that H is idempotent, has rank $T - K$, and hence there exists an orthogonal C such that $C'HC = \begin{bmatrix} I_{T-K} & 0 \\ 0 & 0 \end{bmatrix} \equiv G$. Further we note $HV_r = V_r$, $r = 1 \ldots R$ and $He = e$. Hence, we may write

$$C'HCC'e = C'HCC'\Sigma V_r \gamma_r + C'HCC'\varepsilon \tag{9}$$

or

$$GC'e = GC'\Sigma V_r \gamma_r + GC'\varepsilon$$

and, partitioning $C = [C_1 C_2]$ so that the first $T - K$ rows of (9) become

$$f \equiv C_1'e = C_1' \sum_{r=1}^{R} V_r \gamma_r + C_1'\varepsilon \tag{10}$$

$$= C_1' \sum_{r=2}^{R} \mathscr{Z}_r X \gamma_r + \eta.^2$$

This last inequality comes from noting that $V_r = H\mathscr{Z}_r X$, and hence $C'V_r = C'H\mathscr{Z}_r X = C'HCC'\mathscr{Z}_r X = GC'\mathscr{Z}_r X$, which implies $C_1'V_r = C_1'\mathscr{Z}_r X$. We have also let $C_1\varepsilon \equiv \eta$.

We also note that η is spherically distributed, since $E\eta = 0$, $V\eta = E\eta\eta' = EC_1'\varepsilon\varepsilon'C_1 = \sigma_\varepsilon^2 C_1'C_1 = \sigma_\varepsilon^2 I_{T-K}$, due to the orthogonality of C.

It is the transformed residuals $f = C_1'e$ that we make use of in step two. The transformation C_1' comes from finding an orthogonal set of eigenvectors of $H = I - X(X'X)^{-1}X'$, and hence f depends only on knowledge of X and Y and does not require knowledge of Z.

Step Two

It is clear from (10) that the residuals from step one depend in a very involved way on the interrelation of X and Z through the terms $\mathscr{Z}_r X$. However, under the null hypothesis $H_0 : \beta(t) = \beta$, these terms disappear, and a simpler test is available.

Consider a mechanical regression of f on Z transformed by C_1' (which depends only on X):

$$f = C_1'Z\delta + \psi. \tag{11}$$

[1] In passing we note from (6) that

$$b = \Sigma(X'X)^{-1}X'\mathscr{Z}_r X\gamma_r + (X'X)^{-1}X'\varepsilon$$
$$= \Sigma B_r \gamma_r + (X'X)^{-1}X'\varepsilon.$$

Hence, $Eb = \Sigma B_r \gamma_r$, a weighted sum of the γ_r, and $V(b) = \sigma^2(X'X)^{-1}$.

[2] This latter sum goes from $r = 2$ to R since, if Z_1 (the first col. of Z) is a column vector of all ones, then $\mathscr{Z}_1 = I$ and hence $V_1 \equiv \mathscr{Z}_1 X - XB_1 = X - XB_1$, the least squares residuals of the auxiliary equation $X = XB_1 + V_1$. These residuals must necessarily be zero, since $B_1 = I$ does the trick of minimizing the sum of squares. Hence, $C_1'V_1 = 0 = C_1'\mathscr{Z}_1 X = C_1'X$.

OLS gives

(12) $$d = (Z'C_1C_1'Z)^{-1}Z'C_1f \quad \text{and from (10)}$$

$$= (Z'C_1C_1'Z)^{-1}Z'C_1C_1'\Sigma\mathscr{Z}_rX\gamma_r + (Z'C_1C_1'Z)^{-1}Z'C_1C_1'\varepsilon$$

$$\equiv (Z'QZ)^{-1}Z'Q\sum_{r=2}^{R}\mathscr{Z}_rX\gamma_r + (Z'QZ)^{-1}Z'Q\varepsilon$$

where $Q \equiv C_1C_1'$.

Under the null hypothesis $H_0:\beta(t) = \beta$, $\gamma_r = 0$ for $r = 2\ldots R$, and hence the first term of (12) is 0. That is, under H_0:

(13) $$d = (Z'QZ)^{-1}Z'Q\varepsilon$$

$$= (Z'QZ)^{-1}Z'C_1f.$$

In addition, from (10) we have under H_0 that

(14) $$f = C_1'\varepsilon.$$

Further, we note for future reference that Q is idempotent—since $QQ = C_1C_1'C_1C' = C_1IC_1' = C_1C_1' = Q$—and of rank $T - K$.

Now consider the residuals g of this second step; using (13) and (14),

(15) $$g \equiv f - C_1'Zd$$

$$= C_1'\varepsilon - C_1'Z(Z'QZ)^{-1}Z'Q\varepsilon$$

$$= C_1'[I - Z(Z'QZ)^{-1}Z'Q]\varepsilon$$

$$\equiv N\varepsilon \quad \text{where we let } N = C_1'[I - Z(Z'QZ)^{-1}Z'Q].$$

Now

$$g'g = \varepsilon'N'N\varepsilon$$

$$= \varepsilon'[I - QZ(Z'QZ)^{-1}Z']C_1C_1'[I - Z(Z'QZ)^{-1}Z'Q]\varepsilon$$

$$= \varepsilon'[Q - QZ(Z'QZ)^{-1}Z'Q][Q - QZ(Z'QZ)^{-1}Z'Q]\varepsilon$$

(16) $$\equiv \varepsilon'MM\varepsilon \quad \text{where } M \equiv Q - QZ(Z'QZ)^{-1}Z'Q$$

$$\equiv \varepsilon M\varepsilon$$

since M is seen to be idempotent with $\rho(M) = \operatorname{tr} M = T - K - R$. And hence,

(17) $$g'g \leftrightarrow \sigma_\varepsilon^2\chi^2_{T-K-R}.$$

From (13) we have

(18) $$\mathrm{d} = (Z'QZ)^{-1}Z'Q\varepsilon \equiv B\varepsilon$$

and

$$BM = (Z'QZ)^{-1}Z'Q[Q - QZ(Z'QZ)^{-1}Z'Q]$$
$$= (Z'QZ)^{-1}Z'Q - (Z'QZ)^{-1}Z'Q = 0.$$

Hence, the linear form (18) is distributed independently of the quadratic form (17) and the usual tests of significance on d may take place. Under $H_0: \mathrm{E}d = 0$, and hence a t value for a specific d at $T - K - R$ degrees of freedom in excess of the test level rejects the null hypothesis.

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