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9 Liquidity Constraints in Production-Based Asset-Pricing Models

William A. Brock and Blake LeBaron

9.1 Introduction

Economists have simulated Lucas's (1978) exchange economy asset-pricing model to show how mean reversion could appear. (See Cecchetti, Lam, and Mark 1988; Bossaerts and Green 1988; and Kandel and Stambaugh 1988.) In this paper we study the impact of liquidity constraints on market valuation of firms in general equilibrium rational expectations asset-pricing models. To do this we must introduce production.

Our approach uses the production-based asset-pricing models of Brock (1982) and Cochrane (1987) with liquidity constraints added on the firm side. Model parameters are chosen to line up with data in the style of the "real business cycle" school. Hence we will concentrate on explaining phenomena at business cycle frequencies. Robustness of the results to reasonable variations in parameter values is probed.

We show that a production-based asset-pricing model generates mean reversion that is strengthened for liquidity constrained firms. In other words, variance ratio plots drop more for liquidity constrained firms. This is consistent with Fama and French (1988), Poterba and Summers (1987), and with De Bondt and Thaler's (1985, 1987, 1989) "over reaction" effect. We will show that for this result to be visible credit constraints must be binding very tightly on small firms. The fact that it takes strongly binding credit constraints and

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what amounts to 1,000 years of simulated data to show significant results agrees with some of the recent results putting the 3–5 year mean reversion phenomena on much weaker ground statistically.[1]

Large-firm returns strongly lead small-firm returns at the weekly frequency and weakly lead at monthly and quarterly frequencies. This kind of effect is suggested by Lo and MacKinlay (1988b) as a partial explanation for the striking results on short-term mean reversion found by Lehmann (1988). It is difficult to get this effect out of our model. We are not too unhappy with this aspect of the model, however. This is so because we believe that the lead-lag pattern across small- and large-firm returns found by Lo and MacKinlay is stronger at higher frequencies than at lower frequencies.

We found two other results that are consistent with the real data, but very weak. The model generates slightly higher unconditional variance for liquidity constrained firms, and nonlinearity tests performed on simulated constrained and unconstrained returns are consistent with results of nonlinearity tests performed on Center for Research in Security Prices (CRSP) decile-1 (small) firms and decile-10 (large) firms.

In equilibrium, returns $R_i(t, t+1)$ on firm i over $[t, t+1]$ are constrained by

$$(E) \qquad 1 = E\{R_i(t, t+1)b(t+1)|I(t)\}$$

where $\{b(t)\}$ is the discounting process and $I(t)$ denotes information at time t. In Lucas (1978), $b(t) = bu'[c(t)]/u'[c(t-1)]$, where u is the utility function of the representative consumer. For the economy as a whole the "market" portfolio return over $[t, t, +1]$ is given by $R(t, t+1) = [v(t+1) + c(t+1)]/v(t)$. Here $v(t)$ and $c(t)$ are value of the market portfolio and aggregate consumption, respectively, at date t. The constraint (E) makes interpretation subtle. Economic reasoning in rational expectations general equilibrium asset-pricing contexts is treacherous. "Perhaps it has been good judgment, not merely timidity, which has led aggregate theorists to steer clear of any attempt to 'understand the market'" (Lucas 1978, 1441). However, mean reversion in the market portfolio can be made reasonably intuitive.

Mean reversion at the market level is generated by noting that a relatively high consumption (good times) today ($c[t]$ is high) implies a relatively high $b(t+1) = bu'[c(t+1)]/u'[c(t)]$ because $u'' < 0$. Hence (E) implies that $R(t, t+1)$ must be relatively low. Hence ergodicity (the tendency for relatively high consumption $c[t]$ to revert back to average consumption) will cause $R(t, t+1)$ to revert back to its mean level. This explains mean reversion of returns on the market portfolio. We turn now to explaining why mean reversion is stronger for liquidity constrained firms versus unconstrained firms, ceteris paribus.

