Volume Title: Annals of Economic and Social Measurement, Volume 1, number 4

Volume Author/Editor: NBER

Volume Publisher: NBER

Volume URL: http://www.nber.org/books/aesm72-4

Publication Date: October 1972

Chapter Title: How Much Could Be Gained by Optimal Stochastic Control Policies

Chapter Author: Gregory Chow

Chapter URL: http://www.nber.org/chapters/c9443

Chapter pages in book: (p. 391 - 406)

Annals of Economic and Social Measurement, 1/4, 1972

HOW MUCH COULD BE GAINED BY OPTIMAL STOCHASTIC CONTROL POLICIES

BY GREGORY C. CHOW[1]

After an exposition of stochastic control theory for quadratic welfare and linear system with known parameters, this paper decomposes the gain from the optimal policy over the suboptimal policy of setting a constant growth rate for each policy variable into two parts, and measures them using a simple macroeconometric model of the United States. The first part, the gain of the optimal deterministic control policy over the suboptimal policy, is much smaller than the second, the gain from optimal stochastic control over optimal deterministic control. Total gains of 30 to 40 percent, using first differences of the variables, and of 40 to 80 percent, using levels of the variables, have been found. Sensitivities of these results are also studied.

1. INTRODUCTION

The main purpose of this paper is to measure the possible gain by applying optimal stochastic control policies using an econometric model, as compared with policies that maintain a smooth growth path for each policy variable. At the outset, it should be admitted that our measures will depend on the econometric model used, as do conclusions from quantitative economic studies in general. A pertinent argument by proponents of a nondiscretionary rule is that we do not know the dynamic structure of the economy. The viewpoint of this paper is: if we do know the dynamic structure, and if it resembles the one used, how much can be gained by applying an optimal discretionary policy? For those who believe that our present knowledge is meager, this paper provides an estimate of the potential value of acquiring knowledge of the dynamic structure of our economy. Furthermore, the method outlined here can be applied to other econometric models, and it would be of interest to study the sensitivities of our measures of gain, and of the optimal policies implied, to variations in the models.

In order to proceed, we have adopted the following three assumptions. (1) The welfare cost associated with a policy can be measured by the expected value, as of the beginning of a planning period, of a weighted sum of squared deviations of the economic variables from their specified targets. In other words, the welfare function is quadratic. (2) The relevant econometric model is linear. (3) The parameters of the model are known for certain. Discussion of the possibility of relaxing one or more of these assumptions will be postponed to the last section. It is believed that these assumptions, though restrictive, are good enough approximations to make our quantitative results useful.

[1] For valuable suggestions and comments I would like to thank George de Menil, Ray C. Fair, Stephen Goldfeld, Edwin Mills, Richard E. Quandt, and many other colleagues and students attending the Econometric Research Program Seminar in Princeton at which an earlier version of this paper was presented. Douglas R. Chapman has been extremely helpful not only in programming the calculations, but in being critical on many substantive points. Some initial programming work was performed by Michael K. Smith. Last, but not least, financial support from the National Science Foundation, NSF GS 2799 and NSF GS 32003X, is gratefully acknowledged.

Since these assumptions were also made in the well-known works of Simon (1956) and Theil (1958) on first-period certainty equivalence to be applied to multiperiod decision under uncertainty, the main difference of the present study from their analysis should be stressed.[2] Because their method does not explicitly provide optimal policies for future periods beyond the first, and expected welfare depends on these future policies, it cannot conveniently be applied to calculate the expected welfare associated with the optimal policy, or any other policy, except by simulations that require the generation of random disturbances. On the other hand, the calculation of expected welfare by our method is simple and analytic. Similarly, the application of non-stochastic control theory to a linear econometric model by ignoring the random disturbances, as exemplified by the interesting work of Pindyck (1971), does not yield expected welfare for a given policy; nor are the alleged optimal time paths for the policy variables calculated by Pindyck (1971) truly optimal if random disturbances are included. By allowing for the random disturbances of an econometric model as in stochastic control theory, the present study overcomes these deficiencies.

In my opinion, the literature on the control of stochastic systems is unnecessarily complicated for the researcher who wishes to understand the main ideas and the derivations of the optimal control solution for the case of quadratic welfare and linear model in discrete time. In two previous papers, Chow (1970b, 1972), I have provided simple expositions using the elementary technique of Lagrange multipliers. To make this paper self-contained, and to set the stage for further analysis, I will include, in section 2, an exposition of the main ideas, drawing partly on the previous papers and supplementing them by an elementary exposition using the method of dynamic programming. Building on the basic theory, I will derive in section 3 the gain of the optimal policy over a policy of maintaining a constant rate of growth for each policy variable. The method of section 3 will then be applied to a highly simplified and aggregative econometric model of the U.S. economy in section 4. Conclusions and possible extensions of the present study will be presented in section 5.

2. Basic Ideas and Theory

To begin with, we take as given a linear econometric model in its reduced form:

$$y_t = A_{1t}y_{t-1} + \dots + A_{mt}y_{t-m} + C_{0t}x_t + \dots + C_{nt}x_{t-n} + b_t + u_t \tag{2.1}$$

where y_t is a vector of dependent variables, x_t is a vector of variables subject to control, A_{it} and C_{it} are given constant matrices, u_t is a serially uncorrelated vector with mean zero and covariance matrix V. Exogenous variables in the system which are not subject to control will be treated either as part of b_t (also assumed to be given constants) or as a part of u_t. To simplify anaiysis, the system (2.1) will be rewritten as a first-order system,

[2] For a fuller discussion of this point, see Chow (1972), which discussed other differences as well.

