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Measuring the Output and Prices of the Lottery Sector
An Application of Implicit Expected Utility Theory

Kam Yu

10.1 Introduction

This chapter studies the output and price measurement of the lottery sector using an economic approach. Perhaps as a result of the accumulating effects in jackpots when there are no major prize winners in previous weeks, lottery industries in Canada and elsewhere are growing steadily. In 1997, according to the Survey of Household Spending (SHS), 68.4 percent of all households in Canada bought government-run pool and lottery tickets, with the average expenditure per household equal to $238, which translates to 0.3 percent of total expenditures. Expenditure in gambling, however, has been found to be consistently underreported in the SHS. The actual amount of money spent on gambling, according to revenue reported by the government, is three times the amount reported by households (Marshall 1998, 31). Therefore, the lottery industry has become a significant part of the gross domestic product (GDP) and a more accurate method of measuring its output is needed. Moreover, prices in any game of chance are not currently included in the Consumer Price Index (CPI). If we are able to calculate the real output of a lottery, then an implicit price index can also be computed. This price index can be used both as a deflator in the national accounts and as a subindex in the CPI.

In the theory of consumption under uncertainty, the typical consumer

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is traditionally assumed to follow an optimal decision rule with risk-averse preferences. This leads to the well-known expected utility hypothesis (EUH) in which the degree of risk averseness is often assumed to be decreasing in wealth. A wealthy person is more willing to invest in a risky but high-yielding portfolio than an average person. The EUH has been successfully applied to problems in insurance and financial investment. Its linear structure, however, also implies that a risk-averse expected utility maximizer will never buy lottery tickets, unless the payout prizes are exceedingly large. In reality, we observe that consumers who are fully insured in their houses and cars also engage in a variety of gambling activities. Therefore, we need a different approach other than the EUH. In the past two decades, new theories on economic uncertainty have been developed. For example, Diewert (1995) shows that the real output of a simple gambling sector can be measured using implicit expected utility theory. This theory successfully models consumers' risk averseness involving large portions of their wealth, and at the same time, it captures risk-seeking gambling activities involving small amounts of money. In this chapter, Diewert's model will be generalized from a simple two-outcome lottery to an N-outcome one (the 6/49 lotto has six outcomes with different payouts). The functional form of the estimating equation will be derived and estimated with Canadian data.

The portion of government output in the national accounts of industrialized countries has been increasing over the past several decades. There has been an ongoing debate on the concept and practice of measuring government output. Due to the absence of market prices in government services, statistical agencies traditionally use total factor costs as a proxy for the output. This practice has become less acceptable as the government sector has expanded. The Inter-Secretariat Working Group on National Accounts (1993)[1] recommends that government output should be measured directly whenever possible. In fact, statistical bureaus in Australia, the United Kingdom, and the Netherlands have switched to various forms of direct methods recently. In the case of government lotteries, the price of a lottery ticket is not an appropriate price to measure the output of the lottery. In the absence of a suitable output price, government statisticians usually take lottery total factor cost as a proxy for the value of output and use the CPI to deflate this value into a measure of real output. This chapter proposes a more satisfactory direct method of measuring government services in lotteries. Our results show that by using a direct utility approach, the measured output of Lotto 6/49 in Canada is three times higher than the official statistics. We also find that the estimated price elasticity of demand is found to be very similar to those of other countries.

This chapter also addresses a question raised by Hulten (2001), who contrasted the consumer's perspective in measuring output with the pro-

1. This manual is often referred to as SNA93.

ducer's perspective. From the producer's perspective, the lottery corporations simply provide a service to consumers to redistribute income after each draw.[2] Therefore, output can be interpreted as the fee charged by the lottery corporations to provide the services. In this chapter, we take the view that for any services involving risk, the ex ante welfare of the consumers is more relevant. It seems if we do not take this point of view, the insurance and gambling industries are simply wasteful.

The structure of the chapter is as follows. Section 10.2 examines the classical and new economic theories of uncertainty and some of their applications. In section 10.3, we briefly discuss the gambling sector in Canada and apply the new theory to the economics of a lottery. A money-metric measure of the real output of the sector will be derived. In practice, a two-parameter equation is estimated using a nonlinear regression. The next step is to use the Canadian Lotto 6/49 as an example to test the feasibility of the model. The results are presented in section 10.4. Finally, section 10.5 concludes.

10.2 The Economic Analysis of Risk: A Brief Review

10.2.1 The Expected Utility Hypothesis

The classical analysis of economic uncertainty begins with Friedman and Savage (1948) and Von Neumann and Morgenstern (1953). Their writings form the basis for what is generally known as the expected utility hypothesis. The EUH has been successfully applied to a number of economic problems, such as asset pricing and insurance. It has also been used as the premise in statistical decision theory.[3] In the basic model, the uncertainty is represented by a set of simple lotteries $\mathcal{L}$ over a set of outcomes $\mathcal{C}$. A simple lottery $L \in \mathcal{L}$ in the discrete case can be represented by a vector of outcomes and a vector of probabilities; that is, $L = (p_1, p_2, \ldots, p_N)$, where $\sum_{i=1}^{N} p_i = 1$. This notation means that outcome $C_i \in \mathcal{C}$ will occur with probability p_i, $i = 1, \ldots, N$. A consumer or a decision maker is assumed to have a complete and transitive preference structure $\succsim$ on $\mathcal{L}$. In addition, the preferences are supposed to be continuous and independent. The latter assumption means that for all $L, L', L'' \in \mathcal{L}$ and $0 < \alpha < 1$, we have

$$L \succsim L' \text{ if and only if } \alpha L + (1 - \alpha)L'' \succsim \alpha L' + (1 - \alpha)L''.$$

Therefore, the ranking on L and L' remains unchanged if we mix the lotteries with another one to form compound lotteries. Together, the continuity and independence assumptions imply the existence of an expected utility function $U : \mathcal{L} \to \mathbb{R}$, such that

2. This perspective follows the treatment of insurance services from the viewpoint of a producers' approach to output measurement in risky industries.
3. See, for example, Luce and Raiffa (1957) and Pratt, Raiffa, and Schaifer (1995).

$$U(L) = \sum_{i=1}^{N} u_i p_i, \tag{1}$$

where u_i, $i = 1, \ldots, N$ are utility numbers assigned to the outcomes $C_i \in \mathscr{C}$, respectively. Therefore,

$$L \succsim L' \text{ if and only if } U(L) \geq U(L').$$

The independence assumption, which gives rise to the linear structure of the expected utility function, has been controversial from the beginning. Samuelson (1952) defends the independence axiom by arguing that in a stochastic situation, the outcomes C_i are mutually exclusive and therefore are statistically independent. Consequently, $U(L)$ must be additive in structure. Moreover, using a theorem by Gorman (1968), Blackorby, Davidson, and Donaldson (1977) show that continuity and independence imply that the utility structure under uncertainty is additively separable.

