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SYSTEMATIC (NON-RANDOM) VARIATION MODELS VARYING PARAMETER REGRESSION: A THEORY AND SOME APPLICATIONS

BY T. F. COOLEY AND E. C. PRESCOTT

This paper develops a theory of varying parameter regression, involving transitory and permanent components of parameters. Convenient estimation procedures are then described, and applications to agricultural supply functions and capital markets are reviewed.

I. INTRODUCTION

In recent years economic theory has increasingly abandoned the relative security of static equilibrium and perfect certainty. This development has improved our understanding of the complex behavioral and institutional phenomena which we attempt to describe but it has also highlighted a disturbing deficiency in the econometric techniques by which we give empirical content to our theories. It has become increasingly clear that to assume behavioral and technological relationships are stable over time is in many cases not only heroic, but completely untenable on the basis of economic theory. Recent experience with the analysis of Phillips curves provides powerful evidence to this effect.

Econometricians have been aware of the problem of structural change as is evidenced by the work on the random coefficients model [12, 22, 23] and the problems of testing for structural change [8, 16]. Until the work of Rosenberg [19], however, little more than lip service was given to the fact that the parameters in econometric relationships are likely, in many instances, to vary sequentially over time. We have argued elsewhere [1, 5] that sequential parameter variation may arise because of problems of structural change, mis-specification and problems of aggregation. Perhaps more important, however, is the fact that in many instances theory leads us to expect relationships that change over time. Lucas [14] has argued this point very forcefully and some of the examples presented later in this paper also confirm it.

In this paper we present a summary of work that has appeared in several other places. Our initial concern with the problem of parameter variation (in a narrow sense) was provoked by a peculiar dichotomy between theory and practice in the area of econometric forecasting. Forecasters frequently find it necessary to adjust the intercepts of their models over the forecast period and yet these intercepts are assumed to be constant over the estimation period. This led us to develop an adaptive regression model which assumes that the intercepts are subject to variation over the sample period. This model is developed in [4] and analyzed extensively in [3]. In [1] the assumption of parameter variation was extended to the slope coefficients as well and the model was extensively tested. In [5] the general model was presented and the asymptotic properties of the estimators were developed rigorously. This approach to the problem of parameter variation is summarized in the following section. A subsequent section describes a convenient procedure for estimation which makes computational costs quite reasonable. The final section discusses some applications of this technique.

II. A Theory of Varying Parameter Regression

The regression structure with which we shall be concerned has the following form:

$$y_t = x_t'\beta_t \qquad t = 1, 2, \ldots, T, \tag{2.1}$$

where x_t is a k component vector of explanatory variables, β_t is a k component vector of parameters subject to sequential variation and y_t is the tth observation of the dependent variable. If there is an intercept, as typically will be the case, then

$$x_{t1} = 1 \qquad t = 1, 2, \ldots, T, \tag{2.2}$$

and β_1 represents the intercept. The parameters in the model are assumed to be adaptive in nature, subject to permanent and transitory changes. The hypothesized pattern of variation is:

$$\begin{aligned} \beta_t &= \beta_t^p + u_t \\ \beta_t^p &= \beta_{t-1}^p + v_t \end{aligned} \qquad t = 1, \ldots, T \tag{2.3}$$

where the superscript p denotes the permanent component of the parameters.

The u_t and v_t are identically and independently distributed normal variates with mean vectors 0 and covariance structures known up to different scale factors. A particularly convenient parameterization of this is as follows:

$$\begin{aligned} \operatorname{cov}(u_t) &= (1 - \gamma)\sigma^2\Sigma_u \\ \operatorname{cov}(v_t) &= \gamma\sigma^2\Sigma_v, \end{aligned} \tag{2.4}$$

where Σ_u and Σ_v are known up to scale factors. This assumption implies one of the elements of both Σ_u and Σ_v can be normalized to 1. When an intercept is present and is subject to both permanent and transitory changes, setting $\sigma_{11}^u = \sigma_{11}^v = 1$ is a convenient normalization. The transitory change in the intercept then corresponds to the additive disturbance term in the conventional regression model. Subsequently, for expository purposes, we assume β_{1t} is the intercept and that the above normalization has been made. The unknown parameters are the β_t, and the unchanging elements σ^2 and γ which specify the covariance structure. The objective of the estimation techniques is to estimate σ^2 and γ and the permanent components of the β_t.

