HOW MUCH COULD BE GAINED BY OPTIMAL STOCHASTIC CONTROL POLICIES

BY GREGORY C. CHOW

After an exposition of stochastic control theory for quadratic welfare and linear system with known parameters, this paper decomposes the gain from the optimal policy over the suboptimal policy of setting a constant growth rate for each policy variable into two parts, and measures them using a simple macroeconomic model of the United States. The first part, the gain of the optimal deterministic control policy over the suboptimal policy, is much smaller than the second, the gain from optimal stochastic control over optimal deterministic control. Total gains of 30 to 40 percent, using first differences of the variables, and of 40 to 80 percent, using levels of the variables, have been found. Sensitivities of these results are also studied.

I. INTRODUCTION

The main purpose of this paper is to measure the possible gain by applying optimal stochastic control policies using an econometric model, as compared with policies that maintain a smooth growth path for each policy variable. At the outset, it should be admitted that our measures will depend on the econometric model used, as do conclusions from quantitative economic studies in general. A pertinent argument by proponents of a nondiscretionary rule is that we do not know the dynamic structure of the economy. The viewpoint of this paper is: if we do know the dynamic structure, and if it resembles the one used, how much can be gained by applying an optimal discretionary policy? For those who believe that our present knowledge is meager, this paper provides an estimate of the potential value of acquiring knowledge of the dynamic structure of our economy. Furthermore, the method outlined here can be applied to other econometric models, and it would be of interest to study the sensitivities of our measures of gain, and of the optimal policies implied, to variations in the models.

In order to proceed, we have adopted the following three assumptions. (1) The welfare cost associated with a policy can be measured by the expected value, as of the beginning of a planning period, of a weighted sum of squared deviations of the economic variables from their specified targets. In other words, the welfare function is quadratic. (2) The relevant econometric model is linear. (3) The parameters of the model are known for certain. Discussion of the possibility of relaxing one or more of these assumptions will be postponed to the last section. It is believed that these assumptions, though restrictive, are good enough approximations to make our quantitative results useful.

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Since these assumptions were also made in the well-known works of Simon (1956) and Theil (1958) on first-period certainty equivalence to be applied to multiperiod decision under uncertainty, the main difference of the present study from their analysis should be stressed. Because their method does not explicitly provide optimal policies for future periods beyond the first, and expected welfare depends on these future policies, it cannot conveniently be applied to calculate the expected welfare associated with the optimal policy, or any other policy, except by simulations that require the generation of random disturbances. On the other hand, the calculation of expected welfare by our method is simple and analytic. Similarly, the application of non-stochastic control theory to a linear econometric model by ignoring the random disturbances, as exemplified by the interesting work of Pindyck (1971), does not yield expected welfare for a given policy; nor are the alleged optimal time paths for the policy variables calculated by Pindyck (1971) truly optimal if random disturbances are included. By allowing for the random disturbances of an econometric model as in stochastic control theory, the present study overcomes these deficiencies.

In my opinion, the literature on the control of stochastic systems is unnecessarily complicated for the researcher who wishes to understand the main ideas and the derivations of the optimal control solution for the case of quadratic welfare and linear model in discrete time. In two previous papers, Chow (1970b, 1972), I have provided simple expositions using the elementary technique of Lagrange multipliers. To make this paper self-contained, and to set the stage for further analysis, I will include, in section 2, an exposition of the main ideas, drawing partly on the previous papers and supplementing them by an elementary exposition using the method of dynamic programming. Building on the basic theory, I will derive in section 3 the gain of the optimal policy over a policy of maintaining a constant rate of growth for each policy variable. The method of section 3 will then be applied to a highly simplified and aggregative econometric model of the U.S. economy in section 4. Conclusions and possible extensions of the present study will be presented in section 5.

2. BASIC IDEAS AND THEORY

To begin with, we take as given a linear econometric model in its reduced form:

\[ y_t = A_1 y_{t-1} + \ldots + A_m y_{t-m} + C_0 x_t + \ldots + C_m x_{t-m} + b_t + u_t \]

where \( y_t \) is a vector of dependent variables, \( x_t \) is a vector of variables subject to control, \( A_i \) and \( C_n \) are given constant matrices, \( u_t \) is a serially uncorrelated vector with mean zero and covariance matrix \( V \). Exogenous variables in the system which are not subject to control will be treated either as part of \( b_t \) (also assumed to be given constants) or as a part of \( u_t \). To simplify analysis, the system (2.1) will be rewritten as a first-order system.

\[ \text{For a fuller discussion of this point, see Chow (1972), which discussed other differences as well.} \]
which will be redesignated as

\[ y_t = A_{t-1} y_{t-1} + C x_t + b_t + u_t. \]  

(2.3)

Note that the newly defined \( y_t \) includes current and (possibly) lagged dependent variables as well as current and (possibly) lagged control variables, whereas \( x_t \) remains the same as before.