We ignore general equilibrium feedback from both unconstrained and constrained firms into the discounting process. In our model, constrained and unconstrained firms are exactly the same in every respect except that constrained firms cannot borrow to finance investment. Production functions for both types are hit by aggregate and idiosyncratic production shocks. Hence returns are contemporaneously correlated. Unconstrained firms are always forward-looking in choosing investment projects for the future. Therefore in maximizing market value today they choose investment projects that trade off systematic risk and expected return optimally. Constrained firms are constrained by past shocks. They are limited in borrowing by current income. Hence they will invest less than or equal to their desired level for future periods. Therefore they cannot achieve the optimal trade-off between systematic risk and expected return. This lack of optimality causes the market to push up their expected return.

Mean reversion is magnified by liquidity constraints. On the one hand, a low-valued constrained firm (relative to unconstrained) is relatively low valued because the constraint is binding relatively tightly. Since production shocks are independently and identically distributed (i.i.d.), the low-valued constrained firm has an even-money chance of being loosely constrained next period. Hence a good aggregate shock will bounce value back more than the value of an unconstrained firm under the same conditions. This is so because relaxation of the constraint allows the constrained firm to more closely mimic an unconstrained firm and thus achieve a higher market value. On the other hand, if the constrained firm is high valued this is so because the constraint is not binding (although it may bind in the future) so that it can mimic an unconstrained firm. But a negative shock in the future may cause the firm to be constrained for several periods and, hence, lose a lot of value.

The paper is organized as follows. Section 9.1 contains the introduction. Section 9.2 presents stylized facts and the simplest model that can illustrate the basic economic mechanism that we focus on. The basic economic mechanism is the role of liquidity constraints on the firm side in transforming production shocks of short memory into shocks of value and returns of longer memory. Section 9.3 expands upon the simple model. The fourth section presents results of computer simulation of the model. To our knowledge this is the first computer simulation study of production-based asset-pricing models. In addition we add liquidity constraints. The results show that the model is capable of generating mean reversion patterns as measured by Cochrane (1988) and Poterba and Summers (1987). Variance ratio plots across constrained and unconstrained firms compare fairly well with those generated by decile-1 (small) firm returns and decile-10 (large) firm returns. The model does this for parameter values appropriate to the time scale of the returns under comparison. Section 9.5 contains time-series plots illustrating the results. The paper ends with section 9.6, which is a summary with speculations and suggestions for future research.

9.2 The Simplest Model

In this section we present the simplest model that we can think of that generates returns consistent with the observed differences in return dynamics between the small firms and the large firms. The model will be an asset-pricing model of Lucas's (1978) type with financing constraints. It will show how financing constraints can turn nonpersistent production shocks into highly persistent ex post asset returns. This simple economic mechanism shows promise to rationalize a wide spectrum of stylized facts on differences in return behavior in smal and large firms.

Before presentation of the model we list the "stylized facts." Decile 1 (decile 10) denotes the smallest (largest) 10% of CRSP firms. A nice survey of the main stylized facts that we are interested in explaining is De Bondt and Thaler (1989). References are given for the facts that are not in De Bondt and Thaler.

9.2.1 Stylized Facts

1. Unconditional variance, skewness (positive), kurtosis, are larger for decile 1 than for decile 10 (see table 9.1).

2. Mean reversion is stronger for small firms. Mean reversion is essentially zero for largest firms after World War II. It is stronger during periods that include the Great Depression and is weaker during the postwar period. See Lo and MacKinlay (1988a), Fama and French (1988), Poterba and Summers (1987), as well as De Bondt and Thaler (1989). The top part of figure 9.1 displays the variance ratios for the largest and smallest CRSP deciles from monthly returns data, 1926–86.[2] In this panel we see that the variance ratios fall more for the smaller firms than the large firms. The lower panel presents the difference of the large- versus small-firm variance ratios along with simulations taken from resampled (with replacement) data taken from the respective returns series. One hundred simulations are performed on data sets of the same size as the monthly CRSP series. Variance ratios are constructed for both the scrambled large-firm series and small-firm series and the difference is