The proposed structure has several significant advantages over the classical constant parameter techniques and other varying parameter techniques. Since parameter changes are likely to come from a variety of sources, it is reasonable to assume that some of them may persist while others may not. This structure is sufficiently general that it will encompass parameter variation from a wide variety of sources. Furthermore, this specification of the covariance process in terms of γ enables us to estimate the relative variance of the permanent and transitory changes. It is thus, somewhat more general than the model developed by Rosenberg [19] and the requisite assumptions are less restrictive.

Because the process generating the parameters is non-stationary, it is impossible to specify the likelihood function. For the purpose of estimation, however, we are interested in specific realizations of the parameter process. The likelihood function conditional on the value of the parameter process at some point in time is well defined so we can treat specific realizations of the parameter process as random parameters to be estimated.[1] The most convenient procedure for forecasting is to focus on the value of the parameter process one period past the sample. In this case it follows that:

$$\beta^p_{T+1} = \beta^p_T + v_T \tag{2.5}$$

$$= \beta^p_t + \sum_{s=t+1}^{T+1} v_s,$$

$$\beta_t = \beta^p_{T+1} - \sum_{s=t+1}^{T+1} v_s + u_t, \tag{2.6}$$

and (2.1) can be rewritten as:

$$y_t = X_t'\beta + \mu_t, \tag{2.7}$$

where

$$\beta = \beta^p_{T+1}, \tag{2.8}$$

and

$$\mu_t = x_t'u_t - x_t' \sum_{s=t+1}^{T+1} v_s. \tag{2.9}$$

It is easily verified that μ is distributed normally with mean zero and covariance matrix:

$$\operatorname{cov}(\mu) = \sigma^2[(1-\gamma)R + \gamma Q] \equiv \sigma^2\Omega(\gamma) \tag{2.10}$$

where R is a diagonal matrix with

$$r_{ii} = (x_i'\Sigma_u x_i), \tag{2.11}$$

and Q is a matrix such that

$$q_{ij} = \min(T-i+1, T-j+1)x_i'\Sigma_v x_j. \tag{2.12}$$

More generally if one is concerned with the value of the permanent part of the parameter vector in period t, that is in β^p_t, the appropriate formulae for the q_{ij} are

$$q_{ij} = \min\{|t-i|, |t-j|\}x_i'\Sigma_v x_j \tag{2.13}$$

if both i and j exceed or are less than t. Otherwise, $q_{ij} = 0$. This generalization is useful in situations where one is not forecasting future values of the dependent

[1] It is worth noting that this treatment of the parameter process avoids some of the difficulties inherent in recursive estimation schemes (Kalman Filtering). There is no need to assume known relative variances.

variable y_t but rather attempting to draw inference about the path of the coefficients. This is of interest because economic theory sometimes suggests movements in the coefficients and such information is needed to test the validity of the theory. Alternatively, systematic drifts in the coefficients may suggest that the model is subject to specification errors of a particular kind and the information contained in the parameter changes may be useful in modifying the theory.

The full model can be rewritten as:

$$Y = X\beta + \mu, \tag{2.14}$$

where β is the k component vector

$$\beta = \begin{bmatrix} \beta^p_{1.T+1} \\ \beta^p_{2.T+1} \\ \vdots \\ \beta^p_{k.T+1} \end{bmatrix}; \tag{2.15}$$

X is the $T \times k$ matrix:

$$\begin{bmatrix} x_{11} & \dots & x_{k1} \\ x_{21} & \dots & x_{k2} \\ \vdots & & \\ x_{T1} & \dots & x_{Tk} \end{bmatrix}; \tag{2.16}$$

and Y is the T component vector of the y_t. From (2.10) it follows that Y is distributed as:

$$Y \sim [X\beta, \sigma^2\Omega(\gamma)]. \tag{2.17}$$

If γ were known, then the estimation would be a trivial application of Aitkens generalized least squares (GLS) analysis because R and Q are functions of the observed exogenous variables. The parameter γ, however, plays a crucial role in the analysis and is unlikely to be known in most econometric applications. The parameter γ tells us how fast the β's are adapting to structural change. If γ is large (close to 1), then the permanent changes are large relative to the transitory changes.

