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6 Multiperiod Securities and the Efficient Allocation of Risk: A Comment on the Black-Scholes Option Pricing Model

David M. Kreps

6.1 Introduction

Over the past six years, a great stir in academic financial theory (sometimes spilling over into practice) has been caused by the option pricing model originally advanced by Black and Scholes (1973) and by Merton (1973b).[1] The reason for this stir is that strong results are derived from what seem at first to be weak assumptions. While the weakness of these assumptions is illusory, the model does make an important point: The ability to trade securities frequently can enable a "few" multiperiod securities to span "many" states of nature. In the Black-Scholes model there are two securities and uncountably many states of nature, but because there are infinitely many trading opportunities and, what is crucial, because uncertainty resolves "nicely," markets are effectively complete. Thus the punchline: Perhaps even though there are far fewer securities than states of nature, nonetheless there is a complete (or nearly complete) set of contingent claims markets. Perhaps, therefore, risk is allocated efficiently.

The purpose of this paper is to explore this idea and to attempt to see what is important in determining the number of securities "needed" to have complete markets. In this regard, the following two questions will be addressed to some extent: (1) The Black-Scholes model has been criticized on the grounds that it takes as given that which any good

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economist would want endogenously determined: equilibrium prices of the few multiperiod securities. This is a valid criticism because those prices are the critical data in determining whether markets are complete. To what extent, then, is it reasonable to suppose that equilibrium prices will have the property required for complete markets? (2) In what sense, if any, is the Black-Scholes result robust? It will be seen that the property required for complete markets concerns the very delicate fine structure of the model. Other models that approximate the Black-Scholes model in a standard sense do not possess this property. Do these other models have "approximately complete" markets? One hopes that the answer is yes. Otherwise, one either must be able to discern the critical fine structure or must discard the conclusions of the Black-Scholes model for practical purposes.

The paper is divided into two parts. The first part contains an analysis of the basic issues in the spirit of Radner (1972). In section 6.2 a multiperiod exchange economy with uncertainty is formulated. The economy is specified by a finite state space Ω, a collection of agents, a finite set of dates $t = 0, 1, \ldots, T$ at which agents consume, and an exogenously specified information structure, which describes what information (all) agents know at each date. Formally, the information structure is a sequence of nondecreasingly finer partitions of Ω, $\{F_t;\ t = 0, \ldots, T\}$. The interpretation is that at date t, all agents know which cell of F_t contains the true state and no more. There is a single consumption good which serves as numeraire. Finally, there are N "long-lived" securities that allow agents to trade consumption between dates and states. Each security is a contingent claim to consumption at the terminal date T. Markets where these securities can be exchanged for each other and for the consumption good open at each date t, with $p = \{p_n(t, \omega);\ n = 1, \ldots, N, t = 0, \ldots, T, \omega \in \Omega\}$ the price process of the securities. A definition of an equilibrium for this economy is given, exactly as in Radner (1972). Every such equilibrium is given an alternate characterization, as an equilibrium in a Debreu-style economy where a (possibly incomplete) set of contingent claims markets opens at date zero.

The basic question is posed and answered in section 6.3: Under what conditions will the corresponding Debreu-style economy be one with a complete set of markets (so that the equilibrium allocation is Pareto efficient)? A necessary and sufficient condition for this is: For $t < T$ and $A \in F_t$, let $K(t,A)$ be the cardinality of $\{A' \in F_{t+1};\ A' \subseteq A\}$. Then it is necessary and sufficient that for every t and $A \in F_t$, the span of the conditional support of $p(t+1)$ given $\omega \in A$ has dimension $K(t,A)$. Therefore, a necessary condition is that N, the number of securities, must be at least $K = \max\{K(t,A)\}$. This is illustrated by a simple example that makes the basic point: With N securities and T trading dates ($t = 0, \ldots, T-1$), up to N^T states of nature can be spanned.

While it is necessary for complete markets that $K \leq N$, this is not sufficient. The necessary and sufficient conditions involve the equilibrium prices p, and this is clearly less than satisfactory on economic grounds. A refinement is given in section 6.4 that is more satisfactory. Fixing everything except the terminal payoffs of the securities (that is, fixing the state space, information structure, and agents), "almost every" selection of K or more securities (determined by their terminal payoffs) gives an economy with a complete markets equilibrium. (This presumes that an equilibrium with a complete set of contingent claims markets exists.) Here, "almost every" means a generic result in the sense of Radner (1979). Thus, in determining whether markets are "likely" to be complete in an economy with long-lived securities, the crucial comparison is K versus the number of securities N. This section closes with several embellishments on the basic model.

The qualitative insight to be gained from the analysis in sections 6.2, 6.3, and 6.4 is clear: A few securities that are frequently traded *may* span a very large dimensional space of contingent claims. Markets *may be* complete, and it is *possible* that risk is allocated efficiently. But how are these "may be's" to be converted into more positive statements? What value of K is appropriate for modeling purposes? Might it be that K is very much larger than N, and yet markets are approximately complete and risk is allocated approximately efficiently? These questions concerning the robustness of the analysis are extremely difficult for two related reasons. The analysis does not indicate what (if anything) will suffice for "approximately complete markets" and "approximately efficient allocations." The analysis identifies K as the crucial piece of data, and K is a property of the fine structure of the model. If conditions necessary for "approximate completeness/efficiency" involve the datum K, then one is unlikely to be able to apply this analysis with any confidence—the task of discerning the "true" value of K defies the imagination.

The following sort of result is therefore sought. If one economy approximates a second idealized economy in a coarse sense and if the idealized economy has "complete markets," then the first economy has approximately efficient equilibrium allocations. The key is to make the sense of approximation as coarse as possible, in order to make the analysis as robust as possible. The remainder of the paper is devoted to discussion of this type of result and in particular to convergence to the Black-Scholes model that dominates the financial literature. The issues raised are very delicate and difficult mathematically, and therefore the analysis given is preliminary at best.

Section 6.5 concerns the idealized economy to which other economies will converge: the Black-Scholes model. The use of continuous time creates difficulties. Both the sense in which this model represents a Radner equilibrium and the sense in which it is a "complete markets" equilibrium are not straightforward. These difficulties are resolved as in Harrison and Kreps (1979), and the section closes with brief discussion on the inadequacies of this resolution.

A convergence result is proved in section 6.6. Within a certain framework, sequences of models that converge to the Black-Scholes model have asymptotically efficient equilibrium allocations. The mode of convergence required is such that a sequence can converge without convergence of the "fine structure" of the economies: In each economy along the sequence, K is very much larger than N. (In fact, $K = \infty$ and $N = 2$ for each economy.) This shows that for approximate efficiency, the K versus N comparison may be misleading.

This convergence result is a step in the right direction, but it suffers from some severe deficiencies. Chief among these is that the framework of the result is very restrictive—the state space, information structure, and agents are all fixed along the sequence. (What changes along the sequence are the dates at which trading takes place and, perhaps, the equilibrium prices.) It *ought* to be the case that this sort of convergence result holds in a much less restrictive setting. But when one attempts to obtain analogues in wider contexts, difficulties arise. For example, if the state space changes along the sequence, then so does the commodity space and so (perforce) must the agents. How then is one to define "asymptotic efficiency"? Section 6.7 discusses where these difficulties lie, why in some sense they cannot be completely overcome, and how they *might* be partially finessed. "Answers" are not provided in this section. Rather, the aim of the discussion is to indicate limitations of both the result in section 6.6 and any possible extension and to spur research that will culminate in an approximation theory more adequate than that which is given here.

Section 6.8 presents a brief summary of the main points of the paper, together with a list of weaknesses and questions left unanswered by the analysis.

The general topic addressed here has a long history in the literature, and a review of pertinent contributions may help put things in perspective. The mode of analysis of a multiperiod exchange economy follows Radner (1972) and his definition of an equilibrium of plans, prices, and price expectations. This definition is implicit as well in the simpler setting of Arrow (1964). Arrow (1964) and Guesnerie and Jaffray (1974) discuss circumstances under which a Radner economy has a "complete" set of markets—Arrow analyzes a two-period economy, and Guesnerie and Jaffray extend Arrow's idea (that at each date there should be a complete set of financial claims for the next date) to a multiperiod setting. When a Radner economy does not have "complete markets," inefficiencies may result, and these may be inefficiencies even relative to the existing market structure. On this and other points concerning incomplete Radner economies, see Hart (1975). Several papers, noting that there are "fewer securities than states," have discussed the role of options on those securities for completing markets. On this point, see Breeden and Litzenberger (1978), Friesen (1979), and Ross (1976).

There is a chunk of literature that seeks to show how efficient allocations can arise with few securities using arguments very different from those used here. In these papers, agents are assumed to be "sufficiently alike" (for example, identical subjective probability estimates, no nonmarket income, and HARA class utility functions with identical risk cautiousness) so that complete markets are unnecessary. See, for example, Wilson (1968).

When some of the information may be privately held and/or information is endogenously generated and is costly, a host of difficulties arise: Grossman (1977) and Grossman and Stiglitz (1976) are two excellent examples of the huge literature on this topic.