$$
(2.2)\quad \begin{bmatrix} y_t \\ \vdots \\ y_{t-m+1} \\ x_t \\ \vdots \\ x_{t-n+1} \end{bmatrix} = \begin{bmatrix} A_{1t}\ldots A_{mt} & C_{1t}\ldots C_{mt} \\ \ldots I \quad 0 & 0 \ldots 0 \\ 0 \ldots 0 & 0 \ldots 0 \\ 0 \ldots 0 & \ldots I \quad 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ \vdots \\ y_{t-m} \\ x_{t-1} \\ \vdots \\ x_{t-n} \end{bmatrix} + \begin{bmatrix} C_{0t} \\ \vdots \\ 0 \\ I \\ \vdots \\ 0 \end{bmatrix} x_t + \begin{bmatrix} b_t \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} u_t \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

which will be redesignated as

$$
(2.3)\quad y_t = A_t y_{t-1} + C_t x_t + b_t + u_t .
$$

Note that the newly defined y_t includes current and (possibly) lagged dependent variables as well as current and (possibly) lagged control variables, whereas x_t remains the same as before.

The performance of the system will be measured by the deviations of y_t, as defined in (2.3), from the target vectors a_t $(t = 1, \ldots, T)$. The vectors a_t will have the same dimension as y_t, and since the latter include lagged variables, the elements of a_t have to be consistently specified through time. Specifically, welfare cost is measured by

$$
(2.4)\quad W = E \sum_{t=1}^{T} (y_t - a_t)' K_t (y_t - a_t)
$$

where the expectation E is conditional on the initial condition y_0, again in the notation of (2.3), and K_t are known, symmetric (usually diagonal), positive semi-definite matrices, with zero elements as a rule corresponding to lagged (endogenous and control) variables.

The main idea of control is to steer y_t close to the target a_t by choosing appropriately the control variables x_t. It will be fruitful to think of x_t as composed of two parts, $\bar{x}_t$ which is deterministic, and x_t^* which is random, both from the vantage point of the decision process at the beginning of period 1. That is to say, $\bar{x}_t$ $(t = 1, \ldots, T)$ can be specified once and for all in period 1, whereas x_t^* $(t = 1, \ldots, T)$ may depend on the random elements u_t which are observable, at least indirectly, if the parameters in the system are known. Similarly, the time series y_t under control will be viewed as the sum of two parts, the first being

$$
(2.5)\quad \bar{y}_t = A_t \bar{y}_{t-1} + C_t \bar{x}_t + b_t \qquad (\bar{y}_0 = y_0)
$$

which is deterministic, and the remainder $y_t^* = y_t - \bar{y}_t$ being

$$
(2.6)\quad y_t^* = A_t y_{t-1}^* + C_t x_t^* + u_t \qquad (y_0^* = 0)
$$

which is random, and independent of the first part because u_t are. Accordingly, the welfare cost is decomposed into two parts,

$$
(2.7)\quad W = \sum_{t=1}^{T} (\bar{y}_t - a_t)' K_t (\bar{y}_t - a_t) + E \sum_{t=1}^{T} y_t^{*\prime} K_t y_t^* = W_1 + W_2
$$

and the control problem is also separated into a deterministic control problem of minimizing W_1 with respect to $\bar{x}_t$ and a stochastic control problem of minimizing W_2 with respect to x_t^*.

One elementary way to solve the deterministic control problem[3] is to introduce the Lagrange multipliers λ_t, and differentiate the Lagrange expression

$$L_1 = \frac{1}{2}\sum_{t=1}^{T} (\bar{y}_t - a_t)'K_t(\bar{y}_t - a_t) - \sum_{t=1}^{T} \lambda_t'[\bar{y}_t - A_t\bar{y}_{t-1} - C_t\bar{x}_t - b_t] \tag{2.8}$$

to yield

$$\frac{\partial L_1}{\partial \bar{x}_t} = C_t'\lambda_t = 0 \qquad (t = 1, \ldots, T) \tag{2.9}$$

$$\frac{\partial L_1}{\partial \bar{y}_t} = K_t(\bar{y}_t - a_t) - \lambda_t + A_{t+1}'\lambda_{t+1} = 0 \qquad (t = 1, \ldots, T; \lambda_{T+1} = 0). \tag{2.10}$$

Equations (2.10), (2.9), and (2.5) will be used, in that order, to express λ_t as a linear function of $\bar{y}_t$, to solve for $\bar{x}_t$, $\bar{y}_t$, and λ_t as linear functions of $\bar{y}_{t-1}$, and using the last, to express λ_{t-1} as a linear function of $\bar{y}_{t-1}$, and so forth, beginning with $t = T$. Thus, by (2.10),

$$\lambda_T = K_T\bar{y}_T - K_T a_T + A_{T+1}'\lambda_{T+1} = H_T\bar{y}_T - h_T \tag{2.11}$$

where

$$H_T = K_T, \tag{2.12}$$

$$h_T = K_T a_T. \tag{2.13}$$

By (2.9), (2.11) and (2.5),

$$C_T'\lambda_T = 0 = C_T'[H_T\bar{y}_T - h_T] = C_T'[H_T A_T\bar{y}_{T-1} + H_T C_T\bar{x}_T + H_T b_T - h_T], \tag{2.14}$$

implying

$$\bar{x}_T = G_T\bar{y}_{T-1} + g_T \tag{2.15}$$

where

$$G_T = -(C_T'H_T C_T)^{-1}C_T'H_T A_T; \tag{2.16}$$

$$g_T = -(C_T'H_T C_T)^{-1}C_T'(H_T b_T - h_T). \tag{2.17}$$

Using (2.5) and (2.11) respectively, in conjunction with (2.15), we solve for $\bar{y}_T$ and λ_T as functions of $\bar{y}_{T-1}$:

$$\bar{y}_T = (A_T + C_T G_T)\bar{y}_{T-1} + b_T + C_T g_T; \tag{2.18}$$

$$\lambda_T = H_T(A_T + C_T G_T)\bar{y}_{T-1} + H_T(b_T + C_T g_T) - h_T. \tag{2.19}$$

