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The Role of Time Series Analysis in Econometric Model Evaluation

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1. Introduction

The purpose of this paper is to consider the role of modern methods of time series analysis in the evaluation of econometric models. Despite the fact that econometric models are frequently based on time series data, classical regression and related methods are almost always used in parameter estimation and hypothesis testing. An approach to econometric model evaluation which draws heavily on time series methods generally, and spectral methods in particular, is summarized in this paper.

This paper is organized as follows. In the next section the general approaches of classical econometrics and time series analysis are contrasted. This comparison provides the motivation for the view that time series methods can play an important role in econometric model evaluation. The following section introduces several basic concepts of univariate time series analysis including the power spectrum and shows how these can be used in model evaluation. Section 4 is devoted to multivariate time series analysis. Section 5 contains an analysis of aggregate consumption data which illustrates the use of time series techniques to evaluate a simple model. The paper concludes with a brief summary of the potential role of time series methods in econometric model evaluation.

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2. Evaluation of Dynamic Econometric Models

Econometrics is concerned with drawing inferences about economic relationships from observed data. The general approach of classical econometrics to the problem of inference is succinctly summarized by Johnston (1972, pp. 5–6)* as follows.

> The first step in the process is the specification of the model in mathematical form, for . . . the *a priori* restrictions derived from economic theory are not usually sufficient to yield a precise mathematical form. Next we must assemble appropriate and relevant data from the economy or sector that the model purports to describe. Thirdly we use the data to estimate the parameters of the model and finally we carry out tests on the estimated model in an attempt to judge whether it constitutes a sufficiently realistic picture of the economy being studied or whether a somewhat different specification has to be estimated.

Goldberger (1964, p. 4) also emphasizes the crucial importance of the specification of the model.

> Once we have a specification of a parent population we may rely on the rules and criteria of statistical inference in order to develop a rational method of measuring a relationship of economic theory from a given sample of observations. In many cases we may rely also on previous theoretical or empirical knowledge about the values of parameters of the population. Such a priori information is a characteristic feature of economic theory.

Thus the traditional econometric approach begins with the presumption that economic theory or "previous" empirical knowledge is sufficient to specify a hypothetical model. Appropriate estimation methods are determined by the hypothetical model, including the stochastic specification of the disturbance process.

Grenander & Rosenblatt (1957, pp. 115–116) suggest that modern time series methods have been developed to deal with rather different situations.

> One difficulty in many of the applications of time series is that there is very little theory built up from experience so that one is not led to well specified schemes. In such fields it seems more promising to use empirical data to form confidence regions for the models than to test sharply defined models whose validity is questionable to say the least.

> These testing problems may have some theoretical interest, but they are seldom relevant to problems arising in practice.
>
> In engineering and in the physical sciences there has been a strong demand for realistic methods to analyse stationary time series. We believe that the general approach taken by research workers in these fields is more promising and in closer contact with reality than some of the earlier techniques developed by theoreticians. ... This approach consists in not specifying the model very much, and instead of dealing with a finite number of parameters one considers the spectral density or some similar nonparametric concept.

The importance of prior knowledge, or lack thereof, is also emphasized by Priestley (1971, p. 295).

> ... we may contrast the problem of using input/output data to estimate (or "identify") the transfer function of a linear time-invariant "black box" in the context of control systems ... with the problem of estimating a linear relationship between two economic time series .... Although the two problems are often statistically identical, in the former case one usually has substantial knowledge of the physical mechanism underlying the "black-box", whereas in the latter case one has virtually zero prior knowledge.

We need not agree with Priestley's implicit evaluation of economic theory to conclude that the appropriate methods of analysis will depend upon how much prior knowledge is available.

A simple example is sufficient to demonstrate the crucial point that appropriate estimation and hypothesis testing procedures depend on the model that is specified. Suppose that we wish to test the hypothesis that there is no relationship between the two time series $\{y_t\}$ and $\{x_t\}$. If the alternative to this null hypothesis is the simple distributed lag model,

$$y_t = \phi y_{t-1} + \psi x_t + u_t, \qquad t = \pm 1, \pm 2, \ldots, \tag{1}$$

where $\{u_t\}$ is a sequence of independent identically distributed $N(0, \sigma^2)$ random variables and x_t and u_s are independent for all t and s, then we can simply test the hypothesis that $\psi = 0$ against the alternative that $\psi \neq 0$. It would surely be inefficient to employ a more general model if (1) is indeed the correct formulation. But if a more general alternative is appropriate, a test of the hypothesis that $\psi = 0$ in (1) may not be very informative about the relationship between $\{y_t\}$ and $\{x_t\}$ since it will be based on an incorrectly specified model.

This example illustrates the importance of model evaluation or validation. No matter what model is specified initially, it is important to verify to the extent possible that none of the basic assumptions of the model are violated.

This process of model validation can be carried out by relaxing certain maintained hypotheses in such a way that they are testable. This usually involves embedding the model in a more general framework and then testing the hypothesis that the more general formulation is unnecessary. Returning to the distributed lag example, we could examine the adequacy of (1) by asking if a more general formulation such as

$$y_t = \sum_{j=1}^{p} \phi_j y_{t-j} + \sum_{k=0}^{r} \psi_k x_{t-k} + u_t + \sum_{l=1}^{q} \theta_l u_{t-l} \tag{2}$$

is unnecessary. It might be objected that this process confounds hypothesis testing and hypothesis searching. This is certainly possible, especially if the original maintained hypotheses are subsequently rejected. But the alternative of never examining maintained hypotheses is equally unattractive in many cases.

Time series methods can be useful for econometric model evaluation precisely because they have been developed, at least in part, to deal with situations in which little prior knowledge is available. To the extent that an econometric model can be embedded in a more general time series framework, time series methods can be used to determine if the more general formulation is necessary. Stated the other way around, if the assumptions of a structural econometric model place restrictions on a more general time series model, the time series model will provide a vehicle to test the validity of those restrictions, and hence the adequacy of the econometric model.

Returning again to the distributed lag model, suppose (2) is proposed as an alternative to (1). The classical econometric approach to (2) would be to assume that economic theory or other prior knowledge determines the values of p, q, and r, that x_t and u_s are independent for all values of t and s, and that the disturbance term u_t is a serially independent (normal) random variable. Such a complete parametric specification of the model relating $\{y_t\}$ and $\{x_t\}$ leads directly to classical (likelihood) estimation and hypothesis testing procedures. This is not to say that either the theory or the application of these (likelihood) methods is trivial in this case; this is indeed an important topic in time series analysis.

The time series approach to modeling typically involves a slightly weaker set of assumptions. It might be appropriate, for example, to assume that $\{u_t\}$ is a sequence of independent and identically distributed $N(0, \sigma^2)$ random variables and that (2) is the correct specification for some finite but unknown values of the parameters p, q, and r. This is the modeling approach that underlies the Box & Jenkins (1970) procedures. Alternatively, one might assume that $\{u_t\}$ is a sequence of independent identically distributed $(0, \sigma^2)$ random variables and p, q, and r are infinite. This is essentially the approach pursued by Parzen (1974). In this latter case, any finite parameter model is viewed as an approximation to the correctly specified, infinite-parameter model. If prior knowledge is even more limited, it may be appropriate to impose no more structure on the relationship between $\{y_t\}$ and $\{x_t\}$ than the restriction that the two processes have the representation

$$x_t = \theta_{11}(L)u_{1t} + \theta_{12}(L)u_{2t}, \tag{3}$$

$$y_t = \theta_{21}(L)u_{1t} + \theta_{22}(L)u_{2t}, \tag{4}$$

where u_{jt} are mutually and serially uncorrelated processes and $\theta_{ij}(L)$ are polynomials of the form

$$\theta_{ij}(L) = \sum_{k=0}^{\infty} \theta_{ijk}L^k$$

in the lag operator L defined by $L^k x_t = x_{t-k}$. Notice that the system (3)–(4) includes (1) and (2), as well as all the intermediate cases discussed above, as special cases. In particular, if $\theta_{12}(L) \equiv 0$, then (4) can be rewritten as

$$\begin{aligned}\theta_{11}(L)y_t &= \theta_{11}(L)\theta_{21}(L)u_{1t} + \theta_{11}(L)\theta_{22}(L)u_{2t} \\ &= \theta_{21}(L)x_t + \theta_{11}(L)\theta_{22}(L)u_{2t}.\end{aligned} \tag{5}$$

If $\theta_{11}(L)$, $\theta_{22}(L)$, and $\theta_{21}(L)$ are polynomials of degree p, q, and r, respectively, then (5) and (2) are equivalent.

It is now well known that the standard linear econometric model can be embedded in a more general time series model. In particular, Zellner & Palm (1974), Zellner (1975), Wallis (1977), and others have shown that under certain conditions the standard linear econometric model is a special case of multivariate autoregressive, moving-average (ARMA) models of the type studied by Quenouille (1957), Parzen (1969), Hannan (1970), and others. Following Zellner, let z_t denote a vector of variables generated by[1]

$$\Phi(L)z_t = \Theta(L)u_t, \tag{6}$$

where $\{u_t\}$ is a sequence of independent identically distributed $N(0, \Sigma)$ vector random variables of unobserved disturbances, and $\Phi(L)$ and $\Theta(L)$ are matrices of polynomials in the lag operator L, i.e.,

$$\Phi(L) = \sum_{j=0}^{p} \Phi_j L^i, \tag{7}$$

$$\Theta(L) = \sum_{j=0}^{q} \Theta_j L^j, \tag{8}$$

where Φ_j and Θ_j are matrices of coefficients.