Throughout this paper only exchange economies are considered. Extending the analysis to questions of production and productive efficiency involves nontrivial complications, even in the simple models of the first half of the paper. These problems are roughly those pointed to in the "spanning" literature: see Diamond (1967), Stiglitz (1972), and the Bell Journal Symposium on the Optimality of Competitive Capital Markets (1974). Because of the multiperiod setting here, where agents are constantly changing their portfolio holdings, the papers of Grossman and Stiglitz (1980) and Hart (1979) are especially important.

6.2 Equilibrium in a Multiperiod Exchange Economy

Consider the following model of an exchange economy with uncertainty. There is a finite number of states of the world, indexed by $\omega \in \Omega$. There is a finite number of time periods, indexed by $t = 0,1,\ldots,T$. All agents in this economy have access to the same information which is exogenously specified. This information is represented by a sequence of partitions of Ω, $\{F_t;\ t = 0,\ldots,T\}$. The interpretation is that at time t agents know which cell of F_t contains the true state. Information increases through time: F_{t+1} is at least as fine as F_t. For simplicity, it is assumed that F_0 is trivial and that F_T is the discrete partition. The σ-field of events generated by F_t is denoted $\boldsymbol{F}_t$.

There is a single consumption good which cannot be stored. This good is consumed at each date, and the amount consumed at date t can vary across cells of F_t; thus, the consumption space for agents is $X = X_{t=0}^{T} R^{(\Omega,\boldsymbol{F}_t)}$, where $R^{(\Omega,\boldsymbol{F}_t)}$ is the space of $\boldsymbol{F}_t$ measurable real valued functions on Ω. The notation $x = (x(0),\ldots,x(T))$ is used for a generic element of

X, with $x(t, \omega)$ denoting the value of $x(t)$ in the state ω. Vectors x will be interpreted as net trade (rather than total consumption) vectors for agents.

The agents in this economy are indexed by $i = 1, \ldots, I$. Each agent is characterized by a subset $X^i \subseteq X$, representing feasible net trades for agent i, and by a complete and transitive binary relation $\gtrsim^i$ on X^i, representing agent i's preferences among net trades. It is assumed throughout that each X^i is "comprehensive upwards," in the sense that if $x \in X^i$ and if $x' \in X$ are such that $x'(t, \omega) \geq 0$ for all t and ω, then $x + x' \in X^i$. Moreover, it is assumed that each X^i contains the origin, and that each $\gtrsim^i$ is strictly increasing in the sense that for x and $x' \neq 0$ as above, $x + x' >^i x$.

There are N assets or securities in this economy. These are claims to (state contingent) consumption at date T. They are indexed by $n = 1, \ldots, N$. Security n entitles the bearer (on date T) to $d_n(\omega)$ units of the consumption good at date T if the state is ω. The net supply of these securities is zero. It is assumed that for every state there is one of these securities that pays off a nonnegative amount in every state and a strictly positive amount in that state.

At each date $t \leq T$ and in every state, markets open in which these N securities can be traded for one another and for the consumption good. The price (in units of the consumption good) of security n at date t in state ω will be denoted by $p_n(t, \omega)$. These markets are frictionless—there are no transaction costs and no restrictions on short sales. A *price system* is a vector stochastic process $p = \{p_n(t, \omega); n = 1, \ldots, N, t = 0, \ldots, T, \omega \in \Omega\}$ with $p(t)$ F_t measurable for each t.

The agent's problem in this economy is to manage a portfolio of these N securities in order to obtain for himself the best possible net trade vector of state contingent consumption. This is formalized as follows. A trading strategy is an N dimensional vector stochastic process $\theta = \{\theta_n(t, \omega)\}$ such that $\theta(t)$ is F_t measurable for each t. The interpretation is that $\theta_n(t, \omega)$ is the number of shares of security n held from date t until $t+1$ in state ω. (For $t = T$, $\theta_n(T, \omega)$ is the number of shares from which the dividend is received.) The constraint that $\theta(t)$ is F_t measurable is the natural information constraint. If prices are given by $p = \{p_n(t)\}$, then the strategy θ results in the following net trade vector in state contingent consumption:

(2.1) $x(\theta, p) = (x(0; \theta, p), \ldots, x(T; \theta, p))$, where

$$x(0;\theta,p) = -\theta(0) \cdot p(0) \text{ (the dot means dot product),}$$

$$x(t;\theta,p) = (\theta(t-1) - \theta(t)) \cdot p(t) \text{ for } t = 1, \ldots, T-1, \text{ and}$$

$$x(T;\theta,p) = (\theta(T-1) - \theta(T)) \cdot p(T) + \theta(T) \cdot d.$$

A net trade bundle $x \in X$ is said to be feasible for agent i at prices p if $x \in X^i$ and if there exists a trading strategy θ such that $x \leq x(\theta,p)$. The set of feasible net trade bundles for i at prices p is denoted $X^i(p)$. Note that this definition contains an implicit assumption of free disposal and that $x \in X^i(p)$ implies that x satisfies the appropriate budget constraints on net trades.

An equilibrium for the economy described above is a price system p and, for $i = 1,\ldots,I$, net trade bundles x^i and trading strategies θ^i such that

(2.2*a*) $$x^i \leq x(\theta^i,p) \text{ and } x^i \in X^i \text{ for all } i,$$

(2.2*b*) $$x^i \text{ is } \gtrsim^i \text{ maximal among all } x \in X^i(p) \text{ for each } i,$$

and

(2.2*c*) $$\sum_i \theta^i = 0.$$

Condition (*a*) says that x^i is a feasible net trade for i and that x^i is feasible if i adopts the trading strategy θ^i. Condition (*b*) says that taking prices p as given, agent i can do no better than x^i. Condition (*c*) is the market clearing condition. It says that securities markets clear exactly. Note that this together with (2.2*a*) and (2.1) imply that $\Sigma_i x^i \leq 0$, or markets for the consumption good clear. This is an equilibrium of plans, prices, and price expectations in the sense of Radner (1972), assuming rational expectations on the part of agents as to the prices that will prevail at subsequent dates contingent on states.[2]

The following alternative characterization of an equilibrium will be useful. Fix a price system p. Define

$$M' = \{x \in X : x = x(\theta,p) \text{ for some trading strategy } \theta\}.$$

Suppose it is true that

(2.3) $$M' \cap \{x \in X : x \geq 0, x \neq 0\} = \phi.$$

(As shall be claimed in the following proposition, (2.3) is necessary for any equilibrium where agents' preferences are strictly increasing in the sense above.) Then define

(2.4) $$M = \{x \in X : x = m' + (r,0,\ldots,0) \text{ for } m' \in M' \text{ and } r \in R\}$$

and

(2.5) $$\pi(m' + (r,0,\ldots,0)) = r \text{ for } m' \in M' \text{ and } r \in R.$$

Clearly, M is a subspace of X. Moreover, (2.3) guarantees that $\pi : M \rightarrow R$ is a well-defined, strictly positive linear functional.

Proposition 1. If $\{p, (x^i, \theta^i)_{i=1}^I\}$ is an equilibrium, then (2.3) holds, and

(2.6) $x^i \in M \cap X^i, \pi(x^i) \leq 0$, and x^i is $\gtrsim^i$ maximal in $\{x \in M \cap X^i : \pi(x) \leq 0\}$.

Conversely, if p satisfies (2.3) and if there exist x^i satisfying (2.6) (for M and π defined from p) and $\Sigma_i\, x^i = 0$, then there exist θ^i such that $\{p, (x^i, \theta^i)\}$ is an equilibrium.

The proof is straightforward and is left to the reader with one hint. In the converse half, suppose x^i and p are given. By strict monotonicity of $\gtrsim^i$, there exist $\overline{\theta}^i$ such that $x^i = x(\overline{\theta}^i, p)$. Let $\theta^i = \overline{\theta}^i$ for $i \neq 1$ and $\theta^1 = -\Sigma_{i \neq 1}\, \overline{\theta}^i$—then verify that $x(\theta^1, p) = x(\overline{\theta}^i, p) = x^1$. Of course, $\Sigma_i\, \theta^i = 0$.

The interpretation of this proposition is clear. Fix an equilibrium $\{p, (x^i, \theta^i)\}$, and define M and π from p by (2.4) and (2.5). Imagine an economy in the style of Debreu (1959) where at date zero agents can purchase any net trade bundle $x \in M$ at the price $\pi(x)$. Note well that if $M \neq X$, this is not an economy with a complete set of contingent claims markets. Of course, agent i faces two constraints in this Debreu-style economy: The x he selects must lie in X^i and must satisfy the budget constraint $\pi(x) \leq 0$. Then (2.6) says that prices π are equilibrium prices in this economy with corresponding equilibrium allocation (x^i).

Interpreting the converse half is a little trickier. Fix a Debreu-style economy with contingent claims markets for claims in some M, and let π be equilibrium prices and (x^i) the corresponding equilibrium allocation. The proposition does not guarantee that there are prices p that give the same equilibrium allocations in an economy with the given long-lived securities. Rather, *if* there are prices p that give rise to M and the equilibrium π via (2.4) and (2.5), then they are equilibrium prices (with corresponding allocation (x^i)).

6.3 Complete Markets Equilibria

Suppose that for an equilibrium price system p, the corresponding space M is X. Then the equilibrium allocation (x^i) is an equilibrium allocation for a Debreu-style economy with a complete set of contingent claims markets and therefore is Pareto efficient. Thus, it is natural to seek conditions that yield $M = X$.