Using (2.10), we can write the log likelihood function of the observations as:

$$L(Y;\beta,\sigma^2,\gamma,X) = -\frac{T}{2}\ln 2\pi - \frac{T}{2}\ln\sigma^2 - \frac{1}{2}\ln|\Omega(\gamma)| - \frac{1}{2\sigma^2}(Y - X\beta)'\Omega(\gamma)^{-1}(Y - X\beta). \tag{2.18}$$

We can maximize (2.18) partially with respect to β and σ^2 to obtain the estimators conditional on γ:

$$B(\gamma) = [X'\Omega(\gamma)^{-1}X]^{-1}X'\Omega(\gamma)^{-1}Y \tag{2.19}$$

$$s^2(\gamma) = \frac{1}{T}[(Y - X\beta(\gamma))'\Omega(\gamma)^{-1}(Y - X\beta(\gamma))]. \tag{2.20}$$

These are substituted in (2.18) to determine the concentrated likelihood function as:

$$L_c(Y:\gamma) = -\frac{T}{2}\ln 2\pi - \frac{T}{2}\ln s^2(\gamma) - \frac{1}{2}\ln|\Omega(\gamma)| - \frac{T}{2} \tag{2.21}$$

$$= -\frac{T}{2}(\ln 2\pi + 1) - \frac{T}{2}\ln s^2(\gamma) - \frac{1}{2}\ln|\Omega(\gamma)|.$$

Thus, globally maximizing the log likelihood function (2.18) is equivalent to maximizing this concentrated likelihood function. Note that γ, because it is the fraction of parameter variation due to permanent changes, is restricted to fall within the range

$$0 \leq \gamma \leq 1. \tag{2.22}$$

The strategy of estimation then, is to divide the range for γ into a number of points

$$\gamma_i \quad i = 1, 2, \ldots, n,$$

for every γ_i evaluate (2.21) and choose as the estimator of γ, say g, the value such that:

$$L_c(Y; g, X) \geq L_c(Y; \gamma_i, X) \quad \text{all } i. \tag{2.23}$$

The estimates of β and σ^2 are determined from (2.19) and (2.20) above as $B(g)$ and $s^2(g)$ respectively.[2]

To apply the technique Σ_u and Σ_v, which along with γ and σ^2 specify the covariances of the permanent and transitory changes, must be known up to scale factors. Clearly, this is often not the case, but estimation of the model with these elements treated as unknown parameters would be impractical computationally.[3] Unless one has *a priori* knowledge to the contrary, it is reasonable to assume the relative importance of permanent and transitory changes is the same for all random parameters. This implies Σ_u and Σ_v are equal. Similarly, if one has no reason to assume that random changes in parameters are correlated, one might assume these matrices are diagonal; that is

$$\Sigma_u = \Sigma_v = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & \sigma_{22} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdot & \cdots & \sigma_{kk} \end{bmatrix}. \tag{2.24}$$

With these assumptions all one need specify is the relative variability of the different parameters.[4] In a well studied economic relationship, it should be possible to specify reasonable values for these elements. It will be seen in Section III below

[2] Usually $(T/T - k)s^2(g)$ would be a better estimate of σ^2 than the maximum likelihood estimator for it would be unbiased if $g = \gamma$.

[3] An additional problem is that the properties of the estimators have not been developed when additional unknown parameters are present.