In spite of its solid theoretical foundation and normative implications, some applications of the EUH do not conform well with real behavior.[4] The most serious challenge is the Allais (1953) paradox, which can be illustrated by the following example. It involves decisions over two pairs of lotteries. The outcomes are cash prizes $(C_1, C_2, C_3) = (\$2{,}500{,}000; \$500{,}000; 0)$. In the first pair, the subjects are asked to choose between $L_1 = (0, 1, 0)$ and $L_1' = (0.10, 0.89, 0.01)$. That is, L_1 is getting \$500,000 for sure, while L_1' has a 10 percent chance of winning \$2,500,000, an 89 percent chance of winning \$500,000, and a 1 percent chance of winning nothing. The second part involves choosing between $L_2 = (0, 0.11, 0.89)$ and $L_2' = (0.10, 0, 0.90)$. Allais claims that most people choose L_1 and L_2'. This contradicts the EUH, because if we denote u_{25}, u_{05}, and u_0 to be the utility numbers that correspond to the three prizes, then $L_1 \succ L_2$ means that

$$u_{05} > 0.1u_{25} + 0.89u_{05} + 0.01u_0.$$

Adding $0.89u_0 - 0.89u_{05}$ to both sides of the above inequality gives

$$0.11u_{05} + 0.89u_0 > 0.1u_{25} + 0.9u_0.$$

This implies people should choose L_1' instead of L_2'.

The linear structure of the EUH also implies that a risk-averse consumer will never gamble, even for a fair game, no matter what the degree of risk aversion the consumer has.[5] Friedman and Savage (1948) try to correct this problem by proposing a utility function u with concavity varying with wealth level. This ad hoc fix does not solve the problem for small gambles, because both the normal wealth level and the payout prizes are far out in the concave

4. See, for example, Machina (1982), Rabin (2000), and Rabin and Thaler (2001).

5. See Diewert (1993, 425). Rabin and Thaler (2001) provide numerical illustrations on the absurdity of some implications of the EUH. Also see comments by Watt (2002) and the response from Rabin and Thaler.

section of u, given the insurance-buying behavior of the typical consumer. Cox and Sadiraj (2001) propose a new expected utility of income and initial wealth model, which assumes that the outcomes are ordered pairs of initial wealth and income (prize). Their model may have applications in other areas, but they concede that "the empirical failure of lottery payoffs is a failure of expected utility theory" (16). The EUH may be a good theory in prescribing how people should behave, but it fails as a model to describe how people actually behave. Therefore, in order to model a small gamble like the Lotto 6/49, we need a preferences structure that is more flexible than the EUH.

10.2.2 Nonexpected Utility Theories

Most of the theories developed to resolve the Allais paradox involve replacing or relaxing the independence axiom.[6] For example, by taking a general approach to the idea of a mean function, Chew (1983) replaces the independence axiom with the betweenness axiom. Instead of discrete probabilities on events in $\mathscr{C}$, let $\mathscr{L}$ now denote the set of probability distribution functions. The betweenness axiom assumes that for all F and G in $\mathscr{L}$, $F \sim G$ requires that

$$\alpha F + (1 - \alpha)G \sim F, \qquad 0 < \alpha < 1, \tag{2}$$

where $F \sim G$ means $F \succsim G$ and $G \succsim F$; that is, the consumer is indifferent between the lotteries F and G. This means that if a consumer is indifferent between lotteries F and G, then every convex combination of F and G is indifferent to them as well. As a consequence, the indifference curves are still straight lines. The independence axiom in EUH, on the other hand, can be characterized as

$$F \sim G \Rightarrow \alpha F + (1 - \alpha)H \sim \alpha G + (1 - \alpha)H, \qquad 0 < \alpha < 1, \tag{3}$$

for any $H \in \mathscr{L}$. We can see that equation (3) reduces to equation (2) if $H = F$. The involvement of a third lottery H in equation (3) implies that the indifference curves are parallel straight lines. This additional restriction gives rise to the Allais paradox. The betweenness axiom together with other regularity conditions imply that preferences can be represented by a general mean function $M : \mathscr{L} \rightarrow \mathbb{R}$, such that[7]

$$M(F) = \phi^{-1}\left(\frac{\int \alpha \phi dF}{\int \alpha dF}\right), \tag{4}$$

where ϕ is a strictly monotonic and increasing function, and α is a continuous and nonvanishing function, both on the domain of F. In equation (4), ϕ is similar to the Von Neumann-Morgenstern utility function u in equa-

6. For surveys of the nonexpected utility theories, see Epstein (1992), Machina (1997), and Starmer (2000).
7. For details, see Chew (1983), Dekel (1986), and Epstein (1992).

tion (1), while α is an additional weighting function. The mean function M can be interpreted as the certainty equivalent of F.[8] This two-parameter generalization of the EUH is less restrictive and can be used to resolve the Allais paradox.

Other developments in nonexpected utility theory include, for example, Kahneman and Tversky's (1979) prospect theory, Gul's (1991) theory of disappointment aversion, and the rank-dependent utility theory.[9] In Gul's analysis, for example, a lottery is decomposed into an elation component and a disappointment component. A weak independence axiom is defined in terms of the elation/disappointment decompositions of lotteries. The combination of disappointment aversion and a convex Von Neumann-Morgenstein utility function may represent preferences that are risk averse to even chance gamblers and gamblers facing large losses with small probabilities, but also to risk-loving gamblers facing large prizes with small probabilities. Basically, this provides the fanning out effect to avoid the Allais paradox (Machina 1997).

Using the contingent commodity approach of Arrow (1964) and Debreu (1959), Diewert (1993) develops an implicit utility function as follows:

$$\sum_{i=1}^{N} p_i \phi_u(x_i) - \phi_u(u) = 0, \tag{5}$$

where $\phi : \mathbb{R}^2 \to \mathbb{R}$ is function of the utility u and x_i. In this formulation, $x_i = f(y_i)$, where y_i is a choice vector in the state of nature i, $i = 1, \ldots, N$, and f is the consumer's certainty utility function.[10] The function $u = F(y_1, y_2, \ldots, y_N)$ is the consumer's overall state-contingent preference function. Notice that u is implicitly a solution of equation (5). For aggregation purposes, if we assume that the consumers have homothetic preferences, equation (5) reduces to

$$\sum_{i=1}^{N} p_i \gamma\left(\frac{x_i}{u}\right) - \gamma(1) = 0, \tag{6}$$

where γ is an increasing and continuous function of one variable.

A common property of nonexpected utility theories is that they can represent consumers with first-order risk aversion, which implies that the risk premium of a small gamble is proportional to the standard deviation of the gamble.[11] For a consumer with an expected utility function, on the other hand, second-order risk aversion is exhibited, where the risk premium is

8. In the context of equation (1), the certainty equivalent $\mu(L)$ of lottery L is defined as $u[\mu(L)] = U(L)$. For a risk-averse decision maker, the risk premium of L is the difference between the expected value of L and $\mu(L)$.

