Volume Title: The Theory and Empirical Analysis of Production

Volume Author/Editor: Murray Brown, editor

Volume Publisher: NBER

Volume ISBN: 0-870-14486-3

Volume URL: http://www.nber.org/books/brow67-1

Conference Date:

Publication Date: 1967

Chapter Title: Recent Empirical Studies of the CES and Related Production Functions

Chapter Author(s): Marc Nerlove

Chapter URL: http://www.nber.org/chapters/c1477

Chapter pages in book: (p. 55 - 136)

RECENT EMPIRICAL STUDIES OF THE CES AND RELATED PRODUCTION FUNCTIONS

MARC NERLOVE

> ". . . this bottle was *not* marked 'poison,' so Alice ventured to taste it, and finding it very nice (it had, in fact, a sort of mixed flavour of cherry-tart, custard, pine-apple, roast turkey, toffee, and hot buttered toast), she very soon finished it off."
>
> —*Alice's Adventures in Wonderland*

IN their paper which first popularized the now famous constant-elasticity-of-substitution production function, Arrow, Chenery, Minhas, and Solow (ACMS) suggest three important areas in which knowledge of the elasticity of substitution plays a crucial role: [1] (1) The stability or instability of certain growth paths implied by some models, notably the Harrod-Domar model, depends on the value of the elasticity of substitution. (2) The effects of varying factor endowments on the pattern of trade and relative factor prices depends crucially on the nature of variation in the elasticity of substitution between factors among different industries.[2] Finally, (3) ACMS reiterate the traditional importance of the elasticity of substitution for relative shares over time. The last point has been stressed by Kravis (1959) and more elaborately by Solow

NOTE: This paper is based on research done under Grant NSF-GS-142 from the National Science Foundation to Stanford University and with the partial support of NSF-GS-818 to Yale University. I am indebted to G. Chow, Zvi Griliches, G. S. Maddala, and Peter Mieszkowski for helpful comments, although they are not, of course, responsible for errors.

[1] Arrow *et al.* (1961; see References at end of paper for bibliographic details). Although no Russian has yet stepped forward to claim credit for discovery of the CES function, it is safe to say that ACMS first *popularized* the function in the English-language literature. They mention its earlier use by Solow (1956) and Swan (1956). Whitaker (1964) attributes its first use to Dickinson in a 1954 paper in the *Review of Economic Studies*. In any case, the claim of Brown and de Cani (1963) to have derived the function independently seems well substantiated.

[2] See especially Minhas (1962).

(1964). It seems clear, however, that the most important implication of any aggregate production function will be in the linking of changes in factor supplies and output, in the aggregate, over time and thus to the understanding of economic growth.

Nelson (1965) has recently questioned the relevance of the CES production function in this connection. He shows that the rate of growth in total output is approximately given in the two-factor case by

$$(1)\quad \frac{\dot{V}}{V} = \frac{\dot{A}}{A} + b_0\frac{\dot{L}}{L} + (1 - b_0)\frac{\dot{K}}{K} + \frac{1}{2}b_0(1 - b_0)\frac{\sigma - 1}{\sigma}\left[\frac{\dot{K}}{K} - \frac{\dot{L}}{L}\right]^2,$$

where b_0 denotes the initial share of labor in total output. The final term shows the effect of the CES function as opposed to the Cobb-Douglas form on growth, i.e., of assuming a constant but nonunitary elasticity of substitution. Note that when either the rates of growth in capital and labor are equal or when $\sigma = 1$, the expression for the rate of growth in total output reduces to that implied by the Cobb-Douglas function. $\dot{A}/A$ here represents the effects of *neutral* technical change. What Nelson shows is that ". . . for the analysis of growth over short periods of time, and in situations where the growth of capital stock is not greatly different from that of the labor supply . . . little is to be gained from going to the CES model." [3] For example, between 1947 and 1960, capital stock grew about 3 per cent a year faster than the labor force. Assuming labor's share at about two-thirds and an elasticity of substitution of one-half rather than 1 would reduce the annual rate of growth less than 3 per cent per year below the rate predicted by the Cobb-Douglas model, i.e., by

$$\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{3}\cdot\left(\frac{-\frac{1}{2}}{\frac{1}{2}}\right)(.03)^2 = -\frac{1}{9}\cdot\frac{9}{10{,}000} = -.0001.$$

The implication which Nelson draws from this is that, no matter what the elasticity of substitution may be taken to be, increases in capital per worker or per man-hour explain only a small fraction of the growth in productivity in the postwar period and that the low degree of expla-

[3] Nelson (1965, p. 6).

nation is not sensitive to the choice of any particular value of the elasticity of substitution.

Three points might be made in partial reply to Nelson's assertion of the irrelevance of the elasticity of substitution: First, as Nelson recognizes, over long periods or when the capital grows much more rapidly than the labor force, differences of the elasticity of substitution from 1 may play a much more significant role in determining the final outcome. Second, the result refers to capital and labor inputs as conventionally measured and makes no allowance for growth in the *effective* stock of capital or labor force due to improvements in quality or investment in human capital. If technical change is primarily of the capital-embodied type and there is little or no change in the quality of the labor force, conventional measures of inputs may greatly understate the discrepancy between the rates of growth of the two inputs and thus lend unwarranted support to Nelson's contention. On the other hand, investment in human capital and the consequent growth in the quality of the labor force tend to offset embodied technical change and to return us to Nelson's position. The net outcome, it seems, is an empirical question; to argue the irrelevance of the elasticity of substitution a priori contains elements of prejudgment. Third, the analysis refers only to the aggregate relationship between outputs and inputs. As aggregate output and therefore income grow, we would expect a shift of demand from primary goods to manufactures and services. Differences in the elasticities of substitution among industries may lead to significant effects on the rate of growth possible with given growth in factor supplies.[4] High elasticities of substitution in primary production and lower elasticities in secondary and tertiary industries coupled with the assumed shift in demand will lead to a redistribution of the labor force along the lines observed in developed economies in recent decades. Unless, however, technical change is biased in the tertiary sector, or there are other offsetting effects leading to increased elasticities of substitution between capital and labor, the aggregate elasticity of substitution must fall and ultimately growth must be slowed by even a relatively small discrepancy between the rates of growth of capital stock and labor force.

One may conclude, then, that despite Nelson's persuasive argument, the elasticity of substitution is not a priori essentially irrelevant to the problem of growth. In particular, differences in the elasticity of substi-

[4] See Arrow *et al.* (1961, p. 241).

tution among industries may profoundly affect the pattern and rate of development. There are, of course, numerous other applications in which the elasticity of substitution plays a crucial role. All this is well known and has been said many times before; its repetition here is to set recent empirical work on measurement of the elasticity of substitution in proper perspective.

In subsequent pages, we consider a selection of recent cross-section and time-series studies of the CES and related production functions. The major finding of this survey is the diversity of results: Even slight variations in the period or concepts tend to produce drastically different estimates of the elasticity. While there seems little rhyme or reason for most of these differences, a number of possible sources of bias exist and may account for at least some of the discrepancies. In addition to these effects, simultaneous equations difficulties also arise; they are discussed both within the traditional profit-maximizing framework [following Kmenta (1964) and Maddala-Kadane (1965)] and to a limited extent within the framework of an aggregate model. Finally the question of identifying biased technical change and a nonunitary elasticity of substitution is discussed. It is shown that the "impossibility theorem" of Diamond and McFadden (1965) does not invalidate most of the recent work of David and van de Klundert (1965). The impressive and useful work of Dhrymes and Kurz (1964) and McFadden (1965), extending and modifying the CES function in application to the electric power industry, must regretfully be left to one side in the present paper. In addition, the unpublished work of D. M. O'Neill, which came to my attention after completion of this paper, is not discussed.

Cross-Section Studies

Since publication of the 1961 ACMS paper, there have been a number of attempts to estimate CES or related production functions from cross-section data. Minhas (1960–63) describes in detail the data and methods used to arrive at the intercountry CES functions presented in Arrow *et al.* (1961), and the estimates he gives are identical. Fuchs (1963) has recomputed these same regressions with a shift variable, 1 for what he calls the developed countries, and zero for what he calls the underdeveloped countries. Independently of ACMS, Minasian (1961) estimated the elasticity of substitution between capital and labor from logarithmic regressions of values added per unit of labor on the wage rate for two-digit manufacturing industries in the United States using *Annual Survey of Manufactures* data for 1957 by state. Solow (1964), using identical data for 1956 by Census regions, has derived an alternative set of elasticities for most of the same two-digit industries. Liu and Hildebrand (1965) have estimated a production function which includes the CES as a special case, again from *Annual Survey of Manufactures* data for 1957 by states. All employees and production workers only are considered separately. Dhrymes (1965) has calculated elasticities of substitution from two different estimating equations derived from the CES formulation for two-digit industries in 1957. He finds corresponding estimates significantly different. Finally, Murata and Arrow (1965) have repeated the earlier intercountry comparisons of Minhas and ACMS using United Nations and International Labor Organization data for two periods: 1953–56 and 1957–59. These results are more comparable with those of Minasian, Solow, and Liu and Hildebrand as they refer to two-digit industries or combinations thereof rather than the three-digit industries considered originally by ACMS. Arrow *et al.* (1961) also include a number of results comparable with those for two-digit industries based on an analysis of United States and Japanese data alone.

Findings of the major studies are summarized in Table 1. The findings of Murata and Arrow (1965) are reported in Table 2. Table 11 below repeats the summary for corresponding two-digit industries in more convenient form and gives analogous results based on time series data.

Before turning to the interpretation of the Liu-Hildebrand results, which require some rather extensive explanation, let us compare the results obtained by Arrow *et al.* (1961), Fuchs (1963), and Arrow-Murata (1965), on the one hand with those of Minasian (1961) and Solow (1964), on the other.

The results reported by Arrow *et al.* (1961) comprise multicountry comparisons for a number of three-digit industries and U.S.-Japanese comparisons for a number of two-digit industries. The data for the first were developed on the basis of censuses of manufactures for nineteen countries in different years between 1950 and 1955. The second comparison is made on the basis of data derived from input-output studies.[5] Fuchs

[5] See Minhas (1960–63, pp. 24–25 mimeo. version); and Arrow *et al.* (1961, p. 239).

TABLE 1

Cross-Section Estimates of the Elasticity of Substitution Between Capital and Labor in Manufacturing Industries 60

Industry	Arrow, *et al.* (1961)	Fuchs[a] (1963)	Minasian (1961)	Solow (1964)	Liu-Hildebrand (1965)[b] All Employees	Liu-Hildebrand (1965)[b] Production Workers	Dhrymes (1965)[c] Elasticity of Substitution from Regression I	Dhrymes (1965)[c] Elasticity of Substitution from Regression II
Food and kindred products			0.58 (.16)	0.69 (.22)	2.15[d]	1.29[d]	.560 (.122)	0.972 (.132)
Dairy products	0.72 (.07)	0.90 (.08)						
Fruit and vegetable canning	0.86 (.08)	1.09 (.10)						
Grain and mill products	0.91 (.10)	1.32 (.17)						
Bakery products	0.90 (.07)	1.07 (.11)						
Sugar	0.78 (.12)	0.90 (.18)						
Tobacco	0.75 (.15)	1.22 (.21)	3.46 (.52)	1.96 (.30)				
Textile mill products			1.58 (.35)	1.27 (.15)	1.65	2.08	.676 (.115)	1.033 (.153)
Spinning and weaving	0.81 (.07)	0.98 (.10)						

(continued)

TABLE 1 *(continued)*

Industry	Arrow, *et al.* (1961)	Fuchs[a] (1963)	Minasian (1961)	Solow (1964)	Liu-Hildebrand (1965)[b] All Employees	Liu-Hildebrand (1965)[b] Production Workers	Dhrymes (1965)[c] Elasticity of Substitution from Regression I	Dhrymes (1965)[c] Elasticity of Substitution from Regression II
Knitting mills	0.79 (.06)	0.95 (.08)						
Apparel and related products				1.01 (.13)	1.43	2.38[d]	.538 (.134)	1.029 (.181)
Lumber and wood products	0.86 (.07)	1.08 (.14)	0.94 (.11)	0.99 (.09)	1.00	0.91	.779 (.076)	1.101 (.111)
Furniture and fixtures	0.89 (.04)	1.04 (.09)	1.09 (.23)	1.12 (.11)	0.92[d]	0.96[e]	.696 (.079)	1.394 (.060)
Pulp, paper, and products	0.97 (.10)	0.91 (.18)	1.60 (.35)	1.77 (1.01)	1.06[d]	0.72[d]	.203 (.062)	0.638 (.078)
Printing and publishing	0.87 (.06)	1.02 (.09)		1.02 (.21)			.681 (.125)	1.106 (.061)
Chemicals and products				0.14 (.95)	1.24	0.88	.309 (.096)	1.030 (.063)
Basic chemicals	0.83 (.07)	1.11 (.10)						
Misc. chemicals	0.90 (.06)	1.06 (.09)						
Fats and Oils	0.84 (.09)	1.06 (.18)						

(continued)

TABLE 1 *(continued)*

Industry	Arrow, *et al.* (1961)	Fuchs[a] (1963)	Minasian (1961)	Solow (1964)	Liu-Hildebrand (1965)[b] All Employees	Liu-Hildebrand (1965)[b] Production Workers	Dhrymes (1965)[c] Elasticity of Substitution from Regression I	Dhrymes (1965)[c] Elasticity of Substitution from Regression II
Petroleum and coal products			−0.54	1.45	Not calculated		.113	1.311
			(1.06)	(.71)			(.111)	(.083)
Rubber products			0.82	1.48	1.44	1.39	.403	1.037
			(.29)	(.88)			(.088)	(.144)
Leather and leather goods	0.86	0.98	0.96	0.89	0.79	0.93	.508	1.126
	(.06)	(.10)	(.29)	(.27)			(.149)	(1.117)
Stone, clay, and glass products			0.59	0.32	1.28[d]	1.44[d]	.491	0.887
			(.25)	(.46)			(.110)	(.077)
Clay products	0.92	0.66						
	(.10)	(.20)						
Glass	1.00	1.27						
	(.08)	(.10)						
Ceramics	0.90	1.08						
	(.04)	(.13)						
Cement	0.92	1.31						
	(.15)	(.22)						
Primary metal products			0.92	1.87	0.99[d]	1.00[d]	.095	0.968
			(.24)	(1.25)			(.061)	(.136)
Iron and steel	0.81	0.76						
	(.05)	(.11)						

(continued)

TABLE 1 *(concluded)*

Industry	Arrow, *et al.* (1961)	Fuchs[a] (1963)	Minasian (1961)	Solow (1964)	Liu-Hildebrand (1965)[b] All Employees	Liu-Hildebrand (1965)[b] Production Workers	Dhrymes (1965)[c] Elasticity of Substitution from Regression I	Dhrymes (1965)[c] Elasticity of Substitution from Regression II
Nonferrous metals	1.01	0.94						
	(.12)	(.20)						
Fabricated metal products	0.90	1.01		0.80	0.70[d]	0.45[e]	.401	0.950
	(.09)	(.17)		(.29)			(.135)	(.149)
Nonelectrical machinery			0.31	0.64	0.60[d]	0.44[e]	.121	0.245
			(.21)	(.45)			(.071)	(.702)
Electrical machinery	0.87	1.03	1.26	0.37	0.79[d]	1.10[d]	.194	0.620
	(.12)	(.21)	(.33)	(.54)			(.109)	(.350)
Transportation equipment			2.04	0.06	2.01[d]	1.91[d]		
			(.49)	(.82)				
Instruments and related products				1.59	1.24[e]	1.65[d]		
				(.15)				

[a]Countries broken into two groups; shift variable — 1 for Group I, 0 for Group II introduced. Group I: United States, Canada, New Zealand, Australia, Denmark, Norway, United Kingdom, Ireland, Puerto Rico. Group II: Colombia, Brazil, Mexico, Argentina, El Salvador, Southern Rhodesia, Iraq, Ceylon, Japan, India.

[b]Computed at 1957 share of capital in value added. See Tables 4-6 for the method and results.

[c]Regression I is logarithmic regression of value added per unit of labor on the wage rate; regression II is logarithmic regression of value added per unit of capital on the rate of return to capital; state data from 1957 *Census of Manufactures*.

[d]Capital-labor ratio coefficient in regression is more than twice its standard error.

[e]Coefficient of capital-labor ratio nearly twice its standard error.

TABLE 2

Intercountry Estimates of the Elasticity of Substitution for Two-Digit Manufacturing Industry, Murata-Arrow (1965)[a]

	Data for 1953-56:				Data for 1957-59:			
	Estimate Based on:				Estimate Based on:			
Industry	Log Regression of Value Added per Employee on Wage Rate	Log Regression of Wage Rate on Value Added per Employee[b]	R^2	Degrees of Freedom	Log Regression of Value Added per Employee on Wage Rate	Log Regression of Wage Rate on Value Added per Employee[b]	R^2	Degrees of Freedom
Food, beverages and tobacco	.722 (.054)	.799 (.075)	.903	19	.725 (.054)	0.801 (.074)	.906	19
Textiles	.793 (.049)	.851 (.062)	.932	19	.827 (.069)	0.931 (.084)	.888	18
Clothing, footwear, and made-up textiles	.660 (.067)	.775 (.102)	.851	17	.804 (.043)	0.841 (.054)	.956	16
Wood products and furniture	.818 (.068)	.920 (.083)	.890	18	.919 (.074)	1.025 (.080)	.896	18
Paper and paper products	.904 (.050)	.955 (.055)	.947	18	.788 (.061)	0.874 (.078)	.901	18
Printing and publishing	.836 (.075)	.951 (.090)	.879	17	.926 (.063)	0.999 (.068)	.927	17
Leather and leather goods	.711 (.059)	.801 (.083)	.888	18	.699 (.050)	0.761 (.072)	.919	17

(continued)

TABLE 2 *(concluded)*

Industry	Data for 1953-56: Estimate Based on: Log Regression of Value Added per Employee on Wage Rate	Data for 1953-56: Estimate Based on: Log Regression of Wage Rate on Value Added per Employee[b]	Data for 1953-56: R^2	Data for 1953-56: Degrees of Freedom	Data for 1957-59: Estimate Based on: Log Regression of Value Added per Employee on Wage Rate	Data for 1957-59: Estimate Based on: Log Regression of Wage Rate on Value Added per Employee[b]	Data for 1957-59: R^2	Data for 1957-59: Degrees of Freedom
Rubber products	.829	.889	.933	15	.768	1.000	.768	16
	(.058)	(.069)			(.106)	(.137)		
Chemicals, petroleum, and coal	.838	.887	.946	16	.834	0.988	.844	17
	(.050)	(.060)			(.087)	(.104)		
Stone, clay, glass	.847	.896	.945	20	.859	0.920	.934	20
	(.046)	(.054)			(.051)	(.060)		
Primary metals	.856	.943	.909	17	.873	0.946	.923	16
	(.066)	(.077)			(.063)	(.072)		
Metal products	.917	.970	.945	18	.922	1.011	.912	17
	(.052)	(.056)			(.069)	(.076)		

Source: Exchange rate from *Year Book of Labor Statistics, 1963,* International Labor Organization, Geneva, 1963. Value added (local currency), wages and salary payments (local currency), number of employees from *The Growth of World Industry, 1938-1961,* United Nations, 1963.

[a]Countries used were selected from the following list in each case: Australia, Belgium, Canada, Denmark, El Salvador, Finland, India, Iraq, Ireland, Japan, Luxembourg, Mexico, New Zealand, Norway, Pakistan, Philippines, Portugal, Puerto Rico, Singapore, Sweden, United Arab Republic, United Kingdom, and United States.

[b]Approximate standard errors based on Taylor's series expansion; see L. R. Klein, *A Textbook of Econometrics,* Evanston, Ill. 1953, p. 258.

(1963) has simply re-used the three-digit industry data but calculated new regressions, introducing a shift variable to account for differences in intercept between two groups of countries:

Group I	*Group II*
United States	Colombia
Canada	Brazil
New Zealand	Mexico
Australia	Argentina
Denmark	El Salvador
Norway	Southern Rhodesia
United Kingdom	Iraq
Ireland	Ceylon
Puerto Rico	Japan
	India

Aside from the fact that it seems improbable to find Japan in Group II and Puerto Rico in Group I, the two groups might be taken to reflect differences in development. Note that in every case but clay products, iron and steel, and nonferrous metals, the elasticity of substitution estimated allowing for a shift exceeds that obtained originally by Arrow *et al.* (1961) and in only two cases (glass, and iron and steel) can the estimate be considered significantly different from 1.