Define for $t < T$ and $A \in F_t$

(3.1)
$$K(t,A) = \text{cardinality}\{A' \in F_{t+1} : A' \subseteq A\},$$
$$\text{and } K = \max\{K(t,A); t < T, A \in F_t\}.$$

In words, $K(t,A)$ is the number of "subcells" of A in F_{t+1}. This is a measure of the amount of information that might be received by date $t+1$ if at date t the event A is known to prevail: If $K(t,A) = 1$, then no new information will be received. If $K(t,A) = 2$, then new information of an either/or type will be received, and so on.

Proposition 2. Let p be an equilibrium price system, and let M be defined from p by (2.4). A necessary and sufficient condition for $M = X$ is that for each $t < T$ and $A \in F_t$,

$$\text{dimension}\{\text{span}\{p(t+1,\omega);\omega \in A\}\} = K(t,A). \tag{3.2}$$

A paraphrase of this condition is that the conditional support of $p(t+1)$ given that $\omega \in A$ consists of $K(t,A)$ linearly independent vectors. There are at most $K(t,A)$ vectors in this conditional support (because $p(t+1)$ is F_{t+1} measurable). Thus, $K(t,A)$ is an upper bound on dim $\{\text{span}\{p(t+1,\omega);\ \omega \in A\}\}$. The condition is that this upper bound is hit in every instance. The proof of this proposition involves straightforward induction on T. Rather than work through the details, a full example will be given which should make both the proposition and its proof transparent.

Example. Suppose that there are six states $\{\omega_1,\ldots,\omega_6\}$ and four dates $t = 0,\ldots,3$. The exogenous information structure is given by the partitions

$$F_0 = \{\Omega\}, F_1 = \{\{\omega_1,\omega_2\},\{\omega_3,\omega_4,\omega_5,\omega_6\}\},$$
$$F_2 = \{\{\omega_1,\omega_2\},\{\omega_3,\omega_4\},\{\omega_5,\omega_6\}\},$$
$$F_3 = \{\{\omega_1\},\{\omega_2\},\{\omega_3\},\{\omega_4\},\{\omega_5\},\{\omega_6\}\}.$$

Thus, $K(1,\{\omega_1,\omega_2\}) = 1$ while $K(2,\{\omega_1,\omega_2\}) = 2$. Suppose that there are two securities whose dividends at date 3 are as in table 6.1.

Consider two possible equilibrium price systems arising from these data, as depicted in figures 6.1*a* and 6.1*b*. The column vectors in these event trees give the prices of the two securities as a function of the date and state. For example, in figure 6.1*a* the column vector $(9, 4.2)'$ which is starred is interpreted to mean $p_1(2,\omega_4) = .9$ and $p_2(2,\omega_4) = 4.2$. Note that the tree structure corresponds to the information structure.

Does $M = X$ in either or both cases? The answer is yes if and only if for every $t > 0$ and $A \in F_t$, the vector $x = (x(0),\ldots,x(T))$ that is given by $x(s) = 0$ for $s \neq t$ and $x(t) = 1_A$ is in M. That is, there must exist a trading strategy that produces one unit of consumption in event A at date t and nothing at any other date-event pair. Begin by asking if this is true for $t = 1$ and for every $A \in F_1$. In each case the answer is yes—the two possible values of $p(1)$ are linearly independent; thus, there exist (θ_1,θ_2) and (θ_1',θ_2') such that $(\theta_1,\theta_2)\cdot(p_1(1),p_2(1))' = 1_{\{\omega_1,\omega_2\}}$ and $(\theta_1',\theta_2')\cdot(p_1(1),p_2(1))' = 1_{\{\omega_3,\omega_4,\omega_5,\omega_6\}}$. This clearly suffices. Now proceed to ask the question for $t = 2$. For $A = \{\omega_1,\omega_2\}$ there is no problem in either case. But matters are not so simple for $A = \{\omega_3,\omega_4\}$. In case a it can be done: First, solve

Table 6.1

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
Payoff of security #1—$d_1(\cdot)$	1	1	1	1	1	1
Payoff of security #2—$d_2(\cdot)$	1	2	3	6	4	5

$$(\theta_1,\theta_2)\cdot(.9,4.2)' = 1 \text{ and } (\theta_1,\theta_2)\cdot(.909,4.1)' = 0.$$

This can be done because the two column vectors are linearly independent. Let (θ_1^*,θ_2^*) be the solution. Next, solve

$$(\theta_1,\theta_2)\cdot(.81,1.26)' = 0 \text{ and } (\theta_1,\theta_2)\cdot(.81,3.75)' = (\theta_1^*,\theta_2^*)\cdot(.81,3.75)'.$$

This can be done by the first step—the solution, denote it $(\theta_1^{**},\theta_2^{**})$, is just a scalar multiple of (θ_1',θ_2'). Then the strategy of starting with $(\theta_1^{**},\theta_2^{**})$ at date zero, changing to (0,0) at date one if $\{\omega_1,\omega_2\}$ occurs and to (θ_1^*,θ_2^*) if $\{\omega_3,\omega_4,\omega_5,\omega_6\}$ occurs, and then consuming everything at date two yields one unit of consumption at date two if and only if $\{\omega_3,\omega_4\}$ occurs.

But consider case *b*. One cannot solve

$$(\theta_1,\theta_2)\cdot(.9,4.2)' = 1 \text{ and } (\theta_1,\theta_2)\cdot(.909,4.242)' = 0,$$

because the two column vectors are linearly dependent. Thus, if one consumes one unit at date two and nothing at date three when $\{\omega_3,\omega_4\}$ occurs, one must consume something either at date two or at date three when $\{\omega_5,\omega_6\}$ occurs.

By inductively applying this sort of logic, one can see that $M = X$ in case *a* (as predicted by proposition 2), but that $M \neq X$ in case *b*.

Example *a* makes the basic idea clear. In this economy there are six states of nature and only two securities, yet markets are complete. This is because the process of learning which of the six states is the true state takes place not all at once but in three steps. Agents can revise their portfolios after each step in the learning process. At each step, at most *two* "signals" are possible. And the equilibrium prices of the two securities are "well behaved"—they are "linearly independent" in a fashion that enables agents to take full advantage of new information as it is received.

6.4 Genericity of the case $M = X$ with K or more securities

Condition (3.2) in proposition 2 can be viewed as two nested conditions. First, the number of securities N must be at least as large as $\max_{t,A} K(t,A) = K$. In addition to this, the equilibrium prices must be "sufficiently independent." In case *b* of the example, $N = K$ $(= 2)$, but because $p(2,\omega_3)$ and $p(2,\omega_5)$ are not independent, markets are not com-

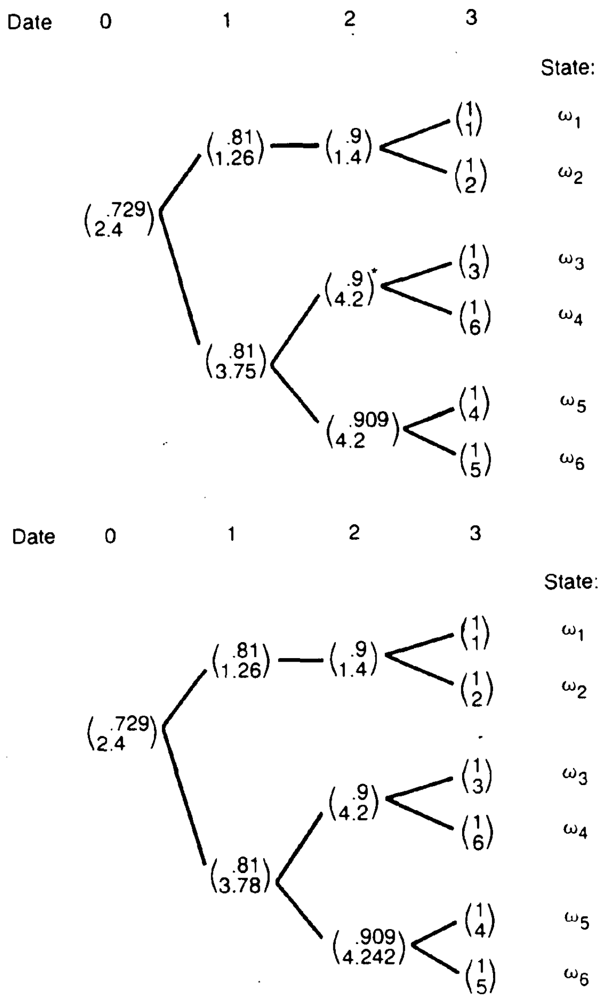

Fig. 6.1*a* (*top*) and *b* (*bottom*)

plete. This second part of (3.2) is less than satisfactory on economic grounds, because it involves endogenous data, the equilibrium prices p. It cannot be completely dispensed with—not *every* set of K or more securities will have equilibrium prices that satisfy (3.2). Consider, for example, K securities whose dividends at date T are scalar multiples of one another. (That is, $d_n = r_n d_1$ for $r_n \in R$.) But a result almost this strong is possible. Fix the economic setting; that is, fix the state space, information structure, and agents. Suppose N securities are selected at "random." By a selection of N securities is meant a selection of a point d from the set $(R^{(\Omega, F_T)})^N$, which hereafter is denoted by D. A subset of D will be called *sparse* if its closure has Lebesgue measure zero. If the selection of d is done "randomly enough," there is zero probability that the outcome will land in a given sparse set. Following the terminology of Radner (1979), a result that holds off of a sparse set is called *generic*. The next proposition therefore gives the title of this section.

