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6 Investing for the Short and the Long Term

Stanley Fischer

The expected real monthly return on Treasury bills is serially correlated, by some estimates following a random walk.[1] Expected real returns on stocks have a different dynamics. This means that the relative risk characteristics of stocks and bills differ depending on how long they are held.

For example, suppose that the expected real returns on Treasury bills are highly serially correlated and that expected real returns on stocks are less serially correlated. It is known that the variance of unexpected real returns on stocks, looking ahead one month, is about one hundred times the variance of the unexpected real return on bills. Stocks are of course much riskier than bills in the short run of a month.

Now consider an investor making a long-term portfolio decision to allocate his wealth between two mutual funds—a bill fund and a stock fund—with the proceeds being automatically reinvested in the fund in which they originate. Given the assumed serial correlation properties of asset returns, the longer the investment period, the less risky are stocks relative to bills.

Three questions are taken up in this chapter: (1) How does the term structure of risk arising from differences in the dynamics of asset returns affect optimal investment behavior? (2) What is the evidence on the dynamics of returns on stocks and bills in the United States? (3) Given the returns dynamics estimated in the paper, how do optimal portfolios change with the length of the holding period?

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In section 6.1 I distinguish between the *horizon* of the investor and the *portfolio holding period* and briefly review known results on the effects of the horizon on the investment decision. In section 6.2 I set out the dynamics of asset returns and uncertainty about those returns as a function of the length of the holding period of an asset. Optimal mean-variance portfolios for investors choosing between two assets each with returns following a first-order autoregressive process are calculated in section 6.3. The dynamics of returns on stocks and bills are described in section 6.4, using United States data since 1926. Section 6.5 presents the results of simulations of optimal portfolios for holding periods of different lengths, given the dynamics described in section 6.4. Section 6.6 contains comments on the applicability of the analysis to pension investing. Concluding remarks are in section 6.7.

6.1 The Investment Horizon and the Portfolio Holding Period

Consider an investor maximizing the intertemporal utility function

$$(1) \qquad V(\quad) = \theta E_0 \int_0^T e^{-\delta t} U[C(t)]\,dt + (1-\theta) E_0 B[W(T)],$$

where $C(t)$ is the rate of consumption at time t, $U[C(t)]$ is the instantaneous utility function, δ is the discount rate, $W(T)$ is real wealth at time T, $B(\quad)$ is a utility of bequests function, and θ is a constant, $0 \leq \theta \leq 1$.

Suppose first that θ is equal to zero, so that the individual maximizes only the expected utility of bequests $E_0 B[W(T)]$. In this case T is the *investment horizon*. The function $B(\quad)$ is also called the terminal utility of wealth function. Research on growth and turnpike portfolios (for example, Hakansson 1971, 1974; Leland 1972; Merton and Samuelson 1974; Ross 1974) has examined the effects of the length of the investor's horizon on optimal portfolio composition. The main question here is whether as the horizon lengthens all investors tend to hold the same portfolio—the general answer is no. There is a further question whether investors should or might want to maximize the expected growth rate of the value of the portfolio (subject to no short sales); again, the answer is in general no.

The investor's *holding period* is the interval of time between successive portfolio actions.[2] At one extreme, the investor may have an arbitrarily short holding period, engaging in continuous trading to rebalance the portfolio. At the other extreme, the investor may make his only portfolio decision at time zero and thereafter not be able to adjust the portfolio's composition. The important point is that the investor does not respond within the holding period to changes in actual and desired portfolio composition resulting from the behavior of asset returns.

For any given holding period, the individual solves the optimal portfolio problem from the recursion relations

$$J[W(t)] = \max E_t\{J[W(t+\Delta)]\},\ 0 \leq t < T, \tag{2}$$

with $J[W(T-\Delta)] = \max E_{T-\Delta}B[W(T)]$. In (2) $J(\ \ )$ is the indirect or derived utility function, and Δ is the length of the holding period. The maximization is conducted with respect to the composition of the portfolio.

Research on myopia in portfolio choice, by Mossin (1968), Hakansson (1970), and Samuelson (1969), considers the circumstances under which the investor's optimal portfolio is, for any given holding period, independent of the horizon. For utility functions with constant absolute or relative risk aversion, the investor's portfolio decision is independent of the length of the horizon, depending only on wealth. But, as shown by Goldman (1979), the composition of the optimal portfolio is *not* independent of the holding period, even when utility functions have constant relative risk aversion.

The holding period for an individual managing his own portfolio is likely to be finite but not constant. Portfolio rebalancing will be undertaken only at discrete intervals because it is costly. But the interval is not fixed because the need for rebalancing varies with the behavior of asset prices.

An investor who saves through regular contributions to a retirement fund, for which he specifies the breakdown of his portfolio between equities and bonds, may formally be permitted to change the composition of his retirement portfolio only once a year or every few years. However, if such an individual also has discretionary portfolio assets, he can effectively rebalance his portfolio more frequently than the rules of the retirement fund formally permit. He does this by using the discretionary funds to offset movements in portfolio composition in the retirement funds.

When the possibility of consuming at intermediate dates, $t < T$, is reinstated by setting θ in (1) at a value other than zero, the notion of the horizon loses its crispness. Date T is still the horizon in the sense that the individual looks no further ahead than T. But now events that occur at $t < T$ matter not only because they affect the situation at T but also because consumption at t and later depends on the state of the world at time t. Despite the ambiguity, I continue to refer to T as the horizon.

The notion of the portfolio holding period retains its meaning, however. Even if consumption is continuous, optimal portfolio behavior may involve infrequent rebalancing of the portfolio. Inventories of goods and liquid assets are used to finance consumption within the holding period, while the investment portfolio is rebalanced at discrete intervals.

The optimizing problem of the investor-consumer is again solved as in (2), with the aid of recursion relations and an indirect utility function. For any given frequency with which decisions are made, questions about myopia in portfolio behavior receive the same answers as they do without intermediate consumption.

I do not in this chapter analyze optimal investment strategy for an individual faced with costs of portfolio management and given dynamic properties of asset returns. Optimal strategy in such a case will involve a finite but not constant holding period. Instead, I study the simpler problem in which the holding period is given. My analysis focuses on the effects of the length of the holding period on the optimal composition of the portfolio when asset returns are serially correlated.[3] I assume that there are a significant number of individuals for whom the portfolio holding period is on the order of months or even years. For such individuals the distinction between the short-run and the long-run properties of asset returns may be important.

