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Volume Author/Editor: Milton Friedman

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Chapter Author: Milton Friedman

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cations for the first as well. It is, moreover, restricted to an especially simple case: interpolation when only a single related series is used, and when the only values of the given series used in the interpolation are those for one date preceding and one date following the date for which a value is to be interpolated and the only values of the related series used are for these same dates plus the interpolation date itself. Though this may seem a very special case, it has widespread importance since the bulk of all actual interpolation by related series satisfies these conditions.

This paper considers first the methods commonly employed in interpolation by related series, describing them (Section I) and then analyzing the errors of estimation associated with them (Section II). The characteristic feature of these methods is their use of a priori parameters, which is to say that none of them explicitly takes into account the degree of correlation between the interpolated and related series. For simplicity in reference, we may term them noncorrelation methods.

Section III presents a generalization of these methods that is suggested by elementary considerations in the statistical theory of correlated data and compares the errors associated with correlation and noncorrelation methods. Section IV discusses the form in which to express the data and Section V summarizes the conclusions.

I. NON-CORRELATION METHODS OF USING RELATED SERIES

Let X be the series to be interpolated; Y, the related series to be used in interpolation. Although the class of methods considered in this section does not explicitly use any information on the degree of correlation between Y and X, a particular series Y is of course chosen for use in interpolation because its intrayearly movements are believed to be highly correlated with the intrayearly movements of X. This belief may be based on nonquantitative considerations (e.g., that employment in different firms producing the same product or vault cash in different classes of banks in the same geographic area will be affected by common forces and hence are likely to move together). Alternatively, it may be based on an observed high correlation between movements in Y and in X for the time units to be interpolated (e.g., months) but for a different period than that to be interpolated; or between movements in Y and in X for different time units for the same period (e.g., a high correlation between annual observations may be taken as evidence that there is also a high correlation between the unknown intrayearly movement); or between movements in two series other than Y and X but analogous to them for the same time units and for the same period (e.g., a high correlation between vault cash in national and state weekly reporting member banks may be taken as evidence that there is also a high correlation between vault cash in weekly-reporting and nonweekly-reporting member banks).

To make the problem specific, suppose it is to convert X, which is known for one date a year, into a monthly series. The common procedure is to superimpose the intrayear movement of Y on the year-to-year movement of X, using one or another device for eliminating any difference between the year-to-year movements of the two series. Different variants arise from different ways of superimposing the intrayear movement and from different devices for eliminating differences between year-to-year movements. It turns out that most of these variants can be expressed as special cases of a very simple method, which I shall term method M_1.

A. *A Method and Its Main Variants*

1. *Method* M_1. A straight-line trend[2] is computed between the known values of X, also between the corresponding values of Y. The difference between each monthly value of Y and the corresponding trend value of Y is then added to the trend value of X.

Let x_0 and x_{12} be two successive known values of X; $x_1, \cdots, x_i, \cdots, x_{11}$, the unknown monthly values to be estimated by interpolation; and $y_0, y_1, \cdots, y_i, \cdots, y_{11}, y_{12}$, the values of Y for the corresponding annual period. Let x_i^* be an estimate of x_i for $i=1, \cdots, 11$. Then this method involves using the formula

$$\begin{aligned} x_i^* &= x_0 + \frac{i}{12}(x_{12} - x_0) + \left\{y_i - \left[y_0 + \frac{i}{12}(y_{12} - y_0)\right]\right\} \\ &= \frac{(12 - i)x_0 + ix_{12}}{12} + y_i - \left[\frac{(12 - i)y_0 + iy_{12}}{12}\right]. \end{aligned} \tag{1}$$

If we write T_{x_i} and T_{y_i} for the (straight-line) trend values of X and Y at time i, we can write (1) more simply as

$$x_i^* = T_{x_i} + (y_i - T_{y_i}). \tag{1.1}$$

This method involves only "arithmetic" operations for both superimposition of the intrayear movement of Y and for eliminating any difference between year-to-year movements in X and Y. It is used when it can be assumed that the absolute magnitudes of the movements in the two series tend to be equal.