[1] For simplicity, it is assumed that there are no purely deterministic variables in the system.

The multivariate ARMA model imposes a considerable amount of structure on the process. However, it does not distinguish between endogenous and exogenous variables, a basic feature of econometric modeling. Suppose the system of equations is partitioned according to

$$\begin{bmatrix} \Phi_{11}(L) & \Phi_{12}(L) \\ \Phi_{21}(L) & \Phi_{22}(L) \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} \Theta_{11}(L) & \Theta_{12}(L) \\ \Theta_{21}(L) & \Theta_{22}(L) \end{bmatrix} \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix} \tag{9}$$

and the restrictions $\Phi_{21}(L) \equiv 0$, $\Theta_{12}(L) \equiv 0$, and $\Theta_{21}(L) \equiv 0$ are imposed. Then the system simplifies to

$$\Phi_{11}(L)y_t + \Phi_{12}(L)x_t = \Theta_{11}(L)u_{1t}, \tag{10}$$

$$\Phi_{22}(L)x_t = \Theta_{22}(L)u_{2t}. \tag{11}$$

Suppose, in addition, that $\{u_{1t}\}$ and $\{u_{2t}\}$ are independent. Then the first set of equations in this system contains the vector y_t of endogenous variables and the vector x_t of exogenous variables and corresponds to the *structural form* of the standard linear dynamic simultaneous equations econometric model. The second equation, which describes how the exogenous variables are generated, is usually omitted from the econometric model. However, if the model is to be used for *ex ante* prediction, a forecasting model for any stochastic exogenous variables is required. Notice that the independence of $\{u_{1t}\}$ and $\{u_{2t}\}$ implies that $\{x_t\}$ is independent of $\{u_{1t}\}$ and hence is exogenous. Thus under this specification the simultaneous equations econometric model is a block recursive multivariate ARMA time series model.

The structural form of an econometric model is typically used to test hypotheses about economic behavior. The distinguishing feature of the structural form of the simultaneous block is the fact that current values of the endogenous variables appear in more than one equation. That is, if the operator $\Phi_{11}(L)$ is written as

$$\Phi_{11}(L) = \sum_{j=0}^{p_{11}} \Phi_{11j}L^j, \tag{12}$$

then Φ_{110} need not and generally will not be the identity matrix. If the simultaneous block of equations is premultiplied by Φ_{110}^{-1}, then the *reduced form* of the system is obtained. The reduced form, denoted by

$$\bar{\Phi}_{11}(L)y_t + \bar{\Phi}_{12}(L)x_t = \bar{\Theta}_{11}(L)u_{1t}, \tag{13}$$

is generally used for purposes of conditional prediction and control.

For illustrative purposes, consider the reduced-form model

$$y_{1t} = \alpha_1 y_{1t-1} + \alpha_2 y_{2t-1} + u_{1t}, \tag{14}$$

$$y_{2t} = \beta_1 y_{1t-1} + \beta_2 x_{1t} + u_{2t}, \tag{15}$$

with two endogenous variables y_{1t} and y_{2t} and one exogenous variable x_{1t}. This model can be written in operator notation as

$$\begin{bmatrix} 1-\alpha_1 L & -\alpha_2 L \\ -\beta_1 L & 1 \end{bmatrix}\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} - \begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} x_{1t} = \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix}, \tag{16}$$

which is a specific example of the general form given in (13). Conditional one-period ahead forecasts of y_{1t} and y_{2t} are obtained from this model by setting $u_{1t} = u_{2t} = 0$ in (14) and (15) and solving for y_{1t} and y_{2t} in terms of y_{1t-1}, y_{2t-1}, and $\hat{x}_{1t}$, the predicted value of x_{1t}. Such forecasts are referred to as *ex post* forecasts if $\hat{x}_{1t}$ is the actual realized value of this variable and *ex ante* forecasts if the value assigned to x_{1t} is a predicted value of x_{1t}. The prediction error variance for y_{1t} is σ_{11}, the variance of u_{1t}, for both *ex ante* and *ex post* forecasts. The prediction error variance for y_{2t} is σ_{22} for *ex post* forecasts and $\sigma_{22} + \beta_2^2 E(x_{1t} - \hat{x}_{1t})^2$ for *ex ante* forecasts.

From the point of view of model evaluation, the interesting feature of the multivariate ARMA model or reduced-form econometric model is that these models place restrictions on the transfer functions and univariate ARMA representations of the variables. The reduced-form system implies a set of *transfer functions* which relate each of the endogenous variables to lagged values of that endogenous variable and to current and lagged exogenous variables.[2] These transfer functions are obtained by premultiplying the reduced-form equations by $\bar{\Phi}_{11}^*(L)$, the adjoint of $\bar{\Phi}_{11}(L)$. This yields

$$\bar{\Delta}(L)y_t = -\bar{\Phi}_{11}^*(L)\bar{\Phi}_{12}(L)x_t + \bar{\Phi}_{11}^*(L)\bar{\Theta}_{11}(L)u_{1t} = \bar{\Psi}(L)x_t + \bar{\Gamma}(L)u_{1t} \tag{17}$$

where $\bar{\Delta}(L) = |\bar{\Phi}_{11}(L)|$. These transfer functions can be used to predict y_{jt}, given lagged values of y_{jt} and current and lagged values of x_t. However, these predictions will generally be inferior to the predictions from the reduced-form model since they are based on a smaller information set. The reduced-form forecasts use lagged values of all the endogenous variables, whereas the transfer-function forecasts use only the lagged values of one of the endogenous variables.

The transfer functions for the expository model can be obtained as follows. The adjoint of the matrix

$$\Phi_{11}(L) = \begin{bmatrix} 1-\alpha_1 L & -\alpha_2 L \\ -\beta_1 L & 1 \end{bmatrix} \tag{18}$$

is

$$\Phi_{11}^*(L) = \begin{bmatrix} 1 & \alpha_2 L \\ \beta_1 L & 1-\alpha_1 L \end{bmatrix}. \tag{19}$$

[2] What are referred to here as transfer functions are called fundamental dynamic equations by Kmenta (1971, p. 591).

When (16) is premultiplied by $\Phi^*_{11}(L)$, the result is

$$y_{1t} = \alpha_1 y_{1t-1} + \alpha_2\beta_1 y_{1t-2} + \alpha_2\beta_2 x_{1t-1} + u_{1t} + \alpha_2 u_{2t-1}, \tag{20}$$

$$y_{2t} = \alpha_1 y_{2t-1} + \alpha_2\beta_1 y_{2t-2} + \beta_2 x_{1t} - \alpha_1\beta_2 x_{1t-1} + u_{2t} + \beta_1 u_{1t-1} - \alpha_1 u_{2t-1}. \tag{21}$$

The transfer function for y_{1t} clearly illustrates the general proposition that transfer-function forecasts are inferior to reduced-form forecasts. The *ex post* transfer-function forecast error is $u_{1t} + \alpha_2 u_{2t-1}$ with variance $\sigma_{11} + \alpha_2^2\sigma_{22}$, which clearly exceeds σ_{11} (assuming both α_2 and σ_{22} are nonzero).[3] Notice also that the reduced form places restrictions on the orders of the polynomial operators in the transfer functions. The transfer function for y_{2t}, for example, involves $y_{2t-1}, y_{2t-2}, x_{1t}$, and x_{1t-1}. In addition, the composite disturbance term

$$v_{2t} = u_{2t} + \beta_1 u_{1t-1} - \alpha_1 u_{2t-1} \tag{22}$$

is the sum of two moving-average processes and is itself a moving-average process.[4] Thus the reduced-form specification places a number of testable restrictions on the transfer-function relationships.

A link between the multivariate model and univariate autoregressive, moving-average representations for each of the variables in the general linear dynamic econometric model is provided by the *final equations*.[5] If the original system (6) is multiplied by $\Phi^*(L)$, the adjoint of $\Phi(L)$, the system

$$\Delta(L)z_t = \Gamma(L)u_t \tag{23}$$

is obtained, where $\Delta(L) = |\Gamma(L)|$ and $\Gamma(L) = \Phi^*(L)\Theta(L)$. In particular, if $\Gamma_j(L)$ denotes the jth row of $\Gamma(L)$, then z_{jt} can be written as

$$\Delta(L)z_{jt} = \Gamma_j(L)u_t = \sum_{k=1}^{n} \Gamma_{jk}(L)u_{kt}. \tag{24}$$

Since the disturbance process in this equation is a sum of moving-average processes, it is itself a moving-average process and, hence, the process can be

[3] This model provides a simple example of the more general problem considered by Pierce (1975).

[4] For a further discussion of this point, see Ansley, Spivey, & Wrobleski (1977) and Palm (1977).