Proposition 3. Fix the economic setting. Suppose that if in this setting a Debreu-style regime of complete contingent claims markets is set up, then there is an equilibrium with equilibrium allocation $\{x^i\}$. Then if $N \geq K$, there is a sparse set in D such that for all d not in that set, the economy with N long-lived securities paying d admits an equilibrium with $M = X$ and with equilibrium allocation $\{x_i\}$.

Proof. In the Debreu-style economy, there is a linear functional $\phi : X \to R$ that is strictly positive and that satisfies

$$(4.1) \quad x^i \in X^i, \phi(x^i) \leq 0, \text{ and } x^i \text{ is } \gtrsim^i \text{ maximal in } \{x \in X^i : \phi(x) \leq 0\}.$$

(That is, ϕ gives the equilibrium prices.) Normalize ϕ so that $\phi((1,0,\ldots,0)) = 1$. For $t = 0,\ldots,T$ and $A \in F_t$, define $\chi_{t,A}$ by

$$\chi_{t,A}(s) = 0 \text{ for } s \neq t \text{ and } \chi_{t,A}(t) = 1_A.$$

That is, $\chi_{t,A}$ is the claim that pays one unit of consumption at date t in the event A.

For any $d \in D$, define p from d and ϕ as follows. For $t \leq T$ and $\omega \in \Omega$, let $A \in F_t$ be such that $\omega \in A$. Then let

$$(4.2) \qquad p_n(t,\omega) = \sum_{\omega \in A} d_n(\omega)\phi(\chi_{T,\{\omega\}})/\phi(\chi_{t,A}).$$

Two things, once demonstrated, give the result. First, except for d from a sparse subset of D, p so defined satisfies (3.2), and thus $M = X$. Second, for all such d, the linear functional π defined in (2.5) is ϕ.

For the first result, it is necessary to show that except for d from a sparse set, the set $\{p(t+1,\omega);\ \omega \in A\}$ contains $K(t,A)$ linearly independent vectors for every t and $A \in F_t$. Since there are finitely many such pairs (t,A) and since the union of a finite number of sparse sets is sparse, it suffices to show that for every t and A the set of $d \in D$ for which the corresponding $\{p(t+1,\omega);\ \omega \in A\}$ does not contain $K(t,A)$ linearly independent vectors is sparse. Using (4.2), the set $\{p(t+1,\omega); \omega \in A\}$ can be written

$$\{\sum_{\omega \in A'} d(\omega)\phi(\chi_{T,\{\omega\}})/\phi(\chi_{t+1,A'}); A' \in F_{t+1}, A' \subseteq A\},$$

which, letting $\alpha(t,\omega)$ denote the strictly positive scalar $\phi(\chi_{T,\{\omega\}})/\phi(\chi_{t,A})$ is

$$\{\sum_{\omega \in A'} d(\omega)\alpha(t+1,\omega); A' \in F_{t+1}, A' \subseteq A\}.$$

The set of d for which this set of $K(t,A)$ vectors is linearly dependent is clearly closed. That it has Lebesgue measure zero is also apparent as follows: Let $\Upsilon : D \to (R^N)^{K(t,A)}$ be the map

$$\Upsilon(d) = [\sum_{\omega \in A'} d(\omega)\alpha(t+1,\omega)]_{A' \in F_{t+1}, A' \subseteq A},$$

and let λ denote Lebesgue measure on D. Then the measure $\lambda \circ \Upsilon^{-1}$ on $(R^N)^{K(t,A)}$ is absolutely continuous with respect to Lebesgue measure because the $\alpha(t+1,\omega)$ are strictly positive. And the Lebesgue measure in $(R^N)^{K(t,A)}$ of vectors $[(r)_{n=1}^N]_{k=1}^{K(t,A)}$ such that the $(r)_n$ are linearly dependent is zero, if $N \geq K$.

For the second result, it suffices to show that for all strategies θ, $\phi(x(\theta,p)) = 0$. There is nothing to do but grind this out:

$$\phi(x(\theta,p)) = \phi(\chi_{0,\Omega})x(o;\theta,p) + \sum_{t=1}^{T-1} \sum_{A \in F_t} \phi(\chi_{t,A})x(t,A;\theta,p) +$$

$$\sum_{\omega \in \Omega} \phi(\chi_{T,\{\omega\}})x(T,\omega;\theta,p)$$

$$= \phi(\chi_{0,\Omega})(-\theta(0)\bullet p(0)) + \sum_{t=1}^{T-1} \sum_{A \in F_t} \phi(\chi_{t,A})\,[\theta(t-1,A) -$$

$$\theta(t,A)]\bullet p(t,A) + \sum_{\omega \in \Omega} \phi(\chi_{T,\{\omega\}})[(\theta(T-1,\omega) -$$

$$\theta(T,\omega))\bullet p(T,\omega) + \theta(T,\omega)\bullet d]$$

$$= \sum_{t=0}^{T-1} \sum_{A \in F_t} [-\theta(t,A)\bullet p(t,A)\phi(\chi_{t,A}) +$$

$$\sum_{A' \in F_{t+1}, A' \subseteq A} \theta(t,A)\bullet p(t+1,A')\phi(\chi_{t+1,A'})]$$

(note that $d \equiv p(T)$)

$$= \sum_{t=0}^{T-1} \sum_{A \in F_t} \theta(t,A)\bullet[-\sum_{\omega \in A} p(T,\omega)\phi(\chi_{T,\{\omega\}}) +$$

$$\sum_{A' \in F_{t+1}, A' \subseteq A} p(T,\omega)\phi(\chi_{T,\{\omega\}})]$$

$$= \sum_{t=0}^{T-1} \sum_{A \in F_t} \theta(t,A)\bullet[0] = 0. \qquad \text{QED}$$

A remark may help the reader through this maze. If the security prices p are to be the "same" as ϕ, then (4.2) is required. This can be seen as follows. In the Debreu-style economy, for $A \in F_t$ a claim to $d_n 1_A$ (contingent) units of consumption at date T costs $\Sigma_{\omega \in A}\, d_n(\omega)\phi(\chi_{T,\{\omega\}})$ units of date zero consumption, or $\Sigma_{\omega \in A}\, d_n(\omega)\phi(\chi_{T,\{\omega\}})/\phi(\chi_{t,A})$ units of date t, event A consumption. If expectations are rational and an equilibrium is in force, then this must be the price in units of date t consumption of security

n at date t in the event A. This is (4.2). As to the genericity of condition (3.2), the reader may find it helpful to take the economic structure of the example, make up a set of equilibrium prices for a Debreu-style economy with complete markets, and then see what is entailed in picking d so that p defined from (4.2) does not satisfy (3.2).

Several remarks about this proposition and the previous analysis are worth making.

1. A by-product of the proposition is a result concerning the "generic existence of equilibrium": If $N \geq K$ and if the economic setting is such that an equilibrium exists with a Debreu-style regime of complete contingent claims markets, then for all d except from a sparse subset of D, the economy admits an equilibrium.

2. The proposition does not show that for $N \geq K$ and generic d, all equilibria are Pareto efficient. It only shows that there are efficient equilibria. But it seems likely that the stronger result is true, at least for "most" economic settings.

3. The following result complementary to the proposition might be imagined: Fixing Ω and $\{F_t\}$, for every $d \in D$ where $N \geq K$ the set of "communities of agents" that do not admit an equilibrium in which $M = X$ is sparse among all communities of agents. The concept of a community of agents is ambiguous here, but what is intended is something like the treatment in Radner (1979), where agents are parametrized by their subjective probability assessments. Such a result is impossible—as already noted, if the d_n are collinear and Ω has more than one state, then $M = X$ cannot result. It is conjectured that the result is true, however, if this and similar trivially pernicious choices of d are disallowed. (It seems likely that the technology developed in Radner (1979) would work excellently in this context.)

4. In sections 6.2 through 6.4 it has been assumed that there is a single perishable consumption good. It should be clear that the results given hold if there is a finite number of consumption goods, as long as there are spot markets in the consumption goods at each date and if securities pay off in a good whose relative price is strictly positive in the date T spot market.

5. It has been assumed that all securities "live" from date zero to T and that securities pay off only on date T. Clearly, the basic results do not change if securities pay off on other dates as well and/or if securities live for other sets of dates. The important thing is that for any time period t to $t+1$ and event $A \in F_t$, at least $K(t,A)$ securities must be "alive."

6. For the sake of completeness, a result from Harrison and Kreps (1979) is repeated here. Suppose that one is given a state space Ω, a time index set $\{0,\ldots,T\}$, an information structure $\{F_t\}$, and a set of N securities $\{d_n; n = 1,\ldots,N\}$. Moreover, suppose that a price system p is given and it

is claimed that p is the equilibrium price system for an economy as above. That is, the claim is that there exists a population of agents meeting the requirements above such that p is part of an equilibrium in the economy that houses them. Under what conditions is this claim true? For simplicity, assume that one of the securities, say, the first, pays out a strictly positive amount in every state of nature. One necessary and sufficient condition for an affirmative answer is that (2.3) is true. A second is that there exists a probability measure Q on Ω such that $Q(\{\omega\}) > 0$ for every $\omega \in \Omega$, and if $E[\bullet]$ denotes expectation with respect to Q, then

$$E[p_n(t+1)/p_1(t+1)\,|\,F_t] = p_n(t)/p_1(t),$$

for every $t < T$ and n. (Also, $p(T)$ must be proportional to d.) Moreover, $M = X$ if and only if there exists exactly one such probability measure.