2. *Logarithmic variant of* M_1. The preceding method may be applied to the logarithms of the series in question (or to any other transformation; see Section IV). The same formulas apply with x and y replaced by their logarithms.

This variant involves only "geometric" or "relative" operations and is used when it can be assumed that the relative magnitudes of the movements in the two series are comparable. Expressed in terms of the original observations, the formula becomes

$$x_i^* = \left[x_0^{12-i} x_{12}^{i}\right]^{1/12} \cdot \frac{y_i}{\left[y_0^{12-i} y_{12}^{i}\right]^{1/12}} = \frac{L_{x_i}}{L_{y_i}} y_i, \tag{2}$$

where L_{x_i} and L_{y_i} stand for the logarithmic trend values of X and Y respectively at time i. It is readily seen that this method is equivalent to equating the ratio of X to a logarithmic trend with the corresponding ratio for Y.

[2] Note that throughout this paper "trend" is used to designate a time function which connects the known end values of the time interval within which interpolation is performed. The use of a "trend" based on a larger number of observations would take us beyond the special case in which only the terminal values of X are used in interpolation.

3. *Ratio to arithmetic trend variant of* M_1. A variant that combines parts of the preceding two is to use ratios to a straight-line arithmetic trend, so that

$$x_i^* = \left[\frac{(12-i)x_0 + ix_{12}}{12}\right] \frac{y_i}{\dfrac{(12-i)y_0 + iy_{12}}{12}} = \frac{T_{x_i}}{T_{y_i}} \cdot y_i. \qquad (3)$$

This variant uses geometric operations to superimpose the intrayear movement of Y but arithmetic operations to eliminate any difference between year-to-year movements in X and Y.[3]

In practice, (2) and (3) will usually yield approximately the same results since, first, the difference between the geometric and arithmetic trends varies with the magnitude of the annual change (i.e., the ratio of x_{12} to x_0 or y_{12} to y_0) and the change is generally moderate, and, second, the difference between the geometric and arithmetic trends affects the trend values of both series and hence only the difference between these differences affects the final result. In consequence, (3) may frequently be interpreted as an approximation to (2).

Variant (3) can also be interpreted as a modified version of (1). If we add and subtract the trend value of X, T_{x_i}, to and from the right-hand side of (3) and rearrange terms, we can write (3) as

$$x_i^* = T_{x_i} + \frac{T_{x_i}}{T_{y_i}}(y_i - T_{y_i}). \qquad (3.1)$$

The only difference between (1.1) and (3.1) is the factor T_{x_i}/T_{y_i}, which multiplies the deviation of y_i from its straight-line trend value. This factor is to be interpreted as a scale factor used to convert Y values into X values.

This method can also be regarded as a special case of method (1) in its original form. Consider the original X and Y series replaced by their ratios to straight-line arithmetic trends connecting the values for dates for which values of X are known. For these new series, say X' and Y', formula (3) becomes

$$x_i'^* = y_i', \qquad (3.2)$$

and similarly, (1) reduces to (3.2) as well, since all trend values are unity.

4. *Difference from geometric trend variant of* M_1. For logical completeness, we may list also a variant combining parts of (1) and (2) in the opposite order to the way (3) does, that is, using arithmetic operations to superimpose the

[3] This is the method used by Leong in interpolating vault cash in all banks by vault cash in national banks, though he describes his method in a very different and more complicated way. See Y. S. Leong, "An Estimate of the Amount of Money Held by the Banks and of the Amount of Money in General Circulation in the United States," *Journal of Political Economy*, 38 (1930), 176–7.

I should perhaps note explicitly that I have made no attempt to search the literature systematically for references to interpolation techniques that have been used. This and occasional later references are to items I happen to have come across. Similarly, textual references to actual practice are based on such knowledge as I happen to have accumulated as a by-product of other work, rather than on any survey made for the purposes of this paper.