Table 9.1 **Unconditional Moments: Smallest and Largest Deciles**

	Mean	Standard Deviation	Skewness	Kurtosis	BDS(2)
Monthly (26–87)					
Largest	.009	.053	.28	7.45	4.95
Smallest	.017	.114	2.90	20.7	9.08
Monthly (47–87)					
Largest	.009	.03	.00	.88	−.17
Smallest	.014	.07	1.53	13.6	2.71
Annual (26–87)					
Largest	.086	.19	−.948	1.13	
Smallest	.139	.38	−.312	.54	

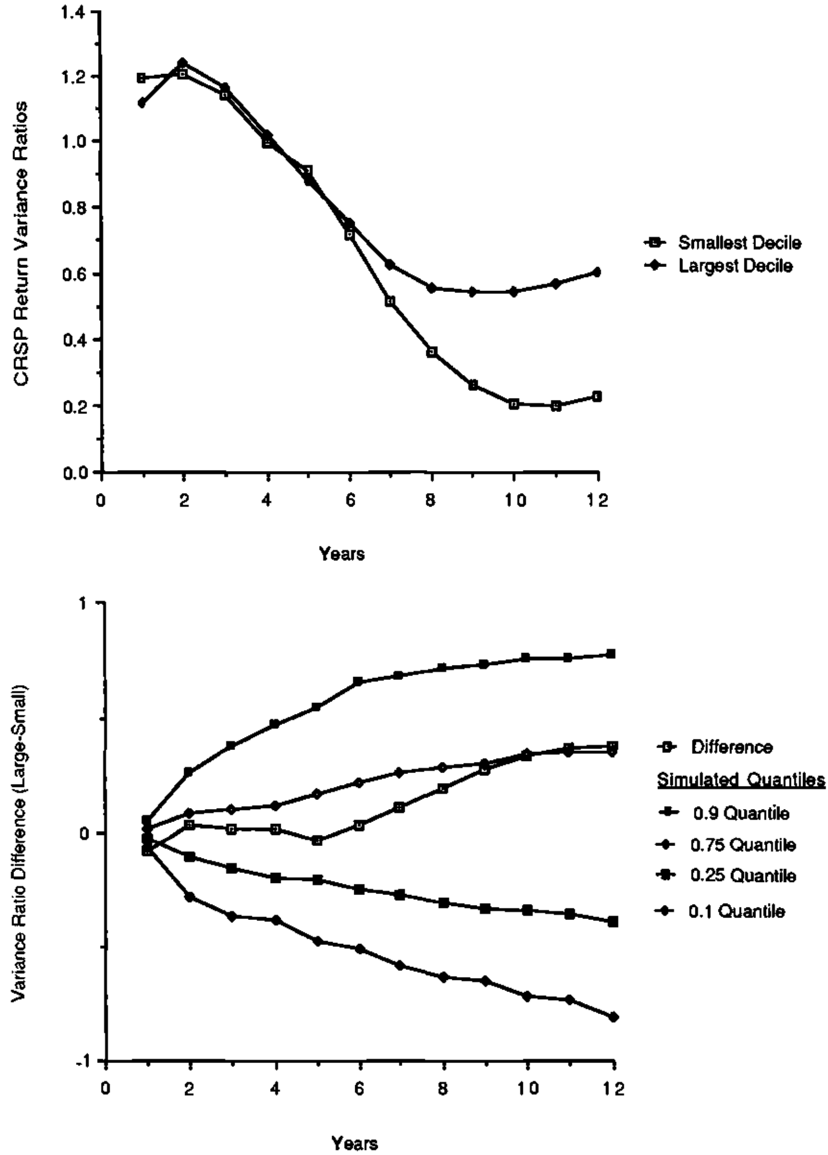

Fig. 9.1 Variance ratios: largest and smallest deciles

taken. This picture shows that in these small samples the variance ratios are not very different statistically. They just barely break through the 0.75 quantile at 10 years.

3. Rejection of i.i.d. returns by the nonparametric (BDS) test for independence (Brock et al. 1988) for monthly CRSP returns from 1926 to the present

is strongest for small firms. Decile-10 returns appear i.i.d. to the BDS test after World War II, but decile-1 returns do not.

4. Form portfolios of stocks that are past losers and past winners over the past few years. Price reversals are more pronounced for loser portfolios than for winner portfolios, with much of the correction for losers occurring in January (De Bondt and Thaler 1989).