[4] This model reduces to the random coefficients model of [12, 22] if $\gamma = 0$.

that the very nature of the process being studied frequently suggests the appropriate specification of the covariance structure, including the off diagonal elements. Even if this is not the case, the loss in estimation efficiency is surprisingly small for sizable errors in specifying the diagonal elements. Further, losses in efficiency resulting from incorrectly assuming zero correlation between changes in parameters are also small.[5]

An alternative assumption is to assume only the intercept and not the slopes are subject to transitory changes. If in addition, the permanent changes are assumed independent then:

$$\Sigma_u = \begin{bmatrix} 1 & . & \cdots & 0 \\ . & 0 & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ 0 & . & \cdots & 0 \end{bmatrix} \qquad \Sigma_v = \begin{bmatrix} 1 & . & \cdots & 0 \\ . & \sigma_{22} & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ 0 & . & \cdots & \sigma_{kk} \end{bmatrix}. \tag{2.25}$$

This structure is very similar to the one considered by [19] and would be the same if γ were assumed known. If $\gamma = 0$, this is the conventional multivariate regression model.

A final special case that has been extensively tested in [3, 4] is to assume the slopes are constant and only the intercept is subject to random changes. Then,

$$\sigma_{ij}^v = \sigma_{ij}^u = 0 \quad \text{for } i \text{ or } j > 1, \quad \text{and} \quad \sigma_{11}^v = \sigma_{11}^u = 1. \tag{2.26}$$

This structure which assumes disturbances have both permanent and transitory components is an alternative to the conventional assumption that errors are subject to a first order auto-regressive process. The latter makes the extreme assumption that the effects of all omitted factors decay exponentially and all at the same rate.

Thus far, we have presented a class of models in which the parameters are subject to sequential stochastic variation. The nature of the process, however, prohibits any simple application of the usual asymptotic results because no consistent estimator exists for the parameter set $(\beta, \gamma, \sigma^2)$. The variances of the β's are bounded away from zero because these parameters are subject to random change in every period.

In [5], we developed the asymptotic properties of the maximum likelihood estimator. First, we observed that if γ were known, $\beta(\gamma)$ would be the efficient estimator in the sense that the Cramer–Rao minimum variance bound for the class of unbiased estimators would be satisfied. We then proved that the estimator g of γ was consistent so that asymptotically $\beta(g)$ is efficient. This consistency argument did not employ the normality of the $\{u_t\}$ and $\{v_t\}$ processes except to conclude the existence of fourth moments of these random variables. Indeed, the only essential use of normality was to write down a likelihood function to be maximized. Thus, even in the absence of normality, the consistency of g implies that $\beta(g)$ converges in probability to $\beta(\gamma)$. But, $\beta(\gamma)$ would be the best linear unbiased estimator of β if γ were known by the Gauss–Markov theorem.

[5] Extensive tests of the robustness of these estimators are reported in [1].

In addition, we developed the asymptotic distribution of the parameter pair $\theta \equiv (\gamma, \sigma^2)$. We let $L^*(Y; X, \theta)$ denote the log likelihood function concentrated on β and

$$I(\theta) = -\frac{1}{T}E[\partial^2 L^*/\partial\theta^2], \tag{2.27}$$

which corresponds to the information matrix. Then asymptotically

$$\sqrt{T}(\hat{\theta} - \theta) \sim N[0, I(\theta)^{-1}]. \tag{2.28}$$

This relationship can be used to test whether γ is significantly different from zero, which implies permanent changes in the coefficients of the regression model.

III. Transformation of the Model

A possible drawback of the estimation scheme presented in the previous section is that it requires the inversion of the $T \times T$ matrix $(1 - \gamma)R + \gamma Q$ for each value of γ_i. In this section, we present a transformation which greatly reduces the number of computations required to obtain the estimators. The strategy of the transformation is to make the matrices R and Q diagonal so that inversion of the covariance matrix is a trivial computation. The elements of R and Q are known since they depend on the exogenous variables and the matrices Σ_u and Σ_v. To eliminate the matrix R, the model can be transformed as follows:

$$\begin{aligned} y_t^* &= y_t/\sqrt{r_{tt}} \qquad t = 1, 2, \ldots, T, \\ x_{ti}^* &= x_{ti}/\sqrt{r_{tt}} \qquad i = 1, 2, \ldots, k, \end{aligned} \tag{3.1}$$

where r_{tt} is the tth diagonal element of the matrix R. This yields a transformed model where Y^* is distributed as