[5] The term *final equation* is taken from Zellner & Palm (1974) and should not be confused with what Kmenta (1971, p. 592) calls the *final form* of the equation system.

written as

$$\Delta(L)z_{jt} = \Psi_j(L)v_{jt} \tag{25}$$

where $\{v_{jt}\}$ is a sequence of independent identically distributed $N(0, \zeta_{jj})$ random variables. The final equations show that the multivariate model with stochastic exogenous variables implies that each variable in the model has a unique univariate autoregressive moving-average representation. Moreover, each variable has the same autoregressive part (unless there are common factors in $\Delta(L)$ and $\Gamma_{jk}(L)$) but generally different moving-average parts.

In order to obtain final equations for the expository model consisting of Eqs. (14) and (15), it is necessary to augment the model with a stochastic equation for the exogenous variable.[6] Suppose that the equation for the exogenous variable x_{1t} is

$$x_{1t} = u_{3t}, \tag{26}$$

where $\{u_{3t}\}$ is a sequence of independent identically distributed $N(0, \sigma_{33})$ random variables. The final equations can be obtained as follows. The adjoint of the matrix

$$\Phi(L) = \begin{bmatrix} 1-\alpha_1 L & -\alpha_2 L & 0 \\ -\beta_1 L & 1 & -\beta_2 \\ 0 & 0 & 1 \end{bmatrix} \tag{27}$$

is

$$\Phi^*(L) = \begin{bmatrix} 1 & \alpha_2 L & \alpha_2\beta_2 L \\ \beta_1 L & 1-\alpha_1 L & \beta_2 - \alpha_1\beta_2 L \\ 0 & 0 & 1-\alpha_1 L - \alpha_2\beta_1 L^2 \end{bmatrix}. \tag{28}$$

Thus the final equations, after cancellation of common factors, are

$$y_{1t} = \alpha_1 y_{1t-1} + \alpha_2\beta_1 y_{1t-2} + u_{1t} + \alpha_2 u_{2t-1} + \alpha_2\beta_2 u_{3t-1}, \tag{29}$$

$$y_{2t} = \alpha_1 y_{2t-1} + \alpha_2\beta_1 y_{2t-2} + u_{1t-1} + u_{2t} - \alpha_1 u_{2t-1} + \beta_2 u_{3t} - \alpha_1\beta_2 u_{3t-1}, \tag{30}$$

$$x_{1t} = u_{3t}. \tag{31}$$

[6] If the exogenous variable x_{1t} is deterministic, this step is not possible, and we would simply leave the model in the ARMAX (see Hannan (1976)) form as shown in (20) and (21). That is, the univariate models would include the exogenous variable x_{1t}, and univariate ARMA models would not exist for the endogenous variables.

This example illustrates the general result that univariate ARMA forecasts are inferior to transfer-function forecasts. The one-step ahead prediction error variance for y_{1t} using (29) is $\sigma_{11} + \alpha_2^2\sigma_{22} + \alpha_2^2\beta_2^2\sigma_{33}$, which exceeds the prediction error variance of the transfer-function forecast for y_{1t}. Notice also that the reduced-form model imposes restrictions on the order of the autoregressive and moving-average parts of the univariate representations of the variables in the model.

As we have seen, a structural dynamic economic model imposes testable restrictions on the transfer functions and final equations of the model. If the simultaneous equations specification is correct, these restrictions should be satisfied by the data. More generally, an estimated econometric model can be used to derive implications about prediction error variances and dynamic properties of the endogenous variables. If the model is correctly specified, these implied properties should be consistent with direct estimates of these same characteristics. The next two sections are concerned with those aspects of univariate and multivariate time series analysis that appear to be most promising for econometric research and model evaluation. No attempt is made to provide an exhaustive review of the literature. Hopefully, however, enough material is included to enable the reader to see how time series methods can be used to evaluate econometric models.

3. Univariate Time Series Analysis

Econometric research is primarily concerned with the estimation of relationships among variables. Nevertheless, the analysis of individual time series is still very important. For example, univariate methods are of interest in connection with: (a) development of benchmark forecasting models, (b) testing restrictions imposed on the data by simultaneous equation models, (c) development of formulas for expectational variables, and (d) modeling disturbance processes in multiple regression models. This section is devoted to a brief discussion of four important areas of univariate time series analysis:

(i) descriptive measures for time series including the covariance function and the power spectrum,
(ii) tests for serial correlation in a time series,
(iii) estimation of the covariance function of a stochastic process, and
(iv) identification and estimation of autoregressive, moving-average models for stationary time series.

Throughout this section, it is assumed that the stochastic processes from which samples are obtained are covariance-stationary. A covariance-stationary process is a process for which the mean, variance and autocovariances do not depend on calendar time. In particular, stationarity rules out any trend in the mean or variance of the series. This is not usually thought to be an especially restrictive assumption when applied to the stochastic disturbance term in a time series regression model. It is a restrictive assumption when applied to economic time series data directly since most economic time series exhibit pronounced trends in the mean. It is usually necessary to transform the raw series by detrending in order to obtain a stationary series. It is assumed throughout this section that an appropriate transformation has been applied to the data, if necessary, to produce a series that satisfies the stationarity assumption.

3.1. Descriptive Measures for Time Series

The (sample) mean and variance are frequently used to summarize important characteristics of a random sample. The *sine qua non* of time series analysis is the (potential) existence of serial correlation in the processes that are sampled. Some descriptive measure of the correlation pattern in a sample (or population) is therefore of interest.

The autocovariance function of the stationary process $\{x_t\}$ with mean μ, defined by

$$\gamma(s) = E(x_{t+|s|} - \mu)(x_t - \mu), \qquad s = 0, \pm 1, \pm 2, \ldots, \tag{32}$$

provides one way to summarize this correlation pattern. An alternative and sometimes very useful way to summarize the correlation pattern is provided by the power spectrum. The power spectrum, provided it exists,[7] is defined by

$$f(\omega) = \sum_{s=-\infty}^{\infty} \gamma(s) \exp(-i\omega s), \qquad -\pi \leq \omega \leq \pi, \tag{33}$$

where $\exp(-i\omega s) = \cos(\omega s) - i \sin(\omega s)$. The descriptive appeal of the power spectrum derives from the fact that [8]

$$\gamma(s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(i\omega s) f(\omega)\, d\omega \tag{34}$$

[7] A sufficient condition for the spectrum to exist is that the autocovariance function be absolutely summable (Fuller, 1976, p. 127).

[8] The function $f(\omega)$ as defined in (33) is the Fourier transform of the autocovariance function $\gamma(s)$. The relationship in (34) is the inverse Fourier transform.

and, in particular,

$$\gamma(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\omega)\, d\omega. \tag{35}$$

Thus although there is a one-to-one relationship between the autocovariance function and the spectrum, and hence the same information is conveyed by both, the spectrum has an analysis of variance interpretation. The quantity $f(\omega)\,d\omega$ measures the contribution to the variance of $\{x_t\}$ of the interval $(\omega, \omega + d\omega)$.

Although it is not obvious, the argument ω has an interpretation in terms of sinusoidal oscillations per unit of time. Recall that the time function

$$y_t = a\cos(\lambda t) + b\sin(\lambda t) \tag{36}$$

is a perfectly periodic function with *period* $p = 2\pi/\lambda$ that is, $y_{t+kp} = y_t$ for all integer values of k. The *frequency* of this periodic function is the fraction $1/p = \lambda/2\pi$ of a cycle that is completed per unit of time. For example, if t is measured in years and the period is ten years, the frequency is one tenth of a cycle per year.[9]

Now suppose a and b are independent, zero-mean random variables with variance 1. Then $\{y_t\}$ is a random variable with mean zero and covariance function

$$\gamma(s) = \cos\lambda s. \tag{37}$$

Strictly speaking, the power spectrum does not exist for this process, but we can consider the approximation[10]

$$f_\beta(\omega) = \sum_{s=-\infty}^{\infty} \beta^s \cos\lambda s \exp(-i\omega s) \tag{38}$$

as a function of β. As the parameter β approaches one, $f_\beta(\omega)$ approaches the spectrum of $\{y_t\}$. The graph of $f_\beta(\omega)$ is shown in Fig. 1 for $\beta = .9$ and .95. The graphs in Fig. 1 indicate that the limiting value of the power spectrum is a sharp spike at the frequency $\lambda/2\pi$. This indicates that all of the variation in $\{y_t\}$ is due to variation at this frequency. This is, of course, precisely

[9] With equally spaced data points, the highest observable frequency of oscillation is one-half cycle per time unit. Cyclical variations with a period shorter than two units of time appear as longer cycles in the discrete data.

[10] As long as $|\beta| < 1$, the power spectrum will be defined since $\beta^s \cos\lambda s$ is absolutely summable.