9.2.2 Asset-Pricing Model with Production and Fixed Cost of Stock Issue

Fazzari, Hubbard, and Petersen (1988a, 1988b) have documented and stressed the influence of financing hierarchies across firm size. These financing hierarchies are due to the relative cost of using different modes of finance across different size classes of firms. Table 2 of Fazzari, Hubbard, and Petersen (1988b) shows financing for firms with assets under \$10 million (over \$1 billion) coming from short-term bank debt, long-term bank debt, other long-term debt, and retained earnings in percentages 5.1 (−.6), 12.8 (4.8), 6.2 (27.9), 75.9 (67.9). New equity issue seemed to be used as a last resort even for large firms. This may be due to higher tax costs. The idea is that relative costs are such that internal funds are used by the smallest, bank loans are used by the next smallest, and bonds and equity are used by the largest. We are going to use an asset-pricing model with financing costs to show:

Persistence result: Independent and identically distributed shocks to profits and production cause non i.i.d. returns when financing costs loom large relative to profits. Thus i.i.d. production shocks lead to i.i.d. returns when financing costs are zero.

Consider the following model, which is an adaptation of Brock (1982, 36) which, in turn, introduces production into the model of Lucas (1978). There are two size classes of firms. Both are all equity financed. Balance sheet dynamics are given by

$$(1) \quad p_i(t)[z_i(t) - z_i(t-1)] - [F + c(t)]I(t) + f_i[x_i(t-1), r(t)] = x_i(t) + d_i(t)z_i(t-1),$$

where $F + c(t)$ denotes fixed cost plus variable cost of issuing new equity at date t, and $I(t)$ is one when new equity is issued $[z_i(t) - z_i(t-1)] > 0$ and 0 otherwise. Here x denotes capital stock, f denotes current profit plus undepreciated capital stock, $r(t)$ denotes stochastic process (assumed i.i.d. for now), d denotes dividend payout per share, and z denotes number of shares of security outstanding. We can view z as a type of long-term bond if the "divident payout" is constant. This interpretation is more consistent with the Fazzari, Hubbard, and Petersen (1987, 1988) evidence because new-share issues are a small part of actual financing.

At first, suppose that small firms, denoted by $i = 1$, choose not to issue new securities because F is too high relative to f. So their balance sheet dynamics are given by

$$(2) \qquad f_i[x_i(t-1), r(t)] = x_i(t) + d_i(t)z_i(t-1).$$

At first we suppose that large firms can issue new securities for free. The idea is that $F + c(t)$ is so small relative to f that to a first approximation we can ignore the floatation costs. Large-firm balance sheet dynamics are given by

$$(3) \; p_i(t)[z_i(t) - z_i(t-1)] + f_i[x_i(t-1), r(t)] = x_i(t) + d_i(t)z_i(t-1),$$

Turn now to the consumer side. As in Lucas (1978), the Euler equation of the consumer is given by

$$(4) \qquad p_i(t-1) = E\{b(t)[p_i(t) + d_i(t)]|I(t-1)\},$$

where $b(t)$ denotes a discounting process (the marginal rate of substitution between date $t-1$ and date t consumption) and $I(t-1)$ denotes information available to condition on at date $t-1$. Let the process $b(t)$ be the constant b, $0 < b < 1$, for now. Let both firms be intial stock market value maximizers. Assume, for all values of r, that f is zero at $x = 0$, $f'(\cdot)$ is infinity at $x = 0$, $f'(\cdot)$ is zero at x equal to infinity, and that f is strictly concave and continuously differentiable.

It is easy to see that the large firms jump to a steady state $\bar{x}_2$ given by the unique solution to,

$$(5) \qquad bEf'_2(x, r) = 1.$$

Net cash flow is given by

$$(6) \qquad n_2(t) = f_2[x_2, r(t)] - x_2.$$

Now multiply both sides of the Euler equation by the number of equities outstanding and use the balance sheet to find

$$(7) \qquad p_i(t-1)z_i(t-1) = v_i(t-1) = E\{[v_i(t) + n_i(t)]|I(t-1))\}.$$

Ex post returns are given by

$$(8) \qquad R_2(t) = \frac{[v_2(t) + n_2(t)]}{v_2(t-1)} = \frac{[\bar{v}_2 + n_2(t)]}{\bar{v}_2},$$

for the large firms, where $\bar{v}_2$ is the constant solution of (6) and (7) for the large firms. Hence we have proved:

PROPOSITION 1. Ex post returns are i.i.d. when production shocks are i.i.d. provided that financing costs are zero so that there are no constraints on borrowing through the stock market.