Having solved for λ_T in terms of $\bar{y}_{T-1}$, we will substitute (2.19) into (2.10) in order to obtain an equation analogous to (2.11):

$$\lambda_{T-1} = K_{T-1}\bar{y}_{T-1} - K_{T-1}a_{T-1} + A_T'\lambda_T = H_{T-1}\bar{y}_{T-1} - h_{T-1} \tag{2.20}$$

[3] Although the result given below is well-known, the simple derivation presented here does not seem to be available in the literature on control.

where

$$(2.21)\quad H_{T-1} = K_{T-1} + A_T' H_T (A_T + C_T G_T)$$

$$(2.22)\quad h_{T-1} = K_{T-1} a_{T-1} - A_T' H_T (b_T + C_T g_T) + A_T' h_T.$$

The development from (2.14) on can now be followed, with $T-1$ replacing T, and so forth. The above solution to this deterministic control problem consists of using the pair of equations (2.16) and (2.21) to obtain $G_T, H_{T-1}, G_{T-1}, \ldots,$ consecutively with (2.12) as the initial condition, and, given H_t, of using the pair of equations (2.17) and (2.22) to obtain $g_T, h_{T-1}, g_{T-1}, \ldots,$ consecutively with (2.13) as the initial condition. Having obtained G_t and g_t, we set the optimal $\bar{x}_t$ by the linear feedback control rule (2.15) on y_{t-1}.

The stochastic control problem of minimizing W_2 subject to (2.6) can also be solved by the method of Lagrange multipliers.[4] However, it may be useful to present an elementary exposition using the method of dynamic programming of Bellman (1957), a method which has often been applied to both deterministic and stochastic control problems. Consider the decision on x_T^* in the last period, when the welfare cost will be

$$(2.23)\quad W_T = E_T y_T^{*\prime} K_T y_T^*$$

where E_T denotes expectation conditional on the information available at the beginning of period T, namely, y_{T-1}^*. To facilitate its generalization to other periods than T, we rewrite (2.23) as

$$(2.24)\quad \begin{aligned} W_T &= E_T y_T^{*\prime} H_T y_T^* \\ &= (A_T y_{T-1}^* + C_T x_T^*)' H_T (A_T y_{T-1}^* + C_T x_T^*) + p_T \end{aligned}$$

where

$$(2.25)\quad H_T = K_T \qquad p_T = E_T u_T' H_T u_T.$$

Minimizing (2.24) with respect to x_T^* by differentiation yields

$$(2.26)\quad x_T^* = G_T y_{T-1}^*$$

where

$$(2.27)\quad G_T = -(C_T' H_T C_T)^{-1} C_T' H_T A_T.$$

Substitution of (2.26) for x_T^* in (2.24) gives the minimum expected cost for the last period as

$$(2.28)\quad \hat{W}_T = y_{T-1}^{*\prime}(A_T + C_T G_T)' H_T (A_T + C_T G_T) y_{T-1}^* + p_T.$$

By the principle of optimality in dynamic programming, the optimal strategy for any period, say, $T-1$, is obtained by minimizing the expected cost from that period on under the assumption that all future controls shall be optimally set. This is to minimize

$$(2.29)\quad E_{T-1}[\hat{W}_T + y_{T-1}^{*\prime} K_{T-1} y_{T-1}^*].$$

[4] For such a solution, see Chow (1972).

Substituting (2.28) for $\hat{W}_T$ in (2.29), we find that the expression to be minimized will have the same form as (2.24), i.e.,

$$E_{T-1}[y_{T-1}^{*\prime}H_{T-1}y_{T-1}^* + p_T] = (A_{T-1}y_{T-2}^* + C_{T-1}x_{T-1}^*)' \times H_{T-1}(A_{T-1}y_{T-2}^* + C_{T-1}x_{T-1}^*) + p_{T-1} \tag{2.30}$$

where

$$H_{T-1} = K_{T-1} + (A_T + C_TG_T)'H_T(A_T + C_TG_T) = K_{T-1} + A_T'H_T(A_T + C_TG_T); \tag{2.31}$$

$$p_{T-1} = p_T + E_{T-1}u_{T-1}'H_{T-1}u_{T-1}. \tag{2.32}$$

Thus, the development from (2.26) on can be followed, with $T - 1$ replacing T, and so forth. In brief, we use the pair of equations (2.27) and (2.31) to obtain $G_T, H_{T-1}, G_{T-1}, \ldots,$ with initial condition (2.25). Given G_t, we obtain the optimal x_t^* by the linear feedback control equation (2.26) on y_{t-1}^*, noticing that the feedback coefficients G_t are identical with those applied to $\bar{y}_{t-1}$ in the deterministic control problem earlier. In the above exposition, we have decomposed the optimal control problem of minimizing the welfare cost (2.7) into a deterministic control and a stochastic control problem. In an ordinary treatment, the solution to the entire problem is given simply as $x_t = G_ty_t + g_t$, where $x_t = \bar{x}_t + x_t^*$ and $y_t = \bar{y}_t + y_t^*$ in our notations.

It is interesting to note the steady-state solution for G_t. If $A_t = A$, $C_t = C$, and $K_t = K$ for all t, G_t may reach a steady-state solution G, obtained by solving

$$G_t = -(C'H_tC)^{-1}C'H_tA; \tag{2.33}$$

$$H_{t-1} = K + (A + CG_t)'H_t(A + CG_t). \tag{2.34}$$

Since the solution is obtained backward in time, starting from $t = T$, G_t will reach a steady-state for small values of t. This means that, when the time horizon is long, and with a time-invariant model, the optimal rule is the same for the early periods, and the terminal condition will affect behavior only for periods close to the end of the time horizon. Note also that the possibility for G_t to reach a steady-state depends on the parameters A and C, but not on the time paths of the target a_t and the combined effect b_t of other exogenous variables, the latter affecting the solution for g_t. As can be seen from (2.17) and (2.22), g_t could reach a steady-state if, in addition, a_t and b_t are constant through time.