6.2 Rates of Return and the Length of the Holding Period

In this section I briefly examine the distribution of per period rates of return on an asset as a function of the number of periods for which it is held. The returns are assumed to follow a stable first-order autoregressive process.

Suppose the rate of return on an asset, r_t, is described by

$$\ell n(1 + r_t) \equiv x_t = \alpha + \beta x_{t-1} + \varepsilon_t, \tag{3}$$

where ε_t is serially uncorrelated and normally distributed with expectation zero and variance σ_ε^2.

Let W_N be the amount obtained by buying one dollar of the asset at the beginning of period 1 and reinvesting the returns for N periods. Then

$$\begin{aligned} \ln W_N &= \ln \Pi_1^N (1 + r_i) \\ &= \Sigma_1^N x_i . \end{aligned} \tag{4}$$

From (3) and (4),

$$\ln W_N = \frac{N\alpha}{1-\beta} + \frac{\beta(1-\beta^N)}{1-\beta}(x_0 - \alpha) + \Sigma_1^N \varepsilon_i \left(\frac{1-\beta^{N+1-i}}{1-\beta}\right). \tag{5}$$

W_N is therefore lognormally distributed, with

$$E(\ln W_N | x_0) = \frac{N\alpha}{1-\beta} + \frac{\beta(1-\beta^N)}{1-\beta}(x_0 - \alpha) \equiv m_N \tag{6}$$

and

$$\operatorname{var}(\ln W_N | x_0) = \frac{\sigma_\varepsilon^2}{(1-\beta)^2}\left\{N - \frac{\beta(1-\beta^N)}{1-\beta}\left[2 - \frac{\beta(1+\beta^N)}{1+\beta}\right]\right\} \equiv s_N^2 . \tag{7}$$

The expectation and variance of terminal wealth, W_N, are given by

$$E(W_N | x_0) = e^{m_N + s_N^2} \tag{8}$$

and

(9) $$\text{var}(W_N|x_0) = e^{2m_N + s_N^2}(e^{s_N^2} - 1).$$

Now define the expected rate of return per period on an N-period investment, $\mu(N)$, by

(10) $$N\mu(N) \equiv m_N.$$

The variance of the per period return, $\sigma^2(N)$, is defined by

(11) $$N\sigma^2(N) \equiv s_N^2.$$

Asymptotically, the per period expected rate of return is just $[\alpha/(1 - \beta)]$, with the additional term in (6) reflecting the effect on expected returns of initial conditions.

The per period variance of returns goes asymptotically to

(12) $$\lim_{N\to\infty} \sigma^2(N) = \frac{\sigma_\varepsilon^2}{(1 - \beta)^2}.$$

For $N = 1$, of course,

(13) $$\sigma^2(1) = \sigma_\varepsilon^2.$$

Thus the variance of the per period rate of return on an asset increases by a factor of $(1 - \beta)^{-2}$ as the number of periods for which it is held rises from one to many. For a highly autocorrelated series, $\beta = .9$, the ratio of the asymptotic to the one-period variance of the per period return is 100.

Table 6.1 shows how the variance of the per period rate of return changes with the number of periods for a first-order autoregressive process, for alternative values of β. The effects of the serial correlation on the variance of the per period return are highly nonlinear in the parameter β.

Table 6.1 **Variance of per Period Returns for a First-Order Autoregressive Process**

N	$\beta = 0.95$	$\beta = 0.9$	$\beta = 0.75$	$\beta = 0.5$
1	1.0	1.0	1.0	1.0
2	2.4	2.3	2.0	1.6
3	4.3	4.0	3.1	2.1
6	12.5	10.4	6.2	2.9
12	36.2	25.1	10.0	3.4
24	92.7	48.6	12.9	3.7
48	186.9	71.6	14.4	3.9
120	304.5	88.6	15.4	3.9

Note: Entries show variances of per period returns for holding period of length N relative to variance for one period.

6.3 Minimum Variance and Optimal Mean-Variance Portfolios

In this section I examine minimum variance and optimal mean-variance portfolios when asset returns follow first-order autoregressive processes like (3). The consumer-investor is understood to be maximizing an intertemporal utility function with indirect utility function that is quadratic in the portfolio return, and with portfolio holding period of length N.

Suppose there are two assets, 1 and 2, with returns described by

$$(14) \qquad x_{i,t} = \alpha_i + \beta_i x_{i,t-1} + \varepsilon_{it}, \qquad i = 1, 2,$$

with $E(\varepsilon_{1t}\varepsilon_{2t}) = \sigma_{12}$ and variances of ε_i denoted σ_i^2.

Define the variance of the per period rate of return for each asset by $\sigma_i^2(N)$, as in (11).[4] Let w be the share of the first asset in the portfolio and define the variance of the portfolio rate of return as

$$(15) \qquad \sigma_P^2(N) = w^2\sigma_1^2(N) + 2w(1-w)\sigma_{12}(N) + (1-w)^2\sigma_2^2(N),$$

where $\sigma_{12}(N)$ is the covariance of the per period rates of return, given by

$$(16) \qquad \begin{aligned} N\sigma_{12}(N) &= E\left[\Sigma_1^N \varepsilon_{1i}\left(\frac{1-\beta_1^{N+1-i}}{1-\beta_1}\right)\Sigma_1^N \varepsilon_{2i}\left(\frac{1-\beta_2^{N+1-i}}{1-\beta_2}\right)\right] \\ &= \frac{\sigma_{12}}{(1-\beta_1)(1-\beta_2)}[N - f(N)] \end{aligned}$$

where

$$\begin{aligned} f(N) \equiv{} & (1-\beta_1\beta_2)[\beta_1(1-\beta_2)(1-\beta_1^{N+1}) + \beta_2(1-\beta_1)(1-\beta_2^{N+1})] \\ & + \beta_1\beta_2(1-\beta_1)(1-\beta_2)[1-(\beta_1\beta_2)^{N+1}]. \end{aligned}$$

6.3.1 The Minimum Variance Portfolio

The minimum variance portfolio is given by

$$(17) \qquad w^*(N) = \frac{\sigma_2^2(N) - \sigma_{12}(N)}{\sigma_1^2(N) - 2\sigma_{12}(N) + \sigma_2^2(N)}.$$

In particular,

$$(18) \qquad w^*(1) = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 - 2\sigma_{12} + \sigma_2^2}$$

and

$$(19) \qquad w^*(\infty) = \frac{(1-\beta_1)^2\sigma_2^2 - (1-\beta_1)(1-\beta_2)\sigma_{12}}{(1-\beta_2)^2\sigma_1^2 - 2(1-\beta_1)(1-\beta_2)\sigma_{12} + (1-\beta_1)^2\sigma_2^2}.$$

The difference between the one-period and asymptotic minimum variance portfolios depends largely on $\beta_2 - \beta_1$:

$$w^*(\infty) - w^*(1) = \frac{(\beta_2 - \beta_1)\{\sigma_1^2(1-\beta_2)[\sigma_2^2 - \sigma_{12}] + \sigma_2^2(1-\beta_1)(\sigma_1^2 - \sigma_{12})\}}{D_1 D_\infty}, \tag{20}$$

where D_1 and D_∞ are the denominators of the expressions in (18) and (19), respectively.