The performance of the system will be measured by the deviations of \( y_t \), as defined in (2.3), from the target vectors \( a_t \) (\( t = 1, \ldots, T \)). The vectors \( a_t \) will have the same dimension as \( y_t \), and since the latter include lagged variables, the elements of \( a_t \) have to be consistently specified through time. Specifically, welfare cost is measured by

\[ W = \mathbb{E} (y_t - a_t)' K a (y_t - a_t) \]  

(2.4)

where the expectation \( \mathbb{E} \) is conditional on the initial condition \( y_0 \), again in the notation of (2.3), and \( K \) are known, symmetric (usually diagonal), positive semi-definite matrices, with zero elements as a rule corresponding to lagged (endogenous and control) variables.

The main idea of control is to steer \( y_t \) close to the target \( a_t \) by choosing appropriately the control variables \( x_t \). It will be fruitful to think of \( x_t \) as composed of two parts, \( \tilde{x}_t \) which is deterministic, and \( x_t^* \) which is random, both from the vantage point of the decision process at the beginning of period 1. That is to say, \( \tilde{x}_t \) (\( t = 1, \ldots, T \)) can be specified once and for all in period 1, whereas \( x_t^* \) (\( t = 1, \ldots, T \)) may depend on the random elements \( u_t \) which are observable, at least indirectly, if the parameters in the system are known. Similarly, the time series \( y_t \) under control will be viewed as the sum of two parts, the first being

\[ y_t = A_{t-1} y_{t-1} + C \tilde{x}_t + b_t \]  

(2.5)

\( \tilde{y}_t = A_{t-1} y_{t-1} + C \tilde{x}_t + b_t \)  

(2.5) \( \tilde{y}_0 = y_0 \)

which is deterministic, and the remainder \( y_t^* = y_t - \tilde{y}_t \), being

\[ y_t^* = A_{t-1} y_{t-1}^* + C x_t^* + u_t \]  

(2.6) \( y_0^* = 0 \)

which is random, and independent of the first part because \( u_t \) are. Accordingly, the welfare cost is decomposed into two parts,

\[ W = \sum_{i=1}^{T} (\tilde{y}_i - a_t)' K a (\tilde{y}_i - a_t) + \mathbb{E} \sum_{i=1}^{T} y_t^* K y_t^* = W_1 + W_2 \]  

(2.7) \[ W = \sum_{i=1}^{T} (\tilde{y}_i - a_t)' K a (\tilde{y}_i - a_t) + \mathbb{E} \sum_{i=1}^{T} y_t^* K y_t^* = W_1 + W_2 \]
and the control problem is also separated into a deterministic control problem of minimizing \( W_1 \) with respect to \( x \), and a stochastic control problem of minimizing \( W_2 \) with respect to \( x \).

One elementary way to solve the deterministic control problem is to introduce the Lagrange multipliers \( \lambda_t \), and differentiate the Lagrange expression

\[
L_1 = \frac{1}{2} \sum_{i=1}^{T} (\hat{y}_i - a_i)^2 + \sum_{i=1}^{T} \lambda_i^T [y_i - A_i \hat{y}_{i-1} - C_i \hat{x}_i - b_i]
\]

to yield

\[
\frac{\partial L_1}{\partial \hat{y}_i} = C_i \lambda_i = 0 \quad (t = 1, \ldots, T)
\]

\[
\frac{\partial L_1}{\partial \hat{y}_i} = K_i (\hat{y}_i - a_i) - \lambda_i + A_i \lambda_{i+1} = 0 \quad (t = 1, \ldots, T; \lambda_{T+1} = 0).
\]

Equations (2.10), (2.9), and (2.5) will be used, in that order, to express \( \hat{y}_i \) as a linear function of \( \hat{x}_i \), to solve for \( \hat{x}_i, \hat{y}_i, \) and \( \lambda_i \) as linear functions of \( \hat{y}_{i-1} \), and using the last, to express \( \hat{y}_{i-1} \) as a linear function of \( \hat{y}_{i-1} \), and so forth, beginning with \( t = T \). Thus, by (2.10),

\[
\lambda_T = K_T \hat{y}_T - K_T \hat{x}_T + A_{T+1} \lambda_{T+1} = H_T \hat{y}_T - b_T
\]

where

\[
H_T = K_T,
\]

\[
h_T = K_T \hat{x}_T.
\]

By (2.9), (2.11) and (2.5),

\[
C_T \lambda_T = 0 = C_T [H_T \hat{y}_T - b_T] = C_T [H_T A_T \hat{y}_{T-1} + H_T C_T \hat{x}_T + H_T b_T - b_T],
\]

implying

\[
\hat{x}_T = G_T \hat{y}_{T-1} + g_T
\]

where

\[
G_T = - (C_T H_T C_T)^{-1} C_T \hat{x}_T;
\]

\[
g_T = - (C_T H_T C_T)^{-1} C_T (H_T b_T - b_T).
\]