9. See, for example, Yaari (1987), Chew and Epstein (1989a), Quiggin (1993), and Diecidue and Wakker (2001).

10. The function f is the counterpart of the Von Neumann-Morgenstein utility function.

11. See Segal and Spivak (1990) and Epstein (1992).

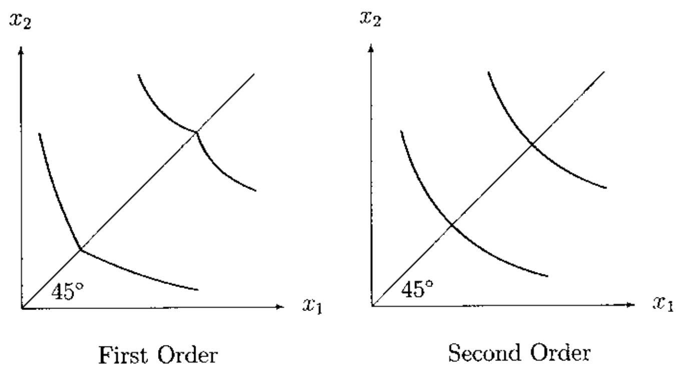

Fig. 10.1 First- and second-order risk aversion

proportional to the variance of the gamble. The difference can be illustrated graphically for the case $N = 2$. In figure 10.1, x_1 and x_2 represent the monetary outcome of states of nature 1 and nature 2, respectively. We assume that $p_1 = p_2 = 1/2$, so the indifference curves are symmetric about the forty-five degree certainty line. First-order risk aversion is represented on the left with a kink at the certainty line, whereas second-order aversion is represented by the smooth indifference curve on the right.[12] At the lower indifference curve, the kink implies a lot of risk aversion, and the consumer would not want to gamble for low levels of income (and will be willing to pay a premium for insurance). At the higher indifference curve, there is now a willingness to engage in small gambles (and the premium the consumer is willing to pay for insurance is now less). Extensive discussion on this point can be found in Diewert (1993).

Intuitively, both standard derivation and variance are statistical measures of spread of the distribution. In other words, they measure how far the random variable is away from the mean over the whole distribution. The standard derivation is conceptually equivalent to measuring the absolute distance between the variable and the mean, $|x - \mu|$, while the variance is equivalent to measuring the square of distance, $(x - \mu)^2$. This explains the kinks for the first-order risk aversion and the smoothness in the second-order risk aversion.

10.2.3 Applications of the New Theories

The EUH has been applied to many areas in economics involving uncertainty. Because observed behavior and experimental results sometimes contradict the theory, it is interesting to see whether the nonexpected utility theories can be successfully applied to those areas. In this section, we

12. Machina (2001) provides a detailed discussion of kinks on an indifference curve.

review some applications of the newly developed theory to intertemporal consumption analysis, asset pricing, and output analysis in insurance and gambling.

Chew and Epstein (1989b) first extend the implicit expected utility to an axiomatic analysis of a two-period intertemporal preferences. They find that in order for the new theory to be admissible, one of the two axioms (consistency and timing indifference, which imply the EUH) has to be relaxed. The application is later extended to the case of multiple-period consumption-saving decision with a recursive structure (Chew and Epstein 1990; Epstein 1992). In traditional consumption-saving analysis, the use of a one-parameter utility function cannot separate intertemporal substitution and the degree of risk aversion. For example, a typical intertemporal utility function is

$$U(c_0, p) = f(c_0) + \beta E \sum_{t=1}^{\infty} \beta^{t-1} f(c_t)$$

and

$$f(c) = \begin{cases} c^{1-\alpha}/(1-\alpha), & 0 < \alpha \neq 1 \\ \log c & \alpha = 1, \end{cases}$$

where c_t is the consumption expenditure in period t, $t = 0, 1, \ldots, \infty$; p is the probability measure of the future (uncertain) consumption vector $(c_1, c_2, \ldots)$; and $\beta \in (0, 1)$ is the discount factor. Here, α serves both as a relative risk-aversion parameter and the reciprocal of the elasticity of substitution. By modifying the recursivity axiom, Chew and Epstein (1990) show that the two concepts can be untangled by a class of utility functions that exhibits first-order risk aversion; for example, the one suggested by Yaari (1987, 113).[13] If the recursivity axiom is not assumed, however, then preferences may be inconsistent; that is, a consumption plan formulated at $t = 0$ may not be pursued in subsequent periods. The situation can be modeled as a noncooperative game between the decision maker at different times, and a perfect Nash equilibrium is taken to describe the behavior.

Using a similar approach, Epstein and Zin (1989) develop a generalized intertemporal capital asset-pricing model (CAPM). This model is used to study the equity premium puzzle in the United States, which has a historical average value of 6.2 percent. Using calibration of preferences by simulation technique, empirical results by Epstein and Zin (1990) show that the use of nonexpected utility function can explain at least a part (2 percent) of the equity premium. Epstein and Zin (1991) also apply the intertemporal CAPM to update the permanent income hypothesis of Hall (1978). In this study, the utility function takes the form

13. The recursivity axiom assumes that the recursive preference structure of a consumer is consistent over time and across states of the world. See Chew and Epstein (1990, 62–63) for details.

$$\tilde{U}_t = W[c_t, \mu(U_{t+1}|I_t)],$$

where μ is the certainty equivalent of the recursive utility $\tilde{U}_{t+1}$ at period $t + 1$, given the information I_t in period t. The separation of intertemporal substitution and risk aversion makes the model more realistic. The resulting estimating equation is the weighted sum of two factors: a relation between consumption growth and asset return (intertemporal CAPM) and a relation between the risk of a particular asset and the return of the market portfolio (static CAPM). They conclude that the expected utility hypothesis is rejected, but the performance of the nonexpected utility model is sensitive to the choice of the consumption measure (nondurable goods, durable goods, services, etc.). Average elasticity of substitution is less than one, and average relative risk aversion is close to one.