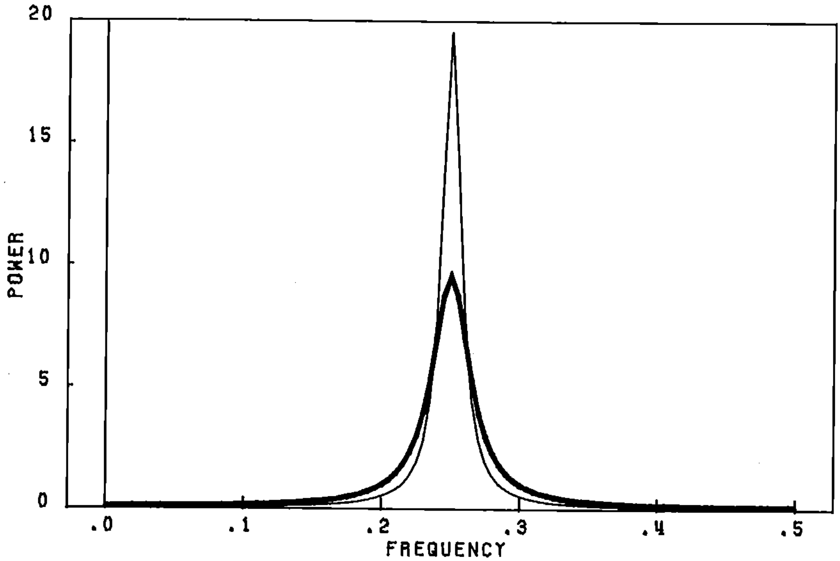

Fig. 1. Power spectrum transformation of $\beta^s \cos(\lambda s)$. Bold curve: $\beta = .90$; light curve: $\beta = .95$.

what would be expected from the definition of $\{y_t\}$ since any one realization of the process is a simple sinusoidal oscillation.[11]

Processes such as that defined in (36) are not very important in economics, except for illustrative purposes. A more useful model for time series, especially for disturbances in regression models, is the first-order autoregressive model,

$$y_t = \rho y_{t-1} + u_t, \qquad |\rho| < 1, \tag{39}$$

where $\{u_t\}$ is a sequence of independent identically distributed random variables. Power spectra of $\{y_t\}$ are shown in Fig. 2 for three different values of ρ. For $\rho = 0$, $\{y_t\}$ is serially uncorrelated so that $\gamma(s) = 0$ for $s \neq 0$. The power spectrum in this case is constant, indicating that all frequencies of oscillation contribute equally to the variance of the uncorrelated sequence. This is an important benchmark case. With $\rho = .7$, the spectrum is a decreasing function of frequency indicating that low-frequency variations contribute more to the variance of $\{y_t\}$ than do high-frequency variations. Conversely, with $\rho = -.7$ the spectrum is an increasing function of frequency and high-frequency variations are dominant.

[11] A realization of $\{y_t\}$ is obtained by fixing the values of the random variables a and b. Once these values are fixed, y_t is determined by (36) for all values of t.

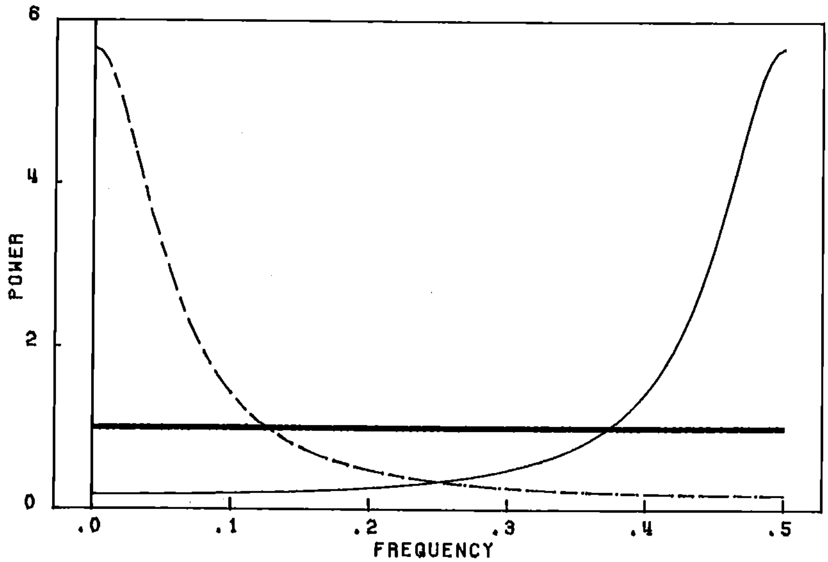

Fig. 2. Power spectrum of $y_t = y_{t-1} + u_t$. Light curve: $\rho = -.7$; bold curve: $\rho = 0$; dashed curve: $\rho = .7$.

A graph of the spectrum reveals at a glance whether or not the series is uncorrelated, contains strong seasonal fluctuations, or exhibits strong business-cycle variation. These characteristics of the series may not be obvious to the untrained eye from an examination of the series itself, the autoregressive, moving-average representation of the process, or the (sample) autocovariance function of the series. Thus the spectrum provides a useful visual aid in describing a time series.

The power spectrum also provides a convenient way to examine the effect of linear (moving-average) operations on a time series. For example, suppose that $\{y_t\}$ is related to $\{x_t\}$ by

$$y_t = \sum_{j=-\infty}^{\infty} w_j x_{t-j}. \tag{40}$$

Then the power spectrum of $\{y_t\}$ is related to the spectrum of $\{x_t\}$ by

$$f_y(\omega) = G(\omega) f_x(\omega), \tag{41}$$

where

$$G(\omega) = |\textstyle\sum w_j \exp(i\omega j)|^2. \tag{42}$$

The function $G(\omega)$ is referred to as the gain of the relationship. The cyclical characteristics of the transformed series $\{y_t\}$ will depend in part on the

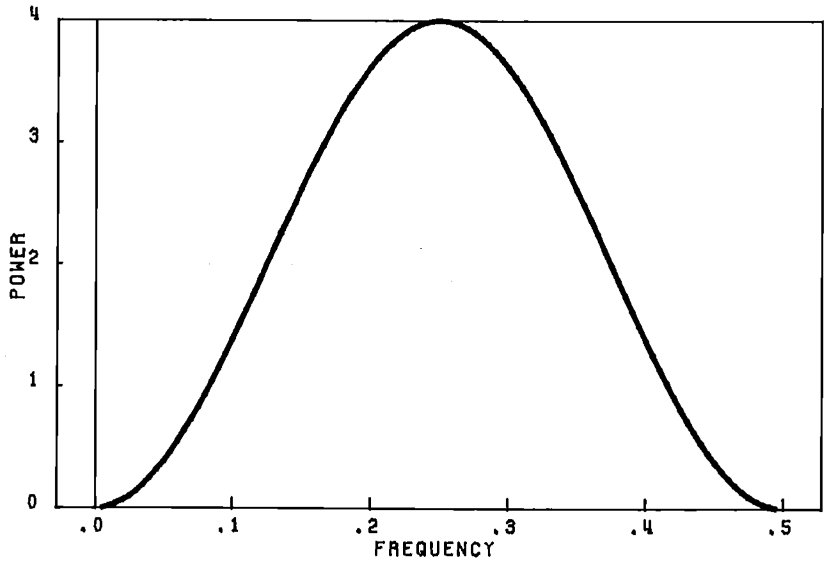

Fig. 3. Gain function for $y_t = x_t - x_{t-2}$.

correlation pattern of the original series and in part on the transformation that is used. An examination of the function $G(\omega)$ of the transformation will reveal the nature of the smoothing that is implicit in the linear operation. Consider, for example, the centered first difference transformation defined by

$$y_t = x_{t+1} - x_{t-1}. \tag{43}$$

In this case, $G(\omega)$ is

$$G(\omega) = 2(1 - \cos 2\omega), \tag{44}$$

the graph of which is shown in Fig. 3. If this centered first difference were applied to a serially uncorrelated process, the resulting series would be serially correlated and would contain relatively little long- or short-run variations but rather pronounced intermediate-run variation.

Spectral methods have been used in this way in economics to study the effect of filtering on the cyclical characteristics of time series,[12] the properties of seasonal adjustment procedures,[13] and the dynamic characteristics of macroeconometric models.[14]

[12] See, for example, Howrey (1968) and Fishman (1969).

[13] For examples of the use of spectral methods to study the properties of seasonal adjustment procedures, see Nerlove (1964), Godfrey & Karreman (1967), and Granger (1978).

[14] Spectral methods have been used to study the dynamic properties of econometric models by Chow (1975), Dhrymes (1970), Howrey (1971, 1972), and Howrey & Klein (1972).

3.2. Tests for Serial Correlation

Tests for serial correlation of the disturbances in a regression model are an extremely important aspect of model evaluation. The Durbin–Watson test is probably the most well known and widely used test for serial correlation in econometrics. As Durbin (1967) and others have noted, however, the Durbin–Watson test is not a very powerful test for serial correlation in a model like

$$u_t = \lambda u_{t-2} + v_t, \tag{45}$$

where $\{v_t\}$ is a sequence of independent and identically distributed random variables. For this model, the first-order serial correlation coefficient is zero but the $\{u_t\}$ sequence is not serially uncorrelated. On the other hand, if the Durbin–Watson test leads to a rejection of the null hypothesis, it is not necessarily appropriate to assume that a first-order autoregressive alternative is appropriate. Indeed, there is some evidence that the Durbin–Watson test is rather powerful over a much wider range of alternatives than the first-order autoregressive model.[15]

Several tests for serial correlation based on an estimate of the spectrum have been proposed in the literature. Durbin (1969), for example, has developed a test based on the cumulated periodogram. The basic idea of this test is that if the time series is uncorrelated, the spectrum will be a constant as in Fig. 2 (with $\rho = 0$) and the normalized cumulative spectrum,

$$F(\omega) = \frac{1}{\gamma(0)} \int_0^{\omega} f(\lambda)\, d\lambda, \tag{46}$$

will trace out a 45° line. If an estimate $F(\omega)$ based on the regression residuals deviates significantly from the 45° line, the null hypothesis of no serial correlation is rejected. It should be obvious that this test, at least in principle, is capable of detecting a wider range of departures from the null hypothesis than the standard Durbin–Watson test.