For both β_1 and β_2 less than one in absolute value, and for zero covariance of asset returns ($\sigma_{12} = 0$), the minimum variance portfolio moves toward or away from stocks as the holding period lengthens, depending only on the sign of ($\beta_2 - \beta_1$). When asset returns are positively correlated ($\sigma_{12} > 0$), the direction of the shift in the minimum variance portfolio apparently becomes less certain. However, for stocks as the riskier asset (so $\sigma_1^2 > \sigma_{12}$) and provided $w^*(1)$ is positive (so $\sigma_2^2 > \sigma_{12}$), the direction of shift is of the same sign as ($\beta_2 - \beta_1$).

The composition of the minimum variance portfolio may be highly sensitive to the length of the holding period. To take a simple example, in which asset 1 should be thought of as stocks, assume that $\sigma_{\varepsilon 1}^2 = 99\sigma_{\varepsilon 2}^2$, $\sigma_{12} = 0$, $\beta_1 = 0$, and $\beta_2 = .9$. Then $w^*(1) = .01$ and $w^*(\infty) = .50$. As the holding period lengthens in this case, stocks take up a larger part of the minimum variance portfolio. It takes a holding period of 19 periods for the optimal share of the first asset in the portfolio to reach 25%. The 47.5% mark is reached only after 131 periods.

6.3.2 Mean-Variance Portfolios

Although the usual justifications for mean-variance portfolio analysis do not apply when portfolio decisions are made for the long term, it is instructive briefly to consider optimal mean-variance portfolios as a function of the decision period. If utility is defined as a function of the mean and variance of the portfolio returns, the optimal proportion of the first asset in the portfolio is

$$w^{**}(N) = A\left[\frac{\mu_1(N) - \mu_2(N)}{\sigma_1^2(N) - 2\sigma_{12}(N) + \sigma_2^2(N)}\right] + w^*(N), \tag{21}$$

where A is a measure of risk tolerance and $\mu_i(N)$ is the expected per period return on asset i.

In (21) we interpret the first asset as the stock, which has a higher expected return than bills. Two forces act on the portfolio as the horizon changes. In the first (excess return) term on the right-hand side of (21), the numerator stays constant as N increases while the denominator increases with N. Thus the asset holder will want to hold less of the stock when the riskiness of the excess return on stocks rises relative to the expected return, as the holding period lengthens. Second, as seen above, the share of the stock in the minimum variance portfolio, $w^*(N)$, changes as the holding period N increases.

For $\beta_2 > \beta_1$, the two portfolio effects—that through the first, excess return, term and that through the minimum variance portfolio—work in opposite directions. Thus the effects of changes in the holding period on the composition of the portfolio will be ambiguous for this mean-variance case, if $\beta_2 > \beta_1$. The net effect on the portfolio will depend on the parameters of the stochastic processes describing asset returns and on the investor's risk tolerance. The presumption is that if $\beta_2 >> \beta_1$, the shift in the portfolio will be toward stocks as the holding period lengthens, but if the serial correlation properties of the returns on the two assets are similar, it is less certain which way the portfolio will shift with the holding period.

6.3.3 Constant Relative Risk Aversion Portfolios

Mean-variance portfolio analysis is difficult to justify when the holding period is long. But it turns out that ambiguities similar to those noted above emerge when utility functions are isoelastic and asset returns follow diffusion processes.

Goldman (1979) has shown, for isoelastic utility functions, that portfolios become less diversified as the holding period lengthens when asset returns are generated by diffusion processes with no serial correlation. Portfolio proportions move away from one-half toward undiversified positions as the holding period lengthens.

When serial correlation of asset returns is introduced, there is an effect additional to that of Goldman on the composition of the portfolio (Fischer 1982). As the relative risk of assets changes with the holding period, the composition of the portfolio changes for that reason as well as the Goldman effect. The net effect depends on the relative strengths of the Goldman effect and the risk-aversion effect.

Portfolio analysis thus cannot unambiguously describe the effects of changes in the holding period on the composition of the portfolio. The effects depend on both the facts—the stochastic processes describing asset returns—and the investor's preferences. In the next section we turn to the facts.

6.4 Asset Returns

Although knowledge of the stochastic processes generating asset returns is essential to portfolio behavior, there is no consensus on what these processes are. Nor are there well-known competing estimates of the stochastic processes. In this section I first present evidence that there are both serial correlation in bill returns and differential returns dynamics of bill and stock returns. Then I present three alternative estimates of the stochastic processes generating asset returns.

Method 1 estimates a simple autoregressive model for real bill returns and then treats the real return on stocks as a function of the anticipated real rate on bills and lagged stock returns. This method has been used by Fama and Gibbons (1982).

Method 2 estimates a complete monthly vector autoregressive model of the economy, including stock and bill returns among the variables in the model. The vector autoregressive model implies the dynamics of stock and bill returns. Because the rate of inflation, growth rate of industrial production, and rate of money growth are included in the model, the dynamics of asset returns is potentially richer than in the simpler constrained processes estimated by method 1.

Both methods 1 and 2 at times imply that the expected real return on bills exceeds that on stocks. Method 3 therefore imposes a constraint, of a type implied by the capital asset pricing model, on the processes generating the returns.

This section ends with a comparison of the alternative estimates of returns.

6.4.1 Differential Returns Dynamics

Simple time-series properties of realized real rates of return on stocks and Treasury bills are suggested by table 6.2. Stock and bill returns are monthly Ibbotson-Sinquefield data from the Center for Research in Security Prices; stock returns are from the Standard and Poor's Composite Index. Real rates of return are calculated from the nonseasonally adjusted consumer price index.[5] Returns are measured as logarithms of one plus the return. Returns for more than one month are compounded for nonoverlapping periods.