Using the implicit utility function as described in equation (5), Diewert (1993, 1995) outlines simple models for measuring the real outputs of the insurance and gambling sectors. Here, we describe the model of a two-state lottery game. This simple model will be extended in the next section into a six-state lottery. The two-state lottery is $L = (p_1, p_2)$, with $p_2 = 1 - p_1$. The corresponding outcomes are

$$x_1 = y - w, \qquad x_2 = y + Rw, \tag{7}$$

where y is the consumer's income, w is the wager, and R is the payout ratio. Assuming homothetic preferences, the implicit utility function ϕ_u can be written as γ in equation (6):

$$\phi_u(z) = \gamma\left(\frac{z}{u}\right).$$

In order to provide a kink in the indifference curve, we employ the following functional form for γ:

$$\gamma(z) = \begin{cases} \alpha + (1 - \alpha)z^{\beta}, & z \geq 1 \\ 1 - \alpha + \alpha z^{\beta}, & z < 1, \end{cases} \tag{8}$$

where $0 < \alpha < 1/2$, $\beta < 1$, $\beta \neq 0$. The implicit expected utility in equation (5) for this game is

$$p_1\phi_u(x_1) + p_2\phi_u(x_2) - \phi_u(u) = 0. \tag{9}$$

Substituting γ in equation (8) into equation (9) as ϕ_u, we have for $x_1 < x_2$,

$$u = [\delta x_1^{\beta} + (1 - \delta)x_2^{\beta}]^{1/\beta}, \tag{10}$$

where $\delta \equiv p_1\alpha/[p_1 + (1 - p_1)(1 - \alpha)]$. Putting equation (7) into equation (10), the consumer's utility maximization problem is

$$\max_w [\delta(y - w)^{\beta} + (1 - \delta)(y + Rw)^{\beta}]^{1/\beta},$$

where $0 \le w \le y$. The first-order condition is

$$\frac{y + Rw^*}{y - w^*} = \left[\frac{1-\delta}{\delta}R\right]^{1/(1-\beta)}$$
$$= \left[\frac{(1-p_1)(1-\alpha)R}{p_1\alpha}\right]^{1/(1-\beta)}$$
$$\equiv b.$$

Solving for the optimal w^*, we have

$$w^* = \frac{y(b-1)}{b+R}.$$

Because y, R, and w^* are observable, we can calculate b in each period. Then, α and β can be estimated with a regression model. Having estimated α and β, we can calculate the consumer's utility level without gambling:

$$u^0 = [\delta y^\beta + (1-\delta)y^\beta]^{1/\beta} = y.$$

Similarly, the utility level with gambling is

$$u^* = [\delta(y-w)^\beta + (1-\delta)(y+Rw)^\beta]^{1/\beta}.$$

The real output of the gambling service is then

$$Q = u^* - u^0.$$

10.3 Modeling the Gambling Sector

10.3.1 Gambling Sectors in Canada

The gambling industry in Canada has been growing in size and in revenue over the last decade. For example, revenue increased from \$2.7 billion in 1992 to \$7.4 billion in 1998, while employment grew from 11,900 in 1992 to 39,200 in 1999. In 1992, government lotteries were the major component in all games of chances, representing 90 percent of all gambling returns. They peaked at \$2.8 billion and have been declining at a moderate rate. On the other hand, video lottery terminals (VLTs) and casinos have grown rapidly. In 1998, revenue from the latter has overtaken government lotteries as the dominant player (Marshall 2000).

Government lotteries are administered by five regional crown corporations: namely, Atlantic Lottery Corporation, Loto-Québec, Ontario Lottery and Gaming Corporation, Western Canada Lottery Corporation, and British Columbia Lottery Corporation. Most of these corporations offer their own local lottery games. The national games, Lotto 6/49, Celebration (a special event lottery), and Super 7, however, are shared by all the corporations through the coordination of the Canadian Interprovincial Lottery Corporation, which was established in 1976 to operate joint lottery games

Table 10.1 Prizes of Canadian Lotto 6/49

Prize	Rule	Probability of winning, π_i	Share of the pool fund
Jackpot	6 numbers	0.0000000715	50%
Second	5 numbers + bonus	0.000000429	15%
Third	5 numbers	0.00001802	12%
Fourth	4 numbers	0.0009686	23%
Fifth	3 numbers	0.01765	$10

across Canada. Lotto 6/49 games are held twice a week on Wednesday and Saturday. Forty-five percent of the sales revenue goes to the prize fund. The fifth prize, which requires matching three numbers out of the six drawn, has a fixed prize of ten dollars. The prize fund, after subtracting the payout for all the fifth prizes, becomes the pool fund. This pool fund is divided among the other prizes by fixed shares, as shown in table 10.1. The prize money is shared equally among the winners of a particular prize category. If there is no winner for the jackpot, the prize money will be accumulated (rollover) to the prize fund of the next draw. About 13.3 percent of the sales revenue is used as the administration and retailing costs. This portion is used by Statistics Canada as the output of the Lotto 6/49 game in the GDP. As a consequence, the lottery corporation retains 41.7 percent (55 percent minus 13.3 percent) of the revenue as profit. This profit margin can be regarded as a tax on the output of the lottery sector. Thus from the final demand perspective, the value of lottery output should be listed as 55 percent of the sales volume, which is about four times the value from the industry accounts perspective (13.3 percent).

10.3.2 The Output of Government Lotteries

In this section, we extend Diewert's (1995) simple model to the measurement problem of a common lottery sector. A typical game of lottery—for example, Lotto 6/49 in Canada—involves choosing six numbers out of forty-nine. Five prizes are awarded, according to the rules listed in table 10.1.[14]

For example, the probability of winning the jackpot for one single ticket is $1/C^6_{49} = 1/13{,}983{,}816 = 0.0000000715$—a one in 14 million chance. The probability of the second prize is six times the probability of the jackpot; that is, $6/13{,}983{,}816 = 0.000000429$, and so on.[15] The following notation is used in the model: w = wager; p_i = probability of winning the ith prize, $i = 1, \ldots, 5$; p_6 = probability of not winning any prize; x_i = state-contingent consumption, $i = 1, \ldots, 6$; y = real disposable income; and R_i = payout for the ith prize, $i = 1, \ldots, 6$.

Buying more than one ticket increases the chance of winning. Therefore,

14. See Ziemba (1986).
15. For details of computing all the probabilities, see Hoppe (1996).

(11) $$p_i = w\pi_i, \qquad i = 1, \ldots, 5,$$

where π_i is the probability of winning the ith prize for one single ticket. Also, we have

(12) $$p_6 = 1 - \sum_{i=1}^{5} p_i = 1 - w\sum_{i=1}^{5}\pi_i$$

and

(13) $$x_i = y + R_i - w, \qquad i = 1, \ldots, 6.$$

We assume a representative consumer with homothetic preferences, so his or her state-contingent preference function $u = F(x_1, \ldots, x_6, p_1, \ldots, p_6)$ can be defined implicitly using equation (6). Using the kinked functional form in equation (8), equation (6) becomes

$$\sum_{i=1}^{5} p_i\left[\alpha + (1-\alpha)\left(\frac{x_i}{u}\right)^{\beta}\right] + p_6\left[1 - \alpha + \alpha\left(\frac{x_6}{u}\right)^{\beta}\right] - 1 = 0.$$