Let us now examine the polar case of the small firms who cannot borrow at all and, hence must finance accumulation to the desired level of capital stock internally. Since the small firms choose not to issue new securities, and since they are not allowed to pay out negative "dividends," one can show that initial "stock market value" maximization implies that the optimal investment path takes the form

$$(9)\qquad x_1(t) = \min\{\bar{x}_1, f_1[x_1(t-1), r_1(t)]\},$$
$$\bar{x}_1 = \operatorname*{argmax}_x \{bE[f_1(x, r_1)] - x\}.$$

It is instructive to graph the stochastic dynamics of (9) assuming that higher r generates higher f. Look at net cash flow and ex post returns:

$$(10)\qquad n_1(t) = f[x_1(t-1), r_1(t)] - x_1(t),$$

$$(11)\qquad R_1(t-1, t) = \frac{[v_1(t) + n_1(t)]}{v_1(t-1)},$$

$$(12)\qquad v_1(t) = E\{bn_1(t+1) + \ldots\} = \text{expected discounted sum of future net cash flow.}$$

The desired level of capital stock is $\bar{x}_1$. When a low shock occurs, the firm wants to accumulate at the maximal rate and pay no dividends. If it gets hit with a very low shock at t it is likely that several periods will pass before it has reached $\bar{x}_1$. Hence there will be spells where zero dividends are paid out. We sum this up into:

Proposition 2. The solution of (9)–(12) transforms i.i.d. production shocks into non-i.i.d. ex post returns.

Unfortunately, we can not hope to explain mean reversion or the phenomenon of large-firm returns leading small-firm returns with this model.

Proposition 3. Returns across size classes of firms are uncorrelated at all leads and lags.

Proof. This follows from the Euler equation:

$$(13)\qquad 1/b = E\{R_i(t, t+1)\},\ i = 1,2.$$

We must show that

$$E\{R_i(t, t+1)R_j(t+L, t+L+1)\} = (1/b)^2, \quad \text{for } L > 0,\ \forall i,j.$$

To do it use iterated expectations and (13). Q.E.D.

Remark. Any formulation where conditional expectations are independent will yield this result. Dependence induced by the discounting process is a useful way to remove this independence.

9.3 The Extended Model: The Case $b(t)$ Stochastic

In order to get mean reversion and large-firm returns leading small-firm returns we must have $b(t)$ stochastic since returns are uncorrelated when $b(t) = b$ for all t. In the case $b(t)$ stochastic the Euler equation for a value-maximizing firm can be written in terms of returns:

$$(14)\qquad 1 = E\{b(t+1)R(t, t+1) \mid I(t)\},$$

where $I(t)$ denotes information available at date t.

It is relatively straightforward to work out some results for the case $u(c) = \ln(c), f(x,r) = rx^d$. Unfortunately, this form of the prodution function implies that capital lasts only one period. Nevertheless, working out the solution is instructive.

Let the economy as a whole choose nonanticipating functions $\{x\},\{c\}$ (as in Brock [1982]) to solve

$$\max E \sum b^{(t-1)} \ln[c(t)], \tag{15}$$

subject to

$$c(t) + x(t) = r(t)[x(t-1)]^d, \quad x(0) \text{ given.} \tag{16}$$

Here the sum runs from $t = 1,2, \ldots, \infty$.