The point of these more general tests for serial correlation, such as the Durbin periodogram test or the Box and Pierce (1970) "portmanteau" test, is that if little or nothing is known about the nature of potential departures from the null hypothesis, a test that is sensitive to a wide range of alternatives is desirable. Test procedures based on spectrum estimates seem to satisfy this requirement rather well.

[15] See, for example, Smith (1976).

3.3. Consistent Estimation of the Covariance Matrix

There are important situations in econometrics in which a consistent estimator of a covariance matrix is required to obtain an asymptotically efficient estimator of the parameters of a regression model. Consider, for example, the model

$$y = X\beta + u \tag{47}$$

where y is a $T \times 1$ vector of observations on the dependent variable, X is a $T \times k$ matrix of observations on the explanatory variables distributed independently of u, β is a $k \times 1$ vector of regression parameters, and u is a $T \times 1$ vector of unobserved disturbances. If $u \sim (0, \Sigma)$ where Σ is not equal to $\sigma^2 I$, the least squares estimator is generally inefficient relative to the Aitken estimator. If Σ is not known the preferred estimator is the feasible Aitken estimator

$$\hat{\beta} = (X'\hat{\Sigma}^{-1}X)^{-1}X'\hat{\Sigma}^{-1}y, \tag{48}$$

where $\hat{\Sigma}$ is a consistent estimator of Σ.

The problem is to obtain a consistent estimator of Σ. In a time series context, $u' = [u_1\, u_2 \ldots u_T]$ and

$$\Sigma = E(uu') = \begin{bmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(T-1) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(T-2) \\ \vdots & \vdots & & \vdots \\ \gamma(T-1) & \gamma(T-2) & \cdots & \gamma(0) \end{bmatrix}, \tag{49}$$

where $\gamma(s) = E(u_{t+|s|}u_t)$. Thus if no restrictions are imposed on the covariance matrix Σ, there are $T + k$ parameters to be estimated from T observations. The traditional solution to this problem as described in the econometrics literature is to impose some restrictions on Σ. For example, it is frequently assumed that $\{u_t\}$ is generated by a first-order autoregressive process so that $\gamma(s) = \sigma^2\rho^s$. This restriction reduces the number of parameters to be estimated and effectively solves the estimation problem.

An alternative approach which is especially attractive when little is known about the form of the disturbance process is based on the following result.[16] As $T \to \infty$ characteristic roots of Σ are equal to the values of the power spectrum at the harmonic frequencies $\omega_j = 2\pi j/T$. The corresponding matrix of characteristic vectors is $W = (w_{jk})$ with elements $w_{jk} = T^{-1/2}$

[16] See, for example, Fuller (1976, Chapter 4).

$\exp(-2\pi ijk/T)$. In other words,

$$\lim_{T\to\infty} W^*\Sigma W = D, \tag{50}$$

where W^* is the conjugate transpose of W and D is a diagonal matrix with diagonal elements $f(2\pi j/T)$. It is easy to verify that W is what is called a unitary matrix, i.e., $W^*W = WW^* = I$, where W^* is the transpose of W with all elements replaced by their complex conjugates. Hence a consistent estimator of Σ is obtained from a consistent estimator of the spectrum using

$$\hat{\Sigma} = W\hat{D}W^*. \tag{51}$$

Since consistent estimation of the spectrum does not require a parametric specification of the disturbance process[17] a feasible Aitken estimator can be obtained in the absence of such a specification.[18]

3.4. Identification and Estimation of Autoregressive, Moving-Average Models

The univariate time series methods described up to this point do not rely on a finite parameter time domain model. In these situations, spectral methods have played a key role. Box & Jenkins (1970) have developed a set of techniques to deal with finite parameter autoregressive, moving-average models in which the orders of the autoregressive and moving-average operators are assumed to be unknown. This specification of the model relaxes the characteristic assumption of classical econometrics that the degree of the polynomial operators is known but is less extreme than the approach taken by Parzen in that a finite parameter model is retained.

In brief, the Box–Jenkins procedures for univariate time series involve an examination of the sample autocovariance and partial autocovariance functions[19] to determine the order of the autoregressive and moving-average operators. Once tentative values have been assigned to these parameters, maximum likelihood estimates of the coefficients are obtained. Finally, various diagnostic checks and overfitting procedures are employed to make sure that the model that is identified is consistent with the data.

[17] For a discussion of power spectrum estimation techniques, see for example, Jenkins & Watts (1968, Chapter 6).

[18] This is the basis of the procedures suggested by Hannan (1963, 1965). As Amemiya & Fuller (1967) show it is possible to develop a regression analogue to Hannan's estimator.

[19] The partial autocovariance function is the covariance of x_t and x_{t-s}, given $x_{t-1}, \ldots, x_{t-s+1}$, regarded as a function of s.

As Zellner (1975) has remarked, Box and Jenkins employ a somewhat informal approach to the model selection problem. Zellner has proposed the use of likelihood ratio tests and posterior odds ratios to aid in the selection of the appropriate model. This emphasis on the model selection aspect of these techniques serves to underscore the distinctive feature of time series approaches to modeling, namely, a model is determined through both *a priori* reasoning and data analysis. Since this modeling procedure makes careful and extensive use of the data, it provides a good way to make sure that an important characteristic of the data has not been overlooked.

4. Multivariate Time Series Analysis

Within the context of econometric modeling univariate time series analysis is useful for estimating the final equations of an econometric model and for modeling disturbance processes. Multivariate time series methods are of importance for the estimation and analysis of transfer functions and distributed lag models. This section begins with a brief introduction to some of the basic concepts of multivariate time series analysis. Following this introductory material, several applications of particular interest in econometrics are reviewed including tests for causality.

4.1. Distributed Lag Models

For expository purposes, consider the bivariate distributed lag model

$$y_t = \sum_{j=-\infty}^{\infty} \beta_j x_{t-j} + u_t, \qquad t = 0, \pm 1, \pm 2, \ldots, \tag{52}$$

where $\{x_t\}$ and $\{u_t\}$ are mutually independent stationary stochastic processes. To ensure that $\{y_t\}$ has a finite variance, we impose the restriction that the distributed lag coefficients $\{\beta_j\}$ are absolutely summable. This rather general model includes the distributed lag models discussed in Section 2 as special cases. This general formulation of the model might be appropriate if there were little theoretical knowledge or prior information about the relationship between $\{y_t\}$ and $\{x_t\}$, so that one has to search for an appropriate model.

Alternatively, statistical analysis of this more general model would provide a way of testing the validity of a simpler specification. One might, for example, want to test the hypothesis that the disturbance process $\{u_t\}$ is serially uncorrelated, given the general linear relationship between $\{y_t\}$

and $\{x_t\}$. Similarly, a test of the hypothesis that the distributed lag relationship is one-sided, i.e., $\beta_j = 0$ for $j < 0$, might be of interest. In some applications it might be appropriate to impose such restrictions at the outset and never investigate their validity; in other cases it might be very important to see if the data are consistent with such assumptions.

The spectral approach to the analysis of this model proceeds as follows.[20] It is not difficult to verify that the model implies the covariance relationships

$$\gamma_{yx}(s) = \sum_j \beta_j \gamma_{xx}(s-j), \qquad s = 0, \pm 1, \pm 2, \ldots, \tag{53}$$

$$\gamma_{yy}(s) = \sum_j \sum_k \beta_j \beta_k \gamma_{xx}(s+k-j) + \gamma_{uu}(s), \qquad s = 0, \pm 1, \pm 2, \ldots, \tag{54}$$

where $\gamma_{xx}(s)$ is the autocovariance function of $\{x_t\}$ and $\gamma_{uu}(s)$ is the autocovariance function of $\{u_t\}$. The spectral and cross-spectral functions are thus given by[21]

$$f_{yx}(\omega) = \sum_{s=-\infty}^{\infty} \gamma_{yx}(s) \exp(-i\omega s) = B(\omega) f_{xx}(\omega), \qquad -\pi \leq \omega \leq \pi, \tag{55}$$

$$f_{yy}(\omega) = \sum \gamma_{yy}(s) \exp(-i\omega s) = |B(\omega)|^2 f_{xx}(\omega) + f_{uu}(\omega), \qquad -\pi \leq \omega \leq \pi, \tag{56}$$

where $f_{xx}(\omega)$ and $f_{uu}(\omega)$ are the power spectra of $\{x_t\}$ and $\{u_t\}$ and $B(\omega)$ is defined by

$$B(\omega) = \sum_j \beta_j \exp(-i\omega j). \tag{57}$$

The first point to notice is that the convolution relationship $\sum \beta_j \gamma_{xx}(s-j)$ is transformed into a product relationship $B(\omega) f_{xx}(\omega)$. This results in important numerical simplifications when the method of moments is used to estimate the parameters. In particular, (55) and (56) can be rewritten as

$$B(\omega) = f_{yx}(\omega)/f_{xx}(\omega), \qquad -\pi \leq \omega \leq \pi, \tag{58}$$

$$f_{uu}(\omega) = f_{yy}(\omega) - |f_{yx}(\omega)|^2/f_{xx}(\omega), \qquad -\pi \leq \omega \leq \pi. \tag{59}$$

In addition, the inverse of (57) is

$$\beta_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} B(\omega) \exp(i\omega j)\, d\omega, \qquad j = 0, \pm 1, \pm 2, \ldots. \tag{60}$$

[20] For a more detailed discussion of the spectral approach to the distributed lag models, see Dhrymes (1971) and Fishman (1969), for example. This section is based in large part on Wahba (1969). The results can be readily generalized to the multivariate distributed lag case.