If we write the CES production function as

$$y = \gamma[\delta x^{-\rho} + (1 - \delta)]^{-1/\rho}, \tag{2}$$

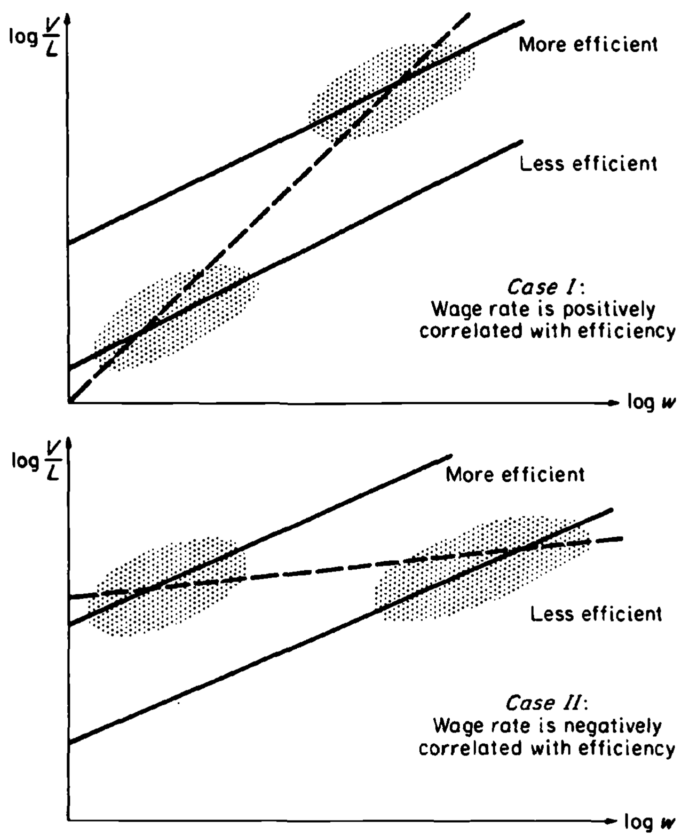
where y = value added per unit labor and x = the capital-labor ratio, the equation estimated by Arrow *et al.* (1961) is

$$\log y = -\frac{1}{1 + \rho} \log \gamma^{-\rho}(1 - \delta) + \frac{1}{1 + \rho} \log w. \tag{3}$$

Now Fuchs' modification amounts to saying that the constant term in (3) varies in a systematic way depending on how developed a country one is considering. However, a very curious result emerges from Fuchs' calculations: The shift variable is 1 for Group I, mainly developed countries, and zero for Group II, mainly underdeveloped countries. Furthermore, the estimated coefficients for the shift variable in the various regressions are nearly all *negative!* If we think of the grouping of countries as reflecting essentially differences in efficiency, this is a most peculiar result, as it would seem to imply lower value added per unit of labor for a given wage rate the more highly developed the country. Fuchs' explanation for this result is that observed wages in less developed countries more often fail fully to reflect labor costs than they do in more highly developed economies. Thus the *observed* wage rate (but not the true unit labor cost) is negatively correlated with efficiency. As shown in the lower panel of Figure 1, this results in an underestimate of the elasticity of substitution. As can be seen it is not even necessary that there be any differences in efficiency at all; the errors of observation in the wage rate would be sufficient to account for Fuchs' result.

FIGURE 1

Effects of Varying Efficiency on the Estimated Elasticity of Substitution



More realistically, however, one might wish to allow far differences in efficiency *positively* correlated with the observed wage rate. In this case, an overestimate of the elasticity of substitution will result, as illustrated in the top panel of Figure 1. In order to test for variations in efficiency, ACMS used capital data (conventionally measured) for four industries and five countries (United States, Canada, United Kingdom, Japan, and India).[6] On the basis of very rough estimates of the relation between the efficiency parameter γ and the wage rate, ACMS argue that the two are positively related, the elasticity of the former with respect to the latter being about 0.3. On the basis of a constant elasticity relation between γ and w, the wage rate, ACMS work out a correction factor for finding the elasticity of substitution, σ, from the regression coefficiant of log w in the regression of log V/L, value added per unit labor, on log w.[7] Let b be the regression coefficient and e the elasticity of γ with respect to w; then

$$\sigma = \frac{b - e}{1 - e}. \tag{4}$$

As long as $e < b < 1$, $\sigma < b$, so that the estimated elasticity of substitution is biased toward 1 (if it is less to start with) if wages and efficiency are positively corrected. This is illustrated in the top half of Figure 1.

The results obtained by Arrow and Murata (1965) may be used to shed light on the question of whether a broadening of the commodity classification tends to increase the estimated elasticity of substitution as Solow (1964) argued.[8] Table 3 compares the range of the findings of ACMS for three-digit industries with the corresponding results of Arrow and Murata. The table shows that a finer classification has very little effect; if anything, the elasticities of substitution are somewhat higher for the more narrowly defined groups! While one cannot definitely reject Solow's very plausible contention, it seems that we cannot account for differences on the basis he suggested.

Minasian (1961), Solow (1964), and Liu and Hildebrand (1965) have used *Survey of Manufactures* data for 1956 and 1957. Solow uses regional aggregates, and the others use state data to investigate

[6] Arrow *et al.* (1961, p. 235).
[7] Arrow *et al.* (1961, p. 237).
[8] Solow (1964, p. 118): "It seems plausible that, in general, elasticities of substitution should be smaller the more narrowly defined the industrial classification, and the larger the degree of aggregation."

TABLE 3

Comparison of Corresponding Elasticities of Substitution for Two-Digit and Three-Digit Industry Classifications

Industry	Two-Digit Results, Range, Murata-Arrow (1965)	Three-Digit Results, Range, Arrow *et al.* (1961)
Food and kindred products; tobacco	.72-.73	.72-0.91
Textiles	.79-.83	.79-0.81
Apparel	.66-.80	
Lumber and products	.82-.92	.86-0.89
Paper	.79-.90	.97
Printing and publishing	.84-.93	.87
Chemicals, coal, etc.	.83-.84	.83-0.90
Rubber, etc.	.77-.83	
Leather, etc.	.70-.71	.86
Stone, clay, glass	.85-.86	.92-1.00
Primary metals	.86-.87	.81-1.01
Fabricated metal products	.92	.90

the elasticities of substitution. The Liu-Hildebrand results are actually based on a production function of which the CES is a special case; therefore, we consider the Minasian and Solow results first. One important difference between the two is that Solow's results are based on a very few regional aggregates while Minasian's utilize the greater number of observations supplied by state data. Another difference is that 1957, the year used by Minasian, was one of recession, whereas 1956, Solow's year, was not.[9] There are a number of cases in which the results differ markedly:

[9] Gordon (1961, pp. 492–501): "By the beginning of 1956 the economy was operating close to full capacity, with bottlenecks appearing in various durable-goods industries . . ." (p. 494). "In the industrial sphere prices rose . . . wages rose rapidly throughout the economy" (p. 496). ". . . Industrial production failed to rise any further after the beginning of 1957; the economy entered into what we have called a turning point zone; and a cumulative contraction developed in the latter half of the year" (p. 496). "A number of deflationary forces were already at work in the first half of the year. Manufacturers' new orders for durable goods were declining. . . . Temporarily offsetting these deflationary

Industry	*Elasticity of Substitution* Minasian	Solow
Tobacco manufactures	3.46	1.96
Petroleum and coal	−0.54	1.45
Rubber and plastics	0.82	1.48
Primary metals	0.92	1.87
Nonelectrical machinery	0.31	0.64
Electrical machinery	1.26	0.37
Transportation equipment	1.04	2.04

The remaining results are more consistent with one another. These differences must either be accounted for by the difference in the level of regional aggregation or by the differential effects of full-capacity operation (1956) as compared with partial utilization (1957). The level of regional aggregation affects the results because of differences in the product mix compared: The mix is likely to be more heterogeneous across states than across Census regions. For example, when we compare Delaware and Ohio in the category rubber and plastics we are getting largely plastics in Delaware and rubber in Ohio. On the other hand, regional aggregation tends to obscure these differences and produce a more homogeneous product mix across regions. There seems to be somewhat more localization in the industries for which Solow and Minasian obtained very different estimates, but it does not seem possible to account for the peculiar pattern of results in this way. Nor does it seem possible to account in detail for the differences found in terms of the differential effect of the 1957 recession which was concentrated primarily in the durable goods area. Certainly one can say that the category "food and kindred products" will be more comparable between the two years than, say, "electrical machinery." But why should "tobacco manufactures" differ so, or "lumber and timber" not? Perhaps all one can conclude is that, when there is a lot of noise in the system, apparently small changes can produce substantial variation in the results.

In his review of Minhas (1960–63), Leontief (1964) wrote (pp. 343–44):

---

forces were the continued expansion in consumers' expenditures on nondurables and on services. . ." (p. 497). ". . . Though brief, the decline [after the middle of 1957] was very rapid; and in terms of the decline in GNP, industrial production, and employment, it was the most severe (although the shortest) of the recessions experienced since World War II." Durable goods expenditures and expenditures on plant and equipment declined most severely (pp. 497–98).

> Minhas . . . proceeded on the assumption that only the variable . . . [value added per unit labor] is subject to random errors, while the [wage rate] . . . is not. Had he instead, in fitting the slopes of these regression lines, allowed also for errors affecting the observed magnitudes of [the wage rate] . . . , all estimated elasticities would necessarily turn out to be larger, since in 23 out of 24 industries examined by him, the magnitude of the . . . [elasticities of substitution] turn out to be less—although in most instances only slightly less—than 1.

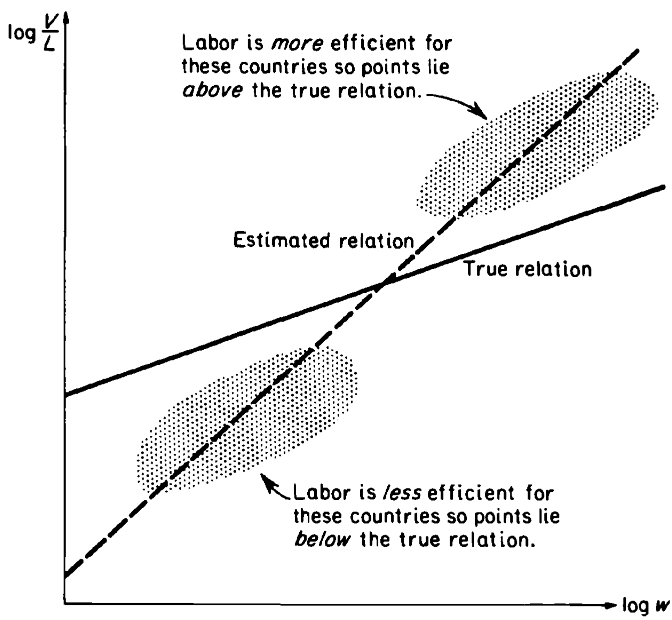
> The inverse proportionality . . . [implied by an elasticity of substitution less than 1] between the number of workers employed per unit of output of a particular industry and the wage rate paid to them by that industry in different countries can be explained in entirely different terms. The assumption that a man-year of labor in one part of the world is equivalent to a man-year in any other part . . . can be questioned.

> The elasticity which Minhas estimates . . . measures . . . not the substitution between capital and labor but the substitution between different grades of labor.

Leontief's point is illustrated by Figure 2.

FIGURE 2

Illustration of the Possible Bias in Estimating a CES Production Function from Intercountry Data



According to Leontief's argument, then, high-wage countries are those with more efficient labor and so the estimated elasticity of substitution is biased toward unity. We would expect far more variation in labor quality among countries than among regions or states within the United States. Consequently, the results obtained by Solow and Minasian for the elasticities of substitution should be *less* than the corresponding estimates obtained by Arrow and Murata (1965), which, as we saw earlier, are little different from those obtained by Minhas and also presented in Arrow *et al.* (1961). Unfortunately, this does not prove to be the case. Only in the category "stone, clay, glass" do the estimated elasticities of Minasian and Solow seem unambiguously less than the corresponding result obtained by Arrow and Murata. The other two cases are "food and kindred products" and "chemicals, etc.," but the former has been combined with "tobacco" and the latter with "petroleum and coal" by Arrow and Murata. The Solow-Minasian results bracket the Arrow-Murata results. The U.S.-Japanese comparisons presented in Arrow *et al.* (1961) alter this finding little. Once again the expected differences do not turn out to be the ones observed.

Solow (1964) suggests a possible explanation (p. 118):

> . . . the earlier observations [used by ACMS] come from a list of countries which included the United States and Canada at the high-wage end and Ceylon, India and Iraq at the low-wage end. Within any one industry the range of wage rates was always very wide, the highest running at least ten or twenty times the lowest. Within my interregional samples, the wage variation is much smaller; never in any industry is the highest wage as much as twice the lowest and almost always the range is much narrower.

But it is difficult to see that the narrow range of wage variation in the interregional or interstate studies explains more than large standard errors; the explanation does not seem to account for *higher* estimated elasticities.

In an unpublished paper, McKinnon (1963b) suggests an alternative possibility, namely, that the differences may be due to the systematic variation of product prices.[10] In all of the cross-section studies considered so far, the assumption has been made that the prices of final commodities across regions or countries were constant while money wages varied. This, of course, must imply variations in the rate of return on

[10] A similar idea is contained in V. Smith (1963), who, however, does not systematically explore its implications.

capital. Certainly, such variations are not plausible *within* a country such as the United States. Furthermore, Minhas (1960–63) shows that among the five countries he considered (United States, Canada, United Kingdom, Japan, and India) there is remarkably little variation in the average rate of return on capital in manufacturing; there are, however, marked differences in the rates of return among countries in individual industrial categories. Suppose, then, we tentatively accept the hypothesis of constant product prices internationally, but reject this hypothesis within the United States. Grounds for this position might be that the product mix within each two-digit category varies less across countries than within the United States, where there is a good deal of regional specialization. (This, of course, is not to deny that product prices do vary internationally. There is considerable evidence that they do. The question is whether or not an explanation of the differences between the United States and intercountry results can be explained on the hypothesis that product prices vary less internationally than interregionally because product mix varies less internationally than interregionally at the two-digit level.)

In order to explain the higher estimates of the elasticity of substitution Solow and Minasian obtain on the basis of the use of money values rather than real values, we need only assume that prices are positively correlated with money wages—eminently plausible. In the case of Solow, he intends to estimate

$$\log \frac{V}{L} = a + b \log w, \tag{5}$$

where V is real value added and w is the real wage rate. Instead, however, he estimates

$$\log \frac{pV}{L} = a' + b' \log pw, \tag{5'}$$

where p is the price index for output. Estimating (5′) rather than (5) is tantamount to leaving out the term $-(b - 1) \log p$ in

$$\log \frac{pV}{L} = a + b \log pw - (b - 1) \log p. \tag{5''}$$

Now if the true elasticity of substitution is less than 1, the coefficient of $\log p$ will be positive. Hence, the omission of $\log p$ will bias the slope of log pw upward if log pw and log p are positively correlated. Indeed the bias is

$$(6) \qquad E[\text{est. } b - b] = (1 - b)\frac{\text{cov}\,[\log pw, \log p]}{\text{var}\,[\log p]}$$
$$= (1 - b)\frac{\text{var}\,[\log p] + \text{cov}\,[\log w, \log p]}{\text{var}\,[\log p]}.$$

Unless, therefore, the logarithms of real wages and prices are actually negatively correlated, the estimated elasticity of substitution will be biased upward. Although Minasian actually estimates a logarithmic regression of labor's share in the wage rate, exactly the same argument goes through. The argument thus accounts for the differences between the intercountry and the interregional results.[11]

The final two columns of Table 1 present results obtained by Dhrymes (1965). Those presented in the first of the two columns are based on logarithmic regressions of value added per unit of labor on the wage rate across states in 1957. Even though these results appear to be based on substantially the same data (individual states, 1957) as those obtained by Minasian, it is clear from the table that they differ very substantially from corresponding estimates obtained by Minasian. For example, Minasian's estimate of σ for pulp, paper, and products is nearly eight times Dhrymes', while his estimate of σ for primary metal products is nearly ten times Dhrymes'. Conversely, Minasian's estimate for rubber products is double Dhrymes'. The only explanation for these gross differences appears to be the slight variation in the basic series employed.

In the second of the two columns in which Dhrymes' results are reported, elasticities of substitution based on logarithmic regressions of value added per unit of capital stock (essentially book value) on the rate of return to capital (computed as a residual) are presented. These estimates are uniformly higher than those based on the more usual estimation procedure. While it is possible to account for these results in terms of errors in the measurement of capital stock (as Dhrymes shows in the first section of his paper), Dhrymes prefers to regard the divergence as a test of the perfect-competition and constant-returns-to-scale hypothesis

[11] Eisner's explanation of the possible reasons for an upward bias in Solow's estimates is closely related to the discussion here. Eisner, however, views the matter somewhat more generally in terms of "permanent" and "transitory" components. See his discussion of Solow (1964, pp. 128–37).

of the standard formulation. In view, however, of the extreme sensitivity of nearly all estimates of elasticities of substitution it is moot whether any such elaborate conclusions might be drawn from the differences among these estimates.

The Liu-Hildebrand results are most interesting, for together with some unpublished work of Bruno (1962), they suggest a production function which is neither the Cobb-Douglas nor its generalization, the CES function, but which includes both as special cases.

What Liu and Hildebrand do is to fit a logarithmic relationship containing the capital-labor ratio as well as the wage rate to explain value added per unit labor. Thus, letting

$$y = \frac{V}{L}$$

$$x = \frac{K}{L}$$

we write the production function $V = F(K, L)$, which we assume is homogeneous of degree 1, as

$$V = LF\left(\frac{K}{L}, 1\right) = Lf(x)$$

or

$$y = f(x).$$

Assuming the wage rate w equals the marginal product of labor (output prices held constant),

$$w = y - xf',$$

we find the relationship fitted by Liu and Hildebrand as

$$\log y = \log a + b \log\left(y - x\frac{dy}{dx}\right) + g \log x. \tag{7}$$

As is well known, the assumption $g = 0$ leads to the CES function. Following Bruno, we integrate (7) to uncover the production function implied:

Equation (7) may be rewritten

$$y = a\left(y - x\frac{dy}{dx}\right)^b x^g.$$

Thus
$$y = a\left(y^2\frac{dz}{dx}\right)^b x^g,$$

where $z = x/y$. After some manipulation we find

$$x^{(1-2b-g)/b}dx = a^{1/b}z^{(1-2b)/b}dz,$$

so that integrating one obtains

$$\frac{x^{(1-g-b)/b}}{(1-g-b)/b} = \frac{a^{1/b}z^{(1-b)/b}}{(1-b)/b} + c,$$

where c is a constant of integration. Substituting $z = x/y$ and simplifying we obtain finally

$$y = [\beta x^{-\rho} + \alpha x^{-m\rho}]^{-1/\rho} \tag{8}$$

where
$$\rho = \frac{1-b}{b}$$

$$m = \frac{g}{1-b}$$

$$\alpha = \frac{1-b}{(1-b-g)a^{1/b}}$$

$$\beta = \frac{-c(1-b)}{a^{1/b}b}.$$

Note that (8) corresponds exactly to the CES formulation when $m = 0$, and this occurs if and only if $g = 0$; i.e., when the capital-labor ratio does not enter the estimating equation.