$$Y^* \sim N\{X^*\beta, \sigma^2[(1 - \gamma)I + \gamma Q^*]\} \tag{3.2}$$

where

$$q_{ij}^* = \frac{q_{ij}}{\sqrt{r_{ii}}\sqrt{r_{jj}}}. \tag{3.3}$$

Now, there exists an orthogonal matrix P whose columns are a set of orthonormal eigenvectors of the matrix Q^* so that

$$P'P = I, \tag{3.4}$$

and[6]

$$P'Q^*P = D. \tag{3.5}$$

D is a diagonal matrix whose elements are the eigenvalues of Q^*. Now let

$$\bar{Y} = P'Y^* \qquad \bar{X} = P'X^* \qquad \bar{\mu} = P'\mu^*. \tag{3.6}$$

Observe that $\bar{Y}$ is now distributed as:

$$\begin{aligned} \bar{Y} &\sim N[\bar{X}\beta, \sigma^2(P'P + \gamma P'QP - \gamma P'P)] \\ &\sim N\{\bar{X}\beta, \sigma^2[I + \gamma(D - I)\}. \end{aligned} \tag{3.7}$$

[6] See Hadley [10, p. 255].

The matrix Q^* is known so that its eigenvalues need only be computed once. After this is done, estimation is relatively inexpensive for each γ_i that is searched.

The computation of the eigenvalues and eigenvectors of Q is a well-studied problem. It is clear that every root of the characteristic equation must be obtained in order to have the matrix D completely specified. We found the use of Householders tri-diagonalization followed by the QR method [17] quite accurate and fast. This calculation need only be done once and the transformation reduces the total number of computations significantly making these estimators less capital intensive than many commonly used non-linear estimation techniques.

IV. Some Applications

The usual objectives of applied econometric research are to gain more precise information about the structure of economic relationships and/or to obtain estimated relationships that are suitable for forecasting. The estimation technique developed in the previous section is particularly well-suited to both of these ends, because it makes it possible to draw inference about the structure of the relationship at every point in time. Thus, there are problems in macroeconomics, finance, economic history, and a variety of other areas that are suitable candidates for varying parameter estimation techniques. The technique developed in this paper, rather than being arcane and impractical is quite easy to use.[7] It has been and is currently being used in a wide variety of applications. In this section we describe some of those applications and discuss the results.

The Estimation of Agricultural Supply Functions 1866–1914

A subject of great interest to economic historians in recent years has been the estimation of agricultural supply functions for certain basic crops for the latter part of 19th century. The issues at stake in these investigations are not simple or easily summarized in a few sentences. The most important objective, however, has been to shed light on the regional specialization in late nineteenth century and early twentieth century agriculture. While it is clear that certain regions have a comparative advantage in the production of certain crops it has frequently been argued by nineteenth century observers that overspecialization in crops (especially cotton) was one of the main sources of unrest which culminated in the Populist revolt of the 1890's. Two important investigations by Fisher and Temin [9] and by DeCanio [6] have examined these issues by estimating models of the supply of wheat and cotton respectively.

The model of suppliers reactions to changing relative prices that was used in these studies belongs to the class of dynamic adjustment models introduced first by Nerlove [15]. The final model of supply is

$$S_t = \alpha + \beta\mu P_{t-1} + (1 - \mu)S_{t-1} \tag{4.1}$$

where S_t represents the proportion of total acreage devoted to the crop in question and P_t represents the price of the crop in question relative to the index of prices of the major alternative crops. The parameter μ represents the speed of adjustment

[7] The program and a write-up may be obtained by writing to either of the authors.

of the supply of the crop in question to changes in the relative price; the parameter $\beta\mu$ represents the short run price elasticity of supply and β the long run price elasticity.[8]

This supply model is a natural candidate for the application of varying parameter estimation techniques. The period of time which it spans is one in which substantial changes in the economic environment took place. As a consequence, the speed of adjustment of the farmers and the price elasticities may have changed substantially. Cooley and DeCanio [2] have applied these techniques to the data for both cotton and wheat. The parameters $(1 - \mu)$, $\beta\mu$ and α were assumed to be subject to both permanent and transitory changes with $\Sigma_u = \Sigma_v$. For the diagonal elements of Σ_u and Σ_v they used the estimated variances of the parameters obtained from maximum likelihood estimation of the relation under the assumption of parameter constancy. The magnitude of the off diagonal elements of Σ_u and Σ_v are suggested by the relationship itself. The parameters $\beta\mu$ and $(1 - \mu)$ are clearly negatively correlated. The expression for the correlation between these two suggests that the magnitude should be between -0.5 and -0.9 (depending on the relative size of β and μ).