3. Deriving the Gain from Optimal Stochastic Control

From the exposition of section 2, one easily sees that the gain from applying an optimal feedback policy, as compared with the rule of maintaining a constant growth rate for each control variable, can be decomposed into two parts. The first is the gain of optimal stochastic control over optimal deterministic control, the latter being a policy which sets the values of all future control variables at the beginning of period one. This gain is measured by the difference between $W_2 = \sum_{t=1}^{T} Ey_t^{*\prime}K_ty_t^*$ for the optimal policy of equations (2.26), (2.27), and (2.31), and that value in the absence of any feedback, i.e., $y_t^* = A_ty_{t-1}^* + u_t$. The second is the gain of optimal deterministic control over the deterministic control rule of a constant growth rate for each policy variable. These two parts will in turn be derived.

The reader will have noticed that the proponents of maintaining a constant growth rate for just one control variable (money supply) have not stated their position sufficiently for a meaningful and rigorous analysis. To complete the specification of a meaningful proposition, we add that all control variables should grow at constant rates, that a quadratic welfare function be used to measure the performance of the economy, and that, for the benefit of the proponents of such a proposition, the particular growth rates be determined sub-optimally in accordance with the given welfare function.[5]

For the first part of the gain, one can evaluate W_2 for the optimal policy as follows. Write

$$W_2 = \sum_{t=1}^{T} Ey_t^{*\prime}K_t y_t^* = \sum_{t=1}^{T} \operatorname{tr} K_t(Ey_t^* y_t^{*\prime}). \tag{3.1}$$

Note that, for the optimal policy given by (2.26) and (2.27), the stochastic model (2.6) becomes

$$y_t^* = (A_t + C_t G_t)y_{t-1}^* + u_t. \tag{3.2}$$

Postmultiply (3.2) by $y_t^{*\prime}$ and take expectation to yield

$$Ey_t^* y_t^{*\prime} = (A_t + C_t G_t)Ey_{y-1}^* y_t^{*\prime} + Eu_t u_t'. \tag{3.3}$$

Premultiply the transpose of (3.2) by y_{t-1}^*, take expectation, and substitute the result for $Ey_{t-1}^* y_t^{*\prime}$ in (3.3) to yield

$$Ey_t^* y_t^{*\prime} = (A_t + C_t G_t)(Ey_{t-1}^* y_{t-1}^{*\prime})(A_t + C_t G_t)' + Eu_t u_t'. \tag{3.4}$$

Equation (3.4) can be used to evaluate $Ey_t^* y_t^{*\prime}$ in (3.1), starting with $Ey_1^* y_1^{*\prime} = Eu_t u_t' = V$. Since the suboptimal policy ignores any information on y_t^*, and it sets $G_t = 0$ in the model (3.2), it will have a stochastic welfare cost W_2 with $Ey_t^* y_t^{*\prime}$ given by

$$Ey_t^* y_t^{*\prime} = A_t(Ey_{t-1}^* y_{t-1}^{*\prime})A_t' + V. \tag{3.5}$$

This completes the evaluation of the welfare gain by using optimal stochastic control, over the best deterministic control policy.

For the second part of the gain, W_1 for the optimal deterministic policy can easily be calculated by definition (2.7). For the suboptimal policy, the control equation is constrained to be, with D denoting a diagonal matrix,

$$\bar{x}_t = D\bar{x}_{t-1} \tag{3.6}$$

or alternatively,

$$\bar{x}_t = G\bar{y}_{t-1} = (0 \ldots 0 \quad D \quad 0 \ldots 0) \tag{3.7}$$

[5] An important motivation of this study is to examine rigorously the policy, mostly attributed to Friedman and widely discussed after Friedman (1968), of increasing money supply at a constant percentage rate. We have pointed out that more is needed to make such a proposition meaningful. For further discussion of Friedman's methodology from the viewpoint of dynamic, stochastic, and quantitative economics, the reader may refer to Chow (1970a).

where the matrix G has zero elements except for the submatrix corresponding to the vector $\bar{x}_{t-1}$ which is imbedded in the vector $\bar{y}_{t-1}$. The problem is to minimize W_1 with respect to $(d_1, d_2, \ldots, d_q) = d$, the diagonal elements of the matrix D. There are various methods to perform this minimization. A good method which we have used for the calculations in section 4 is the gradient method described in Goldfeld, Quandt, and Trotter (1966). Given any guess of the unknown vector d, the value of the function $W_1(d)$ can be calculated using the definition (2.7), the model (2.5), and the control rule (3.7); so can the gradient of $W_1(d)$ at that point, either numerically or analytically. By using a quadratic approximation to the function to be minimized near that point, the Goldfeld–Quandt–Trotter method insures that the matrix of second derivatives used to calculate the unknown for the next iteration is positive definite, even if the function itself is not convex. This method can be used to obtain the suboptimal d, and the associated welfare cost W_1 can be calculated.

From the viewpoint of applications, our analytical framework may be applied to the leveis of economic variables, or to their first differences, depending on the interest of the researcher. The variables in the welfare function and in the control equations may be of either type. If the given econometric model explains the levels, one can create new variables for the first differences by introducing identities, and vice versa.

4. Measuring the Gain for a Simple Macro Model

The econometric model to be used to measure the gain from an optimal stochastic control policy is a very aggregative multiplier-accelerator model that I constructed, Chow (1967), using annual data of the United States covering the years 1931–1940 and 1948–1963. There are four stochastic equations explaining the four dependent variables listed below.