To obtain the elasticity of substitution for the Bruno function (8), we simply fill in the appropriate derivatives in the formula

$$\sigma = -\frac{f'(f - xf')}{xff''}. \tag{9}$$

Differentiating (8) we obtain

$$f' = y^{1+\rho}[\beta x^{-(1+\rho)} + \alpha m x^{-(1+m\rho)}],$$

whence

$$f - xf' = y - y^{1+\rho}[\beta x^{-\rho} + \alpha m x^{-m\rho}]. \tag{10}$$

Now the wage rate is assumed to be equal to $f - xf'$ = the marginal product of labor; hence, raise (8) to the power $-\rho$ and multiply by $y^{1+\rho}$; we obtain

$$y = y^{1+\rho}[\beta x^{-\rho} + \alpha x^{-m\rho}].$$

Substituting in (10),

$$w = y^{1+\rho}\alpha(1 - m)x^{-m\rho}, \tag{11}$$

—which of course is really just the form we started with since (11) implies

$$\log y = \frac{-1}{1+\rho}\log \alpha(1 - m) + \frac{1}{1+\rho}\log w + \frac{m\rho}{1+\rho}\log x.$$

Differentiating a second time

$$\begin{aligned} f'' &= (1+\rho)\frac{f'}{xf}[xf' - f] + (1 - m)\rho\alpha m y^{1+\rho}x^{-(2+m\rho)} \\ &= -(1+\rho)\frac{f'}{xf}w + \rho m w x^{-2}. \end{aligned} \tag{12}$$

Substituting from (10) and (12) in (9) we obtain

$$\sigma = \frac{f'w}{\left[(1+\rho)\frac{f'}{xf}w - \rho m w x^{-2}\right]xf} = \frac{1}{(1+\rho) - \rho m\frac{f}{xf'}}. \tag{13}$$

Since, however,

$$f = w + xf',$$

so that

$$\begin{aligned} \frac{f}{xf'} &= \frac{w}{xf'} + 1 = \frac{w}{\frac{K}{L}r} + 1 = \frac{wL/V}{rK/V} + 1 = \frac{s_L}{s_K} + 1 \\ &= \frac{s_L + s_K}{s_K} = \frac{1}{s_K} \end{aligned}$$

where s_K = capital's share, we arrive finally at the simple formula

$$\sigma = \frac{1}{1 + \rho - \frac{\rho m}{s_K}}. \tag{14}$$

TABLE 4

Summary of Liu-Hildebrand (1965) Regressions for Two-Digit Industries[a]

Industry	All Employees				Production Workers Only			
	b	g	$\bar{R}^2$	N	b	g	$\bar{R}^2$	N
Food and kindred products	0.407 (.177)	.446 (.139)	.548	35	0.282 (.144)	.430 (.148)	.464	35
Textile mill products	0.975 (.175)	.160 (.109)	.695	18	1.427 (.299)	.122 (.156)	.641	18
Apparel and related products	1.071 (.263)	.097 (.086)	.669	18	1.094 (.374)	.211 (.102)	.617	18
Lumber and wood products	0.990 (.135)	.002 (.070)	.943	14	0.989 (.165)	−.033 (.090)	.920	14
Furniture and fixtures	1.258 (.128)	−.154 (.072)	.859	19	1.402 (.177)	−.191 (.102)	.807	19
Pulp, paper, and products	0.386 (.322)	.331 (.050)	.730	28	0.298 (.340)	.304 (.069)	.657	28
Chemicals and products	0.866 (.231)	.201 (.085)	.424	31	0.780 (.254)	.076 (.109)	.309	31
Petroleum and coal products	0.180 (.716)	.282 (.224)	.152	18	−0.027 (.951)	.309 (.283)	.213	18
Rubber products	1.278 (.553)	.018 (.217)	.523	16	1.231 (.286)	−.052 (.132)	.738	16
Leather and leather goods	0.890 (.457)	−.050 (.113)	.368	15	0.926 (.528)	.0003 (.118)	.434	15
Stone, clay, and glass products	0.539 (.177)	.295 (.065)	.611	25	0.568 (.175)	.309 (.069)	.627	25
Primary metal products	0.298 (.704)	.321 (.141)	.234	28	0.187 (.683)	.374 (.154)	.250	28
Fabricated metal products	0.401 (.207)	.178 (.068)	.336	32	0.189 (.208)	.243 (.080)	.298	32
Machinery except electrical	0.222 (.263)	.258 (.100)	.343	25	0.222 (.226)	.204 (.104)	.262	25
Electrical machinery	0.300 (.210)	.278 (.071)	.483	22	0.606 (.233)	.202 (.089)	.494	22
Transportation equipment	1.008 (.448)	.214 (.060)	.504	26	0.998 (.545)	.205 (.073)	.441	26
Instruments and related products	0.601 (.294)	.217 (.116)	.681	12	0.874 (.264)	.196 (.098)	.805	12

N = number of observations.

[a]Summary of estimates of regressions of the form

$$\log V/L = a + b \log w + g \log K/L$$

for two-digit industries from *Census of Manufactures, 1957,* in Liu-Hildebrand (1965, pp. 36-39).

Table 4 gives a summary of the Liu-Hildebrand regressions. Tables 5 and 6 show the details of the calculation of the elasticity of substitution obtained from these regressions by applying (14) together with the share of capital figures given by Solow (1964). These are not strictly comparable for two reasons: First, they refer to 1956 rather than 1957; and, second, as they are based on the relative share of all wage payments, not just those to production workers, the calculations in Table 5 contain a slight error. On the whole, however, the use of Solow's figures is not thought to lead to any substantial difficulty.

If $\rho \geqq 0$ and if $m \geqq 0$ (as it will be if $g \geqq 0$ under the condition that ρ is non-negative), then, clearly, the elasticity of substitution will be greater than the coefficient of log w in the estimating equation corresponding to (7):

$$(15) \qquad y = \log a + b \log w + g \log x$$

$$= \frac{-1}{1+\rho} \log [(1-m)\alpha] + \frac{1}{1+\rho} \log w + \frac{\rho m}{1+\rho} \log x.$$

However, if the "apparent elasticity of substitution," b, is greater than unity (corresponding to a negative value of ρ) then the coefficient of log x will be negative unless m is also negative. Indeed, the true elasticity of substitution, σ, can be shown to be less, equal, or greater than the "apparent elasticity," b, according as g is negative, zero, or positive. In most cases, we may expect g to be positive (it is except for lumber and furniture in the Liu-Hildebrand regressions); hence, the direction of the discrepancy is unaltered. In any event, a simple calculation will serve to obtain the corrected elasticity whatever the slope of log x. The formula illustrates, moreover, that generally the elasticities of substitution tend to decrease as the share of capital rises.

When the value of ρ is positive, so that the slope of log w is less than unity, (15) shows that the coefficient of log w in a regression of log y on log w alone will tend to overstate the true slope of log w because this slope will be biased upward if wages and the capital-labor ratio are positively correlated. However, in this case the slope of log w will understate the *true* value of σ. Thus, assuming the Bruno function to be the correct model, we see that omissions of the capital-labor ratio lead to two offsetting effects. However, when the ratio is included the implied elastic-

TABLE 5

Capital Share in Value Added and Computation of Adjusted Elasticities of Substitution Between Capital and Total Employment Implied by the Liu-Hildebrand Results

Industry (1)	Share of Capital[a] (2)	$1+\rho=\frac{1}{b}$ (3)[b]	$m\rho=\frac{g}{b}$ (4)[b]	Elasticity of Substitution in 1957[c] (5)
Food products	.55	2.4570	1.0958	2.1524
Textile products	.39	1.0256	0.1640	1.6526
Apparel	.39	0.9337	0.0905	1.4253
Lumber	.36	1.0101	0.0020	0.9955
Furniture	.42	0.7949	−0.1224	0.9206
Pulp, paper, etc.	.52	2.5906	0.8574	1.0618
Chemicals	.66	1.1547	0.2320	1.2450
Rubber	.45	0.7824	0.0410	1.4465
Leather	.38	1.1235	−0.0561	0.7867
Stone, clay, glass	.51	1.8552	0.5472	1.2783
Primary metals	.46	3.3557	1.0771	0.9860
Fabricated metals	.42	2.4937	0.4438	0.6959
Nonelectrical machinery	.41	4.5045	1.1621	0.5988
Electrical machinery	.45	3.3333	0.9266	0.7848
Transportation equipment	.43	0.9920	0.2122	2.0060
Instruments	.42	1.6638	0.3610	1.2433

[a]From (1964, p. 117). Solow gives the relative share of wage payments from which this column was computed. Strictly speaking, the share of total payroll should have been used. Thus, the capital share is somewhat larger than it should be, and the use of more appropriate figures would tend to increase all the elasticities of substitution except furniture and leather, which would be decreased by the change.

[b]From Liu-Hildebrand (1965, pp. 36-37).

[c]Equals the reciprocal of column 3 minus (column 4 divided by column 2).

TABLE 6

Capital Share in Value Added and Computation of Adjusted Elasticities of Substitution Between Capital and Production Workers Implied by the Liu-Hildebrand Results

Industry (1)	Share of Capital[a] (2)	$1+\rho=\frac{1}{b}$ (3)[b]	$m\rho=\frac{g}{b}$ (4)[b]	Elasticity of Substitution in 1957[c] (5)
Food products	.55	3.5460	1.5247	1.2923
Textile products	.39	0.7007	0.0854	2.0760
Apparel	.39	0.9140	0.1928	2.3832
Lumber	.36	1.0111	−0.0333	0.9060
Furniture	.42	0.7132	−0.1362	0.9639
Pulp, paper, etc.	.52	3.3557	1.0201	0.7174
Chemicals	.66	1.2820	0.0974	0.8815
Rubber	.45	0.8123	0.0422	1.3918
Leather	.38	1.0799	0.0003	0.9267
Stone, clay, glass	.51	1.7605	0.5439	1.4409
Primary metals	.46	5.3475	1.9999	1.0001
Fabricated metals	.42	5.2910	1.2857	0.4485
Nonelectrical machinery	.41	4.5045	0.9189	0.4418
Electrical machinery	.45	1.6501	0.3333	1.0996
Transportation equipment	.43	1.0020	0.2054	1.0073
Instruments	.42	1.1441	0.2242	1.6499

[a]From Solow (1964, p. 117).

[b]From Liu-Hildebrand (1965, pp. 38-39).

[c]Equals the reciprocal of column 3 minus (column 4 divided by column 2).

ity of substitution turns out to be much greater in most cases than the slope of log w. In many instances, the Liu-Hildebrand regression produces elasticities of substitution larger than those obtained by Solow and Minasian but in other cases not. What remains striking is the diversity of results and their sensitivity to small changes in the specification of the equation fitted or of the data used.

A possible explanation for the extreme variability of the results obtained by Liu and Hildebrand has been suggested by Griliches and Mieszkowski: If the true relation were in fact Cobb-Douglas, say,

$$V = aK^{\alpha}L^{1-\alpha},$$

K/L and w would be connected by the log-linear relation

$$\log w = \log a' + \alpha \log \frac{K}{L}$$

where $a' = a(1 - \alpha)$. Thus we would expect a good deal of collinearity between log w and log K/L if the true relation were Cobb-Douglas. This effect, if present, should tend to show up in large standard errors for the estimated coefficients of both variables, as indeed there are.

The results of all of the studies discussed in this section are summarized in Table 11, below. The diversity, as we have remarked, is striking. A number of possible reasons for such apparent inconsistencies have been advanced in the present section. A number of additional comments on sources of bias are discussed below in connection with the work of Maddala and Kadane (1965) and Kmenta (1964). It seems clear that a number of conflicting factors are operating, perhaps simultaneously, to produce the differences observed. The mixed flavor of the contents of *this* bottle does not seem quite so nice as Alice's.

Time-Series Studies

Even before ACMS published their paper in 1961, there had been considerable interest in the elasticity of substitution and in obtaining estimates of it especially for the economy as a whole. However, as in the case of cross-section work, studies utilizing time series have been greatly stimulated by ACMS. Recent studies of the aggregate elasticity of substitution have been made by Kravis (1959), Arrow *et al.* (1961), Diwan (1963), Kendrick and Sato (1963), Brown and de Cani (1963), Kendrick (1964), Ferguson (1965a), David and van de Klundert (1965). Estimates of the elasticity of substitution for two-digit U.S. manufacturing industries based on time-series data have been obtained by McKinnon (1962 and 1963a), Lucas (1963), Kendrick (1964), Maddala (1965), and Ferguson (1965b). Mention should also be made of the work of McKinnon (1963a) and Maddala (1963) on the elasticity of substitution in certain extractive industries. Solow (1964) reports estimates of the rate of embodied technological progress based on his cross-section estimates of the elasticity of substitution, but these will not be discussed here.

Nearly all time-series studies make some attempt to allow for, and estimate, technological change. Some studies attempt to allow for variations in capacity utilization from year to year, although not with much success.[12] Recently, Diamond and McFadden (1965) have raised serious questions as to the possibility of identifying the production function under technological change. A simplified discussion of their "impossibility theorem" along lines developed by K. J. Arrow is given in this section together with some results indicating under what sort of assumptions identification is possible. The possible bias due to lack of attention to the problem of capacity utilization is also discussed.

Table 7 summarizes recent time-series studies of the elasticity of substitution and technical change in two-digit U.S. manufacturing industries.

Both Ferguson (1965) and McKinnon (1962) use annual data for the postwar period. However, Ferguson uses *Census of Manufactures* and *Annual Survey of Manufactures* material while McKinnon's data are derived from secondary sources. McKinnon allows for a distributed lag in the relationship he estimates, while Ferguson does not. Both allow for technical change; however, McKinnon assumes it must be neutral, whereas Ferguson appears to allow for non-neutral varieties. This last is entirely spurious, and although we have reported in the final column of the section of Table 7 dealing with Ferguson's results what he purports to have found, it can be shown that these results are meaningless (see below): On the whole, Ferguson's estimates of the elasticities of substitution are high, being greater than 1 in nine of nineteen cases

[12] I have argued elsewhere that such allowance is exceedingly important to make: Nerlove (1965, pp. 10–17).

TABLE 7

Time-Series Results on the Elasticity of Substitution, Two-Digit Manufacturing Industries, United States

	Ferguson (1965b), 1949-61				
Industry	Elasticity of Substitution	Estimated Coefficient of Trend	$\bar{R}^2$	Rate of Neutral Technological Progress[a] (per cent)	Biased Technical Change[b]
Food and kindred products	0.24 (.20)	.018 (.003)	.995	2.3	None
Tobacco manufactures	1.18 (.46)	.008 (.01)	.99	−4.4[a]	Capital using
Textile mill products	1.10 (.44)	.003 (.005)	.995	−3.0[a]	Capital using
Apparel and related products	1.08 (.16)	.003 (.001)	.99	−3.8[a]	Capital using
Lumber and timber basic	0.91 (.07)	*	.94	None	None
Furniture and fixtures	1.12 (.05)	*	.98	None	Capital saving
Paper and allied products	1.02 (.06)	*	.96	None	Mixed
Printing and publishing	1.15 (.31)	.001 (.004)	.99	−0.7[a]	Capital saving
Chemicals and allied products	1.25 (.07)	*	.97	None	Mixed
Petroleum and coal	1.30 (.15)	*	.87	None	Mixed
Rubber and plastic	0.76 (.56)	.005 (.007)	.96	2.1	Capital using
Leather and leather products	0.87 (.14)	.005 (.002)	.99	0.4	Capital using
Stone, clay, and glass	0.67 (.47)	.007 (.005)	.97	2.1	Capital using
Primary metals	1.20 (.11)	*	.94	None	Mixed
Fabricated metal products	0.93 (.26)	.002 (.005)	.98	2.9	Mixed
Machinery except electrical	1.04 (.04)	*	.98	None	Mixed
Electrical machinery	0.64 (.36)	.007 (.007)	.99	1.9	Capital saving
Transportation equipment	0.24 (.56)	.018 (.013)	.97	2.4	None
Instruments	0.76 (.29)	.011 (.006)	.99	4.6	Capital using

(continued)

TABLE 7 *(continued)*

	McKinnon (1962), 1947-58				
Industry	Long-run Elasticity of Substitution	Coefficient of Adjustment	Estimated Coefficient of Trend	$\bar{R}^2$	Rate of Neutral Technological Progress[c] (per cent)
Food and kindred products	0.373	0.581 (.326)	0.379 (.378)	.977	2.4
Tobacco manufactures	0.921	0.655 (.297)	−0.240 (.556)	.902	−10.7
Textile mill products	0.162	0.691 (.323)	0.860 (.568)	.988	3.4
Apparel and related products	0.694	0.875 (.215)	0.024 (.203)	.927	0.2
Lumber and timber basic	0.802	0.764 (.322)	−0.016 (.909)	.962	−0.3
Furniture and fixtures	1.021	0.704 (.178)	−0.158 (.313)	.959	24.2
Paper and allied products	0.094	0.822 (.577)	0.907 (1.161)	.911	2.8
Printing and publishing	0.844	0.756 (.284)	−0.051 (.294)	.922	−1.0
Chemicals and allied products	−1.109	0.556 (.310)	2.602 (.886)	.948	4.0
Petroleum and coal	n.o.	n.o.	n.o.	n.o.	n.o.
Rubber and plastic	0.354	0.628 (.091)	0.422 (.099)	.994	2.4
Leather and leather products	0.251	0.669 (.294)	0.470 (.351)	.959	2.2
Stone, clay, and glass	−1.124	0.377 (.349)	0.798 (.382)	.943	2.3
Primary metals	0.033	1.233 (.503)	0.838 (.665)	.526	1.6
Fabricated metal products	0.328	0.704 (.173)	0.104 (.092)	.904	0.5
Machinery except electrical	0.754	0.509 (.245)	−0.103 (.260)	.764	−1.9
Electrical machinery	0.432	0.940 (.398)	0.627 (.741)	.924	2.7
Transportation equipment	0.182	0.863 (.360)	0.954 (.292)	.800	3.1
Instruments	0.379	1.371 (.158)	1.439 (.283)	.989	3.9

(continued)

TABLE 7 *(continued)*

	McKinnon (1963a), 1899-1957[d]		Kendrick (1964), 1953-57	Maddala (1965)[e]	
Industry	Elasticity of Substitution	Rate of Neutral Techno-logical-Progress (per cent)	Elasticity of Substitution	Elasticity of Substitution Estimated Using Stigler's Data	Elasticity of Substitution Estimated Using Residual Share of Capital
Food and kindred products	n.o.	n.o.	0.25	.033- 0.142	.088- 0.423
Tobacco manufactures	n.o.	n.o.	0.88	.089- 0.463	-.142- -0.525
Textile mill products	0.44 (.10)	2.6	0.59	.058- 0.099	.138- 0.216
Apparel and related products	1.44 (.51)	3.5	0.09	-.045- -0.134	-.024- -1.030
Lumber and timber basic	0.56 (.23)	2.0	0.40	.171- 0.262	.251- 0.309
Furniture and fixtures	0.91 (.18)	2.1	1.86	.109- 0.206	.184- 0.442
Paper and allied products	0.94 (.17)	4.4	0.55	.170- 0.225	.260- 0.389
Printing and publishing	0.94 (.51)	0.0	0.18	-.037- -0.102	-.079- -0.400
Chemicals and allied products	1.12 (.24)	2.7	0.65	.101- 0.221	.106- 1.139
Petroleum and coal	n.o.	n.o.	0.51	.273- 0.374	.359- 0.486
Rubber and plastic	n.o.	n.o.	0.35	.186- 0.339	.041- 0.224
Leather and leather products	0.52 (.11)	1.4	0.47	-.010- -1.318	-.052- -0.307
Stone, clay, and glass	1.08 (.30)	-4.0	0.89	.266- 0.400	.539- 1.418
Primary metals	n.o.	n.o.	0.81	.215- 0.266	.327- 0.463
Fabricated metal products	n.o.	n.o.	0.78	.038- 0.405	.062- 0.713
Machinery except electrical	n.o.	n.o.	0.50	.147- 0.247	.334- 0.671
Electrical machinery	0.64 (.19)	2.8	0.80	.108- 0.224	-.026- -4.305
Transportation equipment	n.o.	n.o.	0.65	.052- 0.460	-.008- -2.270
Instruments	n.o.	n.o.	-0.14	.416- 0.583	.577- 1.048

(continued)

TABLE 7 *(concluded)*

	Lucas (1963)[f]		
Industry	Elasticity of Substitution	Estimated Coefficient of Trend	R^2
Food and kindred products	.397 (.056)	.010 (.001)	.934
Tobacco manufactures	.152 (.050)	.031 (.003)	.956
Textile mill products Apparel and related products	.131 (.063)	.017 (.001)	.957
Lumber and timber basic Furniture and fixtures	.480 (.068)	.009 (.001)	.800
Paper and allied products	.505 (.098)	.008 (.001)	.793
Printing and publishing	.488 (.069)	.008 (.001)	.921
Chemicals and allied products	.678 (.089)	.012 (.003)	.975
Petroleum and coal	.375 (.068)	.011 (.002)	.852
Rubber and plastic	.323 (.062)	.018 (.002)	.927
Leather and leather products	.407 (.095)	.007 (.001)	.798
Stone, clay, and glass	-.205 (.107)	.029 (.002)	.956
Primary metals Fabricated metal products	.641 (.193)	.008 (.002)	.600
Machinery except electrical Electrical machinery	.476 (.152)	.013 (.003)	.780
Transportation equipment	Automobiles only .730 (.094)	.018 (.004)	.796
Instruments	n.o.	n.o.	n.o.