6.5 The Black-Scholes Model

The Black-Scholes model is a continuous time, infinite state version of the model in section 6.2 that comes to a similar and striking conclusion: With a "simple enough" information structure, a small finite number (two) of securities can span an infinite dimensional space of contingent claims, owing to the infinite number of trading opportunities. The sense in which this is true is not entirely straightforward, so a review of the model is now presented.

A probability space $(\Omega, \boldsymbol{F}, P)$ is given. On this space is defined a standard (mean zero, variance one) Brownian motion $\{B(t);\ t \in [0,1]\}$. The information available to agents at date t ($t \in [0,1]$) is the history of the Brownian motion up to that date: $\boldsymbol{F}_t = \boldsymbol{F}\{B(u);\ 0 \le u \le t\}$. It is assumed that $\boldsymbol{F} = \boldsymbol{F}_1$ as before.

For simplicity it will be assumed that agents consume only at dates zero and one and that they have endowment of the consumption good only at those dates.[3] A consumption bundle is therefore a pair $x = (r, y)$, where $r \in R$ is consumption at date zero and y is state contingent consumption at date one, an $\boldsymbol{F}$ measurable real valued function. The space of consumption bundles is denoted $X = R \times Y$ and is assumed to be a linear space of such pairs.[4] Agents are described by a feasible net trade set $X^i \subseteq X$ and a preference ordering $\gtrsim^i$ on X^i. It is assumed that each X^i is "comprehensive upward" and that $\gtrsim^i$ is strictly increasing in the following sense: If $x' = (r', y') \in X$ is such that $r' \ge 0$ and $y' \ge 0$ P-a.s., and either $r' > 0$ or $P(y' > 0) > 0$, then for all $x \in X^i$, $x + x' \in X^i$ and $x + x' >^i x$.

There are two long-lived securities in this world, which (as before) are claims to date one consumption. The first yields e^r units of consumption independent of the state, while the second yields $\exp\{\mu + \sigma B(1,\omega)\}$ units in state ω. (Here, r, μ, and $\sigma > 0$ are given constants.) Trading in the two securities can take place at any date between zero and one (as better information about the true state of the world is received), with relative equilibrium prices

$$p_1(t) = e^{rt} \text{ and } p_2(t) = \exp\{\mu t + \sigma B(t)\}. \tag{5.1}$$

At date zero, the two securities and the consumption good are traded at relative prices one apiece.

In saying that these are equilibrium prices, the following is meant. Given the price process $p = \{p(t)\}$, agents seek to manage a portfolio of the two securities so as to obtain the best possible net trade vector. A *trading* strategy is formally represented as a vector stochastic process $\theta = \{\theta_n(t); n = 1,2, t \in [0,1]\}$, where $\theta_n(t,\omega)$ represents the number of shares of security n held at time t if the state is ω. The obvious informational constraint on θ is that for every t, $\theta(t)$ must be F_t measurable. But more qualifications are necessary. One must say what sorts of trading strategies represent actions that agents are physically capable of. Moreover, because no consumption takes place between dates zero and one, because agents' preferences are strictly increasing, and because (by assumption) agents do not receive fresh funds for investment between those two dates, any trading strategy θ should be *self-financing*. That is to say, any changes in the composition of an agent's portfolio at dates $t \in (0,1]$ should involve zero net cost of transaction. Any purchases should be financed by a corresponding sale, and the proceeds from any sale should be reinvested elsewhere. (Date one is included in this constraint as it is imagined that date one consumption takes place *after* date one markets close.)

One possibility is to say that agents can employ any strategy θ such that $t \to \theta_n(t,\omega)$ is of bounded variation for every n and a.e. ω. Such θ correspond to trading strategies that have the representation: The amount held at date t is the difference between a total amount bought during $[0,t]$ and an amount sold during that period. It is clear that such a strategy θ should be called self-financing if $0 = d\theta(t)\cdot p(t)$ for all $t \in (0,1]$. In this case θ yields the net trade vector $x(\theta,p) = (r(\theta,p), y(\theta,p))$ given by

$$r(\theta,p) = -\theta(0)\cdot p(0) \text{ and } y(\theta,p) = \theta(T)\cdot d. \tag{5.2}$$

A second, less generous possibility is to say that agents are capable of employing only *simple* trading strategies, defined as follows. A trading strategy θ is called simple if there exist a finite integer J and dates $0 = t_0 < t_1 < t_2 < \ldots < t_J \leq 1$ such that $\theta(t,\omega)$ is constant over intervals of the form $t \in [t_j, t_{j+1})$. In words, agents rearrange their portfolios only finitely many times, where the number of times and the dates are fixed in advance. A simple strategy θ is self-financing if $[\theta(t_j) - \theta(t_{j-1})]\cdot p(t_j) = 0$ for all $j \geq 1$, in which case (5.2) gives $x(\theta,p)$.

Having defined what strategies agents are capable of, the definition of an equilibrium proceeds exactly as before. An equilibrium is an ensemble $\{p,(x^i,\theta^i)\}$ such that $p(t)$ is F_t measurable for all t, $x^i = x(\theta^i,p) \in X^i$ for each i, x^i is $\gtrsim^i$ maximal in $\{x(\theta,p)\} \cap X^i$ for each i, and $\Sigma_i\, \theta^i = 0$.

Does the Black-Scholes model give prices which for some community of agents are equilibrium prices in this sense? The answer to this question depends on what trading strategies are allowed to agents. If bounded variation strategies as defined above are permitted, then the answer is no. This is because there is a bounded variation strategy θ such that for p the Black-Scholes prices, $r(\theta,p) > 0$ and $y(\theta,p) \geq 0$ P-a.s. That is, the condition analogous to (2.3) does not hold. See Harrison and Kreps (1979), section 6) for the basic idea. Note that this phenomenon is not peculiar to the Black-Scholes model. It occurs in virtually every continuous time model with frictionless markets and two or more securities. But if simple trading strategies only are allowed, then the answer is yes. To show this, show that (2.3) does hold in this case. (A direct proof is not difficult.) The discussion in Kreps (1981, section 6) shows that this is sufficient.

Take then the case where only simple trading strategies are allowed. A result analogous to proposition 1 is immediate. Defining

$$M' = \{x \in X : x = x(\theta,p) \text{ for a simple trading strategy } \theta\},$$

$$M = \{x \in X : x = m' + (r,0) \text{ for } m' \in M' \text{ and } r \in R\}, \text{ and}$$

$$\pi(m' + (r,0)) = r \text{ for } m' \in M' \text{ and } r \in R,$$

it follows that M is a subspace of X and π is a well-defined, strictly positive linear functional on M. Moreover, if $\{p,(x^i,\theta^i)\}$ is an equilibrium, then

$$\pi(x^i) \leq 0, x^i \in X^i \cap M, \text{ and } x^i \text{ is } \gtrsim^i \text{ maximal in } \{x \in X^i \cap M : \pi(x) \leq 0\}.$$

That is, $\{x^i\}$ is an equilibrium allocation in a Debreu-style economy where claims in M can be bought at prices π.

Is this a "complete markets" equilibrium? It can be shown that $M \neq X$ (for any reasonable choice of X), so the answer seems to be no. But there is a sense in which the answer is yes. Suppose that $X = R \times L^2(\Omega,F,P)$; strategies θ must satisfy $\theta_n(t)p_n(t) \in L^2(\Omega,F,P)$ for all t and $n = 1,2$, $X^i = X$ for each i, and each agent's preferences are continuous in the Euclidean $\times$ L^2-norm product topology on X. Then following the results in Harrison and Kreps (1979) and Kreps (1981, especially theorem 5), there exists a strictly positive linear functional $\psi : X \rightarrow R$ such that if $\{p,(x^i,\theta^i)\}$ is an equilibrium (where p are the Black-Scholes prices, of course), then

$$\psi(x^i) \leq 0, x^i \in X^i, \text{ and } x^i \text{ is } \gtrsim^i \text{ maximal in } \{x \in X^i : \psi(x) \leq 0\}. \tag{5.3}$$

(The linear functional ψ turns out to be the unique continuous, strictly positive extension of π from M to all of X, and it is the uniqueness of this extension that yields (5.3).)

This is the sense in which two long-lived securities can yield a complete set of contingent claims for uncountably many contingencies if there are infinitely many trading opportunities. One feels uneasy about both this model and the conclusion arrived at on several grounds, among which are the following:

1. The restriction to simple trading strategies is unnatural. There are other more natural ways to exorcise trading strategies that violate (2.3). For example, the requirement that $\theta(t)\bullet p(t) \geq -L$ for some finite L will suffice for the Black-Scholes model. This can be interpreted as a credit constraint. But no theory has been developed along these lines to the author's knowledge.