Using (2.5) and (2.11) respectively, in conjunction with (2.15), we solve for \( \hat{y}_T \) and \( \lambda_T \) as functions of \( \hat{y}_{T-1} \):

\[
\hat{y}_T = (A_T + C_T G_T) \hat{y}_{T-1} + b_T + C_T g_T;
\]

\[
\lambda_T = H_T (A_T + C_T G_T) \hat{y}_{T-1} + H_T b_T + C_T g_T - h_T.
\]

Having solved for \( \lambda_T \) in terms of \( \hat{y}_{T-1} \), we will substitute (2.19) into (2.10) in order to obtain an equation analogous to (2.11):

\[
\lambda_{T-1} = K_{T-1} \hat{y}_{T-1} - K_{T-1} \hat{x}_{T-1} + A_T \lambda_T = H_{T-1} \hat{y}_{T-1} - b_{T-1}
\]

Although the result given below is well-known, the simple derivation presented here does not seem to be available in the literature on control.
where
\[ H_{T-1} = K_{T-1} + A_x H_T(A_T + C_T G_T) \]
\[ h_{T-1} = K_{T-1} h_T - A_y H_T(b_T + C_T g_T) + A_y h_T. \]

The development from (2.14) on can now be followed, with \( T - 1 \) replacing \( T \), and so forth. The above solution to this deterministic control problem consists of using the pair of equations (2.16) and (2.21) to obtain \( G_T, H_{T-1}, G_{T-1}, \ldots \), consecutively with (2.12) as the initial condition, and, given \( H_T \), of using the pair of equations (2.17) and (2.22) to obtain \( g_T, h_{T-1}, g_{T-1}, \ldots \), consecutively with (2.13) as the initial condition. Having obtained \( G_i \) and \( g_i \), we set the optimal \( \bar{x}_i \) by the linear feedback control rule (2.15) on \( y_{i-1} \).

The stochastic control problem of minimizing \( W \) subject to (2.6) can also be solved by the method of Lagrange multipliers. However, it may be useful to present an elementary exposition using the method of dynamic programming of Bellman (1957), a method which has often been applied to both deterministic and stochastic control problems. Consider the decision on \( x_T \) in the last period, when the welfare cost will be

\[ W_T = E_T g_T \]

where \( E_T \) denotes expectation conditional on the information available at the beginning of period \( T \), namely, \( y_{T-1} \). To facilitate its generalization to other periods than \( T \), we rewrite (2.23) as

\[ W_T = E_T y_T H_T y_T \]

\[ = (A_T y_{T-1} + C_T x_{T-1}) H_T (A_T y_{T-1} + C_T x_{T-1}) + p_T \]

where
\[ H_T = K_T \quad p_T = E_T A_T y_{T-1} \]

Minimizing (2.24) with respect to \( x_T \) by differentiation yields

\[ x_T^* = G_T y_{T-1} \]

where
\[ G_T = -(C_T H_T C_T)^{-1} C_T H_T A_T. \]

Substitution of (2.26) for \( x_T^* \) in (2.24) gives the minimum expected cost for the last period as

\[ \bar{W}_T = y^*_{T-1} (A_T + C_T G_T) H_T (A_T + C_T G_T) y_{T-1} + p_T. \]

By the principle of optimality in dynamic programming, the optimal strategy for any period, say, \( T - 1 \), is obtained by minimizing the expected cost from that period on under the assumption that all future controls shall be optimally set. This is to minimize

\[ E_{T-1} [\bar{W}_T + y_{T-1}^* K_{T-1} y_{T-1}^*] \]

For such a solution, see Chow (1972).
Substituting (2.28) for $\tilde{W}_T$ in (2.29), we find that the expression to be minimized will have the same form as (2.24), i.e.,

$$(2.30) \quad E_{T-1}[\tilde{x}_T^{*} (H_{T-1})_{T-1}^{*} + p_r] = (A_{T-1})_{T-2}^{*} + C_{T-1}X^*_T + \tilde{x}_T^{*}$$

where

$$(2.31) \quad H_{T-1} = K_{T-1} + (A_T + C_T G_T) H_T (A_T + C_T G_T) \quad \text{and} \quad p_r = \tilde{x}_T^{*} = \tilde{x}_{T-1} + \tilde{v}_{T-1} + \tilde{v}_{T-1}^{*}$$

Thus, the development from (2.26) on can be followed, with $T - 1$ replacing $T$, and so forth. In brief, we use the pair of equations (2.27) and (2.31) to obtain $G_T, H_{T-1}, G_{T-1}, \ldots$ with initial condition (2.25). Given $G_T$, we obtain the optimal $\tilde{x}_T^{*}$ by the linear feedback control equation (2.26), noticing that the feedback coefficients $G_T$ are identical with those applied to $\tilde{v}_{T-1}$ in the deterministic control problem earlier. In the above exposition, we have decomposed the optimal control problem of minimizing the welfare cost (2.7) into a deterministic control and a stochastic control problem. In an ordinary treatment, the solution to the entire problem is given simply as $x_T = G_T y_T + g_T$, where $x_T = \tilde{x}_T + \tilde{X}^*_T$ and $y_T = y_T + y_T^{*}$ in our notations.