$y_1 = \Delta C$ = first difference of total personal consumption expenditures, millions of current dollars.

$y_2 = \Delta I_1$ = first difference of gross private domestic investment in producers' durable equipment plus change in business inventories, millions.

$y_3 = \Delta I_2$ = first difference of new construction, millions.

$y_4 = \Delta R$ = first difference of yield of 20-year corporate bonds, annual percentage rate times 10,000.

$x_1 = \Delta M$ = first difference of currency and demand deposits adjusted in middle of the year, millions of current dollars.

$x_2 = \Delta G$ = first difference of government purchases of goods and services, millions.

Five other dependent variables are explained by identities, giving a total of nine structural equations as listed in Table 1. The reduced form equations, corresponding to (1.1), are given in Table 2. There are two exogenous variables, $x_1 = \Delta M$ and $x_2 = \Delta G$, both of which are assumed to be control variables. They

TABLE 1

STRUCTURAL EQUATIONS

	ΔC	ΔI_1	ΔI_2	ΔR	ΔY_1	I_1	I_2	ΔY	Y	ΔC_{-1}	$I_{1,-1}$	$I_{2,-1}$	Y_{-1}	ΔM	ΔG	ΔM_{-1}	P_{-1}	Intercept
(1)	−1	0	0	0	0.3083	0	0	0	0	0.1938	0	0	0	0.4079	0.0783	0	87.7	−4797
(2)	0	−1	0	0	0.2806	0	0	0	0	0	−0.6625	0	0.0090	0	0.1683	0	159	−6125
(3)	0	0	−1	−0.2198	0.1046	0	0	0	0	0	0	−0.5099	0.0410	0	0	0	92.8	−5996
(4)	0	0	0	−1	0.1109	0	0	0	0	0	0	0	0	−0.7389	0.1872	0.3178	0	−973
(5)	1	1	1	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	0
(6)	0	1	0	0	0	−1	0	0	0	0	1	0	0	0	0	0	0	0
(7)	0	0	1	0	0	0	−1	0	0	0	0	1	0	0	0	0	0	0
(8)	0	0	0	0	0.79	0	0	−1	0	0	0	0	0	0	0.79	0	0	0
(9)	0	0	0	0	0	0	0	1	−1	0	0	0	1	0	0	0	0	0

Source: Table 1 of Chow (1967), p. 9.

TABLE 2

REDUCED FORM EQUATIONS

Dependent Variable		Coefficient of ΔC_{-1}	$I_{1,-1}$	$I_{2,-1}$	Y_{-1}	ΔM_{-1}	ΔM	ΔG
(1)	ΔC	0.3744	−0.6173	−0.4751	0.0466	−0.0651	0.9393	0.2697
(2)	ΔI_1	0.1644	−1.2243	−0.4324	0.0514	−0.0592	0.4837	0.3425
(3)	ΔI_2	0.0470	−0.1606	−0.6335	0.0531	−0.0868	0.3007	0.0087
(4)	ΔR	0.0650	−0.2220	−0.1709	0.0168	0.2943	−0.5477	0.2561
(5)	ΔY_1	0.5857	−2.0023	−1.5411	0.1511	−0.2111	1.7236	0.6209
(6)	I_1	0.1644	−0.2243	−0.4324	0.0514	−0.0592	0.4837	0.3425
(7)	I_2	0.0470	−0.1606	0.3665	0.0531	−0.0868	0.3007	0.0087
(8)	ΔY	0.4627	−1.5818	−1.2174	0.1194	−0.1668	1.3617	1.2805
(9)	Y	0.4627	−1.5818	−1.2174	1.1194	−0.1668	1.3617	1.2805

Source: Solution of the structural equations of Table 1.

augment the above nine variables in a newly defined 11×1 vector y_t, in the notation of equation (2.3).

There are two sets of calculations to be presented in Table 3, one obtained by controlling the first differences of the variables, and the second by controlling the variables in their levels.[6] The initial conditions y_0 are those for the year 1964.[7] The target paths a_t of all expenditures variables (in first differences or in levels as the case may be) are set to grow by 5 percent per year starting from their historical values y_0. The target path for the first difference of the rate of interest is set equal to zero for all periods; the target path for the interest rate variable itself is set equal to 43,300 (or 4.33 percent per year times 10^4, its value in 1964). Since the model does not explain the price level P, and lagged price P_{-1} is used in the first three structural equations as a deflation device, we will assume that this exogenous variable grows by 2 percent per year from 120.7, its value in 1964. The time horizon T is 10 years.

The matrix K in the welfare function is assumed to be diagonal. Its non-zero diagonal elements have been chosen according to two major considerations: whether the interest rate variable should be weighted, and whether individual expenditures variables C, I_1, and I_2 should be weighted above and beyond their sum. Total of government expenditures is included in the welfare function, either by itself or as one component of a sum, but money supply is not because there is little rationale for doing so. Of course, the path of money supply is appraised through its effect on expenditures and, in some calculations, through its effect on the behavior of the rate of interest. The weight given to the interest rate, when it is present in the welfare function, is equal to that of an expenditure variable, implying that a deviation of 1 percentage point (or 10,000 in our units) from target is as costly as a deviation of 10,000 million dollars for an expenditure variable. Calculations using other weights for the interest rate variable, and for government expenditures, than those reported in Table 3 have also been performed, but they

[6] For brief comments on the relative merits and limitations of using levels and first differences in the welfare function, see Chow (1970b).

[7] See Chow (1967), p. 12, for data on y_0. The initial conditions will not affect G_t; therefore, the stochastic part of the welfare cost will not be affected.