2. The twin assumptions that each $X^i = X$ and that each $\gtrsim^i$ is Euclidean $\times$ L^2 continuous are hardly palatable. (To some extent, the use of the L^2 topology can be foregone. See Harrison and Kreps 1979, section 7.) It would be nice to be able to widen the class of trading strategies so that $M = X$ *and* p is part of an equilibrium. The extant literature on the Black-Scholes model, especially Merton (1977), suggests that the former can be done by allowing trading strategies that are Ito integrals, and it has been conjectured by Harrison (1978) that the entire program is feasible if a restricted class of Ito integrals is allowed.

3. Most important is that no intuitive feeling has been developed for why two securities suffice to give "complete markets" in this model, nor whether this result is generic in any sense. How does one generalize K to a continuous time setting, and why (if a generalization is possible) does $K = 2$ in this case? The proof of theorem 3 in Harrison and Kreps (1979) is the key step in obtaining the result that markets are "complete" in the Black-Scholes model, and in that proof the key step is the use of the remarkable result of Kunita and Watanabe (1967) that every martingale on the Brownian information structure can be written as a stochastic integral of Brownian motion. Intuitive comprehension of that result is necessary if one is to feel comfortable using the Black-Scholes model.

6.6 A Convergence Result

Begin with the following pieces of the Black-Scholes model: (Ω, F, P), $\{F_t; t \in [0,1]\}$; a finite collection of agents with $X^i = X = R \times L^2(\Omega, F, P)$ and preferences that are Euclidean $\times$ L^2 continuous and strictly increasing. Assume that if these agents are placed in a Debreu-style economy with a complete set of contingent claims markets, then the linear functional $\psi : X \to R$ that is introduced in section 6.5 is an equilibrium set of prices, with corresponding equilibrium allocation (x^i).

Now consider placing these agents in a sequence of economies like those in section 6.2 except that Ω as above remains fixed and so is infinite. Index the economies by $H = 1, 2, \ldots$ In economy H, trading in two securities takes place at the $H + 1$ dates $t = 0, 1/H, \ldots, 1$. The information available at date h/H is $F_{h/H}$ (for F_t as above), and thus "$K = \infty$" in each of these economies. Let $\{p_n(t;H); n = 1,2, t = 0, 1/H, \ldots,\}$ denote the equilibrium prices in economy H, and let $(x^i(H))$ denote the corresponding equilibrium allocation. Assume for simplicity that $d_1(H) = e^r$ for all H, so that $p_1(t;H) > 0$ for all t and H by (2.3), and that $p_1(0;H) = 1$ for all H.

Of course, in economy H it is not true that $M = X$. Thus, the allocation $(x^i(H))$ may be Pareto inefficient. To measure the degree of this inefficiency, the allocations $(x^i(H))$ will be compared with the efficient allocation (x^i). Write $x^i(H) = (r^i(H), y^i(H))$ and $x^i = (r^i, y^i)$. Define $\delta^i(H)$ by

$$\text{(6.1)} \qquad \delta^i(H) = \inf\{\delta > 0 : (r^i(H) + \delta, y^i(H)) \succ^i (r^i, y^i)\}.$$

That is, $x^i(H)$ augmented by $\delta^i(H)$ units of date zero consumption is at least as good as x^i. If agent i prefers $x^i(H)$ to x^i, then $\delta^i(H)$ is set equal to zero. If no δ can be found to make i better off with $x^i(H) + (\delta, 0)$ than with x^i, then $\delta^i(H)$ is set equal to $+\infty$. Define

$$\text{(6.2)} \qquad \Delta(H) = \sum_{i=1}^{I} \delta^i(H).$$

In words, $\Delta(H)$ units of date zero consumption can be distributed among agents after they trade to equilibrium in economy H so that each agent is at least as well off as in the efficient allocation (x^i). If $\Delta(H)$ is "small," then economy H is "nearly" efficient.

Proposition 4. Let $\{p(t); t \in [0,1]\}$ be the Black-Scholes price system. If the equilibrium prices $\{p(t;H)\}$ converge to $\{p(t)\}$ in the sense that

$$\text{(6.3)} \qquad \lim_{H\to\infty} p_2(t;H)/p_1(t;H) = p_2(t)/p_1(t) \text{ in } L^2 \text{ uniformly in } t,$$

then $\lim_{H\to\infty} \Delta(H) = 0$.

Proof. (This proof makes heavy use of the technology of Harrison and Kreps 1979 and Kreps 1981, and it is probably unintelligible to readers unfamiliar with those papers.)

Without loss of generality, it can be assumed that $r = 0$ and $p_1(t) \equiv p_1(t;H) \equiv 1$ for all H. (See Harrison and Kreps 1979, section 7.) In this case (6.3) becomes: $\lim_{H\to\infty} p_2(t;H) = p_2(t)$ in L^2 uniformly in t.

It will suffice to show that for every $x \in X$ such that $\psi(x) \le 0$ there exists a sequence of (self-financing) trading strategies $\theta(H)$ that involve trading at dates $0, 1/H, \ldots, 1$ only and a sequence $\{x(H)\} \subseteq X$ such that

$$\text{(6.4)} \qquad x(\theta(H), p(H)) \ge x(H) \text{ for every } H, \text{ and } \lim_{H\to\infty} x(H) = x.$$

(Limits in X are always in the Euclidean $\times$ L^2 product topology.) For if this is true for all x, it is true in particular for x^i. Thus, as H gets large, it is feasible in economy H for agent i to obtain an $x^i(\theta(H), p(H))$ which is at least as good as some $x(H)$ which in turn is close in terms of $\succsim^i$ to x^i. (Recall the continuity of $\succsim^i$.) By revealed preference, $x^i(H) \succsim^i x(\theta^i(H), p(H))$, and thus $\delta^i(H) \to 0$ as $H \to \infty$.

Fix $x \in X$. From Harrison and Kreps (1979) and Kreps (1981) it is known that there exist simple trading strategies $\theta(\ell)$ that are self-financing for p and $x(\ell)$ ($\ell = 1, 2, \ldots$) such that

$$(6.5) \qquad x(\theta(\ell), p) \geq x(\ell) \text{ for every } \ell \text{ and } \lim_{\ell \to \infty} x(\ell) = x.$$

In Harrison and Kreps (1979), such θ are assumed to satisfy the condition $\theta_n(t)p_n(t) \in L^2$ for each t and n. But in fact one can *add* the condition that $\theta_2(t) \in L^\infty$ for all t, and still there exist $\theta(\ell)$ and $x(\ell)$ as in (6.5). (To see this, review the proof of theorem 2 in Harrison and Kreps 1979 with this additional condition on simple trading strategies, and verify that it remains a valid proof.) For the remainder of this proof, simple trading strategies will be assumed to satisfy this additional condition.

Let θ be any simple trading strategy that is self-financing for p. Define from θ a trading strategy θ^H (for economy H) by

$$\theta_2^H(h/H) = \theta_2(h/H) \text{ for } h = 0, \ldots, H, \theta_1^H(0) = \theta_1(0), \text{ and}$$

$$\theta_1^H(t) \text{ for } t > 0 \text{ defined so that } \theta^H \text{ is self-financing for } \{p(t; H)\}.$$

Note that if θ changes values at dates $0 = t_0 < t_1 < \ldots < t_J \leq 1$, then θ^H changes values at dates $t_0^H, t_1^H, \ldots, t_J^H$, where for $t \in [0,1]$, $t^H = \inf\{u \geq t; u = h/H$ for some integer $h\}$.

To show that (6.4) is true, it will suffice to show that for any simple θ (self-financing for p),

$$(6.6) \qquad \lim_{H \to \infty} x(\theta^H, p(H)) = x(\theta, p).$$

For this combined with (6.5) yields (6.4) by an easy argument. To show (6.6), note first that $\theta^H(0) = \theta(0)$ and therefore $r(\theta^H, p(H)) = -\theta(0) \cdot p(0; H)$. Since F_0 is trivial, the constant $p(0; H) \to p(0)$ by assumption, and thus $r(\theta^H, p(H)) \to -\theta(0) \cdot p(0) = r(\theta, p)$. Next note that

$$y(\theta, p) = \theta(1) \cdot d = \theta(t_J) \cdot d = \theta(t_J) \cdot [d - p(t_J)] + \theta(t_J) \cdot p(t_J)$$
$$= \theta(t_J) \cdot [d - p(t_J)] + \theta(t_{J-1}) \cdot p(t_J)$$

(because θ is p self-financing)

$$= \theta(t_J) \cdot [d - p(t_J)] + \theta(t_{J-1}) \cdot [p(t_J) - p(t_{J-1})] + \theta(t_{J-1}) \cdot p(t_{J-1})$$

$$= \theta(t_J) \cdot [d - p(t_J)] + \sum_{j=0}^{J-1} \theta(t_j) \cdot [p(t_{j+1}) - p(t_j)] + \theta(0) \cdot p(0)$$

(by iterating the above argument)

$$= \theta_2(t_J)[d_2 - p_2(t_J)] + \sum_{j=0}^{J-1} \theta_2(t_j)[p_2(t_{j+1}) - p_2(t_j)] + \theta(0)\cdot p(0)$$

(since $d_1 \equiv p_1(t) \equiv 1$). Similarly,

$$y(\theta^H, p(H)) = \theta_2^H(t_J^H)[d_2(H) - p_2(t_J^H;H)] +$$

$$\sum_{j=0}^{J-1} \theta_2^H(t_j^H)[p_2(t_{j+1}^H;H) - p_2(t_j^H;H)] + \theta^H(0)\cdot p(0;H).$$

For H large enough that $t_{j+1} - t_j > 1/H$ for all j, it follows that $\theta_2^H(t_j^H) = \theta_2(t_j)$ for all j. By assumption, $\theta_2(t) \in L^\infty$ for all t, and by previous argument, $\theta(0)\cdot p(0) = \lim_H \theta^H(0)\cdot p(0;H)$. Thus, $y(\theta,p) = \lim_H y(\theta^H, p(H))$ follows from $\lim_H d_2(H) = d_2$ in L^2 (note that d is proportional to $p(1)$ and $d(H)$ is proportional to $p(1;H)$, and by assumption $d_1 \equiv d_1(H) \equiv 1$), and

$$\lim_{H\to\infty} [p_2(t^H;H) - p_2(t)] = \lim_{H\to\infty} [p_2(t^H;H) - p_2(t^H)] +$$

$$\lim_{H\to\infty} [p_2(t^H) - p_2(t)] = 0 + 0 = 0 \text{ in } L^2 \text{ uniformly in } t,$$

the first by the hypothesis of the proposition and the second because $t^H - t < 1/H$ and geometric Brownian motion is L^2 uniformly continuous. QED.