It is interesting to note the steady-state solution for $G_T$. If $A_T = A, C_T = C$, and $K_T = K$ for all $t$, $G_T$ may reach a steady-state solution $G_T$ obtained by solving

$$(2.33) \quad G_T = -(C_T H_T C_T)^{-1} C_T H_T A_T$$

$$(2.34) \quad H_{T-1} = K + \{A_T + C_T G_T\} H_T (A_T + C_T G_T)$$

Since the solution is obtained backward in time, starting from $T$, $G_T$ will reach a steady-state for small values of $T$. This means that, when the time horizon is long, and with a time-invariant model, the optimal rule is the same for the early periods and the terminal condition will affect behavior only for periods close to the end of the time horizon. Note also that the possibility for $G_T$ to reach a steady-state depends on the parameters $A_T$ and $C_T$, but not on the time paths of the target $a_T$ and the combined effect $b_T$ of other exogenous variables, the latter affecting the solution for $g_T$. As can be seen from (2.17) and (2.22), $g_T$ could reach a steady-state if, in addition, $a_T$ and $b_T$ are constant through time.

3. DERIVING THE GAIN FROM OPTIMAL STOCHASTIC CONTROL

From the exposition of section 2, one easily sees that the gain from applying an optimal feedback policy, as compared with the rule of maintaining a constant growth rate for each control variable, can be decomposed into two parts. The first is the gain of optimal stochastic control over optimal deterministic control, the latter being a policy which sets the values of all future control variables at the beginning of period one. This gain is measured by the difference between $W_T = \sum_{t=1}^{T} b_T y_T^{*} K_T y_T^{*}$ for the optimal policy of equations (2.26), (2.27), and (2.31), and that value in the absence of any feedback, i.e., $y_T^{*} = A_T y_{T-1}^{*} + u_T$. The second
is the gain of optimal deterministic control over the deterministic control rule of a constant growth rate for each policy variable. These two parts will in turn be derived.

The reader will have noticed that the proponents of maintaining a constant growth rate for just one control variable (money supply) have not stated their position sufficiently for a meaningful and rigorous analysis. To complete the specification of a meaningful proposition, we add that all control variables should grow at constant rates, that a quadratic welfare function be used to measure the performance of the economy, and that, for the benefit of the proponents of such a proposition, the particular growth rates be determined sub-optimally in accordance with the given welfare function.5

For the first part of the gain, one can evaluate \( W_2 \) for the optimal policy as follows. Write

\[
W_2 = \sum_{t=1}^T E_t r_t K_t s_t^* - \sum_{t=1}^T r_t K_t E_t r_{t+1} s_{t+1}^*.
\]

Note that, for the optimal policy given by (2.26) and (2.27), the stochastic model (2.6) becomes

\[
y_t^* = (A_t + C_t G_t) y_{t-1}^* + u_t.
\]

Postmultiply (3.2) by \( y_t^* \) and take expectation to yield

\[
E y_t^* y_t^* = (A_t + C_t G_t) E y_{t-1}^* y_{t-1}^* + E u_t u_t^*.
\]

Premultiply the transpose of (3.2) by \( y_{t-1}^* \), take expectation, and substitute the result for \( E y_{t-1}^* y_{t-1}^* \) in (3.3) to yield

\[
E y_t^* y_{t-1}^* = (A_t + C_t G_t) (E y_{t-1}^* y_{t-1}^*), (A_t + C_t G_t) + E u_t u_t^*.
\]

Equation (3.4) can be used to evaluate \( E y_t^* y_{t-1}^* \) in (3.1), starting with \( E y_t^* y_t^* = E u_t u_t^* = V \). Since the suboptimal policy ignores any information on \( y_t^* \), and it sets \( G_t = 0 \) in the model (3.2), it will have a stochastic welfare cost \( W_2 \) with \( E y_t^* y_{t-1}^* \) given by

\[
E y_t^* y_{t-1}^* = A_t E y_{t-1}^* y_{t-1}^* + V.
\]

This completes the evaluation of the welfare gain by using optimal stochastic control, over the best deterministic control policy.