Notes to Table 7

Source: McKinnon (1962) uses annual data 1947-58. Labor share wL/V from Schultze and Tryon, *Prices and Cost in Manufacturing Industries*, Study Paper No. 17, in *The Study of Employment, Growth and Price Levels*, Joint Economic Committee, U.S. Congress, Washington: 1960. L and V are from Levinson: *Postwar Movements of Prices and Wages in Manufacturing Industries*, Study Paper No. 21, in *Study of Employment, Growth and Price Levels*,

Mckinnon (1963a) uses John W. Kendrick: *Productivity Trends in the United States*, Princeton for NBER, 1961. Labor's share times output/$mh = w$.

Ferguson (1965) uses *Census of Manufactures*, 1954 and 1958, and *Annual Survey* volumes for other years. V = value added, current dollars; L = number of employees; $wL/L = W$ = compensation of employees per employee. Capital data used to compute bias in technical change are from Daniel Creamer, "Capital Expansion and Capacity in Postwar Manufacturing" and "Recent Changes in Manufacturing Capacity," in *Studies in Business Economics*, National Industrial Conference Board, New York, 1962.

Maddala (1965) uses time series data 1947-58. V = Federal Reserve Board index of industrial production for the industry in question. L = man-hours worked = persons engaged times average hours worked, Department of Commerce figures. K = Total capital by industry from George J. Stigler, *Capital and Rates of Return in Manufacturing Industries*, Princeton for NBER, 1963, adjusted for capacity utilization by average hours worked. Labor's share is obtained by multiplying employee compensation by ratio of persons engaged in production to the total of full-time equivalent employees and dividing by V. Average wage may then be obtained by dividing by L. Rate of return on capital either from Stigler, *op. cit.*, or by dividing capital's share obtained as a residual from V by K.

Lucas (1963) uses time series data 1931-58. Physical output is given as an index, component of Federal Reserve Board index of industrial production. Labor input was derived from number of full-time equivalent employees (Department of of Commerce) times average hours worked per week, (Kendrick, *op. cit.*). Output price was obtained by dividing gross value added (Department of Commerce) by physical output series. Wage rate was obtained by dividing employee compensation (Department of Commerce) by the labor input series.

n.o. = not obtained.

*Not significantly different from zero and negative. Trend therefore dropped.

[a]These estimates differ from Ferguson's by a factor of $1/(1-\sigma)$ where σ is the elasticity of substitution. Since σ is greater than 1 in some cases an estimate of negative technological progress results. This point seems to have been overlooked by Ferguson.

[b]Value of distribution parameter in CES function is computed directly from the marginal rate of substitution = factor-price-ratio relationship using the estimated value of the elasticity of substitution. An increasing weight attached to capital is equated with "capital-using" innovations; a decreasing weight with "capital-saving" innovations. Many series show a mixed behavior.

[c]Computed by dividing estimated coefficient of trend by $\gamma(1-\sigma)$, where γ is the estimated coefficient of adjustment and σ is the estimated elasticity of substitution. Estimates are scaled by a factor of 2.3026/100 to make results comparable to those using natural logarithms.

[d]Years covered: 1899, 1909, 1919, 1929, 1937, 1948, 1953, 1957.

Notes to Table 7 (concluded)

[e]First estimate of each pair based on regression of log K/L on log w/r, where r is the rate of return on capital either as estimated by George Stigler (*Capital and Rates of Return in Manufacturing Industries*, Princeton for NBER, 1963) or as a residual by deducting labor's share from the total value added. The second estimate of each pair is based on the regression of log w/r on log K/L. It can be shown that if both variables are subject to error (or both endogenous in some larger system) the two estimates bracket the consistent estimate [Maddala (1965, p. 8, Table 1)].

[f]Based on a logarithmic regression of output per unit of labor on the deflated wage rate and (linear) trend, 1931-58 [Lucas (1963, p. 63, Table 5.1, cols. 1-3)].

and insignificantly different from 1 in all but one of the remaining ten cases. McKinnon (1962), using data for a slightly different period and of a slightly different sort, provides a more varied assortment. His long-run elasticities of substitution range from 0.033 to 1.021 not counting the negative values obtained for chemicals and allied products and for stone, clay, and glass. It is extremely unlikely that the slight variation in period could account for the substantial differences between McKinnon's and Ferguson's results; furthermore, McKinnon's allowance for a distributed lag tends to increase the long-run elasticities which he measures rather than reducing them. However, there is an extremely significant difference between McKinnon's regressions and Ferguson's: McKinnon uses deflated data, Ferguson, current-dollar values. As we saw in the previous section, the use of dollar values when capital costs vary relatively little and real wages are not highly negatively correlated with prices tends to bias the estimated elasticities of substitution upward. Thus the differences between the two sets of results may be attributed to Ferguson's failure to deflate.

Maddala (1965) presents direct estimates of the elasticity of substitution based upon two different logarithmic regressions using two types of information on the rate of return to capital: the first of the regressions is that of log K/L on log w/r, whereas the second reverses the roles of the two variables. It can be argued that the two estimates of the elasticity of substitution tend to bracket the true value as the sample size increases. In most instances, however, the bracketing values are rather far apart. The estimates differ quite considerably in most cases from McKinnon's results (1962) for roughly the same period and illustrates the great sensitivity of the estimated elasticity of substitution to the form of the relationship fitted.

Lucas (1963) uses the ACMS method but applies it to data over the long period 1931–58. In most instances the resulting estimates of the elasticity of substitution obtained are higher than the previously cited results of McKinnon and Maddala, if only the "short-run" elasticities of McKinnon are considered.

The estimates in both McKinnon (1963a) and Kendrick (1964) differ from the ones discussed above in being based on data for widely separated points in time: Kendrick's results are based on only two points, 1953 and 1957; McKinnon's on eight points, 1899, 1909, 1919, 1929, 1937, 1948, 1953, and 1957. McKinnon's estimates are obtained from a logarithmic regression of real values added per unit of labor on the real wage rate, while Kendrick simply computes an arc elasticity of substitution by comparing the capital-labor ratio with the relative price ratio in the two years 1953 and 1957. Being based on only two years, Kendrick's estimates are subject to a great deal of uncertainty; furthermore, both 1953 and 1957 were recession years, a downturn occurring about the middle of the year in each case.[13] On the other hand, by comparing the capital-labor ratio with the movement of relative prices directly, Kendrick achieves estimates which are free of the assumption that the elasticity of substitution is constant. Of the years chosen by McKinnon, only the last two were periods of relatively low economic activity. Thus, in comparison with the other results reported in Table 7, those of McKinnon (1963a) are relatively less dominated by recession phenomena, Kendrick's less dominated by special assumptions as to the form of the production function. It is perhaps not surprising that there seems to be relatively little consistency between these results and the others. Maddala's estimates, like those of Lucas and Ferguson, are heavily dominated by recession phenomena.

In this connection, however, it is worthwhile giving some indication of the probable results of little or no adjustment for the effects of variations in aggregate demand. In an earlier paper which was to have been part of this survey, the probable effects of variations in the level of aggregate activity upon estimates of the elasticities of output with respect to various factor inputs were discussed.[14] The effects, however,

[13] Gordon (1961, pp. 486–89 and 492–501).
[14] Nerlove (1965).

on the estimates of the elasticity of substitution between capital and labor as estimated by ACMS or related methods is much more subtle and difficult to specify a priori. Ferguson (1965b, p. 142), for example, argues that

> . . . the use of time-series data to estimate the elasticity of substitution imparts a downward bias that is basically attributable to changes in the quality of labor service, especially during periods of expansion and contraction. . . . In recession periods, an increase in unemployment is normally accompanied by an increase in the quality of labor services because the more efficient workers (at each wage rate) are the ones retained. . . . Thus value added per man-year tends to increase in recession periods. . . . The opposite tends to occur in periods of expansion; so on balance the observed slope is less than the true slope.

In order for Ferguson's argument to be valid, it is necessary that real or money wages, depending on which is used in the regression, be negatively correlated with the errors attributable to changing quality of labor. Thus, wages would have to fall in recessions and rise in expansions. There is some evidence that *real* wages (as deviations from trend) did just this; however, Ferguson's argument suggests that under these circumstances his estimate should be lower in contrast to the cross-section results and to those of McKinnon (1963a), which are based largely on full-capacity years. They are higher.

An additional effect of cyclic phenomena on the estimated elasticity of substitution is the "vintage capital" effect. In a downswing the older, less efficient plants are shut down, and in the upswing they are reopened. Hence, product per worker tends to increase in recessions and fall during recoveries. A similar effect due to the quasi-fixity of labor inputs has been noted by Oi, Okun, and others.[15] Labor is a quasi-fixed factor and not freely variable over the course of the relatively short and mild recessions experienced in the postwar period. On the other hand, output does vary over the course of the cycle so that productivity tends to rise in booms and fall in slumps, i.e., output per unit of labor varies directly with the level of aggregate activity. If money wages are rather rigid in the short run, real wages will tend to rise in recessions and fall in recoveries. The net effect would be to bias downward the estimated short-run elasticity in regressions based upon real value added and real wages. On the other hand, evidence to the contrary for the postwar period

[15] Oi (1962); Okun (1962).

would suggest an upward bias. Apparently, therefore, such explanations cannot account for all the differences observed.

Unfortunately, variations in the price level produce even more complicated effects in regressions based upon current-dollar data. In general, as we saw, price variation tends to bias the estimates upward in comparison with the supposedly true values which occur in the real relationship. If, however, errors of the type described in the preceding paragraph also occur, it is difficult to judge the net outcome.

Table 8 summarizes recent studies of the aggregate elasticity of substitution and rate of technical change. While most of the studies are restricted to the measurement of neutral technological change and constant returns to scale, a number of investigations, notably Brown–de Cani (1963) and David–van de Klundert (1965) attempt estimates of technical change which may be either capital- or labor-augmenting. Ferguson (1965a) allows both nonconstant returns to scale and the possibility of labor-augmenting technical change. For comparison, the early results obtained by Solow (1957) are given at the bottom of the table.

Recently, Diamond and McFadden (1965) have questioned the possibility of identifying both the production function and arbitary forms of technical change. Before discussing the results presented in Table 8, therefore, it seems well to examine the Diamond-McFadden "impossibility theorem" in some detail so as to avoid spending time trying to make sense out of results which may in fact be purely arbitrary. The result demonstrated by Diamond and McFadden (p. 1) is that

> . . . it is, in fact, impossible to measure either the bias or the elasticity [of substitution]; i.e., given the time series of all observable market phenomena for a single economy which has a neo-classical production function, these same time series could have been generated by an alternative function having an arbitrary elasticity or arbitrary bias at the observed points. This statement is subject to the limitations that in the absence of technical change one can measure the elasticity of substitution (and trivially the bias, which is zero) while in the absence of a change in the capital-labor ratio one can determine the bias.

The Diamond-McFadden result is also, as we shall show, subject to the qualification that certain "smoothness" assumptions about the nature of technical change may also produce identification. In what follows, we give a simplified and more transparent derivation of the Diamond-

TABLE 8

Summary of Time Series Results on the U.S. Aggregate Production Function: Elasticity of Substitution and Technical Change

Reference and Period	Estimated Elasticity of Substitution	Assumption as to Nature of Technical Change	Estimated Increase in Efficiency (per cent per annum): Neutral	Labor Augmenting	Capital Augmenting
Kravis (1959), 1900-57	0.64	Restricted: Hicks neutral	Not estimated		
Arrow *et al.* (1961), 1909-49	0.57	Restricted: Hicks neutral	1.83		
Diwan (1963) by ACMS method 1919-58	0.37	Restricted: Hicks neutral	1.4		
From regression of log factor ratio on log factor price ratio, 1919-30 and 1935-58.	.068	Restricted: Hicks neutral	Not estimated		
Kendrick-Sato (1963), 1919-60.	0.58	Restricted: Hicks neutral	2.10		
Brown-de Cani (1963)					
1890-1918	0.35SR 0.55LR	Assumed zero within periods; unrestricted between periods.	Numerical estimates not given. Predominantly labor saving.		
1919-1937	0.08SR 0.31LR		Predominantly capital saving. (This does not agree with the finding of David-van de Klundert.)		
1938-58	0.11SR 0.47LR				
Kendrick (1964), 1953-57	0.62	Assumed zero			
Ferguson (1965a), Assuming constant returns to scale 1929-63	0.67	Restricted: Hicks neutral	1.5		
		Restricted: Harrod neutral		1.5	
1948-63	1.16	Restricted: Hicks neutral	1.9		
		Restricted: Harrod neutral		1.9	

(continued)

TABLE 8 *(concluded)*

Reference and Period	Estimated Elasticity of Substitution	Assumption as to Nature of Technical Change	Estimated Increase in Efficiency (per cent per annum) Neutral	Labor Augmenting	Capital Augmenting
Allowing non-constant returns to scale, 1929-63[a]	0.49	Restricted: Hicks neutral	-0.1		
		Restricted: Harrod neutral		-0.1	
1948-63[b]	0.64	Restricted: Hicks neutral	0.1		
		Restricted: Harrod neutral		0.1	
David-van de Klundert (1965), 1899-1960					
Regression with distributed lag	0.11SR 0.32LR	Unrestricted: Factor augmenting		2.23-2.30[c]	1.51-1.58[c]
Regression without distributed lag	0.16	Unrestricted: Factor augmenting		2.30-2.34[c]	1.44-1.48[c]
For comparison: Solow (1957), 1909-49	Assumed 1.0	Unrestricted[d]	1.5		

SR = short run.

LR = long run.

[a]Estimated return to scale equals 2.53.

[b]Estimated return to scale equals 1.45.

[c]Different estimates use different values of labor's share.

[d]With a Cobb-Douglas function and constant returns to scale nonneutral technological change cannot be distinguished.

McFadden result for CES production functions which is due to K. J. Arrow.

Technological change may be considered to be of the capital-augmenting or the labor-augmenting type if it is equivalent to a change in the units in which capital and labor are measured; thus, instead of the arguments K and L in the function F, we write $E_K K$ and $E_L L$. We assume that factors are paid their marginal products and that the shares of capital and labor S_K and S_L respectively exhaust the total product Y. We observe the following variables:

$$\begin{aligned} (1) \qquad Y &= \text{total output.} \\ K &= \text{measured capital input.} \\ L &= \text{measured labor input.} \\ S_K &= \text{total payment to capital.} \\ S_L &= \text{total payment to labor.} \end{aligned}$$

The importance of the constant returns assumption is that, with it, we can identify the factor payments as the marginal factor products times the "true," not the measured, factor inputs. Without the assumption, still assuming competition, some factors would be earning rents (which might be negative) over and above their marginal productivities. Two neoclassical production functions, F and G, both homogeneous of degree 1, will be said to be "consistent with the data" if

$$\begin{aligned} (2) \qquad Y &= F(E_K{}^F \cdot K, E_L{}^F \cdot L, t) = G(E_K{}^G \cdot K, E_L{}^G \cdot L, t) \\ S_K &= F_1 E_K{}^F K = G_1 E_K{}^G K \\ S_L &= F_2 E_L{}^F L = G_2 E_L{}^G L, \end{aligned}$$

where F_i and G_i, $i = 1, 2$, are the marginal productivities. Note that for CES production functions, equality of the partial derivatives of F and G with respect to each of their arguments implies that all the parameters of the functions are identical. What the Diamond-McFadden result amounts to is that with no further restrictions on the "errors" $E_i{}^F$, $E_i{}^G$, $i = K, L$, there exists more than one neoclassical production function consistent with any given set of observations (1) provided: (a) the capital-labor ratio does vary over time, and (b) the elasticity of substitution is not in fact equal to unity, so that factor shares remain constant over time.

Differentiating the first of equations (2) with respect to t we obtain

$$\begin{aligned} (3) \qquad \dot{Y} &= F_1\{\dot{E}_K{}^F K + E_K{}^F \dot{K}\} + F_2\{\dot{E}_L{}^F L + E_L{}^F \dot{L}\} + F_3 \\ &= G_1\{\dot{E}_K{}^G K + E_K{}^G \dot{K}\} + G_2\{\dot{E}_L{}^G L + E_L{}^G \dot{L}\} + G_3 \end{aligned}$$

where F_3 and G_3 are the derivatives of F and G with respect to the third argument, t. Let

$$(4)\qquad \begin{cases} e_i^F = \dfrac{\dot{E}_i^F}{E_i^F}, \\ e_i^G = \dfrac{\dot{E}_i^G}{E_i^G}, \end{cases} \qquad i = K,\ L,$$

be the rates of capital- and labor-augmenting technical progress, and let

$$(5)\qquad \begin{cases} \mu^F = F_3/Y \\ \mu^G = G_3/Y \end{cases}$$

be the rates of non-factor-augmenting technical progress. Making use of the second and third of equations (2) and (4) and (5) we may rewrite (3) as

$$(6)\qquad \begin{aligned} y = \dot{Y}/Y &= s_K\{e_K^F + k\} + s_L\{e_L^F + l\} + \mu^F \\ &= s_K\{e_K^G + k\} + s_L\{e_L^G + l\} + \mu^G, \end{aligned}$$

where k and l are the rates of growth of the *measured* capital stock and labor force. The functions F and G are "consistent with the data" so the factor shares s_K and s_L are the same in both equations as are k and l. Thus

$$(7)\qquad s_K\{e_K^F - e_K^G\} + s_L\{e_L^F - e_L^G\} + \mu^F - \mu^G = 0,$$

is the fundamental relation connecting the rates of technical change.

First note that if we assume no technical change is factor-augmenting, so that

$$\begin{cases} E_i^F \equiv E_i^G \equiv 1 \\ e_i^F \equiv e_i^G \equiv 0, \end{cases} \qquad i = K, L,$$

then (7) implies

$$\mu^F \equiv \mu^G.$$

Referring back to (2) we see this implies all the corresponding partials of F and G are identical, and the data therefore "identifies" the production function.

On the other hand, suppose we assume that the only technical change is factor-augmenting, so that $\mu^F \equiv \mu^G \equiv 0$. Then (7) becomes

$$(7')\qquad s_K\{e_K^F - e_K^G\} + s_L\{e_L^F - e_L^G\} = 0.$$

If s_K and s_L are not constant, then we can always find numbers e_i^F and e_i^G, $i = K, L$, such that (7′) holds *unless we also require that the differences $e_i^F - e_i^G$, $i = K, L$, be constant as well.* This is practically tantamount to requiring that technological change be describable by exponentially smooth growth in E_K and E_L, but not quite. What it does do is to impose a smoothness condition on technological change which may be best understood in terms of smooth exponential growth in the effectiveness of measured capital and labor inputs. If this condition is imposed, (7′) implies

$$e_i^F = e_i^G, \qquad i = K, L,$$

which in turn implies, as e_i^F and e_i^G are also assumed to be constant, that

$$E_i^F = E_i^G.$$

It follows at once from (2), that the corresponding partials of F and G must be identical. If the "smoothness" of E_K and E_L, or some other restriction, is not assumed, then by suitable choice of these numbers we can make some other CES production function consistent with the observed variables; in particular, given any series E_K^F and E_L^F we can find E_K^G and E_L^G such that G, with some other elasticity of substitution, also explains the observable data.

