Volume Title: Human Capital: A Theoretical and Empirical Analysis, with Special Reference to Education, Second Edition

Volume Author/Editor: Gary S. Becker

Volume Publisher: NBER

Volume ISBN: 0-226-04109-3

Volume URL: http://www.nber.org/books/beck75-1

Publication Date: 1975

Chapter Title: Appendix B: Mathematical Discussion of Relation between Age, Earnings, and Wealth

Chapter Author: Gary S. Becker

Chapter URL: http://www.nber.org/chapters/c3741

Chapter pages in book: (p. 257 - 264)

APPENDIX **B**

Mathematical Discussion of Relation between Age, Earnings, and Wealth

1. This appendix derives some relations between the earnings and wealth profiles that were used in section 2 of Chapter VII. If the function $E(j)$ stands for earnings at age j, and $r(t, E)$ for the instantaneous interest rate at time t and the earnings function E, wealth at age j would be given by

$$W(j) = \int_{t=j}^{t=\infty} E(t)e^{-\int_{q=j}^{q=t} r(q,\, E)\, dq}\, dt. \tag{1}$$

The properties of this very general integral equation are not easily discovered and a number of simplifications are introduced. Interest rates are assumed to be independent of the date or earnings function, so

$$r(t, E) = r. \tag{2}$$

Earnings are assumed to grow at a constant rate for m years and then to equal zero, or

$$\begin{aligned} E(j) &= ae^{bj} \quad 0 \leq j \leq m \\ &= 0 \qquad j > m, \end{aligned} \tag{3}$$

where b is the rate of growth.

Time series earnings are often converted into cohort earnings through an expected labor force period that depends on mortality conditions: cohort earnings would equal time series earnings during this period and zero thereafter. Equation (3) can be so interpreted, with m the expected labor force period, and ae^{bj} earnings during the period. Time series earnings profiles in the United States can be approximated by a simple exponential function, although, as shown in the text, a fuller analysis would certainly have to incorporate a declining rate of growth. The labor force period method of adjusting for mortality, although widely used, is not always accurate and the more appropriate survivorship method is used in the text; the former is, however, a first approximation and its use considerably simplifies the mathematical analysis.

Substituting equations (2) and (3) into (1) gives

$$W(j) = \int_j^m ae^{bt}e^{-r(t-j)}\,dt, \tag{4}$$

and wealth can be explicitly computed as

$$W(j) = \frac{a}{b-r}\left[e^{(b-r)m}e^{rj} - e^{bj}\right], \quad b \neq r \tag{5}$$

$$= ae^{rj}(m-j), \quad b = r. \tag{5'}$$

Several relations between this wealth function and length of life (m), the rate of growth in earnings (b), and the rate of interest (r) are worked out in the following sections. It is assumed that $b \neq r$, although similar results can easily be proved for $b = r$.

2. The peak wealth age—the age at which wealth is maximized—is positively related to m, b, and r. Differentiating equation (5) yields

$$\frac{\partial W}{\partial j} = \frac{a}{b-r}\left[re^{(b-r)m}e^{rj} - be^{bj}\right], \tag{6}$$

and

$$\frac{\partial^2 W}{\partial j^2} = \frac{a}{b-r}\left[r^2e^{(b-r)m}e^{rj} - b^2e^{bj}\right] < 0 \quad \text{if } \frac{\partial W}{\partial j} = 0.$$