For the second part of the gain, \( W_1 \) for the optimal deterministic policy can easily be calculated by definition (2.7). For the suboptimal policy, the control equation is constrained to be, with \( D \) denoting a diagonal matrix,

\[
\tilde{x}_t = D x_{t-1}
\]

or alternatively,

\[
\tilde{x}_t = G_{t-1} = (0 \ldots 0 \ D \ 0 \ldots 0)
\]

An important motivation of this study is to examine rigorously the policy, mostly attributed to Friedman and widely discussed after Friedman (1968), of increasing money supply at a constant percentage rate. We have pointed out that more is needed to make such a proposition meaningful. For further discussion of Friedman's methodology from the viewpoint of dynamic, stochastic, and quantitative economics, the reader may refer to Chow (1970a).
where the matrix $G$ has zero elements except for the submatrix corresponding to
the vector $\tilde{x}_{-1}$ which is imbedded in the vector $\tilde{y}_{-1}$. The problem is to minimize
$W_1$ with respect to $(d_1, d_2, \ldots, d_q) = d$, the diagonal elements of the matrix $D$.
There are various methods to perform this minimization. A good method which
we have used for the calculations in section 4 is the gradient method described in
Goldfeld, Quandt, and Trotter (1966). Given any guess of the unknown vector $d$,
the value of the function $W_1(d)$ can be calculated using the definition (2.7), the
model (2.5), and the control rule (3.7); so can the gradient of $W_1(d)$ at that point,
either numerically or analytically. By using a quadratic approximation to the
function to be minimized near that point, the Goldfeld-Quandt-Trotter method
insures that the matrix of second derivatives used to calculate the unknown for
the next iteration is positive definite, even if the function itself is not convex.
This method can be used to obtain the suboptimal $d$, and the associated welfare
cost $W_1$ can be calculated.

From the viewpoint of applications, our analytical framework may be
applied to the levels of economic variables, or to their first differences, depending
on the interest of the researcher. The variables in the welfare function and in the
control equations may be of either type. If the given econometric model explains
the levels, one can create new variables for the first differences by introducing
identities, and vice versa.

4. MEASURING THE GAIN FOR A SIMPLE MACRO MODEL

The econometric model to be used to measure the gain from an optimal
stochastic control policy is a very aggregative multiplier-accelerator model that
I constructed, Chow (1967), using annual data of the United States covering the
years 1931–1940 and 1948–1963. There are four stochastic equations explaining
the four dependent variables listed below.

\begin{align*}
y_1 &= \Delta C = \text{first difference of total personal consumption expenditures,} \\
     & \quad \text{millions of current dollars.} \\
y_2 &= \Delta I_1 = \text{first difference of gross private domestic investment in producers’} \\
     & \quad \text{durable equipment plus change in business inventories, millions.} \\
y_3 &= \Delta I_2 = \text{first difference of new construction, millions.} \\
y_4 &= \Delta R = \text{first difference of yield of 20-year corporate bonds, annual} \\
     & \quad \text{percentage rate times 10,000.} \\
x_1 &= \Delta M = \text{first difference of currency and demand deposits adjusted in} \\
     & \quad \text{middle of the year, millions of current dollars.} \\
x_2 &= \Delta G = \text{first difference of government purchases of goods and services,} \\
     & \quad \text{millions.}
\end{align*}

Five other dependent variables are explained by identities, giving a total of nine
structural equations as listed in Table 1. The reduced form equations, corre-
spending to (1.1), are given in Table 2. There are two exogenous variables,
$x_1 = \Delta M$ and $x_2 = \Delta G$, both of which are assumed to be control variables. They
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<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(7)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>(8)</td>
<td>0</td>
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<td>(9)</td>
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</tbody>
</table>

Source: Table 1 of Chow (1967), p. 9.
augment the above nine variables in a newly defined $11 \times 1$ vector $y$, in the notation of equation (2.3).

There are two sets of calculations to be presented in Table 3, one obtained by controlling the first differences of the variables, and the second by controlling the variables in their levels. The initial conditions $y_0$ are those for the year 1964. The target paths $a_k$ of all expenditures variables (in first differences or in levels as the case may be) are set to grow by 5 percent per year starting from their historical values $v_0$. The target path for the first difference of the rate of interest is set equal to zero for all periods; the target path for the interest rate variable itself is set equal to 4.33 percent per year times $I^{0.016}$, its value in 1964.

Since the model does not explain the price level $P$, and lagged price $P_i$ is used in the first three structural equations as a deflation device, we will assume that this exogenous variable grows by 2 percent per year from 120.7, its value in 1964.

The time horizon $T$ is 10 years.

The matrix $K$ in the welfare function is assumed to be diagonal. Its non-zero diagonal elements have been chosen according to two major considerations: whether the interest rate variable should be weighted, and whether individual expenditures variables $C$, $I_1$, and $I_2$ should be weighted above and beyond their sum. Total of government expenditures is included in the welfare function, either by itself or as one component of a sum, but money supply is not because there is little rationale for doing so. Of course, the path of money supply is appraised through its effect on expenditures and, in some calculations, through its effect on the behavior of the rate of interest. The weight given to the interest rate, when it is present in the welfare function, is equal to that of an expenditure variable, implying that a deviation of 1 percentage point (or 10,000 in our units) from target is as costly as a deviation of 10,000 million dollars for an expenditure variable. Calculations using other weights for the interest rate variable, and for government expenditures, than those reported in Table 3 have also been performed, but they

*For brief comments on the relative merits and limitations of using levels and first differences in the welfare function, see Chow (1970b).

See Chow (1967), p. 12, for data on $y_0$. The initial conditions will not affect $G_i$; therefore, the stochastic part of the welfare cost will not be affected.