The essential point made by table 6.2 is that the relative riskiness of stock returns falls with the length of the holding period. For data covering the entire 1926–80 period, the per period variance of returns on stocks is 100 times that on bills over a one-month holding period; over a one-year holding period, the variance of returns on stocks is 20 times greater than that on bills. The ratio of variances over 5-year holding periods is only 4.4, though this number should be treated with caution because it is based on only 11 5-year periods. A similar, though less dramatic, pattern holds over the 1948–80 period.[6] I will from this point on work with monthly data for the period 1948–80.

The per period variances in table 6.2 suggest both that stock returns are (approximately) serially uncorrelated and that bill returns are positively serially correlated. If stock returns were i.i.d., the per period variance would be independent of the length of the holding period. As it is, the per

Table 6.2 Real Monthly Returns on Stocks and Bills

	1926–80			1948–80		
	Stocks (1)	Bills (2)	Ratio (1)/(2)	Stocks (1)	Bills (2)	Ratio (1)/(2)
Mean return	.00511	−.00008		.00593	.00003	
Variance of returns per month						
1-month holding period	.362	.00353	102.5	.160	.00098	164.3
2-month holding period	.402	.00547	73.5	.164	.00140	117.2
4-month holding period	.355	.00826	43.0	.192	.00177	108.8
12-month holding period	.381	.01649	23.1	.192	.00330	58.3
60-month holding period	(.188	.04322	4.4)	(.326	.00683	47.7)

Note: The variances should all be multiplied by .01. Stock and bill returns are from the Ibbotson-Sinquefield File, Center for Research in Security Prices, University of Chicago; real returns are calculated using seasonally unadjusted CPI. Parentheses in last row of table are a reminder that statistics are based on only 11 and six data points, respectively.

period variance for stocks increases slightly with the length of the holding period. The per period variance of returns on bills rises more sharply with the length of the period.

Autocorrelation functions for real stock and bill returns are presented in table 6.3. Bill returns are significantly serially correlated, whereas stock returns are not. The autocorrelation function for bills suggests that the stochastic process for bill returns is something other than a first-order autoregression.

I now present three sets of estimates of the stochastic processes generating asset returns. Each method allows for correlation of stock and bill returns; such correlations can have a major impact on portfolio decisions. It will become clear below that method 3 is the preferred estimation method in this chapter, but methods 1 and 2 are included since they either have already appeared in the literature or else are typical of methods currently used to generate expectations.

6.4.2 Estimation Methods

Method 1

The first method of estimating bill and stock returns dynamics is that of Fama and Gibbons (1982). The procedure is first to estimate a simple ARMA model for real bill returns and then to relate stock returns to expected bill returns. The rationale for this approach is that models of capital asset pricing imply that expected real returns on stocks are related to expected returns on bills.

Table 6.4 presents estimates of a twelfth-order autoregressive process for the real bill rate, using data from the period 1948:2–1980:12. The

Table 6.3 **Autocorrelation Functions, Real Monthly Returns on Stocks and Bills, 1948:2–1980:12**

	Lag 1	Lag 2	Lag 3	Lag 4	Lag 5	Lag 6	Lag 7	Lag 8	Lag 9	Lag 10	Lag 11	Lag 12
Real bill rate	.39	.29	.22	.10	.09	.05	.05	.14	.15	.14	.14	.13
Real stock return	.03	−.01	.07	.10	.11	−.07	−.03	−.02	.06	−.03	.02	.08

	Lag 13	Lag 14	Lag 15	Lag 16	Lag 17	Lag 18	Lag 19	Lag 20	Lag 21	Lag 22	Lag 23	Lag 24
Real bill rate	.03	.04	.06	.06	−.01	−.05	−.10	0	0	.01	.05	.04
Real stock return	−.01	−.07	0	.03	.02	−.04	−.05	0	−.05	−.02	−.02	−.04

Notes: Real returns are calculated using seasonally unadjusted CPI; patterns using seasonally adjusted CPI show somewhat higher autocorrelations for real bill rates, no change for stocks. Standard error of coefficients is .05. (There are 395 observations.)

Table 6.4 **Bill and Stock Returns, Method 1, 1948:2–1980:12**

Equation (R1): Real Bill Rate[a]

Lag 1	Lag 2	Lag 3	Lag 4	Lag 5	Lag 6
.25 (1.89)	.10 (1.84)	.11 (2.09)	.04 (0.69)	−.03 (−0.55)	−.06 (−1.20)
Lag 7	**Lag 8**	**Lag 9**	**Lag 10**	**Lag 11**	**Lag 12**
.06 (1.27)	.10 (1.95)	.07 (1.51)	.09 (1.80)	−.11 (−2.30)	−.05 (−1.05)

Equation (R2): Real Stock Rate[b]

$$RS_t = \underset{(2.83)}{.0057} + \underset{(2.33)}{3.03}\ {}_{t-1}RB_t + \underset{(0.38)}{.019}\ RS_{t-1}$$

Note: ${}_{t-1}R_t$ is the expectation of RB_t formed at the end of period $t-1$, using eq. (R1). Numbers in parentheses are t-statistics.

[a]Regressed on constant, 11 seasonal dummy variables, and 12 lags of real bill rate. $\bar{R}^2 = .20$, SEE = .0028, D-W = 1.92, Q = 59.7.

[b]$\bar{R}^2$ = .0097, SEE = .0398, D-W = 1.99, Q = 61.5.

length of the autoregression was chosen to eliminate serial correlation in the residuals, as indicated by the Q-statistic. More parsimonious representations using moving average as well as autoregressive parameters did not improve on the properties of the real bill rate equation.[7]

The real return on stocks is then regressed on the expected return on bills, as computed in regression (R1). The real return on stocks is significantly positively related to the ex ante real return on bills. The share of the variance of realized stock returns accounted for by movements in the expected bill rate is however less than 1%. The standard error of estimate of the stock rate of return over the next month is almost 4%, at a monthly rate. Thus actual movements in real stock returns are hardly at all the result of changes in the expected rate, at least according to the estimates presented in table 6.4.

As a result of the constraints under which the stock and bill returns processes are estimated, the ex ante rates of return on stocks and bills follow very similar stochastic processes. The first-order autocorrelations of ex ante bill and stock returns are both about .7.

Method 2

Method 2 estimates a monthly five-variable vector autoregressive model of the United States economy for the period 1948:2–1980:12. The five variables are the rate of money growth (M1-B), the rate of inflation (CPI), the rate of growth of industrial production, the nominal bill rate, and real stock returns. Variables are not seasonally adjusted.