Proposition 4 shows that it is possible to have K much larger than N (∞ versus 2), and yet equilibrium allocations are "nearly" efficient. This in itself is not remarkable—Wilson (1968) shows that this is possible by making strong assumptions concerning agents' preferences. But here much weaker assumptions about preferences are made. Instead, there are strong assumptions on the ability to trade securities frequently, the way in which uncertainty resolves, and the approximate behavior of equilibrium security prices. It would be preferable, of course, not to make assumptions about equilibrium prices. A possible direction would be to take as given Ω, F, P, $\{F_t\}$, and agents, assume that $\psi : X \to R$ gives equilibrium prices for a Debreu-style economy with complete markets, and then show that (1) for each H, or for H sufficiently large, there is an equilibrium in the long-lived securities economy with two securities that pay off exactly (or approximately) what the Black-Scholes securities pay, and (2) these equilibrium prices converge to the Black-Scholes prices in the sense of (6.3). But even if this is true, it is a result predicated on very strong assumptions.

A number of extensions can be obtained cheaply. The reliance on the exact distributions of the Black-Scholes price processes is unnecessary—the methodology works for any diffusion process covered by theorem 3 in Harrison and Kreps (1979). This includes, for more than two securities, multidimensional diffusions. The diffusion assumption is not particularly necessary, except insofar as it is a case where "markets are complete" in the limit. Other stochastic processes, such as the jump process model of Cox and Ross (1976), could be used. Finally, the use of the space of square integrable claims and the L^2 topology is not necessary—see the discussion in Harrison and Kreps (1979, section 7). Of course, if preferences are continuous in another topology, then (6.3) will have to be modified appropriately.

It is worth noting that what is a flaw in the Black-Scholes model, namely, the need to restrict attention to simple trading strategies, becomes a virtue here. If one takes the view (implicit in proposition 4) that the Black-Scholes model is to be regarded as an ideal approximation to economies with many, but only finitely many, trading dates, then the restriction makes sense. In the "limit" economy, agents should not be able to employ strategies that cannot be approximated (in terms of preference) by strategies available in the economies approaching the limit. The trading strategies of bounded variation that turn nothing into one unit of consumption do not pass this test for agents of the sort discussed here. So from this perspective, these strategies can reasonably be excluded from the set of strategies available in the limit economy.

6.7 Extending the Convergence Result

Perhaps the least satisfactory aspect of proposition 4 is that in it, Ω, F, P, $\{F_t\}$, and the agents do not change along the sequence. A more satisfactory treatment of the problem would cover the following example.

Fix a positive integer H and imagine an economy with 3^H states of nature. Every state ω has H coordinates $(\omega_1, \ldots, \omega_H)$ where each ω_h takes on one of three possible values: -2; 0; 2. Think of each ω_h as being determined by an independent experiment, where the probabilities of the outcomes 2 and -2 are $1/8$ apiece and the probability of outcome 0 is $3/4$. Let Ω^H denote this state space and P^H this probability measure on Ω^H. In this economy, agents consume at dates zero and one and trade at dates $t = 0, 1/H, \ldots, 1$. The information available at date h/H, denoted $F^H_{h/H}$, is the σ-field generated by the first h coordinates of the state. (That is, at date h/H the first h coordinates of the state have been revealed to agents.) Note that this yields $K = 3$. There are two securities traded in this economy, paying the following dividends at date one:

$$d_1^H = e^r \text{ and } d_2^H(\omega) = \exp\left[\sigma\left(\sum_{h=1}^{H} \omega_h\right)/\sqrt{H} + \mu\right].$$

Here $\omega = (\omega_1, \ldots, \omega_H)$, and r, μ, and $\sigma > 0$ are given constants. These are traded together with the consumption good at date zero at relative prices one apiece. The two securities are also traded for each other at dates $1/H, \ldots, 1$ with relative equilibrium prices

$$p_1^H(h/H,\omega) = e^{rh/H} \text{ and } p_2^H(h/H,\omega) = \exp[\sigma(\sum_{g=1}^{h} \omega_g)/\sqrt{H} + \mu h/H].$$

(No matter what values r, μ, and σ take on as long as $\sigma > 0$, for large enough H these prices are equilibrium prices for some agents. That is, for all sufficiently large H they satisfy (2.3).)

Because $K = 3$ and $N = 2$, this economy has incomplete markets. Yet for large H, the equilibrium price system in this economy is "very much like" the Black-Scholes price system. A more precise statement is that as H goes to infinity, the price systems p^H converge weakly to the Black-Scholes price system (in the sense of Billingsley 1968).[5] Does this imply that as H goes to infinity, the equilibrium allocations are asymptotically Pareto efficient?

Proposition 4 offers no concrete guidance on this question. Here, unlike there, as H changes so do state spaces and information structures. Since the state spaces change, so must the commodity spaces, and therefore so must the agents. There is a sense in which proposition 4 offers evidence that the answer to this question is yes: It is possible to define on the Black-Scholes probability space a sequence of vector stochastic processes $\{\hat{p}^H(h/H);\ h = 0,\ldots,H\}$ that have the same distribution as the processes $\{p^H(h/H)\}$ given above and that converge to the Black-Scholes prices in the sense of (6.3).[6] Therefore, if for large H one seeks to approximate in $M'(H)$ (the set of budget feasible net trades in economy H) the individual agents' parts of an allocation that is budget feasible at the Black-Scholes prices (on the presumption that one such allocation is Pareto efficient), then proposition 4 suggests that this is possible.

But for a number of reasons, the quality of this evidence is low.

1. As noted above, the agents must change with H. In what sense can an allocation from the Black-Scholes economy be Pareto efficient for economy H? There can be no sense in which this is true, because for agents in economy H, their piece of any such allocation does not lie in the domain of their preferences. What would make sense is a statement such as:

(7.1) In economy H there is a Pareto efficient allocation $(x^i(H)) \in (X(H))^I$ such that as H goes to infinity, each $x^i(H)$ is approximated by some bundle from $M'(H)$.

2. Assuming that (7.1) is to be sought, how is the notion of "approximate" to be formalized? Proposition 4 suggests the following:

(7.2) For each $x^i(H) = (r^i(H), y^i(H))$ there exists $(s^i(H), z^i(H)) \in M'(H)$ such that $\lim_{H\to\infty} \{|r^i(H) - s^i(H)| + E^H[(y^i(H) - z^i(H))^2]\} = 0$.

This type of criterion worked in proposition 4 because the agents (and P^H) did not change with H, and the agents' preferences were assumed continuous and their net trade sets open in the corresponding topology. Thus, from (7.2) it was easy to conclude that $\Delta(H) \to 0$. Here, because agents change with H, (7.2) alone will not suffice to guarantee "asymptotic efficiency," even if every agents' preferences are Euclidean $\times$ L^2 continuous. An assumption of *equicontinuity* (measured in date zero consumption) will clearly be required. That is, as H changes the agent who "plays role i" in economy H cannot be varying too wildly with H in terms of the continuity of his preferences.

3. Assume that (7.1) is to be sought, formalized as in (7.2). This in general will be false. Because agents change with H, the allocation $(x^i(H))$ that is being "chased" may change with H sufficiently quickly to frustrate convergence. (This was not a problem in proposition 4 because there a single unchanging allocation was being chased.) For example, suppose that for efficiency in economy H it is necessary to trade the contingent claim $y(H)$ given by

$$y(\omega;H) = 1_{\{\omega_1 \neq 2\}}, \text{ where } \omega = (\omega_1, \ldots, \omega_H).$$

Trade in this claim would be necessary for efficiency if, for example, two agents disagreed about the probability distribution of ω_1, even if they agreed about all other probabilities. It can be shown that for this claim $y(H)$ there does *not* exist $z(H) \in Y(H)$ such that for some $s(H)$, $(s(H), z(H)) \in M'(H)$ and

$$\lim_{H\to\infty} E^H[(y(H) - z(H))^2] = 0. \tag{7.3}$$

Thus, if these claims appear in the allocations $(x^i(H))$ being chased, (7.2) cannot hold.