[21] The assumption that $\{x_t\}$ is a stationary stochastic process with an absolutely summable covariance function is very convenient but can be relaxed without undue difficulty (Hannan, 1970, Chapter 8).

However, if β_j is zero for all j outside the interval $-m+1 \leq j \leq m$, then (60) can be replaced by

$$\beta_j = (2m)^{-1} \sum_{s=-m+1}^{m} B(\pi s/m) \exp(i\pi sj/m), \qquad -m+1 \leq j \leq m. \tag{61}$$

Since consistent estimators of the spectrum and cross-spectrum based on the sample auto- and cross-covariance functions are readily available,[22] consistent estimation of $B(\omega)$, $f_{uu}(\omega)$, and β_j is possible using (55), (56), and (61).

For purposes of graphical presentation of the results, several additional statistics are usually presented. The coherence between $\{y_t\}$ and $\{x_t\}$ is defined by

$$C_{yx}^2(\omega) = 1 - f_{uu}(\omega)/f_{yy}(\omega). \tag{62}$$

The coherence at frequency ω is the fraction of the variance of $\{y_t\}$ at frequency ω explained by the linear relationship between $\{y_t\}$ and $\{x_t\}$. The function $B(\omega)$ is generally complex valued. Instead of graphing the real and imaginary parts of this function separately, the usual practice is to define what is called the gain function, given by

$$G(\omega) = |B(\omega)|, \tag{63}$$

and the phase function,

$$H(\omega) = \tan^{-1}[\operatorname{Im} B(\omega)/\operatorname{Re} B(\omega)], \tag{64}$$

where $\operatorname{Im} B(\omega)$ and $\operatorname{Re} B(\omega)$ denote the imaginary and real parts of $B(\omega)$. The interpretation of the gain function follows from (56) which shows that $G(\omega)^2$ is the factor by which the variance in $\{x_t\}$ at frequency ω is translated by the distributed lag model into variance in $\{y_t\}$ at frequency ω. The phase at frequency ω indicates the extent to which oscillations at frequency ω in $\{x_t\}$ lead or lag oscillations in $\{y_t\}$ at the same frequency.

As an illustrative example of these relationships, consider the Koyck distributed lag model introduced in (1). Transforming to the distributed lag form, the model can be expressed as

$$y_t = \sum \beta_j X_{t-j} + v_t, \tag{65}$$

where

$$v_t = \sum_{j=0}^{\infty} \phi^j u_{t-j} \tag{66}$$

[22] See Jenkins & Watts (1968, Chapter 9), for example.

and

$$\beta_j = \psi\phi^j, \qquad j = 0, 1, \ldots. \tag{67}$$

This parametric specification implies that

$$B(\omega) = \psi/(1 - e^{-i\omega}) = \psi(1 - \phi e^{i\omega})/|1 - \phi e^{-i\omega}|^2. \tag{68}$$

Hence the gain and phase functions are

$$G(\omega) = |\psi|/|1 - \phi e^{-i\omega}| \tag{69}$$

and

$$H(\omega) = \tan^{-1}[-\phi \sin\omega/(1 - \phi\cos\omega)]. \tag{70}$$

These two functions are graphed on Fig. 4 for $\psi = 1$ and $\phi = 0.9$. The gain function indicates that $\{y_t\}$ responds more strongly to low-frequency variations in $\{x_t\}$ than to high-frequency variations. The phase function shows that $\{y_t\}$ lags behind $\{x_t\}$ at all frequencies, but by varying amounts of time. On the assumption that the disturbances $\{u_t\}$ are serially uncorrelated, the disturbance spectrum is

$$f_{vv}(\omega) = \sigma_u^2/|1 - \phi e^{-i\omega}|^2. \tag{71}$$

Thus the spectrum of disturbances has the same general shape as the gain function shown in Fig. 4.

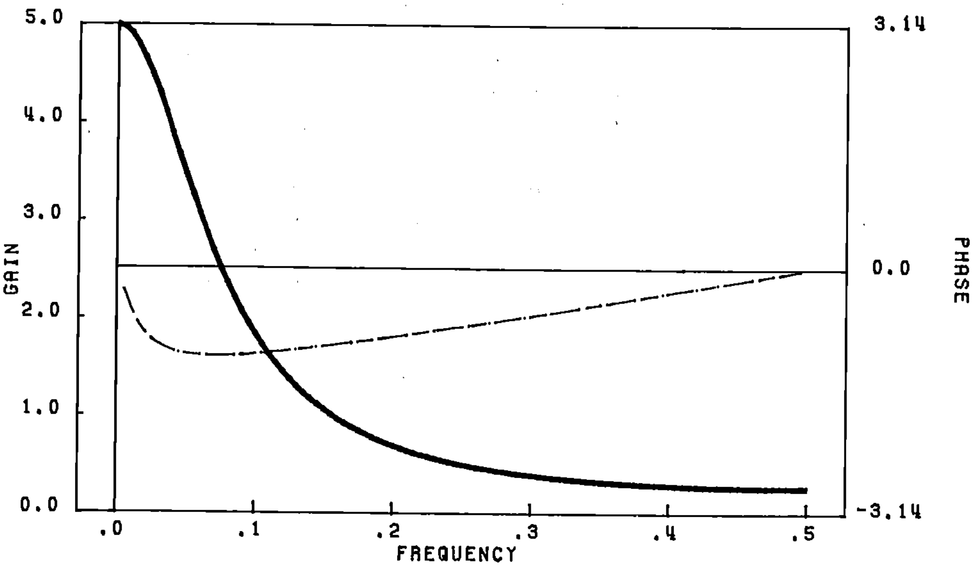

Fig. 4. Gain and phase functions for $y_t = .9y_{t-1} + x_t + u_t$. Bold curve: gain; dashed curve: phase.

Direct estimates of the gain, phase, and disturbance spectrum based on estimates of the auto- and cross-covariance functions can be used to evaluate the adequacy of a particular parametric specification. If the data are consistent with the hypothetical model, the spectral statistics should not differ significantly from the corresponding quantities implied by the model. Conversely, significant differences between direct estimates of the spectral quantities and those implied by the model indicate that the model specification is too restrictive. An example of this type of comparison is provided in Section 5 of this paper.

4.2. Tests for Causality

A distinguishing characteristic of econometric models is the classification of variables as endogenous or exogenous. As shown previously, this is achieved implicitly if not explicitly by imposing certain restrictions on the multivariate autoregressive, moving-average representation of the system. Until quite recently, the validity of such restrictions was not subjected to statistical testing.

In some econometric applications it is not obvious that an exogeneity restriction is valid. Granger (1969), Sims (1972), and others have suggested statistical procedures for testing for causality in bivariate relationships.[23] The basic idea is that if $\{y_t\}$ is causally related to $\{x_t\}$ in the sense that the current value of y depends on current and lagged values of x, then a regression of $\{y_t\}$ on current, lagged, and future values of x should yield insignificant coefficients on future values of x. If the set of coefficients on future values of x is significantly different from zero, the causality assumption is not supported. In terms of the distributed lag model (52), this restriction is $\beta_j = 0$ for $j < 0$. This is a testable restriction on the relationship between $\{y_t\}$ and $\{x_t\}$. Tests for causality can be carried out using an estimate of $B(\omega)$ or using regular least-squares regression methods.[24]

4.3. Band Limited Regression

In econometrics it is frequently possible to use either seasonally adjusted or seasonally unadjusted data in parameter estimation. It is not always clear in principle which should be used nor is it clear that seasonal and nonseasonal

[23] Pierce & Haugh (1977) provide an excellent review of concepts and tests of causality.

[24] Geweke (1978) has extended these results to the case of a complete dynamic simultaneous equation model.

variations are related by the same model.[25] More generally, there are those who argue that different models may be needed to explain long-run and short-run variations in economic time series.

Engle (1974) has recently proposed a method to investigate whether or not the same model is appropriate for all frequencies. Consider the regression model

$$y = X\beta + u, \tag{72}$$

where y is $T \times 1$, X is $T \times k$, and u is $T \times 1$. If this set of equations is multiplied by the matrix W, the matrix introduced in (50), a transformed set of equations,

$$\tilde{y} = \tilde{X}\beta + \tilde{u}, \tag{73}$$

is obtained. If the regression relationship is invariant over all frequencies of oscillation as the time domain regression model assumes, then the coefficient vector β is the same for all T observations in the transformed model. Engle (1974) provides a simple test of this hypothesis.