Solving for u and using equations (11), (12), and (13), we have

(14) $$u(w) = \left[\frac{(1-\alpha)w\sum_{i=1}^{5}\pi_i(y + R_i - w)^{\beta} + \alpha(1 - w\sum_{i=1}^{5}\pi_i)(y - w)^{\beta}}{\alpha + (1-2\alpha)w\sum_{i=1}^{5}\pi_i}\right]^{1/\beta}.$$

The consumer's utility maximization problem is to maximize $u(w)$, subject to the constraint $0 \leq w \leq y$. For notational convenience, we define the following variables as

(15) $$\begin{aligned} d &= y - w, \\ p &= \textstyle\sum_{i=1}^{5}\pi_i, \\ q &= \textstyle\sum_{i=1}^{5}\pi_i(y + R_i - w)^{\beta-1}, \text{ and} \\ r &= \textstyle\sum_{i=1}^{5}\pi_i(y + R_i - w)^{\beta}. \end{aligned}$$

The first-order condition for the utility maximization problem (assuming that a boundary solution does not occur) can be written as

$$\begin{aligned} &\alpha(1-\alpha)r - \beta q(1-\alpha)[\alpha + (1-2\alpha)wp]w \\ &\qquad - \alpha\beta(1 - wp)[\alpha + (1-2\alpha)wp]d^{\beta-1} - \alpha(1-\alpha)pd^{\beta} = 0. \end{aligned}$$

Rearranging terms, we get a quadratic equation in w:

$$\begin{aligned} &\{\beta p[\alpha(1-2\alpha)pd^{\beta-1} - (1-\alpha)(1-2\alpha)^2 q]\}w^2 \\ &\qquad + \{\alpha\beta[\alpha pd^{\beta-1} - (1-\alpha)q - (1-2\alpha)pd^{\beta-1}]\}w \\ &\qquad\qquad + \alpha[(1-\alpha)r - \alpha\beta d^{\beta-1} - (1-\alpha)pd^{\beta}] = 0. \end{aligned}$$

Solving for this quadratic equation gives us the following equation involving the optimal level of the wager w:

$$(16)\qquad w^* = [-\alpha\beta[\alpha p d^{\beta-1} - (1-\alpha)q - (1-2\alpha)pd^{\beta-1}] \pm (\{\alpha\beta[\alpha p d^{\beta-1} - (1-\alpha)q - (1-2\alpha)pd^{\beta-1}]\}^2 - 4\alpha[(1-\alpha)r - \alpha\beta d^{\beta-1} - (1-\alpha)pd^{\beta}] \{\beta p[\alpha(1-2\alpha)pd^{\beta-1} - (1-\alpha)(1-2\alpha)q]\})^{1/2}] \div 2\{\beta p[\alpha(1-2\alpha)pd^{\beta-1} - (1-\alpha)(1-2\alpha)q]\}.$$

Equation (16) is the estimation equation for the parameters α and β, given the data for the other variables. Notice that w appears in the right-hand side of equation (16) through the variables d, q, and r, which were defined in equation (15). But the effects of w on d, q, and r are negligible, because the disposable income y and the sum of y and the payout prizes R_i are so much larger than w, and hence we can simply set w equal to zero in those definitions. Another functional form, the kinked quadratic-generating function,

$$\gamma(z) = \begin{cases} z + \alpha(z-1) + \beta(z-1)^2, & z \geq 1 \\ z, & z < 1, \end{cases}$$

was attempted in addition to equation (8), but the analysis yielded no explicit solution for w.

The output of services provided by Lotto 6/49 is equal to the difference between utility level with the lotteries and utility without the lottery using equation (14); that is,

$$(17)\qquad Q^t = u(w^t) - u(0),$$

where w^t is the observed wager in period t. An implicit price level can also be obtained as

$$(18)\qquad P^t = \frac{(1-\rho)w^t}{Q^t},$$

where ρ is the proportion of the prize fund from the total revenue. In the case of Lotto 6/49, $\rho = 0.45$. The approach here follows the final demand perspective discussed in section 10.3.1. The resulting price index is an implicit cost-of-living index and can be included as a subindex in the CPI.

10.4 Estimating the Output of Government Lotteries

10.4.1 Data

Data on the winning numbers, payout prizes, and sales volume provided by Lottery Canada are available from November 11, 1997, to November 3, 2001, for Lotto 6/49, a total of 419 draws. Monthly data on the CPI and annual data on the number of households, personal disposable income, and participation rates in government lotteries are available from Statistics Canada. The sales volume of each draw is divided by the number of participating households, which gives the average wager per participating household, w_t.

The average personal disposable income per household, adjusted by the CPI, is used as a proxy for y_t.

Figure 10.2 depicts the number of ticket sales for the sample period. We see that there is a downward trend in sales, reflecting the switch from government lotteries to other games, such as VLTs and casinos. Table 10.2 summarizes the average sales, number of winners, and the payout prizes of the observed draws. The biggest jackpot during the sample period was $15 million, won by a single ticket on September 30, 2000. In table 10.2, we also calculate the expected average number of winners using the probabilities in table 10.1. We see that in each prize, the observed average number of winners is slightly smaller than the expected number. One possible explanation of the difference is that some players pay more than one dollar for the same numbers, which often happens in lottery pools. Of the 419 draws, 151 end up with a rollover, which is 36 percent. Given that the expected number of jackpot winners is 1.2 on average, this rollover percentage seems high. In fact, this agrees with previous observations in Canada (Ziemba 1986; Stern and Cover 1989), the United States (Chernoff 1981), and the United Kingdom

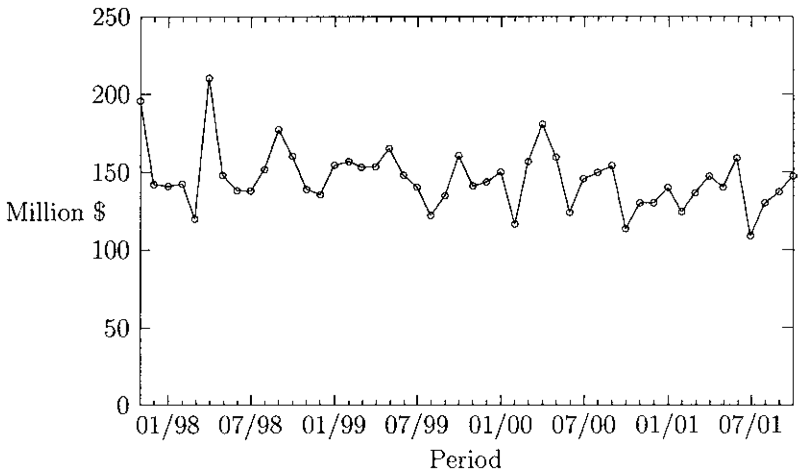

Fig. 10.2 Monthly sales of Lotto 6/49: November 1997 to November 2001

Table 10.2 Descriptive statistics of Canadian Lotto 6/49: November 11, 1997, to March 11, 2001