Absence of non-factor-augmenting technical change or the assumption of exponential factor-augmenting change is sufficient for identification. Suppose, however, we drop the assumption that there is no non-factor-augmenting technical change. We are then back to the general formula (7); only now nonconstancy of the shares no longer implies the equalities $e_i^F = e_i^G$, $i = K, L$, even if e_i^F and e_i^G are assumed to be constant. However, if μ^F and μ^G are also assumed to be constant we have

$$s_K\{e_K^F - e_K^G\} + (1 - s_K)\{e_L^F - e_L^G\} + \mu^F - \mu^G = 0$$

or

$$(e_L^F + \mu^F) - (e_L^G + \mu^G) + s_K\{(e_K^F - e_L^F) - (e_K^G - e_L^G)\} = 0,$$

whence

$$\begin{cases} e_L^F + \mu^F = e_L^G + \mu^G \\ e_K^F - e_L^F = e_K^G - e_L^G \end{cases}$$

Thus the *bias* of technical change is determined uniquely, but neither the technical change itself nor the production function is. It follows that the assumption that all technical change is factor augmenting is quite essential to identification unless we assume it is absent altogether.

Finally, we must show that variation in the factor shares and in the capital-labor ratio is also essential for identification. First, observe that constancy of s_K and s_L allows (7′) to be satisfied for an infinite number of pairs $(e_K{}^F - e_K{}^G,\ e_L{}^F - e_L{}^G)$ the ratio of which is $-s_L/s_K$. Thus, given this restriction we cannot identify the production function. But this should be intuitively clear in any case, since if the factor shares are constant we could not distinguish any production function from the Cobb-Douglas. It is well known that factor-augmenting and neutral technological change are quite equivalent for this function, and the bias is therefore indeterminate.

Next, suppose that the capital-labor ratio does not change over time. This implies

$$L\dot{K} - K\dot{L} = 0$$

or

$$k = l,$$

so that for either F or G we have

$$y = s_K e_K + s_L e_L + \mu.$$

It is not possible to assume e_K, e_L, and μ constant if the observed shares and rate of growth in output do not behave in exactly the right fashion. A fortiori, the production function cannot be identified.

In his paper on two-digit manufacturing industries, Ferguson (1965b) first estimates the elasticity of substitution and the rate of neutral technological change; he then uses his results to compute a nonsmooth estimate of the bias in factor-augmenting technical change (the only other kind permitted in CES functions). This is clearly nonsense on the basis of the Diamond-McFadden impossibility theorem. David and van de Klundert (1965) attempt the same fallacious computation in the second half of their paper. The results reported in Table 8, however, are from the first half of this paper and based on the assumption that all technical change is factor-augmenting and exponential. These assumptions, as we saw, are sufficient to identify the production function and technical change. Ferguson (1965a) assumes either neutral or labor-augmenting exponential technical change; these assumptions are sufficient to identify the production function if constant returns to scale are assumed. If constant returns are not assumed, but the production function is supposed to be homogeneous of degree greater than 1, paying factors their marginal products will more than exhaust the total product, and it is therefore difficult to interpret the factor shares. Under the restrictions imposed by Ferguson, however, identification is achieved.

Brown and de Cani (1963) attempt to estimate all types of technical change (factor-augmenting and nonneutral, non-factor-augmenting change) by estimating different production functions for different "technological epochs." It is only by employing the extreme assumption that there is no technological change of any sort within technological epochs that they can identify all of these factors. In a sense, this is the opposite of our smoothness assumption which "lets us out" of the implications of the impossibility theorem.[16] Brown and de Cani, in effect, assume that everything changes abruptly the instant one passes over from one technological epoch to another. It is moot just how reasonable an assumption this is.

If technical change is not neutral but biased, factor-augmenting, and exponential, then the following results are obtained:

The ACMS estimating equation becomes

$$\log y = -\log E_L(0) + \sigma \log w + e_L(1 - \sigma)t \tag{8}$$

where $E_L(0)$ is the initial value of labor efficiency. Alternatively, the method used by Kravis (1959), Diwan (1963), and Kendrick-Sato (1963), relies on determination of the relationship between the capital-labor ratio and the factor-price ratio. With biased technical change of a factor-augmenting sort, the appropriate estimating equation becomes

$$\log x = (1 - \sigma) \log \frac{E_L(0)}{E_K(0)} + \sigma \log \frac{w}{r} + (1 - \sigma)[e_L - e_K]t \tag{9}$$

where w is the wage rate and r the rate of return on capital. Thus, if the true elasticity of substitution is less than 1, if technical change is biased toward the labor-augmenting type, and if the relative price of labor is rising over time, the estimates of σ obtained by means of (9) will be biased upward if only neutral technical change is assumed ($e_L = e_K$). On the other hand, no bias results from the ACMS method as long as neutral technical change occurs. It follows that the high values of σ

[16] Lest any misinterpretation result, recall that the argument above referred only to *sufficient* conditions for identification.

TABLE 9

Time Series Results on the Elasticity of Substitution in Three Extractive Industries, McKinnon (1963a), Census Years 1870-1958

Industry	Estimated Elasticity of Substitution	Estimated Coefficient of Trend	Rate of Neutral Technical Change	$\bar{R}^2$
Bituminous coal	0.92	.17	4.8%	.93
	(.34)	(.22)		
Anthracite coal	1.23	−.03	0.3	.79
	(.63)	(.25)		
Iron ore	1.06	.29	−12.1	.96
	(.22)	(.23)		

obtained by Kravis, Kendrick and Sato, and by Diwan (his second method), in relation to the low values obtained by Diwan (his first method) and David and van de Klundert, are explicable. The high value obtained by Arrow *et al.*, however, cannot be explained on these grounds. Indeed, the only difference between the result reported by Arrow *et al.* and Diwan (his first method) appears to be in the choice of period. Why a shift of nine or ten years in the period of estimation should produce such a large effect is difficult to surmise.

In closing this section, the time-series studies of McKinnon (1963a) and Maddala (1963) of certain extractive industries are presented without comment in Tables 9 and 10.

Table 11 summarizes the time-series and cross-section results for U.S. two-digit manufacturing industries which have been considered in the above review.

Estimation and Identification Problems

In this section we discuss various simultaneous-equations difficulties which may arise in the estimation of CES production functions along the lines explored by Kmenta (1964) and Maddala and Kadane (1965).

TABLE 10

Estimates of the Elasticity of Substitution in Bituminous Coal Mining Based on Time Series of Cross Sections, Maddala (1963), Census Years, 1919-54

Year	Using Data on Physical Output	Using Data on Deflated Value Added
1919	1.092	1.124
	(.053)	(.051)
1929	1.044	1.042
	(.076)	(.070)
1935	1.141	1.103
	(.083)	(.068)
1939	1.145	1.159
	(.080)	(.068)
1954	1.343	1.326
	(.191)	(.169)
Pooled:		
With year shifts	1.118	1.120
	(.044)	(.038)
Without year shifts	1.213	1.205
	(.046)	(.039)

The form of the CES function usually given is

(1) $$V = \gamma[\delta K^{-\rho} + (1 - \delta)L^{-\rho}]^{-1/\rho}.$$

A slightly more general version allows nonconstant returns to scale while still restricting the function to homogeneity of some degree:[17]

(1′) $$V = \gamma[\delta K^{-\rho} + (1 - \delta)L^{-\rho}]^{-\mu/\rho}.$$

Suppose that V has been defined in real terms. As before, let w and r be the wage rate and rate of return on capital, respectively, and let p be the price of final output. We also introduce the multiplicative resid-

[17] See Kmenta (1964, p. 2), and Brown (1962, p. 10).

TABLE 11

Summary of Time Series and Cross-Section Estimates of the Elasticity of Substitution Between Capital and Labor for Two-Digit Manufacturing Industries

	Time Series Estimates					
Industry	McKinnon (1962)	McKinnon (1963a)	Kendrick (1964)	Ferguson (1965b)	Maddala (1965)	Lucas (1963)
Food and kindred products	0.37	n.o.	0.25	0.24	.03- 0.14	.40
Tobacco manufactures	0.92	n.o.	0.88	1.18	.09- 0.46	.15
Textile mill products	0.16	0.44	0.59	1.10	.06- 0.10	.13
Apparel, etc.	0.69	1.44	0.09	1.08	-.05--0.13	
Lumber and timber	0.80	0.56	0.40	0.91	.17- 0.26	.48
Furniture and fixtures	1.02	0.91	1.86	1.12	.11- 0.21	
Paper, etc.	0.09	0.94	0.55	1.02	.17- 0.23	.51
Printing and publishing	0.84	0.94	0.18	1.15	-.04--0.10	.49
Chemicals, etc.	-1.11	1.12	0.65	1.25	.10- 0.22	.68
Petroleum and coal	n.o.	n.o.	0.51	1.30	.27- 0.37	.38
Rubber and plastics	0.35	n.o.	0.35	0.76	.19- 0.34	.32
Leather, etc.	0.25	0.52	0.47	0.87	-.01--1.32	.41
Stone, clay, glass	-1.12	1.08	0.89	0.67	.27- 0.40	-.21
Primary metals	0.03	n.o.	0.81	1.20	.22- 0.27	.64
Fabricated metal products	0.33	n.o.	0.78	0.93	.04- 0.41	
Nonelectric machinery	0.75	n.o.	0.50	1.04	.15- 0.25	.48
Electrical machinery	0.43	0.64	0.80	0.64	.11- 0.22	
Transportation equipment	0.18	n.o.	0.65	0.24	.05- 0.46	.73[b]
Instruments	0.38	n.o.	-0.14	0.76	.42- 0.58	n.o.

n.o. = not obtained.

[a]Based on a comparison of the United States and Japan only.

[b]Automobiles only.

TABLE 11

	Cross-Section Estimates							
				Liu-Hildebrand (1965)		Murata-Arrow (1965), Intercountry Data		
Industry	Arrow *et al.* (1961)[a]	Minasian (1961)	Solow (1964)	All Employees	Production Workers	1953-56	1957-59	Dhrymes (1965)
Food and kindred products	0.93	0.58	0.69	2.15	1.29	.72	.73	.56-0.97
Tobacco manufactures	n.o.	3.46	1.96	n.o.	n.o.			n.o.
Textile mill products	0.80	1.58	1.27	1.65	2.08	.79	.83	.68-1.03
Apparel, etc.	n.o.	n.o.	1.01	1.43	2.38	.66	.80	.54-1.03
Lumber and timber	0.84	0.94	0.99	0.99	0.91	.82	.92	.78-1.1
Furniture and fixtures	n.o.	1.09	1.12	0.92	0.96			.70-1.39
Paper, etc.	1.14	1.60	1.77	1.06	0.71	.90	.79	.20-0.64
Printing and publishing	1.21	n.o.	1.02	n.o.	n.o.	.84	.93	.68-1.11
Chemicals, etc.	0.90	n.o.	0.14	1.25	.88	.84	.83	.31-1.03
Petroleum and coal	n.o.	−0.54	1.45	n.o.	n.o.			.11-1.31
Rubber and plastics	0.98	0.82	1.48	1.45	139	.83	.77	.40-1.04
Leather, etc.	0.72	0.96	0.89	0.79	0.93	.71	.70	.51-1.13
Stone, clay, glass	1.08	0.59	0.32	1.28	1.44	.85	.86	.49-0.89
Primary metals	n.o.	0.92	1.87	0.99	1.00	.86	.87	.10-0.97
Fabricated metal products	n.o.	n.o.	0.80	0.70	0.45	.92	.92	.40-0.95
Nonelectric machinery	0.97	0.31	0.64	0.60	0.41	n.o.	n.o.	.12-0.25
Electrical machinery		1.26	0.37	0.78	1.10	n.o.	n.o.	.19-0.62
Transportation equipment	1.04	2.04	0.06	2.01	1.91	n.o.	n.o.	n.o.
Instruments	n.o.	n.o.	1.59	1.24	1.65	n.o.	n.o.	n.o.

ual u_0 into (1′). This is equivalent to using γu_0 instead of γ in (1′), and for the sake of simplicity, we do not introduce the residual explicitly until the ends of various derivations. The system consists of six variables: V, L, K, p, w, and r. If we suppose that entrepreneurs maximize profits

$$pV - wL - rK \tag{2}$$

subject to (1′), and taking p, w, and r as given, we can obtain two more equations connecting the six variables. The marginal conditions are

$$\begin{cases} p = \lambda \\ w = \lambda\mu(1-\delta)L^{-(1+\rho)}V^{1+\rho/\mu}\gamma^{-\rho/\mu}, \\ r = \lambda\mu\delta K^{-(1+\rho)}V^{1+\rho/\mu}\gamma^{-\rho/\mu} \end{cases} \tag{3}$$

where λ is marginal cost holding factor prices constant; i.e.,

$$\lambda = \frac{\partial C}{\partial V} = \frac{\partial\{wL + rK\}}{\partial V} \tag{4}$$

where L and K are the equilibrium values satisfying (1′)–(3). Equations (1′)–(3) may be rewritten in a variety of suggestive forms. Note, too, that imperfections in profit maximization may be treated as equivalent to the multiplication of p, w, and r by factors u_1, u_2, and u_3. Again, such factors will be suppressed in what follows until they are needed.

One suggestive form of (1′)–(3) may be obtained directly from the profit-maximizing conditions and the production function by simple manipulation:

$$\begin{cases} \dfrac{V}{L} = a\left(\dfrac{w}{p}\right)^{1/1+\rho} V^{-\rho(1-\mu)/\mu(1+\rho)} \\ \dfrac{V}{K} = b\left(\dfrac{r}{p}\right)^{1/1+\rho} V^{-\rho(1-\mu)/\mu(1+\rho)} \end{cases} \tag{5}$$

where $a = \mu^{-1/1+\rho}\gamma^{\rho/\mu(1+\rho)}(1-\delta)^{-1/1+\rho}$ and
$b = \mu^{-1/1+\rho}\gamma^{\rho/\mu(1+\rho)}\delta^{-1/1+\rho}$.

When there are constant returns to scale $\mu = 1$ and the first of equations (5) becomes the one estimated by ACMS. Not having capital data or information on the rate of return, they did not attempt to estimate the second equation.

If $\mu = 1$ and if w, r, and p are independent of the residuals in (5), which turn out to both be

$$u_0^{\rho/\mu(1+\rho)}$$

and if there are no imperfections in profit maximization, then it will be appropriate to estimate either equation by least squares and so obtain an estimate of $1/(1+\rho) = \sigma$, the elasticity of substitution.[18] Actually, if data are available, it is even simpler to combine the two equations of (5) by dividing one by the other, to obtain

$$\frac{K}{L} = \left[\frac{\delta}{1-\delta}\right]^{1/1+\rho}\left(\frac{w}{r}\right)^{1/1+\rho}. \tag{6}$$

However, this equation does not involve γ and so must hold exactly unless there are imperfections in profit maximization. These in turn might make it impossible to estimate (6) by ordinary least squares.

Equation (6) suggests an alternative method of estimation: If we are considering a particular industry over time in an economy subject to fluctuations in aggregate demand, and thus in the demand for the product of the industry, the output of the industry might be considered as independent of the random effects which cause equations (3) to hold with an error and, hence, of the residuals in (6). Using log V as an instrumental variable would then allow us to obtain consistent estimates of $\sigma = 1/(1+\rho)$ and of $\delta/(1-\delta)$, and hence of δ and ρ. Inserting these estimates in (1′) we obtain

$$V = \gamma z^{\mu}, \tag{7}$$

where

$$z = [\hat{\delta}K^{-\hat{\rho}} + (1-\hat{\delta})L^{-\hat{\rho}}]^{-1/\hat{\rho}}$$

Unfortunately, unless K and L are independent of the residual in the production function, it does not follow that we are able to obtain consistent estimates of μ and γ from a least-squares regression of log V on log z. Kmenta (1964) suggests that this might be the case. Then, however, direct methods for estimating the production function seem more useful.

Although nonlinear, the production function may still be estimated by

[18] An iterative procedure for estimating a similar set of two related equations based on a production function closely related to the CES has been suggested by Hilhorst (1961); see Brown (1962, p. 19).

least-squares.[19] Alternatively, one may follow the procedure outlined by Kmenta and approximate (1′) by the first- and second-order terms in the Taylor series expansion. Write (1′) as

$$\text{(8)} \qquad \log V = \log \gamma - \frac{\mu}{\rho} f(\rho) + \log u_0$$

where $f(\rho) = \log [\delta K^{-\rho} + (1 - \delta) L^{-\rho}]$. Following Kmenta, who relies on the empirical findings of ACMS, we expand $f(\rho)$ around the value $\rho = 0$ which corresponds to the value $\sigma = 1$, i.e., the Cobb-Douglas case:

$$\text{(9)} \quad f(\rho) = -\rho[\delta \log K + (1 - \delta) \log L] + \frac{1}{2} \rho^2 \delta(1 - \delta)[\log K - \log L]^2 + \text{higher-order terms.}$$

Discarding terms of higher order than the second in ρ, we obtain a logarithmic approximation to (1′) in the form

$$\begin{aligned} \text{(10)} \qquad \log V &= \log \gamma - \frac{\mu}{\rho} \{-\rho[\delta \log K + (1 - \delta) \log L] \\ &\quad + \frac{1}{2} \rho^2 \delta(1 - \delta)[\log K - \log L]^2\} + \nu \\ &= \log \gamma + \mu\delta \log K + \mu(1 - \delta) \log L \\ &\quad - \frac{\mu\rho}{2} \delta(1 - \delta)[\log K - \log L]^2 + \nu, \end{aligned}$$

where $\nu = u_0 - (\mu/\rho) -$ the neglected higher-order terms in the Taylor series expansion of $f(\rho)$. If estimates of the coefficients in (10) are available γ, μ, δ, and ρ may be estimated as functions of these in the obvious way. Asymptotic standard errors of the resulting estimates may be obtained along the lines suggested by Klein (1953, p. 258). In this formulation, the term involving the squared logarithm of the capital-labor ratio indicates the departure from the Cobb-Douglas situation (it drops out when $\rho = 0$). Numerical calculations of Kmenta (1964, p. 7), show that, *provided the second-order term is included,* the error resulting from neglect of the higher-order terms is not serious unless *both* the capital-labor ratio and the elasticity of substitution are either very high or very

[19] See the references to Kenney and Keeping and to Davidon cited by Kmenta (1964).

low.[20] If the second-order term is not included, I would surmise that the errors will be substantial even for moderate departures. Thus, even if capital and labor inputs are not correlated with u_0, they will be correlated with v. Since the coefficient of the left-out term will normally be negative, it follows that ordinary least-squares estimates of the Cobb-Douglas will tend to yield an estimate of the elasticity of output with respect to capital which is too low and an estimate of the elasticity with respect to labor which is too high. This is apart from difficulties resulting from the lack of independence between capital and labor and the residual of the production function.

Suppose we are dealing with a sample of firms. Under certain circumstances, it might be plausible to assume that labor and capital were uncorrelated with the residual in the production. If, for example, the residual were a stochastic element from the standpoint of the entrepreneurial decision makers who were forced to decide upon input levels *in advance* of any knowledge of the residual element, then the assumption would follow from expected profit maximization. Outside of agriculture, however, these circumstances seem rather implausible. In a cross section of firms, it is generally more reasonable to assume that the residuals reflect differences among firms, such as the possession of nonmeasured amounts of other factors, and so are known to the decision makers, who then allow for such differences in optimizing input levels, thus producing a correlation between these and the residuals. In other contexts, for example, an industry observed over time, the residual may represent left-out variables, imperfect specification of the production function, or other factors, some of which are likely to be taken account of in the determination of input levels. Thus, in general, one would not expect the conditions for direct estimation of the production function to obtain.