Accordingly, wealth is maximized when

$$re^{(b-r)m}e^{rj} = be^{bj}, \tag{7}$$

and the peak age simply equals

$$\hat{j} = m - \frac{\log b/r}{b - r}. \tag{8}$$

Hence

$$\begin{aligned} \frac{\partial \hat{j}}{\partial m} &= 1 > 0, \\ \frac{\partial \hat{j}}{\partial b} &= -\frac{1}{(b-r)^2}\left[1 - \left(\frac{r}{b} + \log \frac{b}{r}\right)\right] > 0 \\ \frac{\partial \hat{j}}{\partial r} &= -\frac{1}{(b-r)^2}\left[1 - \left(\frac{b}{r} + \log \frac{r}{b}\right)\right] > 0, \end{aligned} \tag{9}$$

since

$$1 < \frac{r}{b} + \log \frac{b}{r}, \quad \text{for all } \frac{b}{r} > 0.$$

A few numerical calculations can illustrate the orders of magnitude involved. If m is taken as 42 years—about the average number spent in the labor force by persons experiencing 1940 mortality rates—r as 8 per cent and b as 3 per cent—roughly the average annual growth in the earnings of 1939 college graduates between ages 30 and 60—$\hat{j}$ would equal 22.4 years, or 40 years if age 18 rather than age 0 were considered the initial year. If b equaled 2.7 per cent—roughly the average growth in earnings of 1939 elementary-school graduates between ages 30 and 60—$\hat{j}$ would equal 20.5 years, or 2 years less than college graduates. If r were 4 per cent, $\hat{j}$ would be 14 and 12 for these college and elementary-school graduates respectively, much lower than when $r = .08$, but still a difference of 2 years. A reduction of m to 36 years—the average time spent in the labor force after age 18 by nineteenth-century slaves—would reduce all peak ages by about 6 years, regardless of the values of b and r.

3. Equations (5) and (6) imply that

$$\frac{\partial W}{\partial j} \Big/ W = \frac{re^{(b-r)m}e^{rj} - be^{bj}}{e^{(b-r)m}e^{rj} - e^{bj}} \tag{10}$$

$$= \frac{re^{(b-r)(m-j)} - b}{e^{(b-r)(m-j)} - 1}.$$

By equation (3)

$$\frac{\partial E}{\partial j} \Big/ E = b; \tag{11}$$

so it follows from (10) and (11) that

$$\frac{\partial W}{\partial j}\Big/ W < \frac{\partial E}{\partial j}\Big/ E. \tag{12}$$

Since earnings reach a peak at age m, later than the peak in wealth, equation (12) implies that the ratio of peak to initial values is greater for earnings than wealth.

The rate of change in wealth is positively related to, as well as less than, the rate of change in earnings, or

$$\frac{\partial\left(\frac{\partial W}{\partial j}\Big/ W\right)}{\partial b} > 0. \tag{13}$$

For

$$\frac{\partial\left(\frac{\partial W}{\partial j}\Big/ W\right)}{\partial b} = \frac{-e^{gx} + 1 + gxe^{gx}}{(e^{gx} - 1)^2} > 0, \tag{14}$$

where $x = m - j$ and $g = b - r$, only if

$$e^{gx}(1 - gx) < 1. \tag{15}$$

If $|gx| \geq 1$, equation (15) clearly holds; if $|gx| < 1$, then

$$\frac{1}{1 - gx} = 1 + gx + (gx)^2 + \cdots, \tag{16}$$

and the infinite series expansion of e^{gx} shows that equation (15) must hold. Therefore equation (13) is proven.

Although the rate of change in wealth is greater the greater the rate of change in earnings, the ratio of peak to initial wealth is a smaller fraction of the ratio of earnings at the peak wealth age to initial earnings the greater the rate of increase in earnings. That is,

$$\frac{\partial\left[\dfrac{W(\hat{j})/W(0)}{E(\hat{j})/E(0)}\right]}{\partial b} = k < 0. \tag{17}$$

Since

$$\frac{W(j)}{W(0)} = \frac{e^{(b-r)m}e^{rj} - e^{bj}}{e^{(b-r)m} - 1}, \tag{18}$$

and

$$\frac{E(j)}{E(0)} = e^{bj}, \tag{19}$$

$$\frac{W(j)/W(0)}{E(j)/E(0)} = \frac{e^{(b-r)(m-j)} - 1}{e^{(b-r)m} - 1}. \tag{20}$$