	Sales	Jackpot	Second	Third	Fourth	Fifth
Average number of winners	16,717,385	1.12	7.13	299	16,036	292,604
Expected number of winners		1.20	7.17	308	16,199	293,287
Prize ($)		3,249,108	133,903	1,976	68	10

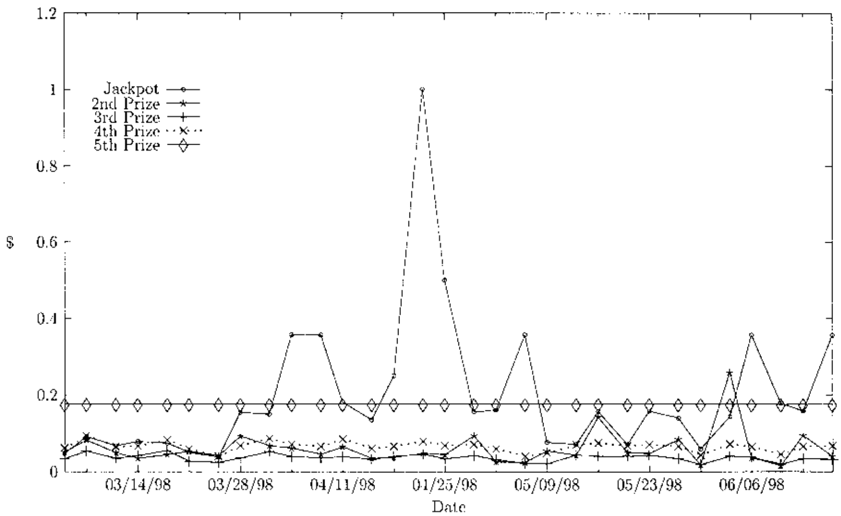

Fig. 10.3 Expected values of the various prizes

(Walker 1998; Simon 1999) that people have "conscious selection" (Cook and Clotfelter 1993); that is, some numbers on average are more popular than the others.[16] For example, the six most popular numbers of Lotto 6/49 in Canada were 3, 5, 6, 9, 12, and 13 in 1986. One possible reason is that a lot of people use their birthdays as their choices. Therefore, numbers starting from thirty-two onward are among the most unpopular numbers.

A lot of attention is concentrated on the jackpot prizes, particularly when there are rollovers and the pool fund becomes very big. Figure 10.3, however, shows that the average expected value is highest for the smallest prize. In the figure, EV1 to EV5 are the products of the payout prized and their respective probability of winning from March 4 to June 17, 1998. Because the payout is fixed at ten dollars, EV5 is constant. In only one draw is EV2 higher than EV5, and EV1 is higher than EV5 in several occasions. The fifth prize has a high expected value because of the relatively high probability of winning. The pleasure and thrill from buying a lottery ticket, nevertheless, comes from buying the big jackpot ticket, which has an extremely low probability of winning. This is why a nonlinear expected utility theory is needed to capture the risk-loving side of consumers.

10.4.2 Estimation and Results

The parameters α and β in equation (16) are estimated by a nonlinear regression equation using the maximum likelihood method. Theoretically,

16. For a general discussion, see also Haigh (1997).

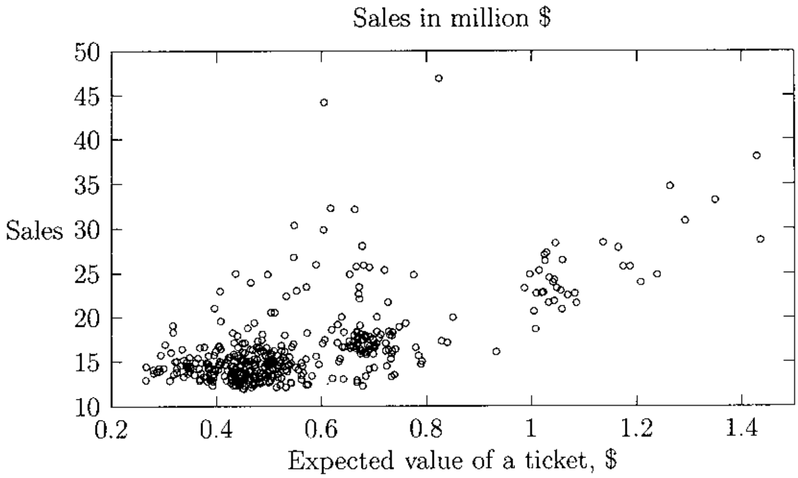

Fig. 10.4 Sales and expected values of Lotto 6/49

demand depends on the expected values of the payout prizes $R_1, \ldots, R_4$, which in turn depend on the sales volume. The actual payout prizes, however, are used in the estimation. Following Walker (1998, 371), we invoke the rational expectations assumption, which implies that consumers do not make systematic mistakes in forecasting the sales. Figure 10.4 is a scatter plot of the sales volume against the ex post expected value of a ticket. It clearly shows the positive relation between the two. The estimated values of α and β are 0.10458 and –31.986, with standard errors equal to 0.003165 and 5.9527, respectively, which implies t-ratios of 33 and –5.4. The estimated values satisfy the constraints $0 < \alpha < 1/2$, $\beta < 1$, $\beta \neq 0$ in equation (8). These estimated values are then used to calculate the money-metric utility $u(w_t)$ and the output level Q^t of the lottery using equations (14) and (17), respectively, for each draw. Outputs are aggregated into monthly results before the implicit price P^t is calculated using equation (18). A fixed-base price index is then calculated using the price level of November 1997 as the base.

Figures 10.5 and 10.6 show the monthly price index and output of the Lotto 6/49 using this procedure. In figure 10.6, the factor cost (13.3 percent of sales revenue) is also included for comparison. Notice that the estimated output using the economic approach is much higher than the official GDP at factor cost, but the former has a steeper downward trend. The average monthly output using the economic approach is \$57.7 million, compared to the official total cost approach of \$19.4 million. We also observe in figure 10.6 that the utility-based output measure is more volatile than the factor cost measure. This is due to the rollover of the prize money when there is no jackpot winner for a particular draw. These rollovers create occasional