A vector autoregressive model (Sims 1980) imposes a minimum of theory in estimating dynamic equations. All variables are modeled as endogenous, lags are made long enough to eliminate any serial correlation of residuals in estimated equations, and no zero restrictions are imposed on coefficients beyond those implied by the choice of variables to include in the model and the length of lag.

The form of the model is

$$\mathbf{X}_t = \sum_{i=1}^{I} \mathrm{A}_i X_{t-1-i} + \mathbf{u}_t, \tag{22}$$

where $\mathbf{X}_t$ is the vector of (in this case five) included variables, the maximal lag length I has to be specified, the coefficients in the A matrices are to be estimated, and $\mathbf{u}_t$ is a white-noise vector of disturbances that may be contemporaneously correlated.

In the model estimated here the lag length was taken to be 12, both to eliminate serial correlation of residuals and to pick up any potential residual seasonal patterns that were not eliminated by the presence of seasonal dummy variables in each of the five equations. The Box-Pierce Q-statistic was used to indicate serial correlation.[8]

The lag coefficients were estimated imposing a Bayes-Litterman prior (Litterman 1980). The prior is that the model is purely first-order autoregressive, with each variable following a random walk. Thus priors are that the coefficient of the first own lag in each equation is unity and all other lag coefficients are zero.[9] Prior estimates of the standard deviations of the lag coefficients are that the standard deviations fall geometrically, with an imposed decay coefficient of .9. The standard deviation for the first own lag coefficient is estimated from a first-order autoregression. Standard deviations on coefficients of all other variables in an equation follow the same decay pattern as those on the own variable, but with standard deviations that are half those on the own variables.

The prior restrictions, which are tighter at the longer lags, reflect a general presumption that economic systems are low-order autoregressions. The priors typically prevent the alternation of coefficients that would be expected in any system in which the regressors are highly collinear.

Summary statistics from the five equations are presented in table 6.5. The regressions themselves contain too many parameters to be presented. The most striking feature of the system is the inability to predict stock returns well using the vector autoregressive approach. The F-statistic for the regression as a whole is not significant at the 5% level, though it is significant at the 10% level.

Table 6.5 **Method 2, Five-Variable Monthly Vector Autoregressive Model**

Equation (R3): *Real Stock Returns*
$\bar{R}^2 = .15$ SEE = .0369 D-W = 2.00
$Q = 42.9$ (significance level = .92)
F-statistics for sums of coefficients on each variable are not significant at 10% level for any of the variables. F-statistics for all coefficients not significant at 5% level.

(R4) *Nominal Bill Returns*
$\bar{R}^2 = .95$ SEE = .00051 D-W = 1.98
$Q = 40.9$ (significance level = .95)
F-statistics show strong significance of lagged bill returns; no other variables significant at 10% level.

(R5) *CPI Inflation Rate*
$\bar{R}^2 = .59$ SEE = .00256 D-W = 1.98
$Q = 58.5$ (significance level = .42)
F-statistics show strong significance of lagged inflation rate; lagged nominal bill rates are significant at 2% level, lagged money growth at 6% level. Sum of coefficients for each of these three variables is positive.

(R6) *Growth Rate of Industrial Production*
$\bar{R}^2 = .85$ SEE = .0116 D-W = 2.05
$Q = 30.4$ (significance level = .998)
Lagged stock returns, lagged industrial production and lagged money growth are all significant at 5% level. Sum of lagged coefficients is positive for all three variables.

(R7) *Growth Rate of Money*
$\bar{R}^2 = .92$ SEE = .0040 D-W = 1.99
$Q = 49.1$ (significance level = .76)
F-statistics show sum of coefficients on lagged nominal interest rates and lagged money growth strongly significant. Lagged stock prices have significance level of .08. Sum of lagged coefficients positive for all three variables. Coefficients on bill rate lagged one and two periods are both negative.

A vector autoregressive model of the type estimated here should be viewed as a statistically sophisticated extension of single-variable time-series forecasting methods. No attempt is made to estimate structural relations. The hypothesis implicit in the use of such models for forecasting purposes is that the underlying economic structure, including policy response functions, is stable. This approach is as vulnerable as more traditional econometric models to the Lucas policy evaluation critique that coefficients will change if policy rules change.

The model was used to form within-sample one-period-ahead forecasts of real rates of return on stocks and bills. These predicted rates are serially correlated. The first autocorrelation of the *real* return on bills (equal to the nominal rate of interest minus the predicted inflation rate) is .61. The first autocorrelation of the predicted return on stocks is .34. However, there is a seasonal pattern in stock returns, resulting in a twelfth-order autocorrelation of .56. The predicted rates of return on stocks have a high standard deviation, equal to 1.3% *per month*. The standard deviation of predicted real bill returns is 0.2% per month.

The high variability of the ex ante stock rate also produces occasions on which the expected return on stocks is lower than the expected return on bills. Rather than attempt to correct this problem by tightening the priors on the lag coefficients in the stock returns equation, I imposed a constraint of a type implied by the capital asset pricing model. This leads to method 3 for estimating bill and stock returns.

Method 3

Method 3 estimates a vector autoregressive model to generate expected real returns on bills and then uses the one-period-ahead forecast of the real bill rate from that model to estimate an equation for the predicted real return on stocks. The assumption is that

$$(23) \qquad {}_{t-1}RS_t - {}_{t-1}RB_t = a_{t-1}s_t^2 + e_t .$$

In (23), the left-hand-side variables are the expected real returns on stocks and bills, respectively. The variable ${}_{t-1}s_t^2$ is the expected or estimated variance of the excess return on the market. The variable e_t is random, and a is a parameter to be estimated.

Equation (23) is not exact, because the capital asset pricing model does not imply a constant value of the parameter a when the opportunity set is changing. The error term is included to reflect such changes. The coefficient a is estimated using the assumption that expectations of stock returns are rational. With rational expectations,

$$(24) \qquad RS_t = {}_{t-1}RS_t + v_t ,$$

where v_t is a serially uncorrelated error term with expectation zero. Substituting (24) into (23), we obtain the estimated equation

$$(25) \qquad RS_t - {}_{t-1}RB_t = a_{t-1}s_t^2 + e_t - v_t .$$

Some comments on (25). First, the structure of the error term in (25), or equivalently the form in which (25) is estimated, is not known or determined by a priori considerations. It is possible that e_t is heteroscedastic[10] and that the implicit assumption made in moving from (24) to (25) about the variance of v_t is inappropriate. Estimation of (25) in the alternative form

$$(25') \qquad \frac{RS_t - {}_{t-1}RB_t}{{}_{t-1}s_t^2} = a + \hat{e}_t - \hat{v}_t .$$

hardly affected the estimate of a, to be reported in table 6.6 below.