5. An Analysis of Aggregate Consumption Data

It may be useful to consider an example to illustrate some of the basic points that have been introduced in connection with univariate and multivariate time series analysis. The aggregate consumption function is chosen for analysis in part because it is so well known and in part because it has been used repeatedly to test new estimation techniques.[26] We begin with a fairly standard time domain analysis of quarterly postwar data on personal consumption expenditure and disposable personal income.[27] Initially, potential problems of simultaneous equation bias are ignored. The main purpose of this exercise is to illustrate the use of time series techniques in model evaluation.

Many empirical investigations of the aggregate consumption function begin with a loosely formulated distributed lag relationship between consumption and income based on: (i) the theoretical proposition that personal consumption expenditure depends largely on disposable personal income,

[25] For a more detailed discussion of these issues, see Plosser (1978).

[26] See, for example, Zellner & Geisel (1970).

[27] The data consist of real quarterly national income and product accounts observations on personal consumption expenditure and disposable personal income from 1954.1 to 1977.1, a total of 93 observations.

and (ii) the empirical finding that a simple linear consumption function with no lagged values of the dependent or independent variables does not provide an adequate explanation of the data.[28] If consumption is simply regressed on income, the result

$$\underset{(5.9)}{\hat{C}_t = 15.86} + \underset{(213.2)}{0.89 Y_t}, \qquad R^2 = .998, \quad \mathrm{DW} = 0.69, \tag{74}$$

is obtained. In this and the following equations, t statistics are shown in parentheses. The low value of the Durbin–Watson statistic indicates a strong possibility that the disturbance process is serially correlated, and this is usually taken as an indication that a more general distributed lag model is appropriate.

On the basis of this preliminary result, the regression equation might be modified to include lagged values of both consumption and income which yields the result

$$\begin{gathered}\hat{C}_t = \underset{(0.9)}{2.00} + \underset{(7.0)}{0.46 Y_t} - \underset{(4.6)}{0.35 Y_{t-1}} + \underset{(11.1)}{0.88 C_{t-1}} \\ R^2 = 0.999, \qquad \mathrm{DW} = 1.96, \qquad \mathrm{DH} = 0.29.\end{gathered} \tag{75}$$

Neither the Durbin–Watson nor the Durbin h statistic indicates that serial correlation of the disturbance process is a potential problem.[29] This form of the consumption function thus satisfies the usual criteria employed in econometric analysis and would accordingly be accepted as a reasonable working hypothesis.[30]

Another way to arrive at (75) is to specify a model in which, in the absence of disturbances, the desired level of consumption is linearly related to current disposable income

$$C_t^* = \alpha + \beta Y_t. \tag{76}$$

Actual consumption in period t is assumed to be equal to desired consumption plus some fraction λ of the discrepancy between actual and desired consumption in the previous quarter plus a random disturbance. Thus

$$C_t = C_t^* + \lambda(C_{t-1} - C_{t-1}^*) + v_t, \tag{77}$$

[28] For a discussion of several plausible theoretical reasons for expecting a distributed lag model, see Kmenta (1971 Chapter 11).

[29] It is well known that the Durbin–Watson statistic is biased toward 2 when a lagged dependent variable is included as a regressor. The Durbin h statistic is a more appropriate test statistic in this case.

[30] The usual caveats about the interpretation of the t statistics and other coefficient estimates should be introduced here, since this equation was obtained after a preliminary test.

where $\{v_t\}$ is a sequence of independent identically distributed random variables. These assumptions lead to the model

$$C_t = \alpha(1 - \lambda) + \beta Y_t - \lambda\beta Y_{t-1} + \lambda C_{t-1} + v_t. \tag{78}$$

The estimated equation (75) is the unrestricted version of (78). Imposing the restriction implied by (78) has very little impact on the coefficient estimates or other summary statistics. Thus, judging from the usual criteria employed in econometric analysis, the data appear to be consistent with the theory leading to (78).

A more careful look at the residuals of the distributed lag function is revealing, however. The autocorrelation and partial autocorrelation functions (ACF and PACF) of the residual series are shown in Fig. 5. Despite the fact that the DW and DH statistics do not indicate that there is a problem with the disturbance process, the estimated autocorrelation and partial autocorrelation functions suggest that significant correlation remains in the residuals. Figure 6 shows the estimated autocorrelation and partial autocorrelation functions for the residuals of the original model (74). These functions indicate that indeed a first-order autoregressive process for the disturbances is not adequate; an autoregressive process of at least third order is indicated. A third-order autoregressive process for the disturbance term in (74) implies that lags of up to order three are needed in the consumption function. The use of an unrestricted third-order lag distribution yields the

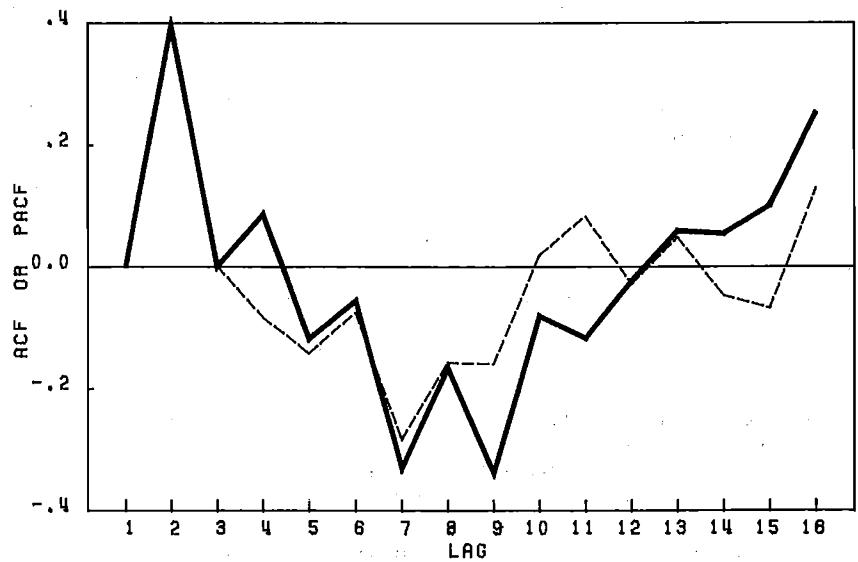

Fig. 5. Estimated ACF and PACF for residuals of Eq. (75). Bold curve: ACF; dashed curve: PACF.

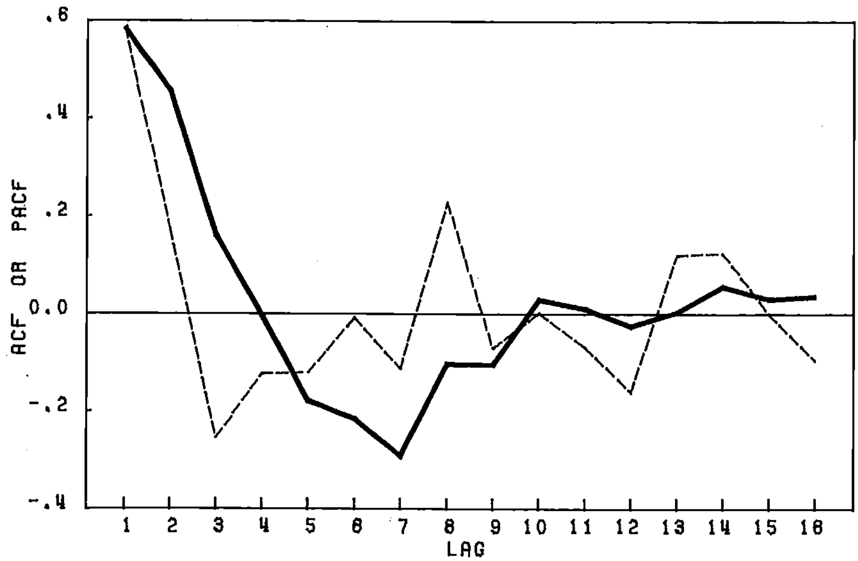

Fig. 6. Estimated ACF and PACF for residuals of Eq. (74). Bold curve: ACF; dashed curve: PACF.

model

$$\begin{aligned}\hat{C}_t = {} & \underset{(1.3)}{2.89} + \underset{(7.9)}{0.51Y_t} - \underset{(3.5)}{0.32Y_{t-1}} - \underset{(2.3)}{0.20Y_{t-2}} + \underset{(2.5)}{0.20Y_{t-3}} \\ & + \underset{(7.5)}{0.86C_{t-1}} + \underset{(2.5)}{0.39C_{t-2}} - \underset{(3.8)}{0.45C_{t-3}}, \\ & R^2 = .999, \qquad \mathrm{DW} = 2.07.\end{aligned} \tag{79}$$

The estimated autocorrelation and partial autocorrelation functions of the residuals in this model do not exhibit evidence of serial correlation. Thus a careful analysis of the residuals of the original equation (74) leads to a third-order model.

The analysis leading to the third-order model illustrates an important point. Standard econometric techniques might lead an investigator to accept a second-order model. However, a more careful analysis of the data indicates that a third-order model is more appropriate. Indeed, the more complicated model did not simply evolve in an *ad hoc* way; rather, it was suggested on the basis of a careful analysis of the residuals.