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provide the same orders of magnitude for the relative gains of the optimal policies over the suboptimal policies.

The first two runs in Table 3 give no weight to the interest rate variable, while the last three give a weight of 1 as specified above. Within these two groups, the runs are presented with increasing number of variables to be controlled. Note that, when the number of variables to be controlled is equal to the number of policy instruments (2 in our case), the deterministic components of these variables will reach their targets exactly, and the deterministic part of the welfare cost is therefore zero. Note also that the welfare cost is measured by a weighted sum of squared deviations in millions, so that 100 \times (10^6) for one expenditure variable would mean a standard deviation of 10 billion dollars.

As far as the five runs using first differences are concerned, the gain of the optimal solution in the stochastic part of welfare cost varies between about 30 to 40 percent, and this part is much more important than the deterministic part. Hence, if the economic model contains stochastic disturbances, one can hardly afford to ignore them in the study of optimal policy. As far as the five runs using the levels of the variables are concerned, the gain of the optimal solution in the stochastic part of welfare cost varies from about 40 to 80 percent, and, again, this part dominates the deterministic part of the welfare cost.

8 All calculations for the stochastic part of welfare cost require the use of the covariance matrix \( V \) of the reduced-form residuals, for which see Chow and Levitan (1969).

9 These calculations require the introduction of three more variables, i.e., the levels of policy instruments \( C, R, \) and \( Y_t \), into our equations of Tables 1 and 2; these variables are explained by simple identities. Furthermore, the control variables are \( M \) and \( G \), rather than \( \Delta M \) and \( \Delta G \).
One might wish to ask why the relative gains of the optimal policies are greater for the calculations using levels of the variables than the corresponding gains using first differences. To answer this question, let us reexamine how the stochastic part of the welfare cost is calculated. For the optimal policy, we choose a linear feedback control equation \( x_t^* = G_t y^*_{t-1} \) in such a way that the system under control

\[
x_t^* = A y_{t-1}^* + C y_t^* + u_t = (A + C G_t) y_{t-1}^* + u_t
\]

will have small weighted sum of variances. More precisely, we choose the matrix \( G_t = -(C'H_t C)^{-1} C'H_t A \) to make the matrix \((A + C G_t)\) small, in the sense of having a minimum \( \text{tr} (A + C G_t)'H_t (A + C G_t) \). This is equivalent to regressing the columns of the matrix \( A \) on the columns of the matrix \( -C \), with the columns of \( G_t \) as regression coefficients in a multivariate regression. For the sub-optimal policy, we set \( G_t = 0 \). The gain from the optimal policy is the gain (in reducing variances) by using a smaller matrix \((A + C G_t)\) in the above stochastic system rather than the matrix \( A \) itself. If the lag structure of the system as reflected in the matrix \( A \) becomes more complicated, with reference to a given matrix \( C \), in such a way that the ratio of \( \text{tr} A'H_t A \) to \( \text{tr} (A + C G_t)'H_t (A + C G_t) \) becomes larger for the optimal \( G_t \), then the gain from optimal control will be greater. Intuitively speaking, the more sparse \( A \) is, \( A = 0 \) being the extreme case, the less will be the effects of the lagged variables on the current state, and thus the less will be the gain from optimal feedback control. If this point is valid, one should expect larger gains from optimal control if he employs a quarterly model instead of an annual one, because there will be more lagged variables and the matrix \( A \) will be bigger in dimension and less sparse.

Granted that the optimal policy is definitely better than the suboptimal policy for a given welfare function, how much of the superiority would remain when judged by a different welfare function? To shed some light on this question, I have calculated the ratios of the suboptimal stochastic welfare costs to the optimal for Runs (1), (2), (4), and (5), using the welfare weights of Run (3) which are unity for \( R \) and \( Y \) only. These ratios are respectively 1.25, 1.32, 1.30, and 1.35 for the calculations in first differences; they are 1.28, 1.25, 1.42, and 1.41 respectively for the calculations in levels. The optimal policies are thus seen to be fairly robust against different welfare functions — recall that the optimal policies of runs (1) and (2) were derived without including the rate of interest in their welfare functions.

The gains from optimal policies having been measured, it would be of interest to examine the nature of the optimal feedback equations as reflected in the matrix \( G_t \). Table 4 shows the matrix \( G_t \) for all the runs of Table 3. In all calculations, the matrices \( G_t \), beginning with \( G_{10} \), converge very rapidly — for three significant figures, Table 4 applies equally well to \( G_5 \). I will confine myself to two observations. First, concerning the relative roles of money supply and government expenditures, other things being equal, optimal money supply will become more active as measured by the absolute values of the feedback coefficients, when the rate of interest occupies a less important position in the welfare function; optimal government expenditures will be more active when this variable occupies a less
important position in the welfare function. Secondly, because our consumption function (see Table 1) is of the form
\[ AC_{t} = 0.3083 \Delta Y_{t,1} + 0.1938 AC_{t-1} + \ldots \]
and our investment function, as derived from the accelerations principle through a stock-flow transformation, is of the form
\[ I_{tr} = 0.2806 \Delta Y_{t,1} + 0.3375 I_{t-1} + \ldots \]