It is well known that the solution is given by

$$x(t) = ey(t), e = bd, y(t) = r(t)[x(t-1)]^d. \tag{17}$$

The value of the "market portfolio" is given by

$$\begin{aligned} v(t) &= bE(\{u'[c(t+1)]/u'[c(t)]\}[v(t+1) + n(t+1)) \\ &= bE\{[y(t)/y(t+1)][v(t+1) + (1-e)y(t+1)]\}, \end{aligned} \tag{18}$$

where net cash flow $n(t + 1) = c(t + 1)$ equals aggregate consumption at date $t + 1$. Conjecture a solution $v(t) = vy(t), v$ constant. We find

$$v = b[v + (1-e)]. \tag{19}$$

Returns are defined by

$$\begin{aligned} R(t,t+1) &= [v(t+1) + n(t+1)]/v(t) \\ &= \{[v + (1-e)]/v\}[y(t+1)/y(t)] \\ &= k[y(t+1)/y(t)] = kr(t+1)[x(t)]^d/y(t) \\ &= ke^d[r(t+1)][(y(t)^{(d-1)})]. \end{aligned} \tag{20}$$

It is well known that (17) converges to a stationary distribution (see Brock [1982] for references). This is easy to see by noting that (17) is linear in logs. The ergodic property inherent in (17) leads to behavior much like "mean reversion" in returns (20). For example, both $E\{R(t,t+1)|t\}$, $\text{var}\{R(t,t+1)|t\}$ fall as $y(t)$ increases. Hence, since a high return last period is associated with a low y this period, and since y converges to a limit distribution, this will cause a tendency for high past returns to be followed by lower future returns. Turn now to the effect of introducing credit-constrained firms.

Look at the valuation equation for firm i,

$$v_i(t) = bE\{u'[c(t+1)]/u'[c(t)][v_i(t+1) + n_i(t+1)]\}, \tag{21}$$

where

$$\begin{aligned} n_i(t+1) &= y_i(t+1) - x_i(t+1) \\ &= a_i(t+1)r(t+1)[x_i(t)]^d - x_i(t+1). \end{aligned} \tag{22}$$

Here $a_i(t)$,(t) are i.i.d. and independent of each other. We will assume that constrained firms at date t maximize market value subject to the constraint

(23) $$x_i(s) \leq y_i(s) \quad \forall s.$$

Unconstrained firms just maximize market value. Both types face the discounting process parametrically. It is easy to find the solutions if we iguore general equilibrium feedback from the firms i to the economy as a whole. We assume that the collectivity of firms to be valued is small relative to the economy as a whole.

9.3.1 Unconstrained Firms

For i in U (U = unconstrained), conjecture $x_i(t) = e_i y(t)$, $v_i(t) = v_i y(t)$ and insert this into the valuation equation (21). Use (17) and simplify to obtain

(24) $$v_i = b[v_i + E\{a_i(t)\}[e_i/e]^d - e_i],$$

(25) $$e_i = (bdE\{a_i(t)\})^{1/(1-d)},$$

(26) $$R_i(t,t+1) = [y(t+1)/v_i y(t)][v_i + a(t+1)(e_i^d) - e_i].$$

9.3.2 Constrained Firms

For i in C(C = Constrained), one can show the optimum x satisfies,

(27) $$x_i(t+1) = \min\{a_i(t+1)r(t+1)x_i(t)^d, e_i y(t+1)\}.$$

Conjecture a solution to the valuation equation of the form

(28) $$\begin{aligned} v_i(t) &= v[i,y_i(t),y(t)] = y(t)j[i,w_i(t)], \\ w_i(t) &= y_i(t)/y(t). \end{aligned}$$

Use (17), (18), and (28) to obtain

(29) $$j[i,w_i(t)] = bE\{j[i,w_i(t+1)] + \max[0,w_i(t+1) - e_i]\},$$

(30) $$w_i(t+1) = a_i(t+1)\{(1/e)\min[w_i(t),e_i]\}^d.$$

Apply standard techniques as in Lucas (1978), for example, to show that (29) has a unique solution. (It is a contraction map since $0 < b < 1$.) Note that $j[i,0] = 0$ and $j[i,w] \leq v_i$. This is so because the value of a constrained problem must be less than or equal to the value of an unconstrained problem, and j is nondecreasing in w. Note that if the dynamics (30) ever hits zero, it stays at zero forever. Let us now look at retums.