Second, (25) is constrained not to allow a constant. When a constant is added on the right-hand side, the constant is small and insignificant, the estimate of a falls a little, but a loses its statistical significance.

Third, it is necessary in (25) to use an estimate of the variance of the excess return on the market.[11] I experimented with variances of lagged realized stock returns over 12, 24, and 36 months. There were no major differences in the estimates of a. The final choice was the 36-month moving variance.[12]

Table 6.6 contains details of the estimated vector autoregressive system and of (R12), which is the estimated version of (25). The vector autoregressive system contains four variables, those of the previous section excluding the real return on stocks. Because real stock returns did not appear significantly in other equations in the five-variable model, the equations for the four-variable model are very similar to those estimated in method 2.

The estimate of a in (R12) is significantly different from zero. The predictive power for real stock returns of an equation like (R12) is of course extremely small. The implied value of the coefficient of relative

Table 6.6 Method 3, Estimated Four-Variable Vector Autoregressive Model and Stock Returns Equation

(R8) *Nominal Bill Returns*

$\bar{R}^2 = .94$ SEE = .00051 D-W = 1.97

$Q = 43.5$ (significance level = .91)

F-statistics show strong significance of lagged bill returns; significance level for inflation variables is .11.

(R9) *CPI Inflation Rate*

$\bar{R}^2 = .58$ SEE = .00259 D-W = 1.99

$Q = 56.0$ (significance level = .51)

Coefficients on lagged bill returns and lagged inflation are strongly significant; significance level for money variables is .08.

(R10) *Industrial Production*

$\bar{R}^2 = .84$ SEE = 0.0119 D-W = 2.04

$Q = 31.6$ (significance level = .998)

Coefficients on lagged industrial production and money are strongly significant.

(R11) *Growth Rate of Money*

$\bar{R}^2 = .91$ SEE = .00408 D-W = 2.00

$Q = 49.5$ (significance level = .75)

Lagged bill rates, industrial production, and money are significant at 5% level. Sum of coefficients on bill rate is positive; first two coefficients are large and negative. Sum of coefficients on industrial production is positive.

(R12) *Real Stock Returns*

$$RS_t = {}_{t-1}RB_t + \underset{(2.94)}{3.42}\,{}_{t-1}s_t^2$$

$\bar{R}^2 = .000046$ SEE = 0.0399 D-W = 1.95

$Q = 59.7$ (significance level = .38)

Variables are: ${}_{t-1}RB_t$ is expected *real* return on bills, equal to nominal rate minus ${}_{t-1}\pi_t$, the expected inflation rate from (R9). ${}_{t-1}s_t^2$ is the variance of real stock returns over the previous 36 months. t-statistics in parentheses.

risk aversion is about 3, corresponding to a utility function of the form $(-W)^{-2}$.

The expected real rate of return on stocks is now highly serially correlated. This is in large part a result of the serial correlation built into the method of creating the variance. The first autocorrelation for expected stock returns is .83. That for bills is 0.62, approximately the same as for method 2. The standard deviation of the expected real rate on stocks is now only 0.27% per month; that for bills is 0.18% per month. Expected stock returns always exceed expected bill returns, though the premium certainly varies. The highest premium is recorded in 1976 and is equal to 1.2% per month. The lowest premium occurs in early 1966 and is only 0.17% per month. If such variation is too large to be plausible, the source of the difficulty is no doubt to be traced to the variance estimator.

The purpose of estimating the alternative forecasting models is to use them in examining portfolio selection over different holding periods. In the next section, I use methods 2 and 3 to simulate the behavior of different portfolios over one- and 60-period holding periods.

6.5 Simulated Portfolio Results

The stochastic processes for bill and stock returns implied by methods 2 and 3 in the previous section were used in simulating the behavior of alternative portfolios over holding periods of one month and 60 months. The utility function was taken to be isoelastic, of the form

$$J(W_t) = W_t^{\gamma}/\gamma, \gamma < 1. \tag{26}$$

For $\gamma = 0$, we have the logarithmic utility function. The smaller is γ, the more risk averse the individual.

Four alternative utility functions were used to evaluate portfolio performance. They were the logarithm, $\gamma = -1.5, \gamma = -4$, and $\gamma = -10$. The last utility function has risk aversion well beyond any that is usually estimated. It is included because the less risk-averse utility functions show little inclination toward portfolio diversification.

The simulation procedure is to set each model off with starting conditions that are equal to historical means of the relevant variables over the estimation period. Drawings of the additive error terms in each equation are then made and first-period values of the variables in the simulation recorded. The process then repeats, with updated values of lagged variables (in the 60-period simulation) and keeps doing so to the end of the holding period.

Portfolios are allocated between bills and stocks, on a grid of 0.05, running from all stocks to all bills. The total return accumulated over the holding period by one dollar invested in each asset and the terminal wealth and utility obtained from each portfolio choice for each utility function are recorded for each simulation. There were 10,000 simulations of the portfolios generated using the stochastic processes of model 2, and 2,500 of the portfolios generated using model 3. The mean of the utility level attained under each portfolio choice for each set of simulations is calculated and taken to be an estimate of expected utility.

Because mean asset returns initially differed over the one-month and 60-month holding periods, the means of the returns on both bonds and stocks were adjusted in the one-month-holding-period simulations to be the same as those in the 60-month simulations. The identity of the reported means of asset returns in one- and 60-month simulations in each table is thus the result of calculation and not chance.

The simulated optimal portfolios in table 6.7 are heavily in stocks for both short and long horizons. Diversification only occurs for utility functions with high risk aversion. The most interesting result in the table, from the viewpoint of this discussion, is that lengthening the holding period shifts the portfolio *toward* bills, rather than away from them, for the highly risk-averse investors. These investors are probably reacting to the increasing riskiness of the excess return on stocks over the return on bills, even though the relative riskiness of stocks is falling. A second factor that may account for the result is that the covariance of bill and stock returns can move investors into bills as the horizon lengthens, even if the relative riskiness of bills is rising (Fischer 1982).

The results of 2,500 simulations made using the dynamics of method 3 estimates are shown in table 6.8. The levels of the optimal portfolios are very similar to those in table 6.7. This is to be expected since the interactions between stock returns and the rest of the system in method 2 were minimal.