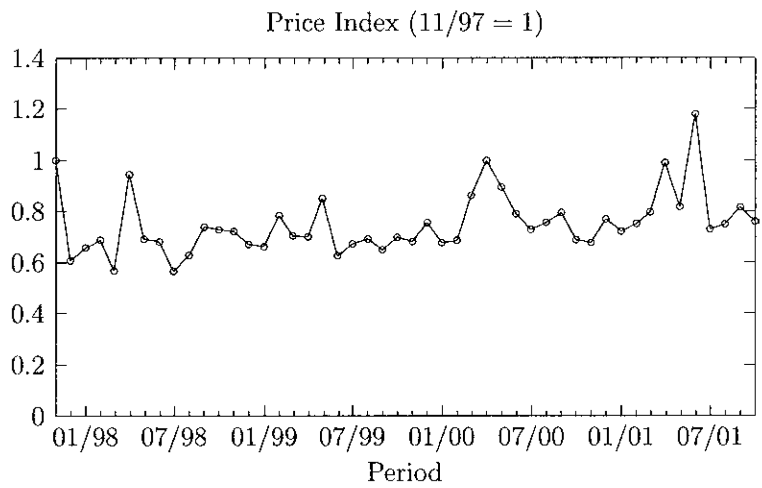

Fig. 10.5 Price index of Lotto 6/49

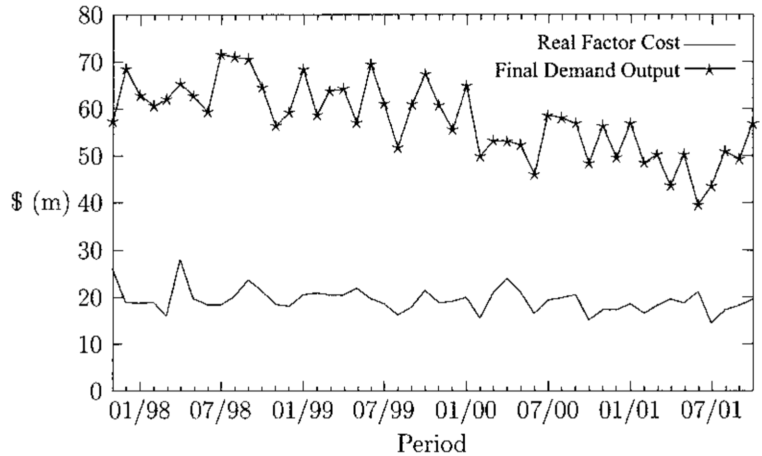

Fig. 10.6 Output of Lotto 6/49

excitement and effectively reduce the unfairness for the next draw. The effects are reflected in the utility measure of the game.[17]

17. This rollover effect creates substantial volatility in both the price and quantity of the final demand consumption of lottery output, and hence, statistical agencies may be reluctant to adopt this approach to measuring the price of lottery services in their CPIs. This problem could be solved by smoothing the raw data. This type of smoothing would automatically occur if statistical agencies adopted a rolling-year methodology for their CPIs; see Diewert (1998) for an explanation of this methodology.

We also estimate the elasticity of demand for the lottery using a simple log-linear model:

$$\log Q = \log P + \log y + T,$$

where T is a trend variable, which is included to capture change in taste over time. The resulting price elasticity of demand is –0.672, with a standard error of 0.017. This result is comparable to the values of –0.66 estimated by Forrest, Gulley, and Simmons (2000), who use a two-stage ordinary least squares estimation, with the difference between the ticket price and the expected value as the effective price of lottery. Using a similar approach, Gulley and Scott (1993) estimated the price elasticities of four state-operated lottos in the United States, with results ranging from –0.40 to –2.5. Farrell and Walker (1999) used cross-sectional data to study the demand for lotteries in the United Kingdom using the Heckman selection model. Their estimated price elasticity was –0.763. Also, Beenstock and Haitovsky (2001) studied the demand for lotto in Israel using time-series data, with the estimated long-run price elasticity equal to –0.65. It is surprising that these results, although differing in methods, nature of data, and countries, show very close estimates of price elasticities of demand.

10.5 Conclusion

The classical expected utility hypothesis fails to capture a consumer's risk-averse behavior in facing big losses with small probabilities and the risk-loving behavior involving large gains with small probabilities. New non-expected utility theories have been developed to overcome that difficulty. In this chapter, we have applied implicit expected utility theory to the problem of measuring outputs of lotteries. The results show that output levels of Lotto 6/49 in Canada is almost three times higher than the official statistics, which uses the total cost of providing the service approach as the output measurement principle. This kind of direct economic approach is recommended by the *System of National Accounts, 1993* (Inter-Secretariat Working Group on National Accounts 1993) for government and nonprofit organization output measurement. The approach taken here is the ex ante welfare measure of the consumers facing risk and uncertainty. The method developed can potentially be applied to other games of chance.[18] The estimated price elasticity of demand for lottery in Canada is close to that of the United Kingdom and Israel in previous studies.

18. See Dubins and Savage (1965) and R. Epstein (1977) for the mathematical analysis of a whole variety of games.

References

Allais, M. 1953. Le comportement de l'homme rationnel devant le risque: Critique des postulats et axiomes de l'ecole Americaine. *Econometrica* 21 (4): 503–46.

Arrow, K. J. 1964. The role of securities in the optimal allocation of risk-bearing. *Review of Economic Studies* 31 (2): 91–96.

Beenstock, M., and Y. Haitovsky. 2001. Lottomania and other anomalies in the market for lotto. *Journal of Economic Psychology* 22 (6): 721–44.

Blackorby, C., R. Davidson, and D. Donaldson. 1977. A homiletic exposition of the expected utility hypothesis. *Economica* 44 (176): 351–8.

Chernoff, H. 1981. How to beat the Massachusetts numbers game. *Mathematical Intelligencer* 3 (4): 166–72.

Chew, S. H. 1983. A generalization of the quasilinear mean with applications to the measurement of income inequality and decision theory resolving the Allais paradox. *Econometrica* 51 (4): 1065–92.

Chew, S. H., and L. G. Epstein. 1989a. A unifying approach to axiomatic non-expected utility theories. *Journal of Economic Theory* 49 (2): 207–40.

———. 1989b. The structure of preferences and attitudes towards the timing of the resolution of uncertainty. *International Economic Review* 30 (1): 103–17.

———. 1990. Nonexpected utility preferences in a temporal framework with an application to consumption-savings behaviour. *Journal of Economic Theory* 50 (1): 54–81.

Cook, P. J., and C. T. Clotfelter. 1993. The peculiar scale economies of lotto. *American Economic Review* 83 (3): 634–43.

Cox, J. C., and V. Sadiraj. 2001. Risk aversion and expected utility theory: Coherence for small- and large-stakes gambles. University of Arizona, Department of Economics. Working Paper no. 01-03, July.

Debreu, G. 1959. *Theory of value.* New Haven, CT: Yale University Press.

Dekel, E. 1986. An axiomatic characterization of preferences under uncertainty: Weakening the independence axiom. *Journal of Economic Theory* 40 (2): 304–18.

Diecidue, E., and P. P. Wakker. 2001. On the intuition of rank-dependent utility. *Journal of Risk and Uncertainty* 23 (3): 281–98.

Diewert, W. E. 1993. Symmetric means and choice under uncertainty. In *Essays in index number theory,* vol. 1, ed. W. E. Diewert and A. O. Nakamura, 335–433. Amsterdam: North-Holland.