The information that would be gleaned from a search of the literature would in any event be limited by the regrettable tendency for estimators simply to describe their procedure as "interpolation by means of . . . " without stating precisely how the interpolation was performed. For example, U. S. Department of Commerce, Office of Business Economics, *National Income, 1954 Edition, A Supplement to Survey of Current Business*, contains on pp. 61–152 an extensive discussion of "sources and methods" which refers repeatedly to interpolation and extrapolation by related series but does not discuss explicitly at any point the technique of interpolation used.

intrayear movement of Y but geometric operations to eliminate any difference between year-to-year movements in X and Y. This gives

$$x_i^* = \left[x_0^{12-i} x_{12}^{i}\right]^{1/12} + \left[y_i - \left(y_0^{12-i} y_{12}^{i}\right)^{1/12}\right] = L_{x_i} + (y_i - L_{y_i}). \tag{4}$$

In practice, (1) and (4) will usually yield approximately the same results for the same reasons as those cited above to explain why (3) can frequently be interpreted as an approximation to (2). However, while (3) is frequently used (though seldom in the explicit form given), (4) is rarely used.

Clearly (4) is a special case of (1) if X and Y are replaced by their differences from a logarithmic trend connecting the values for dates for which X is known.

Hence, (2) can be regarded as a special case of (1) when the original values are replaced by logarithms; (3), when the original values are replaced by ratios to an arithmetic straight-line trend; (4), when the original values are replaced by differences from a logarithmic straight-line trend.

B. *Other Commonly Used Variants*

5. Another method is to proceed in stages. First, series Y, including y_{12}, is converted into a series of relatives to y_0. These relatives cannot be applied directly to x_0 since they would yield a value of x_{12} different from the known value. Consequently, the difference between the ratio of y_{12} to y_0 and the ratio of x_{12} to x_0 is computed and "distributed" arithmetically (or less often, geometrically) among the other relatives.

5a. If the difference between the final relatives is distributed arithmetically, the final result is given by

$$x_i^* = \left[\frac{y_i}{y_0} - \frac{i}{12}\left(\frac{y_{12}}{y_0} - \frac{x_{12}}{x_0}\right)\right] x_0. \tag{5}$$

This method is a modified version of method (1). By algebraic manipulation, (5) can be written as

$$x_i^* = T_{x_i} + \frac{x_0}{y_0}\left[y_i - T_{y_i}\right]. \tag{5.1}$$

The identity of (5.1) and (5) can readily be seen by multiplying out the right-hand sides of the two equations.

The only difference between (5.1) and (1) is the factor x_0/y_0 which multiplies the deviation of y_i from its straight-line trend value and which is to be interpreted as a scale factor. (5) can also be regarded as a special case of (1) in its original form if we replace the original Y series by a new series, $(x_0/y_0)y_i$.

Suppose that relatives had been computed on the terminal value (y_{12}) instead of on the initial value (y_0). The final result would obviously be given by (5.1) except that the scale factor x_0/y_0 would be replaced by x_{12}/y_{12}. It follows that, unlike the preceding variants, the results obtained by this variant depend on whether the initial or terminal value is chosen as the base of the relatives.

Since there is no basis for choosing one instead of the other, this method, despite its extensive use, must be rejected on purely formal grounds.[4]

5b. If the difference between—or rather the ratio of—the final relatives is distributed geometrically, the final result is given by

$$\overset{*}{x_i} = \frac{y_i}{y_0}\left[\frac{x_{12}}{x_0} \Big/ \frac{y_{12}}{y_0}\right]^{i/12} x_0, \tag{5.2}$$

which can be seen to be identical with (2), so, in this form, this method reduces to one already considered.

6. Yet another method is to operate with the ratios of the two series in question. The initial and terminal ratios of X to Y are interpolated to intermediate dates and multiplied by the known values of Y. Different variants arise according to the method of interpolating the ratios.