If factor prices and output price are fixed over all observations, such as would be the case if we were to observe a sample of firms in a perfectly competitive industry with perfect factor mobility, then the production function (1′) and the profit-maximizing conditions as given by (5) determine the output and factor input levels as functions of the produc-

[20] "Very high or very low" means far from 1 in the case of the elasticity of substitution. A very high or very low capital-labor ratio is defined in terms of the value which would make output equal to 1 if the function were Cobb-Douglas (i.e., $\sigma = 1$).

tion function residual u_0 (attached to the parameter γ) and imperfections in profit maximization, say u_1 and u_2 (attached to w/p and r/p, respectively). In logarithmic form, equations (10), the approximate form of (1′), and (5), become

$$
(11)\quad \begin{cases}
x_0 - \mu\delta x_1 - \mu(1-\delta)x_2 + \dfrac{1}{2}\rho\mu\delta(1-\delta)(x_1 - x_2)^2 = k_0 + v_0 \\
\left(1 + \dfrac{\rho}{\mu}\right)x_0 - (1+\rho)x_1 = k_1 + v_1 \\
\left(1 + \dfrac{\rho}{\mu}\right)x_0 - (1+\rho)x_2 = k_2 + v_2
\end{cases}
$$

where
$$
\begin{aligned}
x_0 &= \log V. \\
x_1 &= \log L. \\
x_2 &= \log K. \\
v_i &= \log u_i,\ i = 0, 1, 2. \\
k_0 &= \log \gamma. \\
k_1 &= -\log\left(\frac{w}{p}\right)^{-1}\mu(1-\delta)\gamma^{\rho/\mu}. \\
k_2 &= -\log\left(\frac{r}{p}\right)^{-1}\mu\delta\gamma^{\rho/\mu}.
\end{aligned}
$$

In general, the parameters of (11) are not identified. However, provided one is willing to make sufficiently stringent assumptions regarding the joint distribution of the random variables v_0, v_1, and v_2, one can, as Kmenta (1964) shows, estimate them.

The most statistically convenient assumption that one can make is that the residuals v_0, v_1, and v_2 are independent from observation to observation and that their contemporaneous variance-covariance matrix is diagonal. The econometric content of these assumptions is not clear. If, for example, we are dealing with a cross section of firms, it seems highly unlikely that imperfections in profit maximization as represented by v_1 and v_2 should not be correlated. If we are examining an industry over time, the factors left out of the production function represented by v_0 are likely to be reflected also in the marginal productivity conditions. Furthermore, they are likely to be serially correlated. With no exogenous variables in the system, identification becomes a complex matter, and is only possible by means of rather unrealistic assumptions. When the above assumptions are made, a full-information maximum-likelihood procedure is possible. Kmenta (1964, pp. 24–26), shows that this is equivalent to the following two-stage least-squares procedure:

Define

$$z_1 = \varphi x_0 - x_1$$
$$z_2 = \varphi x_0 - x_2$$
$$z_3 = (x_1 - x_2)^2$$

where $\varphi = (1 + \rho/\mu)(1 + \rho)$.

Since v_1 and v_2 are assumed to be independent, the second and third of equalities (11) imply

$$\text{(12)} \qquad \operatorname{cov}(v_1, v_2) = 0 = \varphi^2 \operatorname{var}(x_0) - \varphi[\operatorname{cov}(x_0, x_1) + \operatorname{cov}(x_0, x_2)] + \operatorname{cov}(x_1, x_2).$$

If the covariances on the right are replaced by their sample values, a consistent estimate of φ may be obtained by solving (12). It can be shown that the roots of (12) are real; the question is which of the two should be used. This may be resolved by reference to the likelihood function: If $\rho > 0$, $0 < \delta < 1$, and $0 < \mu < 1$, it can be shown that the smaller of the two roots will yield the higher value of the likelihood function. Having obtained a consistent estimate $\hat{\varphi}$ of φ, we may form the z_1, z_2, and z_3 defined above and the equation

$$\text{(13)} \qquad x_0 = a_0 + a_1 z_1 + a_2 z_2 + a_3 z_3 + \epsilon$$

where
$$\begin{aligned} a_0 &= k_0/(1 - \mu\varphi) \\ a_1 &= -\mu\delta/(1 - \mu\varphi) \\ a_2 &= -\mu(1 - \delta)/(1 - \mu\varphi) \\ a_3 &= \rho\mu\delta(1 - \delta)/2(1 - \mu\varphi) \\ \epsilon &= v_0/(1 - \mu\varphi) \end{aligned}$$

Thus, if we had estimates of a_i, $i = 0, 1, 2, 3$, we could obtain estimates of ρ, μ, δ and k_0. Since, by the second and third of equations (11)

$$\text{(14)} \qquad z_i = \frac{k_i + v_i}{1 + \rho}, \; i = 1, 2,$$

and

$$\text{(15)} \qquad z_3 = (x_1 - x_2)^2 = \frac{(k_1 - k_2 + v_1 - v_2)^2}{(1 + \rho)^2},$$

it follows, by the assumption that ν_0, ν_1, and ν_2 are independent, that z_i, $i = 1, 2, 3$, are independent of ϵ. Hence, the ordinary least-squares estimates of a_i, $i = 0, 1, 2, 3$, in (13) are consistent. Indeed, Kmenta (1964) shows they are precisely the maximum-likelihood estimates if $\hat{\varphi}$ as determined by (12) is used.

If prices vary from observation to observation a wider variety of estimation possibilities is opened up, and with the choice comes a greater risk of error. Equations (1′) and (3) may be used to determine any three of the six variables V, L, K, p, w, and r except in the degenerate case when $\mu = 1$ which implies that supply is perfectly elastic so that the scale of operation must be indeterminate under conditions of perfect competition.

To obtain various useful formulations from (1′) and (3), it is convenient to proceed by finding first the cost function, then equating marginal cost to price, and finally inserting the result in (3) to obtain the supply function and two derived demand functions for factors of production. These, in fact, will be the reduced form when p, w, and r are treated as exogenous, with V, L, and K endogenous. From (3) and the definition of total factor costs we have

$$\begin{aligned}
(16)\qquad \frac{C}{pV} &= \frac{w}{p}\frac{L}{V} + \frac{r}{p}\frac{K}{V} = \frac{w}{p}\left\{a^{-1}\left(\frac{w}{p}\right)^{-1/1+\rho} V^{\rho(1-\mu)/\mu(1+\rho)}\right\} \\
&\quad + \frac{r}{p}\left\{b^{-1}\left(\frac{r}{p}\right)^{-1/1+\rho} V^{\rho(1-\mu)/\mu(1+\rho)}\right\} \\
&= \left\{a^{-1}\left(\frac{w}{p}\right)^{\rho/1+\rho} + b^{-1}\left(\frac{r}{p}\right)^{\rho/1+\rho}\right\} V^{\rho(1-\mu)/\mu(1+\rho)}.
\end{aligned}$$

Thus, marginal cost is

$$\begin{aligned}
(17)\qquad \lambda = \frac{\partial C}{\partial V} &= p\left\{a^{-1}\left(\frac{w}{p}\right)^{\rho/1+\rho} + b^{-1}\left(\frac{r}{p}\right)^{\rho/1+\rho}\right\} V^{\rho(1-\mu)/(1+\rho)} \\
&\quad \left\{1 + \frac{\rho(1-\mu)}{\mu(1+\rho)}\right\} \\
&= \frac{C}{V}\,\frac{\mu-\rho}{\mu(1+\rho)}.
\end{aligned}$$

Setting marginal cost equal to price we obtain the supply function in implicit form:

$$\left\{a^{-1}\left(\frac{w}{p}\right)^{\rho/1+\rho} + b^{-1}\left(\frac{r}{p}\right)^{\rho/1+\rho}\right\} V^{\rho(1-\mu)/\mu(1+\rho)} = \frac{\mu+\rho}{\mu(1+\rho)}.$$

Solving for p with $\mu = 1$, constant returns to scale, shows that supply is perfectly elastic at a price

$$p = [a^{-1}\, w^{\rho/1+\rho} + b^{-1}\, r^{\rho/1+\rho}]^{1+\rho/\rho},$$

in this case. In general, with $\mu \neq 1$,

$$V = \left[\alpha\left(\frac{w}{p}\right)^{\rho/1+\rho} + \beta\left(\frac{r}{p}\right)^{\rho/1+\rho}\right]^{(1+\rho)/\rho(1-\mu)} \tag{18}$$

where $\alpha = \dfrac{\mu+\rho}{\mu(1+\rho)}\, a^{-1}$

$$\beta = \frac{\mu+\rho}{\mu(1+\rho)}\, b^{-1}.$$

Substituting for V from (18), and $\lambda = p$, in the second and third equations of (3), we obtain the derived demand functions for capital and labor:

$$\left\{\begin{aligned} L &= \frac{\mu(1+\rho)}{\mu+\rho}\,\alpha\left[\frac{w}{p}\right]^{-1/1+\rho} \cdot \\ &\quad \left[\alpha\left(\frac{w}{p}\right)^{\rho/1+\rho} + \beta\left(\frac{r}{p}\right)^{\rho/1+\rho}\right]^{(1+\rho)(\mu+\rho)/\rho(1-\mu)} \\ K &= \frac{\mu(1+\rho)}{\mu+\rho}\,\beta\left[\frac{r}{p}\right]^{-1/1+\rho} \cdot \\ &\quad \left[\alpha\left(\frac{w}{p}\right)^{\rho/1+\rho} + \beta\left(\frac{r}{p}\right)^{\rho/1+\rho}\right]^{(1+\rho)(\mu+\rho)/\rho(1-\mu)} . \end{aligned}\right. \tag{19}$$

If p, w, and r are exogenous, equations (18) and (19) represent the appropriate reduced form; least-squares estimates have all the well-known desirable properties. Of course, the equations are highly nonlinear; so either iterative procedures or a linearization along the lines suggested by Kmenta (1964) and outlined above must be employed. Furthermore, only three parameters enter the equations, and there are more than that number of coefficients; hence, restrictions apply within and across equations. The only way in which the across-equation restrictions may be properly imposed is by specifying the nature of the dependence (or that there is none) between the disturbances in the several equations.

If there are no imperfections in profit maximization, the residual entering all three equations is the same, namely, the residual in the production function itself. In this case, it is useful to combine all three equations into a single estimation equation expressing equilibrium net revenue as a function of the prices

$$\begin{aligned}(20)\qquad \pi &= pV - wL - rK \\ &= \left\{p - \frac{\mu(1+\rho)}{\mu+\rho}[Q]^{\rho/\rho(1-\mu)}\left[\alpha w\left(\frac{w}{p}\right)^{-1/1+\rho}\right.\right. \\ &\qquad \left.\left. + \beta r\left(\frac{r}{p}\right)^{-1/1+\rho}\right]\right\}[Q]^{\mu(1+\rho)/\rho(1-\mu)},\end{aligned}$$

where $Q = \alpha(w/p)^{\rho/1+\rho} + \beta(r/p)^{\rho/1+\rho}$. The terms α and β contain the residual element, which may be factored out. An assumption concerning its distribution then permits maximum-likelihood methods to be employed. The nonlinear equations resulting must, in general be solved by numerical methods.

That there are no imperfections in profit maximization, or, to put the matter more accurately, other factors which cause the marginal conditions to hold only inexactly, seems somewhat implausible. If we allow residual elements in all of the equations (1′) and (3), the residuals in (18) and (19) will all be different. If we maintain the assumption that prices are all exogenous, specifying the joint distribution of the residual elements in (18) and (19) permits employment of maximum-likelihood methods. It is no longer necessary to specify independence, and, indeed, assuming the residuals follow a multivariate normal distribution and are independent, we could compute estimates of covariances of residuals in the supply and derived demand equations. Kmenta (1964, pp. 17–19), suggests a two-stage least-squares procedure which is computationally simpler than the maximum-likelihood procedure described above. It involves first estimating the elasticity of substitution from a logarithmic regression of the capital-labor ratio on the factor-price ratio and using the result in other equations of the system to obtain estimates of the other parameters. I would conjecture that Kmenta's two-stage procedure is asymptotically as efficient as the full maximum-likelihood procedure described above only when the residuals in all three equations may be assumed to be independent; if they are not, the full maximum-likelihood method would appear to offer some advantages which might offset, at least partly, the computational complexity.

Suppose now that output and factor prices are exogenous. We may either suppose that output price is determined endogenously and make use of all three equations (18) and (19), or we may take output price as given as well, so that *real* factor prices are exogenous, and assume that firms minimize costs for a given output rather than maximize profits. This last set of assumptions leads to a model which has applications in the study of regulated industries and seems worth exploring.[21] In this case, we dispense with the supply function altogether; our system now consists of the production function (1′) and quasi-derived-demand functions for the factors which show the dependence of their equilibrium levels on the exogenously determined level of output, e.g., (5). Once again, these equations involve the same parameters, and restrictions must be imposed both within and across equations. A simple way to do this is to consider the cost function rather than the three individual equations. The cost function may be obtained from (16) by substitution of $p = \lambda$ from (17):

$$
\begin{aligned}
(21)\quad C &= p^{1/1+\rho}\{a^{-1}w^{\rho/1+\rho} + b^{-1}r^{\rho/1+\rho}\}V^{(\mu+\rho)/\mu(1+\rho)} \\
&= \left[\frac{C}{V}\frac{\mu+\rho}{\mu(1+\rho)}\right]^{1/1+\rho}\{a^{-1}w^{\rho/1+\rho} + b^{-1}r^{\rho/1+\rho}\}V^{(\mu+\rho)/\mu(1+\rho)} \\
&= \frac{\mu(1+\rho)}{\mu+\rho}[\alpha w^{\rho/1+\rho} + \beta r^{\rho/1+\rho}]^{(1+\rho)/\rho}V^{(1-\mu)/\mu}.
\end{aligned}
$$

The function is highly nonlinear; either iterative procedures must be employed directly or an approximation to the function must be devised along the lines suggested by Kmenta (1964) in connection with the CES function itself. We have

[21] Cf. Nerlove (1963).

$$\text{(22)} \qquad \log C = A^* + \left(\frac{1-\mu}{\mu}\right)\log V + \left(\frac{1+\rho}{\rho}\right) f(\rho) + v^*$$

where $A^* = \log \dfrac{\mu(1+\rho)}{\mu+\rho} + \dfrac{1+\rho}{\rho} \log \left\{ \dfrac{\mu+\rho}{\mu(1+\rho)} \mu^{1/1+\rho} \gamma^{-\rho/\mu(1+\rho)} \right\}$

$$f(\rho) = \log \{ \delta^{1/1+\rho} w^{\rho/1+\rho} + (1-\delta)^{1/1+\rho} r^{\rho/1+\rho} \}$$

and $v^* = \log$ {residual occurring in (21)}.

Expanding $f(\rho)$ in a Taylor's series about $\rho = 0$

$$\begin{aligned}\text{(23)}\ f(\rho) = {} & \rho[\delta \log w + (1-\delta)\log r - \delta \log \delta - (1-\delta)\log(1-\delta)] \\ & + \frac{\rho^2}{2} \{ \delta(1-\delta)[\log w - \log r]^2 + \delta[1 + 2\delta \log \delta \\ & + 2(1-\delta)\log(1-\delta)] \log w + (1-\delta)[1 + 2\,\delta \log \delta \\ & + 2(1-\delta)\log(1-\delta)] \log r - [\delta \log \delta + (1-\delta)\log(1-\delta)]^2 \\ & - [(1-\delta)\log(1-\delta)[1 - \log(1-\delta)] + \delta \log \delta[1 - \log \delta] \} \\ & + \text{(higher-order terms).}\end{aligned}$$

Hence

$$\begin{aligned}\text{(24)}\ \log C = {} & A + \left(\frac{1-\mu}{\mu}\right) \log V + (1+\rho)\delta\{1 + \rho[1 + 2\delta \log \delta \\ & + 2(1-\delta)\log(1-\delta)]\} \log w + (1+\rho)(1-\delta)\{1 + \rho[1 \\ & + 2\delta \log \delta + 2(1-\delta)\log(1-\delta)]\} \log r \\ & + \frac{(1+\rho)\rho\delta(1-\delta)}{2} [\log w - \log r]^2 + v.\end{aligned}$$

where A *is* A^* plus the appropriate function of δ and ρ and v is v^* plus higher-order terms of the Taylor's series expansion. Unfortunately, from the standpoint of estimation, (24) is not as simple a form as (10); indeed, we have three coefficients (of $\log w$, of $\log r$, and of $[\log w - \log r]^2$) to determine only two coefficients ρ and δ so that (24) overidentifies the parameters of the production function. To obtain unique estimates would probably almost be as difficult as estimating (21) directly by iterative methods.

Thus far we have explored the cases in which (a) all prices are taken as exogenous, leaving output and factor input levels to be determined endogenously; and (b) factor prices and output are taken to be exogenous, leaving factor inputs to be determined endogenously on the assumption that the supply function is suppressed. Another interesting case is that in which labor input, the rate of return on capital, and output price are taken as exogenous. This condition might plausibly be assumed in an intercountry comparison in which one assumed immobile labor, mobile capital, and unfettered trade in commodities. (Consideration of more than one industry, however, causes some difficulty with the labor immobility hypothesis, since, while it is plausible to assume immobility among countries, it is harder to swallow immobility among industries in the same country.) In this case, we must solve (1′) and (3) for V, K, and w in terms of L, r, and p. Clearly, the elasticity of substitution is most easily estimated by regressing log V/K on log r/p when returns to scale are assumed to be constant. If the constant returns assumption is not made, however, or if data on capital stock and rate of return are not available, matters become more complicated. In this first instance, a full maximum-likelihood procedure seems to be called for, although doubtless some approximate procedures might be devised. Unfortunately, lack of data is far more serious; even if there are constant returns to scale, lack of capital stock and rate-of-return data will prevent satisfactory estimation of the elasticity of substitution except under other exogenicity assumptions.

One possible approach to the problem of estimation with insufficient data is to ask not what the correct method of estimation should be, but rather how much of a difference it makes if an incorrect method is used. This is the approach taken by Maddala and Kadane (1965). They assume constant returns to scale and suppress the output price level. In addition, they consider only the linear terms of the approximation to the production function, (10), i.e., they consider a system in which a Cobb-Douglas function is used to approximate the CES for purposes of analyzing simultaneous equations effects only. Their system thus consists of only three equations:

$$(25)\qquad \begin{cases} \log V = \log \gamma + \delta \log K + (1 - \delta) \log L + v_0 \\[4pt] \log \dfrac{V}{L} = -\dfrac{1}{1 - \rho} \log \gamma^{\rho}(1 - \delta) + \dfrac{1}{1 + \rho} \log \dfrac{w}{p} + v_1 \\[4pt] \log \dfrac{V}{K} = -\dfrac{1}{1 + \rho} \log \gamma^{\rho}\delta + \dfrac{1}{1 + \rho} \log \dfrac{r}{p} + v_2 \end{cases}$$

Since w/p is endogenous in the case under consideration, estimation of the elasticity of substitution from the second of equations (25) will be subject to simultaneous equations bias.

To assess the effects of simultaneous equations bias analytically, one may proceed as follows: First express w/p and V/L in terms of L and r/p; the two variables considered as exogenous. This may be done by solving (25) for V, K, and w/p. Next, write down the least squares estimates of $\sigma = 1/(1 + \rho)$ from the second of equations (25), i.e.,

$$\hat{\sigma} = \frac{\operatorname{cov}\left[\log \frac{V}{L}, \log \frac{w}{p}\right]}{\operatorname{var}\left[\log \frac{w}{p}\right]} \tag{26}$$

where the covariances are the sample values. Replacing these by the population values tells us what the least-squares estimates will be asymptotically and this in comparison with the true σ will reveal the asymptotic bias. Finally, we express the population covariance and variance in (26) by the corresponding values expressed in terms of the true parameters and the population variances and covariances of the exogenous variables and the residuals in the problem. Unfortunately, one does not come out with very neat answers to the question in this way. Furthermore, the dependence of the simultaneous-equations bias on the population moments of the exogenous variables and residuals in this case means in effect that we cannot ever hope to give a general answer to the question. An alternative approach is to assume various plausible values for the parameters and generate values of the exogenous variables and the residuals according to some stochastic scheme.[22] Maddala and Kadane (1965) have done just this for three cases:

(i) L and r/p exogenous;
(ii) K and L exogenous;
(iii) r/p and x exogenous;

[22] Clearly, this amounts to specifying a distribution for both the exogenous variables and the residuals. Such a specification also makes it possible to complete the analytical treatment of simultaneous-equations bias. Thus, given sufficient energy, I could reproduce analytically all the results obtained by Kadane and Maddala (1965) including those presented below. Furthermore, I could, if I had even more energy, try out alternative distributional assumptions. But electronic computers have nearly infinite energy and the "capital-intensive" approach of Maddala and Kadane seems, after all, the path of least resistance.

and for a variety of assumptions about the contemporaneous variance-covariance matrix of the residuals in equations (25). They assumed serial independence throughout. In case (i) they assumed L was uniformly distributed in the interval (0, 1,000) and r/p in the interval (0, 5), rejecting those values which would have entailed the calculation of the logarithm of a negative number.