A potentially important difference between the usual econometric modeling approach and the time series approach is that the first step in time series analysis is to detrend the series to induce stationarity if that is necessary. For example, Box & Jenkins (1970, p. 378) state that "when the processes are nonstationary it is assumed that stationarity can be induced by suitable differencing." In this case first differences appear to be sufficient.[31] When the model is estimated in terms of first differences with appropriate recognition of the serial correlation in the disturbance process, the result is

$$\begin{aligned} \Delta C_t = {} & \underset{(2.1)}{1.44} + \underset{(7.9)}{0.46}\,\Delta Y_t - \underset{(2.3)}{0.17}\,\Delta Y_{t-2} + \underset{(4.1)}{0.42}\,\Delta C_{t-2} \\ & R^2 = .457, \qquad \mathrm{DW} = 2.10. \end{aligned} \tag{80}$$

If this is rewritten in terms of levels, the coefficients are not dramatically different from the coefficients of the unrestricted third-degree lag model. In order to facilitate comparisons with the corresponding time series results presented subsequently, the first-differenced version of the model will be retained.

The frequency domain properties of (80) provide a further check on the adequacy of this parametric specification. If the model is rewritten as a general (rather than rational) distributed lag model, the result is

$$\begin{aligned} \Delta C_t = {} & 2.48 + (.46 + .02L^2 + .01L^4 + .01L^6 + \cdots)\Delta Y_t \\ & + (1 + .42L^2 + .18L^4 + .07L^6 + \cdots)\hat{u}_t. \end{aligned} \tag{81}$$

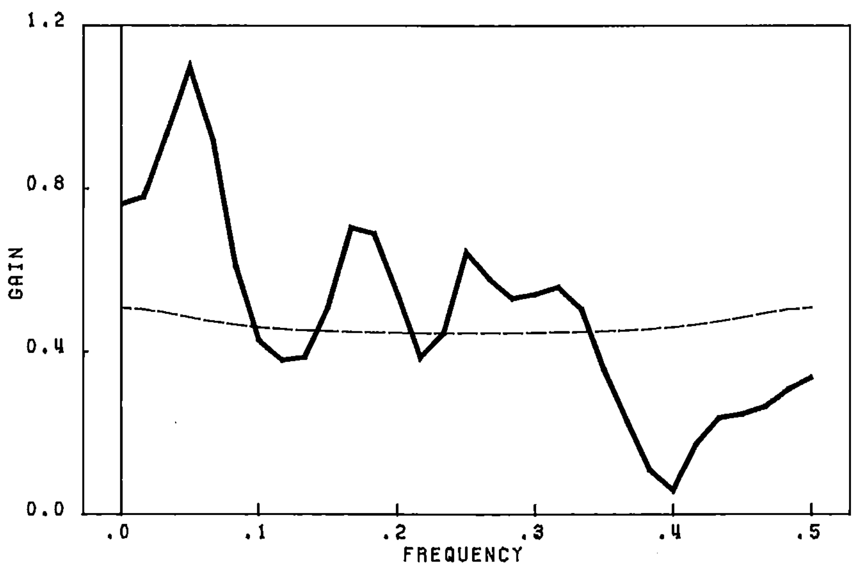

Fig. 7. Estimates of the gain of the consumption–income relationship. Bold line: direct estimate; dashed line: model estimate.

[31] See Box & Jenkins (1970, pp. 174–175) for a discussion of the degree of differencing required to produce stationarity.

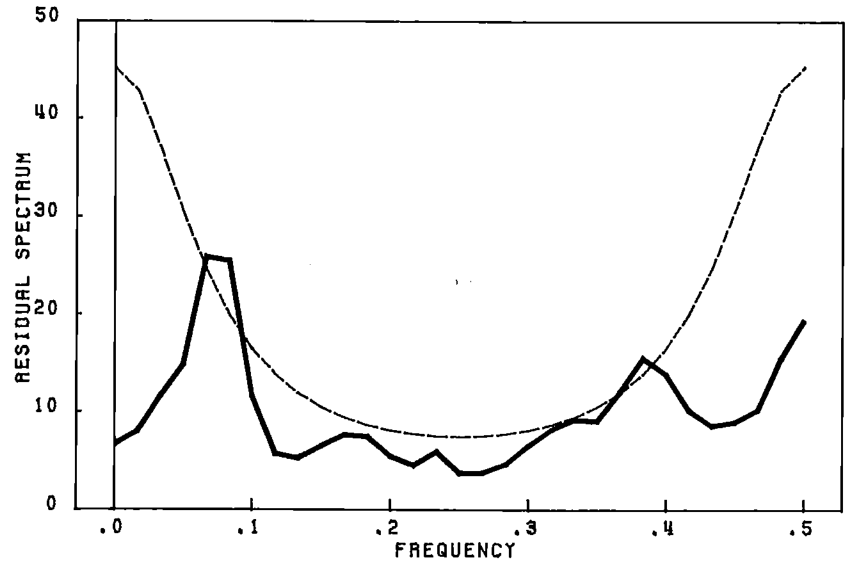

Fig. 8. Estimates of the residual spectrum of the consumption–income relationship. Bold line: direct estimate; dashed line: model estimate.

The gain function implied by this model is shown as a dashed line in Fig. 7 and the implied residual spectrum is shown as a dashed line in Fig. 8. The gain function indicates that the model implies that changes in consumption respond equally strongly to both short-run and long-run changes in income. This result is similar to that reported by Engle (1974) and, as remarked by Engle, does not appear to be consistent with Friedman's permanent income hypothesis. The residual spectrum for this model exhibits a relative predominance of both low- and high-frequency variation.

Direct estimates of the gain function and residual spectrum,[32] based on estimates of the auto- and cross-covariance functions, are also plotted on Figs. 7 and 8. It is clear from visual inspection that while the direct estimates are broadly consistent with the implications of the model, there are some important disparities. In particular, the gain function estimated directly from the data tends to fall off much more sharply at high frequencies than does the gain function implied by (81). Thus the direct estimate of the gain function conforms more closely to the permanent income hypothesis than the result obtained from the parametric model.

[32] These estimates were obtained by replacing $B(\omega)$ by $\hat{B}(\omega) = \hat{f}_{yx}(\omega)/\hat{f}_{xx}(\omega)$ in (63) and (64). Estimates of the spectrum $\hat{f}_{xx}(\omega)$ and cross-spectrum $f_{yx}(\omega)$ were obtained using the Parzen window with truncation point 30. See Jenkins & Watts (1968) for a complete discussion of the estimation procedure.

TABLE 1

DISTRIBUTED LAG COEFFICIENTS AND ESTIMATED t-VALUES

	Spectral estimates		Regression estimates	
	Coefficient	t Value	Coefficient	t Value
ΔY_{t+5}	.046	.92	.054	.96
ΔY_{t+4}	.072	1.44	.094	1.71
ΔY_{t+3}	.148	2.96	.182	3.25
ΔY_{t+2}	−.113	2.26	−.154	2.63
ΔY_{t+1}	.197	3.94	.183	3.14
ΔY_{t}	.384	7.69	.383	6.48
ΔY_{t-1}	.204	4.08	.207	3.57
ΔY_{t-2}	.020	.40	.064	1.04
ΔY_{t-3}	.059	1.18	−.047	.54
ΔY_{t-4}	−.032	.64	.066	.69
ΔY_{t-5}	−.057	1.14	−.084	.97

The distributed lag coefficients and approximate t statistics implied by the spectral estimates[33] are shown in Table 1. The striking feature of these estimates is the fact that the lag distribution is two-sided. Future, as well as current and past, changes in income have a significant effect on current consumption. As a check on these results, regression estimates were obtained, and these are also shown in the table. The two sets of estimates are very similar with respect to both the coefficients and the standard errors, and hence the implied t values.

The conclusion from these estimates is clear. These results do not support the hypothesis that disposable income is an exogenous variable. Rather it appears that income is causally dependent on consumption. This result indicates that any further analysis of the relationship between consumption and income should recognize explicitly the simultaneous equations nature of the problem. This, of course, is not an unexpected result in this case. There may be other situations, however, which are less clear cut for which such statistical evidence would be extremely valuable.

6. Conclusion

This paper has shown how time series analysis differs in outlook from classical econometrics and has summarized briefly some of the basic tech-

[33] These were obtained from (61) with $\hat{B}(\pi s/m)$, in place of $B(\pi s/m)$. See footnote 32 for details of the estimation procedure.

niques of time series analysis that are useful for the evaluation of econometric models. An important distinction between the classical econometric approach and the time series approach to modeling is that the econometric approach typically begins with a strong parametric formulation of the model which provides the basis for the empirical analysis. Potentially blatant contradictions of the assumptions of the model are investigated but diagnostic checking is not pursued vigorously. Time series models, on the other hand, typically begin with a relatively weak, nonparametric formulation of the model. Much more emphasis is placed on data analysis to suggest the types of simplifications that may be appropriate.

This difference in outlook and approach provides the basis for using time series methods to evaluate econometric models. Three steps are involved in the evaluation process. First, certain measurable dynamic characteristics of the structural econometric model are derived. Second, the corresponding dynamic properties are estimated directly from the data. Finally, the direct estimates are compared with those implied by the structural model. Disparities between the direct estimates and implied characteristics provide an indication of model inadequacy.

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