TABLE 3

MEASURING WELFARE COSTS FOR OPTIMAL AND SUBOPTIMAL POLICIES
(10^6 millions)

Run	Welfare Weights on first Differences or levels of: C	I_1	I_2	R	Y_1	Y	G		Using First Differences: $\frac{1}{T}W_2$	$\frac{1}{T}W_1$	$\frac{1}{T}(W_1 + W_2)$	Using Levels: $\frac{1}{T}W_2$	$\frac{1}{T}W_1$	$\frac{1}{T}(W_1 + W_2)$
								Subopt.	547.6	24.1	571.7	639.2	2.1	641.3
(1)					1		1	optimal	388.4	0.0	388.4	388.4	0.0	388.4
								ratio	1.41			1.64		
								Subopt.	586.8	43.5	630.3	778.8	69.9	848.7
(2)	1	1	1			1	1	optimal	451.8	12.5	464.3	417.1	59.7	476.8
								ratio	1.30			1.87		
								Subopt.	358.9	21.6	380.4	460.7	1.0	461.7
(3)				1		1		optimal	257.5	0.0	257.5	257.5	0.0	257.5
								ratio	1.39			1.79		
								Subopt.	564.7	33.4	598.1	701.0	9.6	710.6
(4)				1	1		1	optimal	426.3	12.7	439.0	487.0	7.3	494.4
								ratio	1.32			1.44		
								Subopt.	603.9	50.8	654.7	840.6	73.3	913.9
(5)	1	1	1	1		1	1	optimal	477.4	21.9	499.2	520.3	64.1	584.4
								ratio	1.26			1.62		

provide the same orders of magnitude for the relative gains of the optimal policies over the suboptimal policies.

The first two runs in Table 3 give no weight to the interest rate variable, while the last three give a weight of 1 as specified above. Within these two groups, the runs are presented with increasing number of variables to be controlled. Note that, when the number of variables to be controlled is equal to the number of policy instruments (2 in our case), the deterministic components of these variables will reach their targets exactly, and the deterministic part of the welfare cost is therefore zero. Note also that the welfare cost is measured by a weighted sum of squared deviations in millions, so that 100 (10^6) for one expenditure variable would mean a standard deviation of 10 billion dollars.

As far as the five runs using first differences are concerned,[8] the gain of the optimal solution in the stochastic part of welfare cost varies between about 30 to 40 percent, and this part is much more important than the deterministic part. Hence, if the economic model contains stochastic disturbances, one can hardly afford to ignore them in the study of optimal policy. As far as the five runs using the levels of the variables are concerned,[9] the gain of the optimal solution in the stochastic part of welfare cost varies from about 40 to 80 percent, and, again, this part dominates the deterministic part of the welfare cost.

[8] All calculations for the stochastic part of welfare cost require the use of the covariance matrix V of the reduced-form residuals, for which see Chow and Levitan (1969).

[9] These calculations require the introduction of three more variables, i.e., the levels C, R, and Y_1, into our equations of Tables 1 and 2; these variables are explained by simple identities. Furthermore, the control variables are M and G, rather than ΔM and ΔG.

One might wish to ask why the relative gains of the optimal policies are greater for the calculations using levels of the variables than the corresponding gains using first differences. To answer this question, let us reexamine how the stochastic part of the welfare cost is calculated. For the optimal policy, we choose a linear feedback control equation $x_t^* = G_t y_{t-1}^*$ in such a way that the system under control

$$y_t^* = Ay_{t-1}^* + Cx_t^* + u_t = (A + CG_t)y_{t-1}^* + u_t$$

will have small weighted sum of variances. More precisely, we choose the matrix $G_t = -(C'H_tC)^{-1}C'H_tA$ to make the matrix $(A + CG_t)$ small, in the sense of having a minimum $\operatorname{tr}(A + CG_t)'H_t(A + CG_t)$. This is equivalent to regressing the columns of the matrix A on the columns of the matrix $-C$, with the columns of G_t as regression coefficients in a multivariate regression. For the sub-optimal policy, we set $G_t = 0$. The gain from the optimal policy is the gain (in reducing variances) by using a smaller matrix $(A + CG_t)$ in the above stochastic system rather than the matrix A itself. If the lag structure of the system as reflected in the matrix A becomes more complicated, with reference to a given matrix C, in such a way that the ratio of $\operatorname{tr} A'H_tA$ to $\operatorname{tr}(A + CG_t)'H_t(A + CG_t)$ becomes larger for the optimal G_t, then the gain from optimal control will be greater. Intuitively speaking, the more sparse A is, $A = 0$ being the extreme case, the less will be the effects of the lagged variables on the current state, and thus the less will be the gain from optimal feedback control. If this point is valid, one should expect larger gains from optimal control if he employs a quarterly model instead of an annual one, because there will be more lagged variables and the matrix A will be bigger in dimension and less sparse.

Granted that the optimal policy is definitely better than the suboptimal policy for a given welfare function, how much of the superiority would remain when judged by a different welfare function? To shed some light on this question, I have calculated the ratios of the suboptimal stochastic welfare costs to the optimal for Runs (1), (2), (4), and (5), using the welfare weights of Run (3) which are unity for R and Y only. These ratios are respectively 1.25, 1.32, 1.30, and 1.35 for the calculations in first differences; they are 1.28, 1.25, 1.42, and 1.41 respectively for the calculations in levels. The optimal policies are thus seen to be fairly robust against different welfare functions—recall that the optimal policies of runs (1) and (2) were derived without including the rate of interest in their welfare functions.