6a. If these are interpolated along an arithmetic straight line, the final formula is

$$\overset{*}{x_i} = y_i\left[\frac{12-i}{12}\,\frac{x_0}{y_0} + \frac{i}{12}\,\frac{x_{12}}{y_{12}}\right]. \tag{6}$$

In this form, the method is analogous to method (3), since equation (3) can be written in the form

$$\overset{*}{x_i} = y_i\left[\frac{\dfrac{12-i}{12}\,y_0\,\dfrac{x_0}{y_0} + \dfrac{i}{12}\,y_{12}\,\dfrac{x_{12}}{y_{12}}}{\dfrac{12-i}{12}\,y_0 + \dfrac{i}{12}\,y_{12}}\right]. \tag{3.3}$$

In both (6) and (3.3), the estimate of x_i is obtained by multiplying y_i by a weighted average of the initial and terminal ratios of X to Y, the difference

[4] One "natural" solution to this difficulty is to take a weighted average of the answers obtained by using initial and terminal values as the base, the figure obtained by using the initial year as the base being weighted $(12-i)/12$, the other, $i/12$. It is obvious that this is equivalent to replacing the scale factor x_0/y_0 in (5.1) by a corresponding weighted average of x_0/y_0 and x_{12}/y_{12}, i.e., by

$$\left[\frac{(12-i)}{12}\,\frac{x_0}{y_0} + \frac{ix_{12}}{12y_{12}}\right].$$

Another "natural" solution is to express the figures as relatives to the average of the initial and terminal values. This will mean an initial and terminal discrepancy. A weighted average of these two discrepancies would then be used to "correct" the relatives. This turns out to be equivalent to replacing the scale factor x_0/y_0 in (5.1) by

$$\frac{x_0 + x_{12}}{y_0 + y_{12}}.$$

Either of these seems more satisfactory than (5) which is identical with (5.1). They are put in a footnote both because I know of no instance of their use and because the scale factor implicit in (3) seems more sensible than either of these.

Kuznets' "proportional" method is close to the second of the two alternatives listed in this footnote. However, it uses

$$\frac{x_{12} - x_0}{y_{12} - y_0}$$

as a scale factor. See Simon Kuznets, *National Income and its Composition, 1919–1958*, Volume II. New York: National Bureau of Economic Research, 1941. Pp. 480–3.

being that different weights are used in computing the weighted average. The weights in (3.3) are y_0 and y_{12}, respectively, times the weights in (6). In consequence, methods (6) and (3) will give approximately the same results as long as y_0 and y_{12} are not substantially different.

To show the relation between (6a) and (3) in another way, use an arithmetic straight-line trend for X but compute a trend for Y by the following formula

$$Z_{y_i} = \frac{T_{x_i}}{\dfrac{12-i}{12}\dfrac{x_0}{y_0} + \dfrac{i}{12}\dfrac{x_{12}}{y_{12}}} .$$

Equation (6) can then be written

$$\overset{*}{x}_i = y_i \frac{T_{x_i}}{Z_{y_i}}, \tag{6.1}$$

which is identical with (3) except for the use of a different trend for Y.

As the use of different kinds of trends for X and Y in (6.1) may suggest, this method has an arbitrary element in it. Suppose, instead of interpolating the ratio of X to Y, the interpolation had been performed on the ratio of Y to X, thereby giving a formula

$$\overset{*}{x}_i = \frac{y_i}{\dfrac{12-i}{12}\dfrac{y_0}{x_0} + \dfrac{i}{12}\dfrac{y_{12}}{x_{12}}} . \tag{6.2}$$

Formulas (6) and (6.2) give identical results only if $y_0/x_0 = y_{12}/x_{12}$, in which case the method reduces directly to (3). Since there is no basis for choosing between (6) and (6.2), this method must be rejected as unsatisfactory on purely formal grounds, despite its extensive use.[5]

6b. If the ratios of X to Y are interpolated along a logarithmic straight line, the final result is

$$\overset{*}{x}_i = y_i \left[\left(\frac{x_0}{y_0}\right)^{12-i} \left(\frac{x_{12}}{y_{12}}\right)^{i} \right]^{1/12}, \tag{6.3}$$

which can be seen to be identical with (2).