Table 6.7 Simulated Optimal Portfolios, Method 2

Holding Period	Utility Function: $\ell n\ W$	$-W^{-1.5}$	$-W^{-4}$	$-W^{-10}$
1 month	1	1	1	.45
60 months	1	1	.85	.4

Statistics	Mean Bill Return per Month	Mean Stock Return per Month	Variance of Bill Return per Month (1)	Variance of Stock Return per Month (2)	(2)/(1)
1 month	$.160 \times 10^{-3}$	.00600	$.640 \times 10^{-5}$	.00125	195.3
60 months	$.160 \times 10^{-3}$	.00600	$.341 \times 10^{-4}$	.00163	47.9

Notes: Entries in first two rows are shares of stocks in optimal portfolio. There were 10,000 replications.

Table 6.8 Simulated Optimal Portfolios, Method 3

Holding Period	Utility Function: $\ell n\ W$	$-W^{-1.5}$	$-W^{-4}$	$-W^{-10}$	
1 month	1	1	.75	.35	
60 months	1	1	.80	.35	
Statistics	**Mean Bill Return per Month**	**Mean Stock Return per Month**	**Variance of Bill Return per Month (1)**	**Variance of Stock Return per Month (2)**	**(2)/(1)**
1 month	.00013	.00546	$.633 \times 10^{-5}$	.00162	256.4
60 months	.00013	.00546	$.330 \times 10^{-4}$	.00159	48.1

Notes: Entries in first two rows are shares of stocks in optimal portfolios. There were 2,500 replications.

However, the effects of the holding period on the optimal portfolio are now different from those in table 7. For all but one utility function, there is no change in the portfolio as the holding period changes. For the utility function $(-W)^{-4}$, the portfolio actually moves toward *stocks* as the holding period lengthens. This is more in accord with the intuition suggested by the discussion of Section 6.1, but it is not a strong effect. The effect is not a quirk of rounding, though. A search for optimal portfolios over a finer grid located the optimum for a one-month holding period at a share of .745 for stocks; for a 60-month holding period the optimum was .795.

There are two main conclusions from these simulations.

1. The differential dynamics of asset returns does not cause optimal portfolios to change dramatically with the length of the holding period. The direction of movement depends on the stochastic process generating portfolio returns. Because the stochastic process for method 3 is more soundly based, the results for this method should receive more weight. These indicate that the portfolio moves, if at all, toward stocks as the holding period lengthens.

2. For the specified utility functions, and given the historical behavior of stock and bill returns, portfolios are heavily in stocks.[13] Indeed, for utility functions consistent with estimated coefficients of risk aversion, portfolios are entirely in stocks.[14]

6.6 Pension Investments

Individuals investing in pension or retirement funds are investing for a long horizon. In some cases they are also, formally, investing for a long holding period, since the portfolio proportions may be changed only at discrete intervals, typically a year. The possibility that optimal portfolios differ depending on the holding period is relevant to such investing.

If the investor has other discretionary assets, he can use them to offset movements in the composition of the pension or retirement portfolio within the holding period for the latter portfolio. He may be able effectively to rebalance the portfolio continuously. Given the composition of the retirement portfolio, the individual's discretionary portfolio will hedge against changes in the retirement portfolio composition. But for those for whom the pension fund is the only asset, the holding period may be of the order of a year or several years.

Pension funds looking to create desirable long-term stock portfolios may also be concerned about the term structure of risk, something of which they are of course aware in the case of bonds. It is quite possible that some stocks may have relatively better long-term than short-term risk characteristics—though that cannot be demonstrated at the aggregate level of this chapter.

6.7 Summary

This chapter introduces the notion of the differential term structure of risk between stocks and bonds and then estimates stochastic processes for the generation of bill and aggregate stock returns. The stochastic process estimates are to be regarded as tentative, for it is clear that there are major problems in estimating these returns. Despite the difficulty, estimates of such processes are essential for making informed portfolio choices.

The raw data and the estimated processes show more serial correlation of bill returns than of stock returns. But estimated bill returns are not sufficiently highly serially correlated relative to stock returns to make them anywhere near as risky as stocks for even long holding periods.

The estimated returns processes are then used in stochastic simulations to estimate optimal portfolio proportions over different holding periods. There are two interesting findings. First, optimal portfolios change little as the holding period changes. The direction of movement depends on the estimated dynamic process for stock returns. Indeed, one of the implicit findings of this chapter is the lack of agreed or acceptable estimates of these dynamic processes. Second, and very striking, optimal portfolios for what are thought of as typical utility functions are very heavily in stocks.

Notes

1. The random walk hypothesis is not rejected by Nelson and Schwert (1977), Garbade and Wachtel (1978), and Fama and Gibbons (1982).

2. The terminology is slightly awkward. An alternative term is the portfolio decision period, which however is potentially misleading since for certain utility functions the investor keeps the portfolio composition fixed, and thus need make only one investment decision. Goldman (1979) uses the term "revision period."

3. Goldman has analyzed this question when asset returns are not serially correlated.

4. Note that x_i is the logarithm of one plus the rate of return, so that the variance is that of the logarithmic returns on the portfolio.

5. Use of the seasonally adjusted price index does not much affect the results.

6. There is one period in which the pattern seen in table 6.2 is absent, in that relative riskiness is independent of the holding period. This is the 1953–71 period—the period over which Fama (1975) showed the real interest rate on bills was constant.

7. There is a question about the interpretation to be placed on the coefficients in regression (R1). Suppose, as is assumed by Fama and Gibbons (1982), that the stochastic process generating real bill returns is one between expected real rates. Thus,

$$_{t-1}RB_t = a + b_{t-2}RB_{t-1} + e_t, \qquad \text{(F1)}$$

where a and b are constants and $_{t-i}RB_{t-i+1}$ is the expected bill rate. Given that, under rational expectations,

$$RB_t = {}_{t-1}RB_t + v_t, \qquad \text{(F2)}$$

where v_t is serially uncorrelated, there is an error in variables problem when (F1) is estimated using realized bill rates of return. The estimated coefficient $\hat{b}$ is biased downward from the true b if (F1) is estimated as a first-order autoregression.