The gains from optimal policies having been measured, it would be of interest to examine the nature of the optimal feedback equations as reflected in the matrix G_t. Table 4 shows the matrix G_1 for all the runs of Table 3. In all calculations, the matrices G_t, beginning with G_{10} converge very rapidly—for three significant figures, Table 4 applies equally well to G_5. I will confine myself to two observations. First, concerning the relative roles of money supply and government expenditures, other things being equal, optimal money supply will become more active, as measured by the absolute values of the feedback coefficients, when the rate of interest occupies a less important position in the welfare function; optimal government expenditures will be more active when this variable occupies a less

TABLE 4

OPTIMAL FEEDBACK CONTROL MATRIX G

Run		ΔC_{-1}	$I_{1,-1}$	$I_{2,-1}$	Y_{-1}	M_{-1}	M_{-2}	C_{-1}	R_{-1}	$Y_{1,-1}$	G_{-1}
					Using First Differences						
(1)	ΔM	−0.3398	1.1616	0.8941	−0.0877	0.1225	−0.1225				
	ΔG	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000				
(2)	ΔM	−0.3459	0.9073	0.7028	−0.0688	0.1121	−0.1121				
	ΔG	0.0002	0.2047	0.1136	−0.0119	0.0038	0.0038				
(3)	ΔM	−0.0336	0.1149	0.0885	−0.0087	0.3997	−0.3997				
	ΔG	−0.3256	1.1130	0.8567	−0.0840	−0.2947	−0.2947				
(4)	ΔM	−0.2727	0.9337	0.7198	−0.0706	0.1821	−0.1821				
	ΔG	−0.0888	0.2929	0.2167	−0.0214	−0.0719	0.0719				
(5)	ΔM	−0.2547	0.6713	0.5206	−0.0510	0.1909	−0.1909				
	ΔG	−0.0879	0.4319	0.2881	−0.0290	−0.0712	0.0712				
					Using Levels of Variables						
(1)	M	−0.3398	1.1616	0.8941	−0.0877	1.1225	−0.1225			−0.5802	0.3602
	G	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000			0.0000	0.0000
(2)	M	−0.3612	0.7339	0.5206	−0.2291	1.096	−0.096	−0.6526			0.4463
	G	0.0116	0.2228	0.2179	−0.3170	0.0156	−0.0156	0.3757			0.2791
(3)	M	−0.0336	0.1149	0.0885	−0.2525	1.3997	−0.3997		1.2195		0.0000
	G	−0.3256	1.1130	0.8567	−0.6056	−0.2947	0.2947		−1.2967		1.0000
(4)	M	−0.2762	0.9460	0.7298	−0.0715	1.1785	−0.1785		0.2515	−0.5007	0.2444
	G	−0.0675	0.2206	0.1615	−0.0160	−0.0530	0.0530		−0.2602	−0.0796	0.1286
(5)	M	−0.2822	0.5693	0.4109	−0.2130	1.1624	−0.1624	−0.5172	0.2865		0.3169
	G	−0.0634	0.3786	0.3217	−0.3321	−0.0469	0.0469	0.2471	−0.2719		0.4024

important position in the welfare function. Secondly, because our consumption function (see Table 1) is of the form

$$\Delta C_t = 0.3083\ \Delta Y_{1,t} + 0.1938\ \Delta C_{t-1} + \dots$$

and our investment function, as derived from the accelerations principle through a stock-flow transformation, is of the form

$$I_{1t} = 0.2806\ \Delta Y_{1,t} + 0.3375\, I_{1,t-1} + \dots$$

or

$$\Delta I_{1,t} = 0.2806\ \Delta Y_{1,t} - 0.6625\, I_{1,t-1} + \dots,$$

a compensatory policy would be to react negatively to ΔC_{t-1} but positively to $I_{1,t-1}$ and $I_{2,t-1}$, as indicated by the coefficients in Table 4, and suggested by the coefficients in the reduced-form equation for ΔY in Table 2. If the above formulations of the consumption and investment functions are correct, monetary and fiscal policies should react differently to lagged consumption and to lagged investment expenditures as recommended here.

To provide a very crude check on the wisdom of actual government policies during the period 1948–1963 (the post World War II sample period of our model), I am reporting the following two regressions of ΔM_t and ΔG_t on ΔC_{t-1}, $(I_1 + I_2)_{t-1}$, ΔM_{t-1}, and linear trend t (t = 1948, ..., 1963), placing in parentheses below each regression coefficient its ratio to the standard error.

Observed Feedback Control Regressions
1958–1963

	ΔC_{-1}	$(I_1 + I_2)_{-1}$	ΔM_{-1}	t	Intercept	s	R^2
ΔM	-0.3183	0.0957	0.0898	-135.8	266,539.0	2,673	0.23
	(-1.66)	(0.58)	(0.30)	(-0.31)	(0.31)		
ΔG	-0.2115	0.7270	-0.1343	-2004.0	3,886,846.0	6,401	0.29
	(-0.46)	(1.85)	(0.19)	(-1.89)	(1.90)		

In these regressions, the two investment variables are combined to avoid too much multicollinearity; the trend and the intercept are used to represent g_t in the feedback control equation. Allowing for their standard errors, one can say that the coefficients of ΔC_{-1} and $(I_1 + I_2)_{-1}$ do tend to be respectively negative and positive as in an optimal policy, and that the first three coefficients (or four, if the coefficient of investment expenditures counts as two) are not very different from the optimal coefficients in Table 4—discount the row of zeros in the optimal equations for ΔG in run (1) because ΔG is given too much weight, and discount the row for ΔM in run (3) because ΔR is given too much weight. If uncertainties were ignored, and the above coefficients were used for the matrix G_t to calculate the stochastic part of welfare, one would obtain:

Run	1	2	3	4	5
$\frac{1}{T}W_2$ (regression)	566.1	536.2	288.8	583.0	553.1
$\frac{1}{T}W_2$ (suboptimal)	547.6	586.8	358.9	564.7	603.9
Ratio	0.97	1.09	1.24	0.97	1.09

Thus, a set of (non-stochastic) feedback control equations based on historical observations would not compare unfavorably with the suboptimal policy of $G_t = 0$. However, the above estimates of welfare gains are biased in favor of the observed regression policy because the standard errors s of the regressions and of the regression coefficients, which have been ignored in these calculations, would increase the variances of the system.[10] One cannot say, from this very crude analysis, that monetary and fiscal policies in the period 1948–1963 were destabilizing.