With these assumptions the technique was used to estimate supply functions for 17 states for wheat and for 10 states for cotton. In addition, for each of the supply functions the path of the coefficients was traced out. The results obtained were quite impressive. Significant parameter variation was found in practically all of the supply functions. The results also indicated substantially higher speeds of adjustment for wheat and slightly higher speeds of adjustment for cotton than were previously reported.[9] In addition, when the paths of the parameters were traced out they exhibited a pattern which was quite consistent with economic theory. In several of the deep south states and even some of the western states where the Populist movement was strongest the speeds of adjustment and the short run elasticity showed substantial secular declines. For other states these parameters tended to remain constant or decline only slightly over time. For nearly all states there were cyclical changes in the parameters. The most significant cyclical change occurred for most states during the depression of the 1890's. During this period the speeds of adjustment and the short run price elasticity declined.

Capital Market Application

Capital market theory as developed by Sharpe [20] and Lintner [13] predicts a linear relationship between the expected rate of return of a stock in period t and the expected market rate of return. Letting R_t denote the rate of return for some stock and Rm_t for the market, the implied relationship is

$$R_t = \alpha + \beta_t Rm_t + u_t \tag{4.2}$$

where u_t is an additive disturbance necessitated by the fact that actual rather than expected rate of returns are observed. The intercept α is the average risk free rate

[8] Both Fisher and Temin and DeCanio found it necessary to include a time trend in order to obtain plausible results.

[9] One of the puzzling features of the Fisher and Temin study was that the speeds of adjustment they found were quite small.

of return and β_t is the covariance between the return of the stock and the market rate of return.

Numerous empirical studies (c.f. [7]) assumed that β_t is constant over time and estimated the relationship using conventional regression analysis. Tests for normality of the u_t were rejected and some concluded that the disturbances had infinite variance. Such conclusions are unwarranted given the assumption that the risk characteristics of a firm as summarized by β_t do not change over time. This assumption, however, is unreasonable. Theory predicts that it will not be constant because of changing technology in an industry, changes in management or accounting practices, and diversification. Sunder [21] did not assume constancy of the β_t but rather that they were subject to random changes. He then applied our varying parameter regression procedures to draw inference about the path of the β_t coefficients. For this case the natural assumption is that

$$\Sigma_u = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \Sigma_v = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.3}$$

since there should be only transitory change in the intercept and only permanent change in the β coefficient. For many stocks on the New York Stock Exchange, Sunder found significant variation in the β_t coefficient over time. He also used our procedures to test whether changes in accounting practices affect prices and risk classes as predicted by capital theory The results obtained were quite consistent with theory.

Given that the risk class changes over time, a natural application of the regression technique is to estimate the current value of the β_t. It is the current values that one needs to select an efficient portfolio (that is one which maximizes the expected rate of return for a given risk level). Efficiency is lost, however, by the current practice of using standard regression analysis to estimate the β coefficients since it does not provide the optimal estimates of the current β_t.

Other Applications

In addition to the applications outlined above varying parameter regression has been used in a variety of other contexts. Its usefulness in improving forecasting accuracy has been examined in the context of a three equation model in [1] and it is currently being applied to the behavioral equations of the Wharton Quarterly Forecasting Model. Roll [18] has used it in a study of the relation between interest rates on monetary assets and commodity price index changes. Hedrick [11] has used it in a study of the dynamics of labor supply functions.

Other applications that are currently in progress include an analysis of the movements of Phillips curve over time, analysis of seasonal adjustment procedures and an investigation of the dynamics of aggregate supply. It seems clear that varying parameter regression techniques have a wide variety of potential applications and that their use will become increasingly necessary in areas where the assumption of parameter constancy is not viable.

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