Compare this with the following situation. Suppose that for efficient allocation in economy H only the following two claims are required (in addition to those in $M(H)$):

$$y'(\omega;H) = \sup_h \{\exp[\sum_{g=1}^{h} \omega_g/\sqrt{H}]; h = 0, \ldots, H\}$$

and

$$y''(\omega;H) = [d_2^H(\omega) - a]^+$$

For these claims the statement corresponding to (7.3) is true, and there is some hope that (7.2) may prove to be true.

What distinguishes $y'(H)$ and $y''(H)$ from $y(H)$? Why does (7.3) hold for the first two and not the third? Recall that the price systems p^H converge to the Black-Scholes prices in a very coarse fashion, in the weak topology.[7] The claims $y'(H)$ and $y''(H)$ depend on the state only via "coarse" features of the price history. More precisely, they are given by weak topology continuous functions of prices.[8] This is not true of the claims $y(H)$. They depend on the "fine features" of the price history.

Since convergence takes place in the weak topology, it is reasonable to expect that at *best* sequences of claims corresponding to weak topology continuous functions will be approximated by marketed claims. In general, (7.1) and (7.2) will be false, unless (perhaps) the allocations $(x^i(H))$ being chased "settle down" in this fashion. Since $(x^i(H))$ is meant to be a Pareto efficient allocation for the agents in economy H, it seems likely that in order to ensure that the $(x^i(H))$ "settle down" it will be necessary to assume that agents "settle down." (No general formulation of this can be offered here. But the reader may wish to ponder the following example that seems to work: For $i = 1, \ldots, I$ fix real numbers α^i and a utility function $u^i : R \times R \rightarrow R$. In economy H, let agent i's preferences for (r, y) be given by the index $E^H[u^i(r, \alpha^i d_2^H(\omega) + y(\omega))]$. That is, agents are expected utility maximizers whose date one endowments are proportional shares of the second security.)

4. In proposition 4 the information structure does not change with H—at time h/H agents possess all the Brownian information to that date. Thus, in the proposition, trading strategies θ can have $\theta(h/H)$ depending on more than the history of prices up to time h/H. In the sequence of economies given above, this extra information is not available. It can be shown in the setting of proposition 4 that for some claims this extra information is extraneous: For a claim whose value depends continuously on finitely many values of prices, (6.4) remains true when agents can base portfolio holdings on past price information only. This suggests that with some further restrictions on the allocations $(x^i(H))$ being chased, (7.1) and (7.2) will not be rendered false by the changing information structure. But what those restrictions are in general remains an open question. (A good place to start is probably with the work of Aldous 1978 concerning the relation between weak convergence à la Billingsley 1968 and "convergence of information.")

The somewhat disjointed discussion of this section can be summarized as follows. Proposition 4 is a first step toward a general theory of approximation of the sort discussed in the introduction. But it takes too many things as fixed. A more satisfactory theory would subsume the example with which this section began. Proposition 4 suggests that such a theory can be created, but subject to the very important qualifications noted above.

6.8 Concluding Remarks

To sum up what has been said: Frequent trading makes it *possible* for a few securities to span many states of nature. Whether markets are "perfectly" complete depends critically on the fine structure of the way in which uncertainty resolves. But the condition required for complete markets is not "nearly" required for "approximately" complete markets.

If equilibrium prices approximate an ideal model in a fairly coarse sense and if that ideal model has perfectly complete markets, then markets in the original model will give nearly efficient equilibrium allocations. Thus, if actual security prices behave "like" those in the Black-Scholes model (meaning here the general class of diffusion process models for which markets are complete), risk is allocated approximately efficiently.

A number of caveats to this argument have already been noted. The analysis in the second half of the paper relies on unpalatable assumptions concerning agents' net trade sets and preferences. The approximation analysis takes equilibrium prices as exogenously given, which is certainly an unhappy state of affairs. And the approximation result that is derived is preliminary at best—a more satisfactory theory will require qualifications that may turn out to be unpalatable. To this list the following more general caveats should be added:

1. The final conclusion given above rests on a very large supposition. Do actual security prices behave (even coarsely) "like" those in the Black-Scholes model? One can point to incidents where sudden bits of news have caused security prices to jump discontinuously, which the Black-Scholes prices do not do. In Merton (1976) it is argued that such jumps *may* be unimportant for the efficient allocation of risk because they may be "diversifiable" components of uncertainty. But to make this argument, it is at least necessary to assume that agents hold portfolios that are "diversified" enough to make such risk negligible. This in turn requires strong assumptions on preferences. Moreover, "continuous sample paths" are not (as is sometimes naively believed) sufficient for Black-Scholes type behavior: Harrison (1978) observes that if prices act in precisely the Black-Scholes model except that the diffusion coefficient changes with, say, the political party of the occupant of the White House (and if it is impossible to make book on the results of presidential elections), then sample paths are continuous yet markets are not complete. The question of whether prices do behave approximately like Black-Scholes prices (even coarsely) is very difficult, and nothing here should be construed as an assertion that they do.

2. For efficient allocation of risk, *all* uncertainty must be "spanned." In the Black-Scholes model, the only uncertainty is security price uncertainty. But phenomena such as differential information, moral hazard, individual uncertainty about future tastes, etc., represent uncertainty the resolution of which is not reflected (completely) in any security price. At best, there are complete markets only in uncertainty which is so reflected.

3. Adding production decisions to the story causes major difficulties. A firm contemplating a new and uncertain production process cannot (necessarily) observe prices for claims contingent on the outcome of that uncertainty—the problems addressed in the "spanning" literature (Diamond 1967; Stiglitz 1972; Ekern and Wilson 1974; Leland 1974; Merton

and Subrahmanyam 1974; Radner 1974; Grossman and Stiglitz 1980) all arise. Note that adding firms is "easy" only when they are "competitive" (see Grossman and Stiglitz 1980), and Hart (1979) indicates that with short sales, "competitive" firms will be difficult to find.

4. Still, suppose security prices do behave "nearly" like the Black-Scholes prices. Then at least, it seems, markets are "nearly" complete for purposes of pure exchange in the security price uncertainty. Even this is suspect. The arguments used here put tremendous strain on the assumptions of rational expectations and zero transaction costs. In a world with transactions costs and even slightly "irrational" expectations, there will be a place for markets where agents can purchase at the outset sundry "standard" packages of claims contingent on security price histories. The CBOE need not go out of business owing to the arguments put forward here.

Notes

1. Besides these two seminal papers, the following make significant contributions from the perspective taken here: Cox and Ross (1976), Harrison and Kreps (1979), Kreps (1981), Merton (1977), and Ross (1978). Smith (1976) provides a survey of the literature through 1976. Diffusion models were introduced into financial theory in Merton (1971, and 1973a).

2. Two technical points are worth making. First, unlike in Radner (1972), no bound is placed on the magnitude of θ. This is not necessary here, as general existence of equilibria will not be an issue. Second, the definition of an equilibrium of plans, prices, and price expectations presumes that agents will carry out plans that they embark on (or, more to the point, they believe that they will carry them out). Implicit in this is an "unchanging tastes" assumption, which can be used to motivate restrictions on preferences, notably weak separability across states. See Donaldson, Rossman, and Selden (1978). If agents' preferences "changed" in the sense of Hammond (1976), the analysis here would be significantly different.

3. See Harrison and Kreps (1979, section 7) for a discussion of this restriction.

4. If $y = y'$ P-a.s., then y and y' are assumed to be indistinguishable as time one contingent claims. Note that for the first time the probability measure P has entered the story. It will continue to do so, and the reader should note where and how it does.

5. To be more formal about this, define $p_1^H(t) = e^{rt}$ and $p_2^H(t,\omega) = p_2^H(h/H,\omega)$ for $t \in [h/H,(h+1)/H)$. Then weak convergence in $D^2[0,1]$ (with the Skorohod topology) to the process given in (5.1) follows from Donsker's theorem and the continuous mapping theorem. See Billingsley (1968) for definitions and details.

6. More generally, a theorem of Skorohod (1956) ensures that weak convergence and *almost sure* convergence are compatible in roughly this sense: If $p(H)$ converges weakly to $\hat{p}$, then there exists a probability space on which are defined processes $\hat{p}(H)$ and $\hat{p}$ such that each $\hat{p}(H)$ has the same distribution as $p(H)$ and $\hat{p}$ has the same distribution as p, and $\hat{p}(H)$ converges a.s. to $\hat{p}$. Note that the convergence criterion in (6.3) is neither necessary nor sufficient for almost sure convergence in the Skorohod topology on $D^2[0,1]$. Moreover, replacing (7.3) by a.s. convergence in the Skorohod topology would be insufficient for purposes here, for roughly the same reason that a.s. convergence does not imply L^p convergence for random variables. Therefore, in a general treatment of the convergence problem, convergence in the weak topology would not be the "correct" criterion.

7. Caveat emptor: As noted above in note 6, convergence in the weak topology is apt to turn out to be too weak a criterion for the results being sought. Throughout this section, the weak topology is used for purposes of discussion, to indicate the *general* sort of convergence/topology that one would like to use in extending proposition 4.

8. That is, they correspond to a function $f : D^2[0,1] \rightarrow R$ that is continuous in the Skorohod topology.

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