Retums, $R_i(t,t+1)$ are given by

(31) $$\begin{aligned} R_i(t,t+1) &= \frac{y(t+1)\{j[i,w_i(t+1)] + m_i(t+1)\}}{y(t)j[i,w_i(t)]}, \\ m_i(t+1) &= \max(0,a_i(t+1)\{(1/e)\min[w_i(t),e_i]\}^d - e_i). \end{aligned}$$

Although (31) is a complicated expression, it is not hopeless to compare constrained retums with unconstrained retums ceteris paribus. Let $w_i(t)$ tend to zero. Notice that unconstrained returns (26) do not depend upon $y(i,t)$. The only date t data they depend upon is $y(t)$. So let a firm take a very negative shock relative to the whole economy so that $w_i(t)$ is small. Then if the firm is constrained its returns will "tend" to be high from $[t,t+1]$ relative to the returns of a constrained firm. But at each point in time the pricing equation (21) must be satisfied. Hence for conditional expected returns of a constrained firm to be higher than that for the unconstrained firm, the constrained firm must put more of its payoff on states of the world where marginal utility is low. This effect is not easy to get out of the analytics, but it appears in the computer results reported in Section 9.4 below. We are still working on an analytic formulation of this effect.

9.4 Computer Results

Some of the models presented in the previous sections are simulated in this section. The models are designed to look like real business cycle models and generate reasonable numbers for aggregate fluctuations. From the simulations of aggregate fluctuations, policy and pricing functions are developed for the "small" firms (unconstrained and constrained). With these functions, complete returns series can then be simulated and compared.

The first part of the exercise is to solve for the optimal policy function for the economy as a whole. The economy is assumed to chose nonanticipating $\{x\},\{c\}$ to maximize

$$E\sum_{t=1}^{\infty} b^{t-1}\frac{1}{\gamma}c(t)^{\gamma}, \tag{32}$$

where

$$\begin{aligned} y(t) &= r(t)x(t-1)^{d} + (1-\delta)x(t-1), \\ c(t) &= y(t) - x(t). \end{aligned} \tag{33}$$

Here $\{r\}$ is i.i.d. with finite mean and variance, and is bounded away from zero and infinity. This can be easily solved numerically on a discrete grid, given the parameters and the stochastic process $r(t)$. The solution to problem (32) gives a policy function for $c(t) = g[y(t)]$, and therefore gives us the means to simulate the $c(t)$ and $y(t)$ processes.

With the aggregate $c(t)$ process our small firms taking the aggregate as exogenous can be priced as follows:

$$\begin{aligned} y_i(t) &= r(t)a_i(t)x_i(t-1)^{d} + (1-\delta)x_i(t-1) \\ x_i(t) &= y_i(t) - \theta[y_i(t),y(t)]. \end{aligned} \tag{34}$$

Here $\theta(y_i,y)$ is the optimal policy for the firm given its output and the aggregate output levels. The price of the small firm is a function of both its output level and the aggregate consumption level.

$$p[y_i(t),y(t)] = \max E\{\Gamma[y(t),y(t+1)]\theta[y_i(t+1),y(t+1)] + p[y_i(t+1),y(t+1)]\}, \tag{35}$$

$$\Gamma[y(t),y(t+1)] = \left[\frac{c(t+1)}{c(t)}\right]^{\gamma-1}. \tag{36}$$

Here $\theta(y_i,y)$ is the optimal policy for the firm given its output and the aggregate output levels. In the unconstrained case θ may take on any value positive or negative. For the constrained firms we force the firms to pay out a minimum dividend, $\bar{c}$, in every period. If they cannot meet this payment they will pay out dividends of zero. More formally this puts the following constraint on θ.

$$\theta \geq \begin{cases} \bar{c}, & \text{if } y_i \geq \bar{c}; \\ 0, & \text{otherwise.} \end{cases}$$

These models will now be simulated for several different parameter values. For the first set of simulations we will use ($d = 0.4; \bar{c} = 0.4; b = 0.9; \delta = 0.2; \gamma = 0$). This is the case of logarithmic utility. The two shock processes are simulated with discrete random variables. The aggregate shock, $r(t) = \{0.9, 0.95, 1, 1.05, 1.1\}$, each with probability 0.2. The small-firm shocks for both the unconstrained and constrained cases will be $a_i(t) = \{0, 0.8, 1, 1.2, 2\}$, each with probability 0.2, and $a_i(t)$ is independent of $r(t)$. The structure of the stochastic shocks for production will remain the same for all the examples in this paper. Some experiments have been done to see that minor changes in these distributions will have no effect on the results here. It remains to be seen what effect major changes to the shock processes will have.