———. 1995. Functional form problems in modeling insurance and gambling. *Geneva Papers on Risk and Insurance Theory* 20 (1): 135–50.

———. 1998. High inflation, seasonal commodities, and annual index numbers. *Macroeconomic Dynamics* 2 (4): 456–71.

Dubins, L. E., and L. J. Savage. 1965. *How to gamble if you must.* New York: McGraw-Hill.

Epstein, L. G. 1992. Behavior under risk: Recent developments in theory and applications. In *Advances in economic theory,* vol. 2, ed. J.-J. Laffont, 1–63. Cambridge: Cambridge University Press.

Epstein, L. G., and S. E. Zin. 1989. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica* 57 (4): 937–69.

———. 1990. "First-order" risk aversion and the equity premium puzzle. *Journal of Monetary Economics* 26 (3): 387–407.

———. 1991. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: An empirical analysis. *Journal of Political Economy* 99 (2): 263–86.

Epstein, R. A. 1977. *The theory of gambling and statistical logic.* New York: Academic Press.

Farrell, L., and I. Walker. 1999. The welfare effects of lotto: Evidence from the UK. *Journal of Public Economics* 72 (1): 99–120.

Forrest, D., O. D. Gulley, and R. Simmons. 2000. Elasticity of demand for UK National Lottery tickets. *National Tax Journal,* pt. 1 #53 (4): 853–63.

Friedman, M., and L. J. Savage. 1948. The utility analysis of choices involving risk. *Journal of Political Economy* 56 (4): 279–304.

Gorman, W. M. 1968. The structure of utility functions. *Review of Economic Studies* 35 (4): 367–90.

Gul, F. 1991. A theory of disappointment aversion. *Econometrica* 59 (3): 667–86.

Gulley, O. D., and F. A. Scott, Jr. 1993. The demand for wagering on state-operated lotto games. *National Tax Journal* 45 (1): 13–22.

Haigh, J. 1997. The statistics of the National Lottery. *Journal of the Royal Statistical Society* A 160 (2): 187–206.

Hall, R. E. 1978. Stochastic implications of the life cycle-permanent income hypothesis: Theory and evidence. *Journal of Political Economy* 86 (6): 971–87.

Hoppe, F. M. 1996. Mathematical appendix to the *Maclean's Magazine* article by John Schofield; November 4, 1996. McMaster University, Department of Mathematics. Available at http://icarus.mcmaster.ca/fred/Lotto/.

Hulten, C. R. 2001. Total factor productivity: A short biography. In *New developments in productivity analysis,* ed. C. R. Hulten, E. R. Dean, and M. J. Harper, 1–47. Chicago: University of Chicago Press.

Inter-Secretariat Working Group on National Accounts. 1993. *System of National Accounts, 1993.* Paris: Organization for Economic Cooperation and Development.

Kahneman, D., and A. Tversky. 1979. Prospect theory: An analysis of decisions under risk. *Econometrica* 47 (2): 263–91.

Luce, R. D., and H. Raiffa. 1957. *Games and decisions: introduction and critical survey.* New York: John Wiley and Sons. Repr., Mineola, NY: Dover Publications, 1989.

Machina, M. J. 1982. "Expected utility" analysis without the independence axiom. *Econometrica* 50 (2): 277–323.

———. 1997. Choice under uncertainty: Problems solved and unsolved. In *The environment and emerging development issues,* vol. 1, ed. P. Dasgupta and K.-G. Mäler, 201–55. Oxford: Clarendon Press.

———. 2001. Payoff kinks in preferences over lotteries. *Journal of Risk and Uncertainty* 23 (3): 207–60.

Marshall, K. 1998. The gambling industry: Raising the stakes. *Services Indicators* 5 (4): 29–38.

———. 2000. Update on gambling. *Perspectives on Labour and Income* 12 (1): 29–35.

Pratt, J. W., H. Raiffa, and R. Schlaifer. 1995. *Introduction to statistical decision theory.* Cambridge, MA: MIT Press.

Quiggin, J. 1993. *Generalized expected utility theory: The rank-dependent model.* Norwell, MA: Kluwer Academic.

Rabin, M. 2000. Risk aversion and expected-utility theory: A calibration theorem. *Econometrica* 68 (5): 1281–92.

Rabin, M., and R. H. Thaler. 2001. Risk aversion. *Journal of Economic Perspectives* 15 (1): 219–32.

Samuelson, P. A. 1952. Probability, utility, and the independence axiom. *Econometrica* 20 (4): 670–8.

Segal, U., and A. Spivak. 1990. First order versus second order risk aversion. *Journal of Economic Theory* 51 (1): 111–25.
Simon, J. 1999. An analysis of the distribution of combinations chosen by UK National Lottery players. *Journal of Risk and Uncertainty* 17 (3): 243–76.
Starmer, C. 2000. Developments in non-expected utility theory: The hunt for a descriptive theory of choice under risk. *Journal of Economic Literature* 38 (2): 332–82.
Stern, H., and T. M. Cover. 1989. Maximum entropy and the lottery. *Journal of the American Statistical Association* 84 (408): 980–5.
Von Neumann, J., and O. Morgenstern. 1953. *Theory of games and economic behavior.* Princeton, NJ: Princeton University Press.
Walker, I. 1998. The economic analysis of lotteries. *Economic Policy* 13 (27): 359–401.
Watt, R. 2002. Defending expected utility theory. *Journal of Economic Perspectives* 16 (2): 227–30.
Yaari, M. E. 1987. The dual theory of choice under risk. *Econometrica* 55 (1): 95–115.
Ziemba, W. T. 1986. *Dr Z's 6/49 lotto guidebook.* Vancouver: Dr. Z Investments.

Comment

Alan G. White

Overview: Methods, Data, and Results

In chapter 10 of this volume, Kam Yu presents an economic approach to measuring the output and prices of a hard-to-measure sector—that of the lottery sector. Yu applies implicit expected utility theory by developing a money metric of utility of playing the Canadian Lotto 6/49 game.

Yu argues that the lottery is becoming an increasingly important component of gross domestic product (GDP) in Canada. He notes that according to the 1997 Survey of Household Spending (SHS), over two-thirds of families in Canada purchased lottery tickets, and average expenditure on lottery tickets was approximately $238. Given that expenditure on gambling is likely underreported in the SHS, the lottery industry may be a more important and significant component of GDP than currently measured, necessitating a more accurate method for measuring its output.

In the theory of consumption under uncertainty, a risk-averse consumer maximizes an expected utility function in which risk averseness is often assumed to be decreasing in wealth. Although this theory has been applied to problems in insurance and investment decisions, it predicts that a risk-averse expected utility maximizer would never purchase a lottery ticket unless the payout is extremely large. This, however, is not consistent with reality, where the purchase of lottery tickets and gambling among consum-

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