Appendix A

UTILITY MAXIMIZATIONS

1. DISCRETE TIME

To maximize utility subject to the full wealth and production function constraints, form the Lagrangian expression

$$L = U(\phi_0 H_0, \ldots, \phi_n H_n, Z_0, \ldots, Z_n) + \lambda\left[R - \sum \frac{C_i + C_{1i} + W_i TL_i}{(1+r)^i}\right], \tag{A-1}$$

where $C_i = P_i M_i + W_i TH_i$ and $C_{1i} = F_i X_i + W_i T_i$. Differentiating L with respect to gross investment in period $i - 1$ and setting the partial derivative equal to zero, one obtains

$$\begin{aligned} Uh_i \frac{\partial h_i}{\partial H_i}\frac{\partial H_i}{\partial I_{i-1}} &+ Uh_{i+1}\frac{\partial h_{i+1}}{\partial H_{i+1}}\frac{\partial H_{i+1}}{\partial I_{i-1}} + \ldots + Uh_n \frac{\partial h_n}{\partial H_n}\frac{\partial H_n}{\partial I_{i-1}} \\ &= \lambda\left[\frac{(dC_{i-1}/dI_{i-1})}{(1+r)^{i-1}} + \frac{W_i(\partial TL_i/\partial H_i)(\partial H_i/\partial I_{i-1})}{(1+r)^i}\right. \\ &\quad + \frac{W_{i+1}(\partial TL_{i+1}/\partial H_{i+1})(\partial H_{i+1}/\partial I_{i-1})}{(1+r)^{i+1}} + \ldots \\ &\quad \left. + \frac{W_n(\partial TL_n/\partial H_n)(\partial H_n/\partial I_{i-1})}{(1+r)^n}\right]. \end{aligned} \tag{A-2}$$

But $\partial h_i/\partial H_i = G_i$, $\partial H_i/\partial I_{i-1} = 1$, $\partial H_{i+1}/\partial I_{i-1} = (1 - \delta_i)$, $\partial H_n/\partial I_{i-1} = (1 - \delta_i)\ldots(1 - \delta_{n-1})$, $dC_{i-1}/dI_{i-1} = \pi_{i-1}$, and $\partial TL_i/\partial H_i = -G_i$. Therefore,

$$\begin{aligned} \frac{\pi_{i-1}}{(1+r)^{i-1}} &= \frac{W_i G_i}{(1+r)^i} + \frac{(1-\delta_i)W_{i+1}G_{i+1}}{(1+r)^{i+1}} + \ldots \\ &\quad + \frac{(1-\delta_i)\ldots(1-\delta_{n-1})W_n G_n}{(1+r)^n} + \frac{Uh_i}{\lambda}G_i \end{aligned}$$

$$+ (1 - \delta_i)\frac{Uh_{i+1}}{\lambda}G_{i+1} + \ldots$$

$$+ (1 - \delta_i)\ldots(1 - \delta_{n-1})\frac{Uh_n}{\lambda}G_n. \qquad \text{(A-3)}$$

2. CONTINUOUS TIME

Let the utility function be

$$U = \int m_i f(\phi_i H_i, Z_i)\, \mathrm{d}i, \qquad \text{(A-4)}$$

where m_i is the weight attached to utility in period i. Equation (A-4) defines an additive utility function, but any monotonic transformation of this function could be employed.[1] Let all household production functions be homogeneous of degree one. Then $C_i = \pi_i I_i$, $C_{1i} = q_i Z_i$, and full wealth can be written as

$$R = \int e^{-ri}(\pi_i I_i + q_i Z_i + W_i TL_i)\, \mathrm{d}i. \qquad \text{(A-5)}$$

By definition,

$$I_i = \dot{H}_i + \delta_i H_i, \qquad \text{(A-6)}$$

where $\dot{H}_i$ is the instantaneous rate of change of capital stock. Substitution of (A-6) into (A-5) yields

$$R = \int e^{-ri}(\pi_i \delta_i H_i + \pi_i \dot{H}_i + q_i Z_i + W_i TL_i)\, \mathrm{d}i. \qquad \text{(A-7)}$$

To maximize the utility function, form the Lagrangian

$$L - \lambda R = \int [m_i f(\phi_i H_i, Z_i) - \lambda e^{-ri}(\pi_i \delta_i H_i + \pi_i \dot{H}_i + q_i Z_i + W_i TL_i)]\, \mathrm{d}i, \qquad \text{(A-8)}$$

or

$$L - \lambda R = \int J(H_i, \dot{H}_i, Z_i, i)\, \mathrm{d}i, \qquad \text{(A-9)}$$

[1] Robert H. Strotz has shown, however, that certain restrictions must be placed on the m_i. In particular, the initial consumption plan will be fulfilled if and only if $m_i = (m_0)^i$. See "Myopia and Inconsistency in Dynamic Utility Maximization," *Review of Economic Studies*, 23, No. 62 (1955–56).

where

$$J = m_i f(\phi_i H_i, Z_i) - \lambda e^{-ri}(\pi_i \delta_i H_i + \pi_i \dot{H}_i + q_i Z_i + W_i T L_i). \quad \text{(A-10)}$$

Euler's equation for the optimal path of H_i is

$$\frac{\partial J}{\partial H_i} = \frac{\mathrm{d}}{\mathrm{d}i} \frac{\partial J}{\partial \dot{H}_i} \quad \text{(A-11)}$$

In the present context,

$$\frac{\partial J}{\partial H_i} = U h_i G_i - \lambda e^{-ri} \pi_i \delta_i + \lambda e^{-ri} W_i G_i,$$

$$\frac{\partial J}{\partial \dot{H}_i} = -\lambda e^{-ri} \pi_i,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}i} \frac{\partial J}{\partial \dot{H}_i} = -\lambda e^{-ri} \dot{\pi}_i + \lambda e^{-ri} r \pi_i. \quad \text{(A-12)}$$

Consequently,

$$G_i[W_i + (U h_i/\lambda) e^{ri}] = \pi_i(r - \dot{\pi}_i + \delta_i), \quad \text{(A-13)}$$

which is the continuous time analogue of equation (1-13).