5. Concluding Remarks

In this paper, I have set forth a theoretical framework for measuring the welfare gains by following an optimal stochastic control policy as compared with a suboptimal policy which only permits a constant rate of growth for each policy

[10] One should at least incorporate the two observed feedback control equations as *stochastic* equations in the system by taking into account the random disturbances in them, but this would still leave out the possibly random nature of the regression coefficients. On the other hand, large residuals in the historical feedback regressions might be due to policies designed for non-economic reasons (such as financing and spending for the Korean War), and compensations for their effects later on.

variable, and have provided numerical measures of gains using a macro-economic model. I have found that the stochastic part of welfare cost, which takes into account the random disturbances of an econometric model, is much larger than the deterministic part which does not. If first differences enter the welfare function, the gain varies between 30 to 40 percent; if levels of the variables enter the welfare function, the gain varies between 40 to 80 percent, both in terms of weighted sum of squares of deviations from targets. By examining how the optimal stochastic control policy works in this framework, I have indicated that the complexities of the lagged structure of the system as reflected in the matrix A, for a given matrix C of the effects of the policy variables, will tend to make the gain from the optimal policy greater, implying, for example, that a model using quarterly observations is likely to yield larger gains than an annual model.

It has also been found that the gain from an optimal policy, which is derived from a given welfare function, is fairly robust against (reasonable) variations in the welfare weights for its evaluation. The characteristics of the optimal feedback policies have been examined. Historical feedback relationships have been crudely estimated by regressions, and they do not suggest that monetary and fiscal policies in the United States were destabilizing in the period 1948–1963.

Let me now comment briefly on the possibility of relaxing the three main assumptions stated at the beginning of this paper. As suggested by Athans (1971), for example, an analysis using a quadratic welfare function and a linear stochastic model can be applied to a problem involving non-quadratic welfare and non-linear model by, first, solving the deterministic version of the latter problem, a version that substitutes zero for random disturbances, with whatever method available (such as Pontryagin's minimum principle or dynamic programming), and, second, controlling the deviations of y_t from the optimal path obtained above after the linearization of the original model around this optimal path. This suggestion deserves further investigation, especially in view of the large differences between the optimal stochastic solutions and the optimal deterministic solutions that we have found in this paper.

If the model is linear and the welfare function quadratic, but the parameters are unknown and treated as random, it is well-known, and can easily be shown using techniques parallel to those of equations (2.23) to (2.32), that the optimal feedback equations will remain linear with matrices

$$G_t = -[E(C'H_tC)]^{-1}E(C'H_tA)$$

$$H_{t-1} = K_{t-1} + E(A + CG_t)'H_t(A + CG_t)$$

replacing those of equations (2.27) and (2.31) respectively, *provided* that the random matrices A and C have density functions which are unchanged during the planning period. If this proviso is accepted, as it is reasonable in many applications when the prior information on A and C at the beginning of the planning period dominates the additional information to be collected during the planning period, one has an analytical solution to the optimal control problem after evaluating the mathematical expectation of the product of any two elements of the matrices A and C by Bayesian methods, as is done in Chow (1971). I hope to report on the results of this approach in the near future.

Princeton University

REFERENCES

Athans, Michael, "The Role and Use of the Stochastic Linear-Quadratic-Gaussian Problem in Control System Design," *IEEE Transactions on Automatic Control*, Vol. AC-16, No. 6, (December 1971), pp. 529–552.

Bellman, R. E., *Dynamic Programming*, Princeton University Press, Princeton, New Jersey (1957).

Chow, G. C., "Multiplier, Accelerator, and Liquidity Preference in the Determination of National Income in the United States," *The Review of Economics and Statistics*, Vol. XLIX, No. 1, (February 1967), pp. 1–15.

———, "Friedman on Money," *The Journal of Finance*, Vol. XXV, No. 3, (June 1970), pp. 687–689 (a).

———, "Optimal Stochastic Control of Linear Economic Systems," *Journal of Money, Credit, and Banking*, Vol. II, No. 3 (August 1970), pp. 291–302 (b).

———, "Multiperoid Predictions from Stochastic Difference Equations by Bayesian Methods," Econometric Research Program Memorandum No. 123, Econometric Research Program, Princeton University, Princeton, New Jersey (April 1971).

———, "Optimal Control of Linear Econometric Systems with Finite Time Horizon," *International Economic Review*, Vol. 13, No. 1 (February 1972), pp. 16–25.

Chow, G. C., and Levitan, R. E., "Nature of Business Cycles Implicit in a Linear Economic Model," *The Quarterly Journal of Economics*, Vol. LXXXIII, (August 1969), pp. 504–517.

Friedman, M., "The Role of Monetary Policy," *The American Economic Review*, Vol. LVIII, No. 1 (March 1968), pp. 1–17.

Goldfeld, S. M., Quandt, R. E., and Trotter, H. F., "Maximization by Quadratic Hill-Climbing," *Econometrica*, Vol. 34, No. 3 (July 1966), pp. 541–551.

Pindyck, R., "Optimal Policies for Economic Stabilization," paper presented before the meetings of the Econometric Society at New Orleans, Dec. 1971, Mimeo.

Simon, H., "Dynamic Programming Under Uncertainty with a Quadratic Criterion Function," *Econometrica*, (January 1956), pp. 74–81.

Theil, H., *Economic Forecasts and Policy*, North-Holland Publishing Company, Amsterdam, 1958.