The assumptions made about the contemporaneous residual variance-covariance matrix were as follows:

A. All the disturbances are uncorrelated with "technical" disturbances (those in the production function itself) having a higher variance than the "economic" disturbances [those in the second and third equations of (25)].

B. As in A, except the relative variances of the "technical" and "economic" disturbances are reversed.

C. All disturbances are positively correlated and the variance of the "technical" disturbance is greater than the variances of the "economic" disturbances.

D. As in C except the relative variances of the "technical" and "economic" disturbances are reversed.

E. The "economic" disturbances are highly correlated but independent of the "technical" disturbance. The "economic" disturbances have higher variance than that of the "technical" disturbance.

All residuals were assumed to follow a multivariate normal distribution. For each of three values of the true elasticity of substitution, $\sigma = 0.4$, $\sigma = 0.9$, and $\sigma = 1.6$, and values of $\gamma = 1$ and $\delta = .3$, samples were drawn (how many, they don't say) and two alternative estimates of the elasticity of substitution were computed, one from the regression of log V/L on log w/p (the ACMS case) and the other from the regression of log w/p on log V/L. The results they obtained for L and r/p exogenous are reproduced in Table 12.

Table 12 shows that the true elasticity of substitution seems to be seriously underestimated by the ACMS method except when σ is actually rather close to one. The alternative regression tends to overestimate in most cases, but not as seriously. Simultaneous equations difficulties were aggravated, as we might expect, by dependence among the residuals in the various equations. On the basis of their findings in this and other cases, Maddala and Kadane recommend the regression of log w/p on

TABLE 12

Mean Estimates of the Elasticity of Substitution and Mean Variance of Estimate[a]

(various values of true elasticity of substitution and residual variance-covariance matrix assumed)

True Elasticity of Substitution and Type of Regression	Assumption about Residual Variance-Covariance Matrix				
	A	B	C	D	E
$\sigma = 0.4$					
(i) Log $\frac{V}{L}$ dependent					
Mean estimated $\hat{\sigma}$	0.32	0.10	0.31	0.15	0.11
Mean variance of estimate	(.0018)	(.0034)	(.0027)	(.0038)	(.0009)
(ii) Log $\frac{w}{p}$ dependent					
Mean estimated $\hat{\sigma}$	0.41	0.43	0.51	0.63	0.38
Mean variance of estimate	(.0040)	(.0061)	(.0018)	(.0462)	(.0363)
$\sigma = 0.9$					
(i) Log $\frac{V}{L}$ dependent					
Mean estimated $\hat{\sigma}$	0.88	0.83	0.86	0.77	0.76
Mean variance of estimate	(.0012)	(.0040)	(.0015)	(.0026)	(.0023)
(ii) Log $\frac{w}{p}$ dependent					
Mean estimated $\hat{\sigma}$	0.89	0.91	0.89	0.82	0.80
Mean variance of estimate	(.0010)	(.0050)	(.0035)	(.0028)	(.0030)
$\sigma = 1.6$					
(i) Log $\frac{V}{L}$ dependent					
Mean estimated $\hat{\sigma}$	1.36	0.60	1.29	0.76	0.58
Mean variance of estimate	(.0243)	(.1037)	(.0357)	(.0711)	(.0947)
(ii) Log $\frac{w}{p}$ dependent					
Mean estimated $\hat{\sigma}$	1.62	1.81	1.81	2.12	1.53
Mean variance of estimate	(.0283)	(.2025)	(.0930)	(.2442)	(.1982)

[a]Obtained by Maddala and Kadane (1965) in Monte Carlo studies when labor force and rate of return on capital are assumed to be exogenous

log V/L, rather than the other way around, since these estimates seem to be more robust in the face of simultaneous equations misspecification.

On the whole, there is little that can be done about difficulties in estimation due to lack of relevant data. On the other hand, as we have seen in the course of this section, simultaneous-equations problems and problems of nonlinearity do arise even when all relevant data are available. In general, the best method for attacking the estimation problem in such situations seems to be the maximum-likelihood approach. Although this method typically leads to simultaneous nonlinear equations which must be solved for the estimates, computational techniques are available and may definitely be considered feasible on modern electronic computers.[23] In short, the estimation of the CES production function in a simultaneous-equations context is just one further illustration of the computer revolution underway in the field of econometric methods.

References

Arrow, K. J., H. B. Chenery, B. S. Minhas, and R. M. Solow (1961), "Capital-Labor Substitution and Economic Efficiency," *Review of Economics and Statistics,* August, pp. 225–250.

Brown, M. (1962), "The Constant Elasticity of Substitution Production Function," Report No. 6219, *Econometric Institute,* Rotterdam, Holland, June.

Brown, M., and J. S. de Cani (1963), "Technological Change and the Distribution of Income," *International Economic Review,* September, pp. 289–309.

Bruno, M. (1962), "A Note on the Implications of an Empirical Relationship between Output per Unit of Labour, the Wage Rate, and the Capital-Labour Ratio," Stanford University, July (mimeo.).

David, P. A., and Th. van de Klundert (1965), "Non-Neutral Efficiency Growth and Substitution between Capital and Labor in the U.S. Economy, 1899–1960," *American Economic Review,* June, pp. 356–394.

Dhrymes, P. J. (1965), "Some Extensions and Tests for the CES Class of Production Functions," *Review of Economics and Statistics*, November, pp. 357–366.

———, and M. Kurz (1964), "Technology and Scale in Electricity Generation," *Econometrica,* July, pp. 287–315.

Diamond, P. A., and D. McFadden (1965), "Identification of the Elasticity of Substitution and the Bias of Technical Change: An Impossibility Theorem," unpublished.

[23] See, for example, Traub (1964).

Diwan, R. K. (1963), "An Empirical Estimate of the Constant Elasticity of Substitution Production Function," paper presented at meeting of Econometric Society, Copenhagen, July 1963; abstract in *Econometrica,* October 1964, pp. 662–663.

Ferguson, C. E. (1965a), "Substitution, Technical Progress, and Returns to Scale," *American Economic Review, Proceedings,* May, pp. 296–305.

——— (1965b), "Time-Series Production Functions and Technological Progress in American Manufacturing Industry," *Journal of Political Economy,* April, pp. 135–147.

Fuchs, V. R. (1963), "Capital Labor Substitution, A Note," *Review of Economics and Statistics,* November, pp. 436–438.

Gordon, R. A. (1961), *Business Fluctuations,* New York.

Hilhorst, J. G. M. (1961), "Production Functions for Manufacturing Industry," *Statistische en Econometrische Onderzoekingen,* Central Bureau of Statistics, Den Haag, pp. 180–204.

Kendrick, J. W. (1964), Comment on Solow (1964), in *The Behavior of Income Shares,* Princeton for NBER, pp. 140–142.

Kendrick, J. W., and R. Sato (1963), "Factor Prices, Productivity, and Economic Growth," *American Economic Review,* December, pp. 974–1003.

Klein, L. R. (1953), *A Textbook of Econometrics,* Evanston, Ill.

Kmenta, J. (1964), "On Estimation of the CES Production Function," *Social Systems Research Institute, University of Wisconsin, Paper No. 6410,* October.

Kravis, I. (1959), "Relative Income Shares in Fact and Theory," *American Economic Review,* December, pp. 917–949.

Leontief, W. (1964), "An International Comparison of Factor Costs and Factor Use" [review of Minhas (1960–63)], *American Economic Review,* June, pp. 335–345.

Liu, T. C., and G. H. Hildebrand (1965), *Manufacturing Production Functions in the United States, 1957,* Ithaca, N.Y.

Lucas, R. E. (1963), "Substitution Between Labor and Capital in U.S. Manufacturing, 1929–58," unpublished Ph.D. dissertation, University of Chicago.

McFadden, D. (1964), "Notes on Estimation of the Elasticity of Substitution," Institute of Business and Economic Research, University of California, Berkeley, November (mimeo.).

McKinnon, R. I. (1962), "Wages, Capital Costs, and Employment in Manufacturing: A Model Applied to 1947–58 U.S. Data," *Econometrica,* July, pp. 501–521.

——— (1963a), "The CES Production Function Applied to Two-Digit Manufacturing and Three Mining Industries for the United States," unpublished.

——— (1963b), "Factor Price Changes and Production Function Estimation," Stanford University (mimeo.).

Maddala, G. S. (1963), "Technological Change in the Bituminous Coal Industry, 1919–54," unpublished Ph.D. dissertation, University of Chicago.

——— (1965), "Differential Industry Effects and Differential Factor Effects of Technological Change," Memo. 36, Research Center in Economic Growth, Stanford University, March.

Maddala, G. S., and J. B. Kadane (1965), "Specification Errors in the Context of the CES Production Function," paper presented at meeting of the Econometric Society, New York, December.

Minasian, J. R. (1961), "Elasticities of Substitution and Constant-Output Demand Curves for Labor," *Journal of Political Economy*, June, pp. 261–270.

Minhas, B. S. (1962), "The Homohypallagic Production Function, Factor-Intensity Reversals, and the Heckscher-Ohlin Theorem," *Journal of Political Economy*, April, pp. 138–156.

——— (1960–63), "An International Comparison of Factor Costs and Factor Use." (Ph.D. dissertation, Stanford University, November 1960), *Contributions to Economic Analysis No. 31*, Amsterdam: North-Holland, 1963.

Murata, Y., and K. J. Arrow (1965), unpublished results of estimation of elasticities of substitution for two-digit industries from intercountry data for two periods, June.

Nelson, R. R. (1965), "The CES Production Function and Economic Growth Projections" (mimeo.).

Nerlove, M. (1963). "Returns to Scale in Electricity Supply," pp. 167–198, in C. Christ *et al.* (ed.), *Measurement in Economics: Studies in Mathematical Economics and Econometrics in Memory of Yehuda Grunfeld*, Stanford.

——— (1965), "Notes on the Production Relations Included in Macro-Econometric Models," Stanford University, June (mimeo.).

Oi, W. Y. (1962), "Labor as a Quasi-Fixed Factor," *Journal of Political Economy*, December, pp. 538–555.

Okun, A. M. (1962), "Potential GNP: Its Measurement and Significance," *Proceedings of the Business and Economic Statistics Section of the American Statistical Association*.

Smith, V. (1963), "On Production Functions of Constant Elasticity of Substitution," Purdue University (mimeo.).

Solow, R. M. (1956), "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics*, pp. 65–94.

——— (1957), "Technical Change and the Aggregate Production Function," *Review of Economics and Statistics*, pp. 312–320.

——— (1964), "Capital, Labor and Income in Manufacturing," in *The Behavior of Income Shares*, Princeton for NBER, pp. 101–128.

Swan, T. W. (1956), "Economic Growth and Capital Accumulation," *Economic Record*, pp. 334–361.

Traub, J. L. (1964), *Iterative Methods for the Solution of Equations,* Englewood Cliffs, N.J.

Whitaker, J. K. (1964), "A Note on the CES Production Function," *Review of Economic Studies,* April, pp. 166–167.

COMMENT

EDWIN MANSFIELD

The past decade has witnessed an enormous increase in the amount of attention devoted by economists to the study of production functions. Faced with the large number of studies that have resulted, Nerlove, in this very interesting and useful review article, has chosen to concentrate his attention primarily on the recent work on the CES production function. His paper consists largely of a description of the findings and limitations of recent studies of the elasticity of substitution. His major finding is the diversity of the results; small differences in period and concept seem to produce markedly different estimates. Nerlove makes an attempt to account for some of these discrepancies, but does not get very far. The net impression one obtains from the paper is that there are a large number of biases, none very well understood, which are operating simultaneously to produce very inconsistent and untrustworthy results.

Turning from the elasticity of substitution, I want to take up two questions regarding the rate of technological change. First, to what extent are the estimated rates of technological change in various industries consistent from one study to another. Although Nerlove shows that the estimates of the elasticity of substitution vary greatly, he says nothing in this regard about the estimated rates of technological change. Despite the well-known and important deficiencies in these estimates,[1] they are all that are available for many purposes, and it is important to know whether there is considerable agreement from one study to another, the estimated rate of technological change being consistently higher in some industries than in others.

To find out, I looked first at the studies by Ferguson [2] and McKin-

[1] The rate of technological change is measured by the rate of growth of the residual, which, as Nerlove points out, absorbs various specification errors. For example, besides technological change, the residual contains the effects of inputs not explicitly included in the production function.

[2] C. E. Ferguson, "Time-Series Production Functions and Technological Progress in American Manufacturing Industry," *Journal of Political Economy,* April 1965.

non [3] cited by Nerlove. Then I added the results obtained by Solow,[4] Massell,[5] and myself.[6] The findings, shown in Table 1, indicate that there is considerable diversity in the results, despite the fact that all the studies pertain to much the same period. An extreme case of this diversity is the tobacco industry, where the estimated annual rate of technological change varied from −10.7 per cent to 39.2 per cent. Moreover, the furniture industry is only slightly less unstable, the estimated annual rate of technological change varying from zero to 24.2 per cent.

Some of the factors responsible for these differences are easy to find. Solow's estimates are rates of *capital-augmenting* technological change; consequently, they would be expected to be two or three times as large as the others. My estimates and Massell's are based on the Cobb-Douglas production function, whereas the others use the CES. Solow's estimates and mine assume that technological change is capital-embodied whereas the others assume that it is disembodied. Ferguson uses current-dollar values, whereas the others use deflated data. Massell and I, and to some extent Solow, attempt to correct for underutilization of capital whereas the others do not. Ferguson tries to include nonneutral technological change, whereas the others assume all technological change is neutral.[7]

Although the estimates for a particular industry vary considerably, their rank ordering may remain much the same. To see whether this is so, I ran rank correlations between the estimates in each pair of studies. The results, shown in Table 2, indicate that there generally is some positive correlation between an industry's rank in one study and its rank in another, but that the correlation is never high. The closest agreement seems to exist among Ferguson, McKinnon, and me. Solow and Massell agree with nobody else—including one another.

Having said this, I must add a few words of caution. It is important that we put these findings in perspective and that we refrain from drawing unduly pessimistic conclusions. The results are not as discouraging

[3] R. I. McKinnon, "Wages, Capital Costs, and Employment in Manufacturing: A Model Applied to 1947–58 U.S. Data," *Econometrica,* July 1962.

[4] R. M. Solow, "Capital, Labor, and Income in Manufacturing," in *The Behavior of Income Shares,* Princeton for NBER, 1964.

[5] B. Massell, "A Disaggregated View of Technical Change," *Journal of Political Economy,* 1961.

[6] E. Mansfield, "Rates of Return from Industrial Research and Development," *American Economic Review,* May 1965.

[7] Also, McKinnon allows for a distributed lag in the relationship he estimates, while the others do not.

TABLE 1

Rates of Neutral Technological Change, U.S. Manufacturing, Postwar Period, Results of Five Studies

(per cent)

Industry	McKinnon, 1949-61	Ferguson,[a] 1949-61	Massell, 1946-57	Solow,[b] 1949-58	Mansfield, 1946-62
Food	2.4	2.3	1.4	8.2	4.7
Tobacco	-10.7	-4.4	0.8	39.2	n.a.
Textiles	3.4	-3.0	1.6	7.9	n.a.
Apparel	0.2	-3.8	0.9	n.a.	3.0
Lumber	-0.3	None	3.8	0.0	n.a.
Furniture	24.2	None	1.0	9.0	1.9
Paper	2.8	None	2.3	8.3	3.4
Chemicals	4.0	None	3.5	5.9	3.7
Petroleum and coal	n.a.	None	1.9	n.a.	n.a.
Rubber	2.4	2.1	1.0	7.4	n.a.
Leather	2.2	0.4	1.1	7.3	n.a.
Glass	2.3	2.1	2.5	5.2	1.5
Primary metals	1.6	None	0.4	n.a.	n.a.
Fabricated metals	0.5	2.9	0.3	3.2	n.a.
Machinery	-1.9	None	2.0	0.0	[c]
Electrical machinery	2.7	1.9	3.7	3.9	3.6
Transportation equipment	3.1	2.4	2.4	n.a.	n.a.
Instruments	3.9	4.6	1.0	0.0	8.3
Printing	-1.0	-0.7	2.4	4.3	n.a.

n.a. = not available.

Source: See footnotes 2-6.

[a]Unlike the others, Ferguson tries to include nonneutral technological change as well in his paper.

[b]Rates of capital-augmenting technological change.

[c]Less than zero.

as they may seem. As noted above, several of these studies contain important defects, which are recognized by their authors and which can be overcome by disaggregation, deflation, correction for underutilization of capacity, etc. Perhaps the principal conclusion to be drawn from Tables 1 and 2 is that these defects can cause more havoc than is commonly recognized.

TABLE 2

Coefficients of Rank Correlation Between Estimates of Rates of Technological Change, U.S. Manufacturing

	Ferguson	McKinnon	Mansfield	Massell	Solow
Ferguson	1.00	0.37	0.54	0.05	-0.34
McKinnon		1.00	0.47	0.18	0.25
Mansfield			1.00	0.00	-0.03
Massell				1.00	-0.33
Solow					1.00

Source: Table 1.

Finally, I want to turn to an important question bearing on the work of the National Commission on Automation and others concerned with policy in the area of technological change. To what extent has there been an increase since World War II in the rate of technological change? It is easy to find statements by economists and others asserting that the rate of technological change in the postwar period is much more rapid than that before the war. It is also possible to find statements asserting the opposite. Although there is considerable evidence that the rate of increase of output per man-hour has been higher than before the war, the evidence with respect to total factor productivity has been less clear cut. For example, Kendrick and Sato find that the average annual rate of increase of total factor productivity in the private domestic economy was 2.14 per cent during 1948–60, as contrasted with 2.08 per cent during 1919–60.[8]

Nerlove's Table 7 seems to provide a small amount of new evidence on this score. It contains estimates of the rate of technological change in various two-digit manufacturing industries during 1899–1957 and 1947–58. The results, taken from published and unpublished work by McKinnon,[9] provide no evidence that the rate of technological change has been higher in most industries in the postwar period. Table 7 shows

[8] J. Kendrick and R. Sato, "Factor Prices, Productivity, and Growth," *American Economic Review,* December 1963.

[9] McKinnon, *op. cit.,* and "The CES Production Function Applied to Two-Digit Manufacturing and Three Mining Industries for the United States," unpublished, 1963.

that the estimated postwar rate of technological change was lower than that during 1899–1957 in five of the ten industries for which a comparison can be made. These industries are apparel, lumber, paper, printing, and electrical machinery. In the remaining five industries—furniture, chemicals, leather, textiles, and glass—the estimated postwar rate of technological change was higher than that for 1899–1957.

Moreover, if the results for the postwar period that Ferguson, Massell, or I obtained are used in place of McKinnon's, the results are the same.[10] In at least one-half of the industries for which a comparison can be made, the estimated postwar rate of technological change was lower than that for 1899–1957. These results are suggestive, but extremely tentative. Without a more complete description of McKinnon's unpublished study, one cannot be sure that it is entirely comparable with any of the studies of the postwar period, including his own.

EVSEY D. DOMAR

If it was found by Nelson, as quoted by Nerlove, that sizable changes in the elasticity of substitution produce very small effects on the other variables, it should follow that relatively small changes in the other variables should exert strong effects on the elasticity of substitution. The data being what they are, why is it surprising then that the magnitude of the elasticity of substitution derived in the several studies jumps all over the place?

ZVI GRILICHES

There is an alternative statistical interpretation of the Hildebrand and Liu results. The basic point to note is that there is an identification problem here. There are two equations in the system, say

$$y = ak + u \tag{1}$$

$$y = sw + v \tag{2}$$

where y is the logarithm of output per man, k is the logarithm of capital per man, w is the logarithm of the wage rate, and u and v are disturbances. The first equation is the approximate production function (for σ

[10] Because Solow's results are rates of capital-augmenting technological change, it would be incorrect to compare them with McKinnon's results for 1899–1957.