By equation (8)

$$m - \hat{j} = \frac{\log b/r}{b - r},$$

so

$$k = \frac{e^{\log b/r} - 1}{e^{(b/r)m} - 1} = \frac{b/r - 1}{e^{(b-r)m} - 1}. \tag{21}$$

Hence

$$\frac{\partial k}{\partial b} < 0$$

only if

$$e^{gm} - 1 - gme^{gm} < 0,$$

or only if

$$e^{gm}(1 - gm) < 1. \tag{22}$$

Equation (22) is simply equation (15) again; therefore (17) has been proven.

4. The equation

$$\frac{\partial W}{\partial x}\Big/ W = \frac{ge^{gx}}{e^{gx} - 1} \tag{23}$$

gives the rate of decline in wealth as the number of remaining years in the labor force (x) declines. Equations (10), (11), and (23) imply that

$$\frac{\partial W}{\partial x}\Big/ W + \frac{\partial W}{\partial j}\Big/ W = b = \frac{\partial E}{\partial j}\Big/ E, \tag{24}$$

or

$$\frac{\partial W}{\partial x}\Big/ W = \frac{\partial E}{\partial j}\Big/ E - \frac{\partial W}{\partial j}\Big/ W. \tag{25}$$

The difference between the rates of change in earnings and wealth with respect to age is simply equal to the rate of decline in wealth as the number of remaining years declines.

Equation (23) indicates that wealth declines more rapidly the fewer the years remaining, and declines infinitely fast as these years approach zero. As they go to infinity—life becomes indefinitely long—the rate of decline in wealth approaches $b - r$ if $b > r$, and 0 if $b < r$. Therefore, equations (23) and (24) imply that

$$\lim_{x \to \infty} \frac{\partial W}{\partial j} \Big/ W = \min (b, r). \tag{26}$$

The rate of change in wealth with age approaches the rate of change in earnings only if the latter were less than the discount rate; otherwise the discount rate would be approached, a somewhat surprising result.

5. According to the definition used in the text, the rate of "depreciation" at age j is

$$D(j) = -\frac{\partial W(j)}{\partial j}, \tag{27}$$

while the rate of "appreciation" is $-D(j) = \dfrac{\partial W(j)}{\partial j}$. The average rate during the whole period of labor force participation is given by

$$\begin{aligned}\bar{D} &= \frac{1}{m} \int_0^m D(j)\, dj = \frac{-1}{m} \int_0^m \frac{\partial W}{\partial j}\, dj \\ &= \frac{1}{m} [W(0) - W(m)] = \frac{1}{m} W(0),\end{aligned} \tag{28}$$

since $W(m) = 0$.

Average depreciation divided by average earnings gives the ratio

$$d = \frac{\bar{D}}{\bar{E}} = \frac{\frac{1}{m} W(0)}{\bar{E}} = \frac{\int_0^m Ee^{-rj}\, dj}{\int_0^m E\, dj}, \tag{29}$$

which is the ratio of the present values at the initial age of earnings discounted at the market rate to earnings discounted at a zero rate. This ratio is obviously positively related to the market rate, approaching zero for an infinite, and unity for a zero, rate.

"Permanent" earnings are defined either as

$$E_p(j) = E(j) - D(j), \tag{30}$$

or as

$$E_p(j) = rW(j), \tag{31}$$

so

$$E(j) = D(j) + rW(j), \tag{32}$$

and, therefore, equation (29) can be written as

$$d = \frac{\bar{D}}{\bar{E}} = \frac{\frac{1}{m} W(0)}{r\bar{W} + \frac{1}{m} W(0)}. \tag{33}$$

Hence d would be smaller the smaller the ratio of initial to average wealth. Section 2 of Chapter VII implies that the latter, in turn, would be smaller the faster the rate of increase in earnings because the rate of increase in wealth is positively related to the rate of increase in earnings.