or
\[ \Delta I_{t} = 0.2806 \Delta Y_{t,1} - 0.6625 I_{t-1} + \ldots \]
a compensatory policy would be to react negatively to \( AC_{t-1} \) but positively to \( I_{t-1} \) and \( I_{t-2} \), as indicated by the coefficients in Table 4, and suggested by the coefficients in the reduced-form equation for \( \Delta Y \) in Table 2. If the above formulations of the consumption and investment functions are correct, monetary and fiscal policies should react differently to lagged consumption and to lagged investment expenditures as recommended here.

To provide a very crude check on the wisdom of actual government policies during the period 1948-1963 (the post World War II sample period of our model), I am reporting the following two regressions of \( \Delta M \) and \( \Delta G \) on \( AC_{t-1} \), \( I_{t-1} + I_{t-2} \), \( \Delta M_{t-1} \), and linear trend \( t \) (\( t = 1948, \ldots, 1963 \)), placing in parentheses below each regression coefficient its ratio to the standard error.

### Table 4
**Optimal Feedback Control Matrix G**

| Run | \( AC_{t-1} \) | \( I_{t-1} \) | \( I_{t-2} \) | \( Y_{t-1} \) | \( M_{t-1} \) | \( M_{t-2} \) | \( C_{t-1} \) | \( R_{t-1} \) | \( Y_{t-1} \) | \( G_{t-1} \) |
|-----|----------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| (1) | \( \Delta M \) | -0.3198     | 1.1616     | 0.8941      | -0.0877     | 0.1225      | 0.1225      |             |             |             |
|     | \( \Delta G \) | 0.0000      | 0.0000     | 0.0000      | 0.0000      | 0.0000      | 0.0000      |             |             |             |
| (2) | \( \Delta M \) | -0.359     | 0.9073     | 0.7028      | -0.0688     | 0.1421      | 0.1421      |             |             |             |
|     | \( \Delta G \) | 0.0002      | 0.2047     | 0.1136      | -0.0119     | 0.0038      | 0.0038      |             |             |             |
| (3) | \( \Delta M \) | -0.0336    | 0.1149     | 0.0885      | -0.0087     | 0.3997      | 0.3997      |             |             |             |
|     | \( \Delta G \) | 0.3526      | 1.1130     | 0.8567      | -0.0840     | -0.2947     | -0.2947     |             |             |             |
| (4) | \( \Delta M \) | -0.2727    | 0.9337     | 0.7198      | -0.0706     | 0.1821      | 0.1821      |             |             |             |
|     | \( \Delta G \) | 0.0088      | 0.2929     | 0.2167      | -0.0214     | 0.0719      | 0.0719      |             |             |             |
| (5) | \( \Delta M \) | -0.2547    | 0.6713     | 0.5296      | -0.0510     | 0.1909      | 0.1909      |             |             |             |
|     | \( \Delta G \) | 0.0879      | 0.4319     | 0.2881      | -0.0250     | 0.0712      | 0.0712      |             |             |             |

Using First Differences

Using Levels of Variables

Important position in the welfare function. Secondly, because our consumption function (see Table 1) is of the form
\[ AC_{t} = 0.3083 \Delta Y_{t,1} + 0.1938 AC_{t-1} + \ldots \]
and our investment function, as derived from the accelerations principle through a stock-flow transformation, is of the form
\[ I_{tr} = 0.2806 \Delta Y_{t,1} + 0.3375 I_{t-1} + \ldots \]

or
\[ \Delta I_{t} = 0.2806 \Delta Y_{t,1} - 0.6625 I_{t-1} + \ldots \]

a compensatory policy would be to react negatively to \( AC_{t-1} \) but positively to \( I_{t-1} \) and \( I_{t-2} \), as indicated by the coefficients in Table 4, and suggested by the coefficients in the reduced-form equation for \( \Delta Y \) in Table 2. If the above formulations of the consumption and investment functions are correct, monetary and fiscal policies should react differently to lagged consumption and to lagged investment expenditures as recommended here.

To provide a very crude check on the wisdom of actual government policies during the period 1948-1963 (the post World War II sample period of our model), I am reporting the following two regressions of \( \Delta M \) and \( \Delta G \) on \( AC_{t-1} \), \( I_{t-1} + I_{t-2} \), \( \Delta M_{t-1} \), and linear trend \( t \) (\( t = 1948, \ldots, 1963 \)), placing in parentheses below each regression coefficient its ratio to the standard error.