If one is willing to assert a priori that the true relation is a first-order autoregression, the coefficient b can be identified by estimating a (1, 1) ARMA model for the realized bill rate. It was by using a restriction of this type that Fama and Gibbons concluded that the ex ante bill rate follows a random walk—they were not able to reject the hypothesis that b in (F1) was equal to one. However, separate knowledge of the coefficient b is not needed to form optimal forecasts of the real bill rate when there are errors in variables and no information other than realized bill rates to identify the expected real rate. The optimal forecast is obtained from the appropriate ARMA regression on *realized* bill rates. Thus, from a forecasting viewpoint the interpretation of the coefficients in (R1) is not important.

8. The need for a twelfth-order system arose from the presence of serial correlation in the money growth equation residuals for shorter lags.

9. An exception was made for stocks, for which the prior was that returns were white noise plus a mean.

10. This possibility has been emphasized by Merton (1980) in his exploratory estimation of market returns.

11. Fools rush in, despite the good example of Black (1976). The hope is that this foolishness will encourage those less foolish to do better.

12. The assumption that the estimated stock market variance is formed in this way is obviously crude. In work in progress, Olivier Blanchard and I are attempting to provide a more sophisticated model for the variance.

13. The results of the simulations are consistent with typical estimates of coefficients of relative risk aversion as being around 2. These estimates are based on the market risk premium. In equilibrium, the desired portfolio for the "market" must be the market portfolio, in which Treasury bills play only a small part. Hence the simulated optimal portfolios should have only a small share of Treasury bills.

14. What about taxes, it may be asked. The assumption is that the asset returns are untaxed. Alternative assumptions about taxation could be incorporated in future simulations of optimal portfolios.

Comment Fischer Black

I will start by restating some of Stanley Fischer's points in my own words. Then I will ask some of the questions that his analysis raised in my own mind.

Assume that you have a portfolio containing a single kind of security, like stocks or bills, and that you put all returns from the portfolio back into the portfolio. You reinvest all dividends or interest payments in shares of the same portfolio. Let us look at the variance of the value of this portfolio at the end of a period of fixed length. No matter how long the period is, the variance will be higher for a portfolio of stocks than for a portfolio of bills. Now let us set the length of the period at zero and raise it gradually. The variance will increase for any portfolio. It will increase faster in percentage terms for a portfolio of bills than for a portfolio of stocks. In arithmetic terms, it will increase faster for stocks than for bills.

Assume one person has high fixed transaction costs for going into or out of stocks while another person does not. The first person will face a higher cost of adjusting her portfolio as conditions change. She will not be able to move freely between stocks and bills. Then she will want to hold less in stocks, on average, than the second person. This is true whether or not part of her portfolio is in a defined-contribution pension fund. In equilibrium, the average person will hold the market portfolio of all risky assets. A more risk-averse person will mix the market portfolio with lending, perhaps by holding a portfolio of bills, while a less risk-averse person will mix the market portfolio with borrowing. These points I understand. Other points raised questions in my mind. Some of the questions that came to me are as follows.

Why should an investor be interested in a portfolio strategy with reinvestment of all returns in a single kind of security or a mix of two such strategies? Because a limited number of defined-contribution pension plans currently impose it on their participants? Why not consider a broader class of strategies?

Why should anyone have high fixed transaction costs or a long holding period? Aren't transaction costs on no-load mutual funds negligible? Can't transfers be made frequently between two such funds in a single family of funds? Are the costs of deciding to make such transfers high, at

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the margin? Do most people have their marginal savings in pension claims that are hard to adjust?

Why is the investigator interested in any model other than a generalization of equation (23)? Shouldn't a sensible model say that the expected return on stocks changes in response to changes in risk and past returns and other shocks and then drifts gradually back toward a mean? Shouldn't the mean itself depend on risk and other observables?

Why should we model an individual as caring about his real bequest? If he doesn't have children or care about them, he won't derive utility from bequests. If he cares about his children, won't his utility simply depend on their utility? For this paper, it does not matter much which assumption is made, but doesn't it matter in other contexts?

When one maximizes an expected utility function with consumption at various times and the utility of children, and when transaction costs are zero, investing for the short term and investing for the long term cease to be distinct, as Stanley Fischer noted at the end of the first section of his chapter.

References

Black, F. 1976. Studies of stock price volatility changes. In *Proceedings of the business and economic statistics section*, American Statistical Association, Washington, D.C.

Fama, E. 1975. Short-term interest rates as predictors of inflation. *American Economic Review* 65 (June):269–82.

Fama, E., and Gibbons, M. 1982. Inflation, real returns and capital investment. *Journal of Monetary Economics* 9 (May):297–324.

Fischer, S. 1982. Serial correlation properties of asset returns and optimal investment strategies. Unpublished paper, Department of Economics, MIT.

Garbade, K., and Wachtel, P. 1978. Time variation in the relationship between inflation and interest rates. *Journal of Monetary Economics* 4 (November): 755–66.

Goldman, M. B. 1979. Anti-diversification, or optimal programs for infrequently revised portfolios. *Journal of Finance* 34 (May): 505–16.

Hakansson, N. 1970. Optimal investment and consumption strategies under risk for a class of utility functions. *Econometrica* 38: 587–607.

———. 1971. Multi-period mean-variance analysis: Toward a general theory of portfolio choice. *Journal of Finance* 26:857–84.

———. 1974. Convergence to isoelastic utility and policy in multiperiod portfolio choice. *Journal of Financial Economics* 1:201–24.

Leland, H. 1972. On turnpike portfolios. In *Mathematical methods in* *investment and finance*, ed. G. P. Szego and K. Shell. New York: American Elsevier.

Litterman, R. 1980. A Bayesian procedure for forecasting with vector autoregressions. (Unpublished paper.) Federal Reserve Bank of Minneapolis.

Merton R. 1980. On estimating the expected return on the market. *Journal of Financial Economics* 8: 323–61.

Merton, R., and Samuelson, P. A. 1974. Fallacy of the log-normal approximation to optimal portfolio decision-making over many periods. *Journal of Financial Economics* 1: 67–94.

Mossin, J. 1968. Optimal multiperiod portfolio policies. *Journal of Business* 41: 215–29.

Nelson, C., and Schwert, G. W. 1977. On testing the hypothesis that the real rate of interest is constant. *American Economic Review* 67 (June): 478–86.

Ross, Stephen. 1974. Portfolio turnpike theorems for constant policies. *Journal of Financial Economics* 1: 171–98.

Samuelson, P. 1969. Portfolio selection by dynamic stochastic programming. *Review of Economics and Statistics* 60 (August): 239–46.

Sims, C. 1980. Macroeconomics and reality. *Econometrica* 48 (January): 1–40.