Though this list of methods is by no means exhaustive, I believe it covers the major variations that are extensively used. The important points to be noted are that: first, all the methods can be reduced to method (1) by suitable transformations of the variables; second, methods (1), (2), (3), and (4) applied to the original data transfer to X the full amplitude of the variation in Y; third, methods (3), (5a), and (6a) can be interpreted as introducing "scale factors"

[5] This method is identical to one that Simon Kuznets calls the "ratio method" and that he used most frequently in constructing his national income estimates. See his *National Income*, pp. 479–80. Allyn A. Young also used this method. See his "An Analysis of Bank Statistics for the United States, Part IV, The National Banks," *Review of Economic Statistics*, July 1927, p. 136. Here again various "crosses" could doubtless be found that would eliminate this defect.

designed to convert Y units into X units; they then transfer to X the full amplitude of Y, as converted into X units; and fourth, none of the methods take account of the degree of correlation between X and Y or of the relative amplitude of variation in X and Y.

It may further be worth noting that, among these methods alone, only methods (1), (2), (3), and (4) are satisfactory on formal grounds. Methods (5b) and (6b) reduce to (2); methods (5a) and (6a) in the form in which they are generally used are technically defective since the results depend on a purely arbitrary decision; the variants of (6a) listed in a footnote that are free from this defect are analogous to (3) in their motivation and seem less appealing than (3).

II. ERRORS OF ESTIMATION ASSOCIATED WITH NONCORRELATION METHODS[6]

It will clearly suffice to confine attention to method (1), designated as M_1, in analyzing the errors associated with these noncorrelation methods. The other acceptable methods simply involve applying method (1) to data expressed in a different form—in logarithms, as ratios to an arithmetic trend, or as differences from a geometric trend. In consequence, results for method (1) can be readily translated into corresponding results for the other methods.

A. *Formal specification of* M_1

It will involve no loss of generality to confine our attention to three equally spaced time units, say t_0, t_1, and t_2, for which the values of X are x_0, x_1, x_2, and the values of Y are y_0, y_1, y_2. The values x_0, x_2, and all three values of Y are known. The problem is to estimate the unknown value of X, x_1, by interpolation.

We may further simplify the analysis by expressing our observations as deviations from the corresponding trend values. This mathematical step corresponds to a practical maxim implicit in the preceding section (maxim I): *First interpolate mathematically*. The deviation of the related series from a correspondingly interpolated value can then be used to adjust the interpolated value so obtained. We shall further simplify by using a simple form of mathematical interpolation, namely linear interpolation. Other forms can either be reduced to the linear form by suitable transformations of the data (see Section IV) or require the use of more information than two known observations.

Designate the deviation of X from its trend value by u and the deviation of Y from its trend value by v. We then have

$$\left.\begin{aligned} u_0 &= 0 & v_0 &= 0 \\ u_1 &= x_1 - (1/2)(x_0 + x_2) & v_1 &= y_1 - (1/2)(y_0 + y_2) \\ u_2 &= 0 & v_2 &= 0. \end{aligned}\right\} \tag{7}$$

[6] Curiously enough, the problems considered in this and the next section are formally identical with those involved in judging the circumstances under which a government policy designed to be countercyclical will in fact succeed in reducing instability, and in specifying the optimum mganitude of countercyclical action. In consequence, these sections largely parallel my article "The Effects of 'Full Employment Policy' on Economic Stability: A Formal Analysis," published in my *Essays in Positive Economics*, Chicago, 1953, pp. 117–32.