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In these regressions, the two investment variables are combined to avoid too much multicollinearity; the trend and the intercept are used to represent \( g \), in the feedback control equation. Allowing for their standard errors, one can say that the coefficients of \( \Delta C_{t-1} \) and \( (I_1 + I_2)_{t-1} \) do tend to be respectively negative and positive as in an optimal policy, and that the first three coefficients (or four, if the coefficient of investment expenditures counts as two) are not very different from the optimal coefficient: in Table 4—discount the row of zeros in the optimal equations for \( \Delta G \) in run (1) because \( \Delta G \) is given too much weight, and discount the row for \( \Delta M \) in run (3) because \( \Delta R \) is given too much weight. If uncertainties were ignored, and the above coefficients were used for the matrix \( G \), to calculate the stochastic part of welfare, one would obtain:

<table>
<thead>
<tr>
<th>Run</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{T} W_2 ) (regression)</td>
<td>566.1</td>
<td>536.2</td>
<td>288.8</td>
<td>583.0</td>
<td>553.1</td>
</tr>
<tr>
<td>( \frac{1}{T} W_2 ) (suboptimal)</td>
<td>547.6</td>
<td>586.8</td>
<td>358.9</td>
<td>564.7</td>
<td>603.9</td>
</tr>
<tr>
<td>Ratio</td>
<td>0.97</td>
<td>1.09</td>
<td>1.24</td>
<td>0.97</td>
<td>1.09</td>
</tr>
</tbody>
</table>

Thus, a set of (non-stochastic) feedback control equations based on historical observations would not compare unfavorably with the suboptimal policy of \( G = 0 \). However, the above estimates of welfare gains are biased in favor of the observed regression policy because the standard errors \( s \) of the regressions and of the regression coefficients, which have been ignored in these calculations, would increase the variances of the system.\(^\text{10}\) One cannot say, from this very crude analysis, that monetary and fiscal policies in the period 1948–1963 were destabilizing.

5. CONCLUDING REMARKS

In this paper, I have set forth a theoretical framework for measuring the welfare gains by following an optimal stochastic control policy as compared with a suboptimal policy which only permits a constant rate of growth for each policy.

\(^{10}\) One should at least incorporate the two observed feedback control equations as stochastic equations in the system by taking into account the random disturbances in them, but this would still leave out the possibly random nature of the regression coefficients. On the other hand, large residuals in the historical feedback regressions might be due to policies designed for non-economic reasons (such as financing and spending for the Korean War), and compensations for their effects later on.
variable, and have provided numerical measures of gains using a macro-economic model. I have found that the stochastic part of welfare cost, which takes into account the random disturbances of an econometric model, is much larger than the deterministic part which does not. If first differences enter the welfare function, the gain varies between 30 to 40 percent; if levels of the variables enter the welfare function, the gain varies between 40 to 80 percent, both in terms of weighted sum of squares of deviations from targets. By examining how the optimal stochastic control policy works in this framework, I have indicated that the complexities of the lagged structure of the system as reflected in the matrix A, for a given matrix C of the effects of the policy variables, will tend to make the gain from the optimal policy greater, implying, for example, that a model using quarterly observations is likely to yield larger gains than an annual model.

It has also been found that the gain from an optimal policy, which is derived from a given welfare function, is fairly robust against (reasonable) variations in the welfare weights for its evaluation. The characteristics of the optimal feedback policies have been examined. Historical feedback relationships have been crudely estimated by regressions, and they do not suggest that monetary and fiscal policies in the United States were destabilizing in the period 1948-1963.

Let me now comment briefly on the possibility of relaxing the three main assumptions stated at the beginning of this paper. As suggested by Athans (1971), for example, an analysis using a quadratic welfare function and a linear stochastic model can be applied to a problem involving non-quadratic welfare and non-linear model by, first, solving the deterministic version of the latter problem, a version that substitutes zero for random disturbances, with whatever method available (such as Pontryagin’s minimum principle or dynamic programming), and, second, controlling the deviations of \( y_t \) from the optimal path obtained above after the linearization of the original model around this optimal path. This suggestion deserves further investigation, especially in view of the large differences between the optimal stochastic solutions and the optimal deterministic solutions that we have found in this paper.

If the model is linear and the welfare function quadratic, but the parameters are unknown and treated as random, it is well-known, and can easily be shown using techniques parallel to those of equations (2.23) to (2.32), that the optimal feedback equations will remain linear with matrices

\[
G_t = -[E(CH_tA)]^{-1}E(CH_tA) \\
H_{t-1} = K_{t-1} + E(A + CG_t^t)H(A + CG_t)
\]

replacing those of equations (2.27) and (2.31) respectively, provided that the random matrices \( A \) and \( C \) have density functions which are unchanged during the planning period. If this proviso is accepted, as it is reasonable in many applications when the prior information on \( A \) and \( C \) at the beginning of the planning period dominates the additional information to be collected during the planning period, one has an analytical solution to the optimal control problem after evaluating the mathematical expectation of the product of any two elements of the matrices \( A \) and \( C \) by Bayesian methods, as is done in Chow (1971). I hope to report on the results of this approach in the near future.

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REFERENCES