$s \neq 1$, we could add a k^2 term without affecting the rest of the argument). The second equation is the marginal productivity or ACMS relation. Now fitting a combined relation

$$y = b_1 w + b_2 k + e \tag{3}$$

and finding a significant b_2 coefficient does not contradict the above model [equation (2)]. In fact if k is measured without error, u is independent of v, and the variance of v is not identically zero and if the model [(1) and (2)] is correct, one would expect the b_1 coefficient in (3) to be insignificant (rather than the b_2 coefficient as suggested by Hildebrand and Liu) since *all* the effects of w are contained in k. Moreover, k incorporates also the relevant variance of v. Thus, if k is the correct measure of capital it should not only be significant in (3) but should actually swamp the effect of w. Since, however, k is rarely measured without error and w is related to the "systematic" component of k, the latter variable may perform as a proxy for the correct k measure and not be forced out from (3). The final effect will depend on the relative variances of v, the error in the marginal conditions, and the error of measurement in k. But there is no need to interpret these results as implying a more complicated production function.

RAFORD BODDY, State University of New York at Buffalo

I wish to offer an important reason for the usual but perplexing differences among estimates of the elasticity of substitution, σ. My interpretation of the disparities among the estimates of σ is based on the premise that factor proportions for old plants cannot be varied as easily after the plant is built as at the time of the original investment decision. Only the factor proportions of the newest plants are fully adjusted to current expectations of wage rates and to current capital costs. Over time, factor costs change, and give rise to changes in expectations of future costs. The factor proportions of the newest, best-practice plants change as these costs change. The practice in cross-section and in time series analyses has been to regress aggregate factor proportions, or average-practice coefficients, against recent costs. This leads to biases in estimates of σ which depend on the relation of *average-practice and best-practice* factor proportions.

My interpretation is limited to estimates based upon marginal productivity conditions as distinct from those based on the direct approximations of the production function. In fact, few estimates have yet come from the more direct approach. Most estimates of σ have been related to the equation:

$$\frac{X}{L} = (1 - \kappa)^{-\sigma}\gamma^{\sigma\rho}\left(\frac{w}{p}\right)^{\sigma} \tag{1}$$

I will not develop the argument in terms of (1) but rather in terms of the alternative equation:

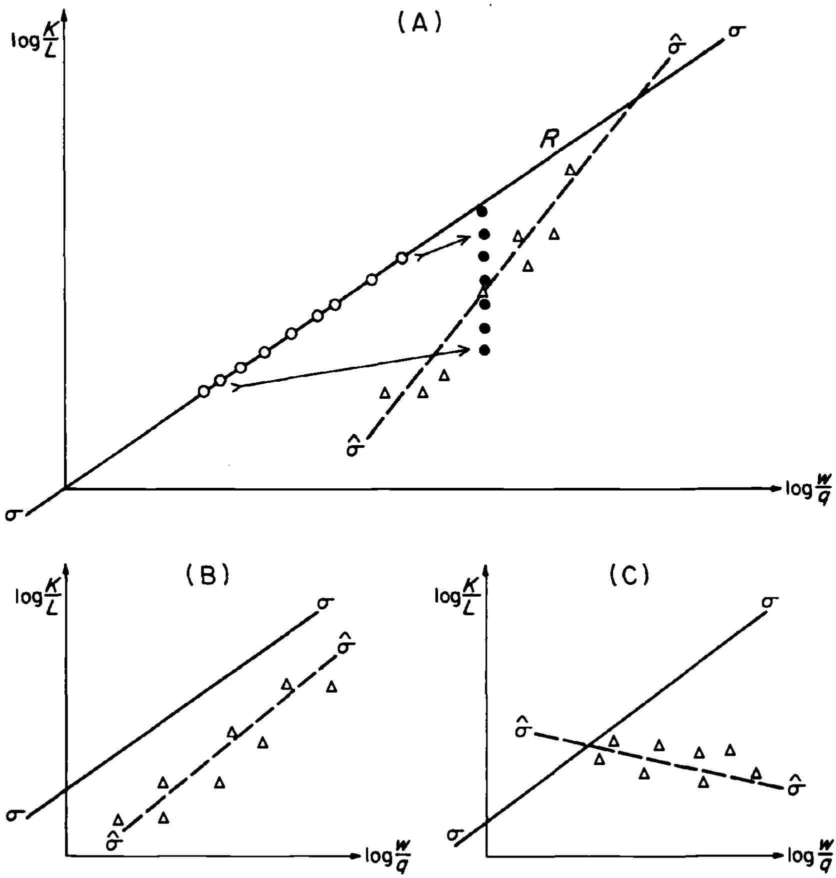
$$\frac{K}{L} = \left(\frac{\kappa}{1 - \kappa}\right)^{\sigma}\left(\frac{w}{q}\right)^{\sigma} \tag{2}$$

where q is the price of capital and w is the wage rate.

Consider plants with identical products, but built at different times in one metropolitan area. Assume that there has been no nonneutral technical change. When building a particular plant, some enterpreneur compared the mean expected wage rates to current capital cost and chose some capital-labor proportion. Assume that expected wage rates depend on past wage rates. With increasing relative wage rates over the period, the coordinates of the logs of the initial factor proportions and expected relative factor prices would lie along the line, R of Figure 1, section A with the coordinates of the newest plant furthest from the origin. The older plants have the lower capital-labor ratios. The slope of the line is σ, the elasticity of substitution.

Given that the older plants are still producing, what will be the average-practice coefficient, $\Sigma K/\Sigma L$, and the average factor prices of the area for the year in which the last plant was built? If all of the capital decisions are analyzed as if they were current ones, the nominal price of capital will be equal for all the plants. For plants in the same business in the single metropolitan area, the current wage rates will tend to be equal, also. The capital-labor ratios are likely to be unequal, with the older plants having the lower values. Variations in the factor proportions of existing plants fall far short of those possible at the time of initial construction. It follows that the average-practice coefficient for the area, $\Sigma K/\Sigma L$, will be less than the capital-labor ratio of the newest plant. The relative size of the two coefficients depends on the relative sizes of the older and newer plants. The logs of the plant factor ratios are the set (●)

FIGURE 1



of Figure 1, section A. The log of the average-practice coefficient for the area is Δ.

The practice in cross-section and time series analyses has been to estimate σ from the regression of average-practice coefficients against recent costs. It is clear from Figure 1, section A that the relation of $\hat{\sigma}$ to σ depends on the relation of average-practice coefficients to the best-practice coefficients of the newest plants.

In cross-section studies for industries in which resources have moved to lower-wage geographic areas, and little new investment has occurred in the higher-wage areas, the estimate of σ is likely to be biased downward. For the same discrepancies on a log scale between average-practice and best-practice coefficients the bias will be greater when there is a narrower spread among regions in recent wage rates. An extreme case arises when the wage rates that determined the average factor proportions of the lower-wage areas are higher than those which would elicit the average-practice coefficient of the high-wage area. The estimate of σ will then be negative (cf. Figure 1, section C).

If the industry is old in all areas and the relative price differences have a long history, the relation of average-practice to best-practice coefficients is likely to be such that $E(\hat{\sigma})$ will be approximately equal to σ. There may be slightly higher standards of obsolescence in the higher-wage areas which would impart a slight upward bias to $\hat{\sigma}$ (cf. Figure 1, section B).

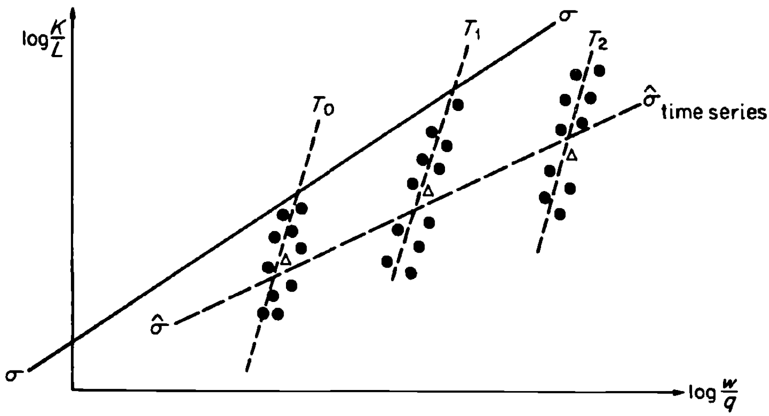
If the recent percentage growth of capacity has been greater in the high-wage areas, or if the wage differences have narrowed, the bias will be upward (cf. Figure 1, section A). Within an economy such as that of the United States, upward biases could occur for a number of reasons. When the workers of an industry are attracted to the higher wages in the expanding areas, the older areas will find themselves faced with higher than expected wage costs and lower quasi-rents. For industries subjected to the recent introduction of regionwide or nationwide wage bargaining the narrowing of the wage differentials would be unfavorable to the areas with the historically lower capital-labor ratios. Again, for the same relative discrepancies between best-practice and average-practice coefficients, the bias will be greater if there is a narrower spread in wage rates.

In Figure 2 below we assess time series estimates through successive cross-section scatters.

Assume that technological change has been neutral. The best-practice coefficients of T_0, T_1, and T_2 would lie along the line with slope σ. Best-practice proportions regressed against the ratio of expected wage rates and current capital costs would provide unbiased estimates of the elasticity of substitution.

Most time series estimates, however, have been for aggregative data. The factor proportions are ratios of total capital to total labor, i.e., average-practice coefficients. Therefore, the aggregate time series estimates of the elasticity of substitution will be unbiased only if the ratio of average-practice to best-practice proportions has not changed systematically. Comparison of best-practice and average-practice coefficients for a

FIGURE 2



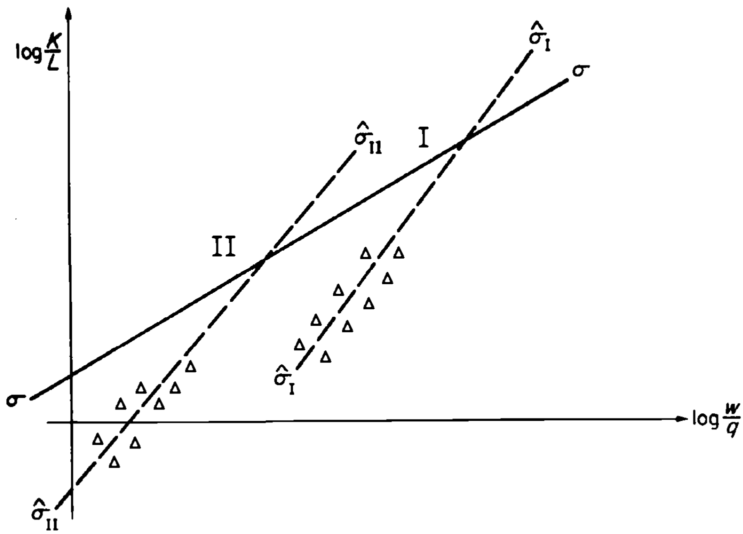
number of industries over extended intervals of time suggests that there has been little systematic change in the ratio.[1] There are some instances, however, where the bias might be significant. If the first observation is of the industry at the time of its inception, and subsequent observations relate to successive periods of growth, the ratio of average-practice to best-practice coefficients will fall over time. There will be a downward bias to $\hat{\sigma}$. Prior to the time when a relatively constant age of capital is achieved, fixed proportions, a constant rate of investment, I_0, and an exponential rate of growth of relative labor costs, w/q, would result in a $\hat{\sigma}$ approximately $(\frac{1}{2})\ \sigma$.

When the industry has begun to decay, the ratio of best-practice to average-practice coefficients again rises as wage costs increase and few, if any, new plants are built. The estimate, $\hat{\sigma}$, is then an estimate of the *ex post* elasticity of substitution that measures the ease of substitution after the plant is built. For time series, the *ex post* elasticity of substitution serves as the lower bound of $\hat{\sigma}$.

The above discussion suggests that not only the standard errors, but the biases to the estimates of $\hat{\sigma}$ are increased for observations with narrow variations in wage rates. Whereas variations in the ratio of average-practice to best-practice coefficients are likely to have little impact on $\hat{\sigma}$ for observations that include high and low wage rates, such variations will lead to significant biases where the sample areas have very similar wage rates. Similarly, time series variations in the average age of the

[1] W. E. G. Salter, *Productivity and Technical Change,* Cambridge, Mass., 1960.

FIGURE 3



capital stock are likely to bias $\hat{\sigma}$ very little if there have been extensive changes in real wage rates, but the bias can be sizable if the variation in wages over the period has been small.

While I do not offer it as the sole explanation, the theory resolves in a simple manner some paradoxical results. We would expect that long-run time series, and cross-section estimates for very diverse countries, would be least subject to the above biases. Generally these estimates of $\hat{\sigma}$ are less than 1. The estimates of $\hat{\sigma}$ for cross sections with less variation in wage rates tend to be greater than 1 and to increase inversely with the narrowness of differences in wage rates. It seems likely that more narrowly defined industries have more narrowly defined regional bases. This may explain why Solow's reasonable deduction,[2] that the elasticity of substitution for two-digit industries should be higher than for three-digit industries, is not borne out by U.S. cross-section estimates. The theory and these results suggest that estimates of $\hat{\sigma}$ from such samples have positive biases.

From the representation in Figure 3 of two cross sections where the wage differences *within* each are small, note the implicit estimates of the relative capital intensity of the "technologies." Because of the bias to $\hat{\sigma}$

[2] Solow, R. M., "Capital, Labor and Income in Manufacturing," *The Behavior of Income Shares*, Studies in Income and Wealth 27, Princeton University Press for National Bureau of Economic Research, 1964.

the intercept is larger for the sample with the lower-capital labor ratios and lower wage rates. The intercept would be the estimate of σ log $\kappa/(1 - \kappa)$. Increases in κ are increases in the capital-using nature of the technology. Cross-section 2, with the lower capital-labor ratios and lower real wage costs appears to have a more capital-using technology! If $\hat{\sigma}$ is biased upward this somewhat paradoxical result should arise for cross-section regressions of one industry at different points in time and for cross sections at one point in time for observations with equally narrow wage differentials. The implications for the simultaneous cross sections explain the estimates of Fuchs [3] and Arrow.[4] The Fuchs estimates for more homogeneous subsamples are represented by cross-sections 1 and 2. The original Arrow estimate (not shown in the figure) corresponds to the regression on all of the observations.

MURRAY BROWN

There are two points I would like to make on Professor Nerlove's excellent review.

1. In the present volume, both Marc Nerlove and Zvi Griliches state that the elasticity of substitution is a second-order parameter in the analysis of middle-range growth. The inference is drawn that if we were living in a world in which the values of this elasticity were 0.5, say, but we analyze the world as if the elasticity were 1.0, the effect on the middle-range rate of growth of such a misspecification would scarcely be noticed. It can be shown, within the same framework used by Nerlove, Griliches, and Richard Nelson,[1] that the elasticity of substitution is a first-order parameter, and hence the misspecification is potentially serious in the analysis of middle-range growth.

There are two relevant equations in the Nelson derivation:

$$d(\dot{O}/O)/dt = b(1 - b)[(e - 1)/e][(\dot{L}/L) - (\dot{K}/K)]^2, \tag{2a}$$

$$\begin{aligned} \Delta O/O \sim \Delta A/A &+ (b_0 \Delta L/L) + (1 - b_0)(\Delta K/K) \\ &+ \tfrac{1}{2} b_0(1 - b_0)[(e - 1)/e][(\Delta K/K) - (\Delta L/L)]^2, \end{aligned} \tag{3}$$

[3] V. R. Fuchs, "Capital-Labor Substitution, A Note," *Review of Economics and Statistics,* November 1963.

[4] K. J. Arrow, H. B. Chenery, B. S. Minhas, and R. M. Solow, "Capital-Labor Substitution and Economic Efficiency," *Review of Economics and Statistics,* August 1961.

[1] *Aggregate Production Functions and Medium-Range Growth Projections,* Santa Monica, Cal., The Rand Corporation, December 1963, pp. 49 ff.

where O is output, L is labor, K is capital, b is the elasticity of production with respect to labor, and e is the elasticity of substitution between capital and labor.[2] Equation (2a) "indicates the rate of change of the output growth equation . . . assuming given and constant $\dot{L}/L$ and $\dot{K}/K$. . . ."[3] Equation (3) depicts the rate of growth of output as a function of the growth of labor; and capital, as a function of technological progress, $\Delta\dot{A}/A$, and as a function of the elasticities of substitution and production. The second equation implies that the rate of growth of output will be greater, the greater the elasticity of substitution.

"With the long run elasticity of substitution equal to about 0.5, Equation (2) suggests that the drag of less than unitary elasticity of substitution should have reduced the annual growth rate of output by <.001 percentage points a year below what a Cobb-Douglas model would have predicted. The effect would scarcely have been noticed."[4]

The difficulty with this is that not only is equation (2) and the last term in equation (3) affected by the misspecification, but all terms in equation (3) are affected. For b, the elasticity of production with respect to labor, is a function of e, the elasticity of substitution, so that when we misspecify, we misspecify not only second-order terms but first-order terms as well. From the CES production function, we find for b:

$$b = h(O/L)^{(1/e)-1}, \tag{4}$$

where h is a constant. Now, combine this with (3), and ignore the second-order term:

$$\Delta O/O \sim (\Delta A/A) + h(O/L)^{(1/e)-1}(\Delta L/L) + [1 - h(O/L)]^{(1/e)-1}(\Delta K/K). \tag{5}$$

Note that e appears in all terms on the right-hand side except the first. If we assume that $e = 1$, and ignore the second-order term still, equation (5) becomes

$$\Delta O/O \sim (\Delta A'/A') + a(\Delta L/L) + (1 - a)(\Delta K/K), \tag{6}$$

where all variables are as before but a is the constant production elasticity with respect to labor and $A' \neq A$. Suppose we focus on the rate of growth in the base period; then (5) becomes

$$\Delta O/O \sim (\Delta A/A) + h(\Delta L/L) + (1 - h)(\Delta K/K). \tag{7}$$

[2] Ibid., p. 52. [3] Ibid. [4] Ibid., p. 53.

Clearly, if $a = h$, then the misspecification would not affect the first-order terms and the Nelson-Nerlove-Griliches conclusion would hold. But—and this is the heart of the story—$a \neq h$ when e is taken as 1.0 rather than 0.5, or any value other than unity for that matter.

It is evident that $a \neq h$ whenever $e \neq 1$, but it can be shown simply as follows. Under the above assumptions, $h \rightarrow a$ as $e \rightarrow 1$; i.e., h becomes the labor production elasticity in the Cobb-Douglas world. In other CES worlds, h must differ from a. Hence the rate of growth forecasted by the specification of $e = 1$ would differ from that forecasted by a specification of $e \neq 1$, and the difference in specifications would affect the first-order terms.

Since b in (3) and (4) is the elasticity of production with respect to labor that is independent of the time period—i.e., it applies to short-run, medium-run, and long-run growth—these results show that the elasticity of substitution is a first-order parameter in the analysis of all three types of growth situations. This controverts the assertion that the elasticity of substitution is a second-order parameter, and hence of negligible importance.

2. The condition that technological change be describable by smooth growth in the factor-augmenting terms in order to identify the elasticity of substitution is inferred by Nerlove from the Diamond-McFadden "impossibility" theorem. This condition is unnecessarily restrictive.[5] There is another kind of technological change that is permitted, which together with the smoothness condition accounts for most of the types of technological change that one would be likely to encounter. To see this, suppose that the growth equation (3) on page 95 is sectionally continuous; i.e., suppose that it can be subdivided into a finite number of parts, in each of which F and G are continuous and F and G have finite limits as the arguments approach either endpoint of the subinterval from the interior. This is a modification of the necessary conditions for identification, and it means that the values of the growth equations are permitted to take on finite jumps. In economic terms, these jumps are structural breaks in the growth equation, and can be estimated by a straightforward application of Chow's analysis of covariance.

[5] An alternative method of deriving an "impossibility" theorem was developed by Ryuzo Sato, "The Estimation of Biased Technical Progress and the Production Function," paper presented to the Econometric Society, December 1964.

Returning to equation (3) and remembering that it is continuous by subregions, Nerlove's results then hold by subregions; i.e., the smoothness condition for identification need only hold within subregions. Hence, if there are structural breaks in the growth equation which are ordinary discontinuities, and if that function is sectionally continuous, and if factor-augmenting technological change is smooth within regions, we obtain different elasticities of substitution (these differences may be interpreted as technological changes, also), and we can identify the production function within regions.