Time series of percentage changes of consumption and returns are presented in figures 9.2 and 9.3, respectively. Figure 9.2 shows the consumption series to be a choppy series reflecting the discrete shocks that it is built from. The mean percentage change in consumption is 2%, which is comparable to that for the United States (see Mehra and Prescott 1985). One problem with comparing our numbers with others' estimated numbers is that the model simulated here is stationary. It is hoped that growth can be worked into this system without affecting the results, but this has not been tested yet.

The returns time series in figure 9.3 shows more blurring around the main bands. This is due to the additional shocks for small firms. Simulated time series of length 1,000 are replicated 100 times to produce the results in figures 9.4–9.10 and table 9.2 below. For these simulations, two sets of shocks are drawn, the $r(t)$ aggregate shocks and the $a_i(t)$ individual shocks. The results for the constrained and unconstrained firms are simulated for the same set of individual shocks, the only difference being the constraint.

Figure 9.4 presents the variance ratio tests used by Cochrane (1988) and Poterba and Summers (1988). Under the hypothesis that the generated returns are i.i.d., the expected value of the variance ratio should be one. (We follow the same bias adjustment of Cochrane 1988 and Poterba and Summers 1987.)

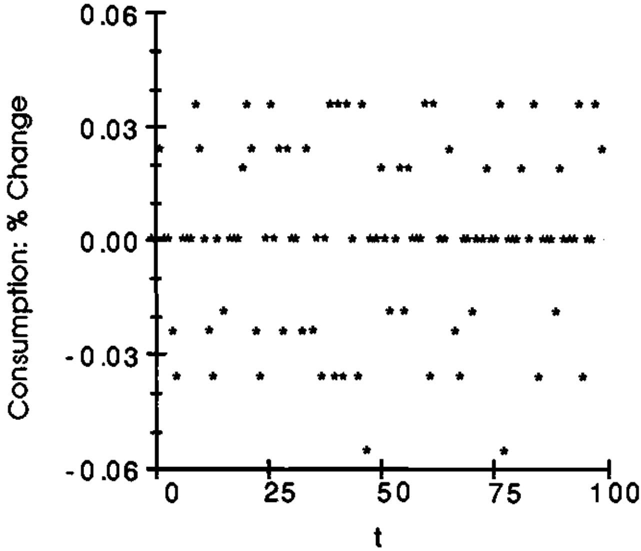

Fig. 9.2 Simulated consumption: percentage change

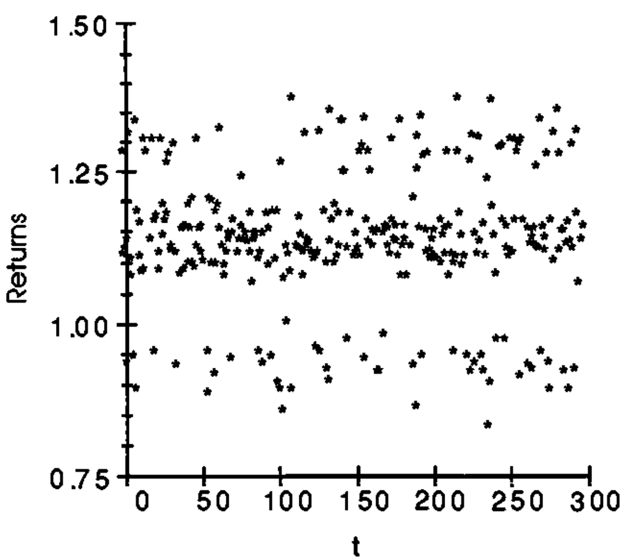

Fig. 9.3 Simulated returns

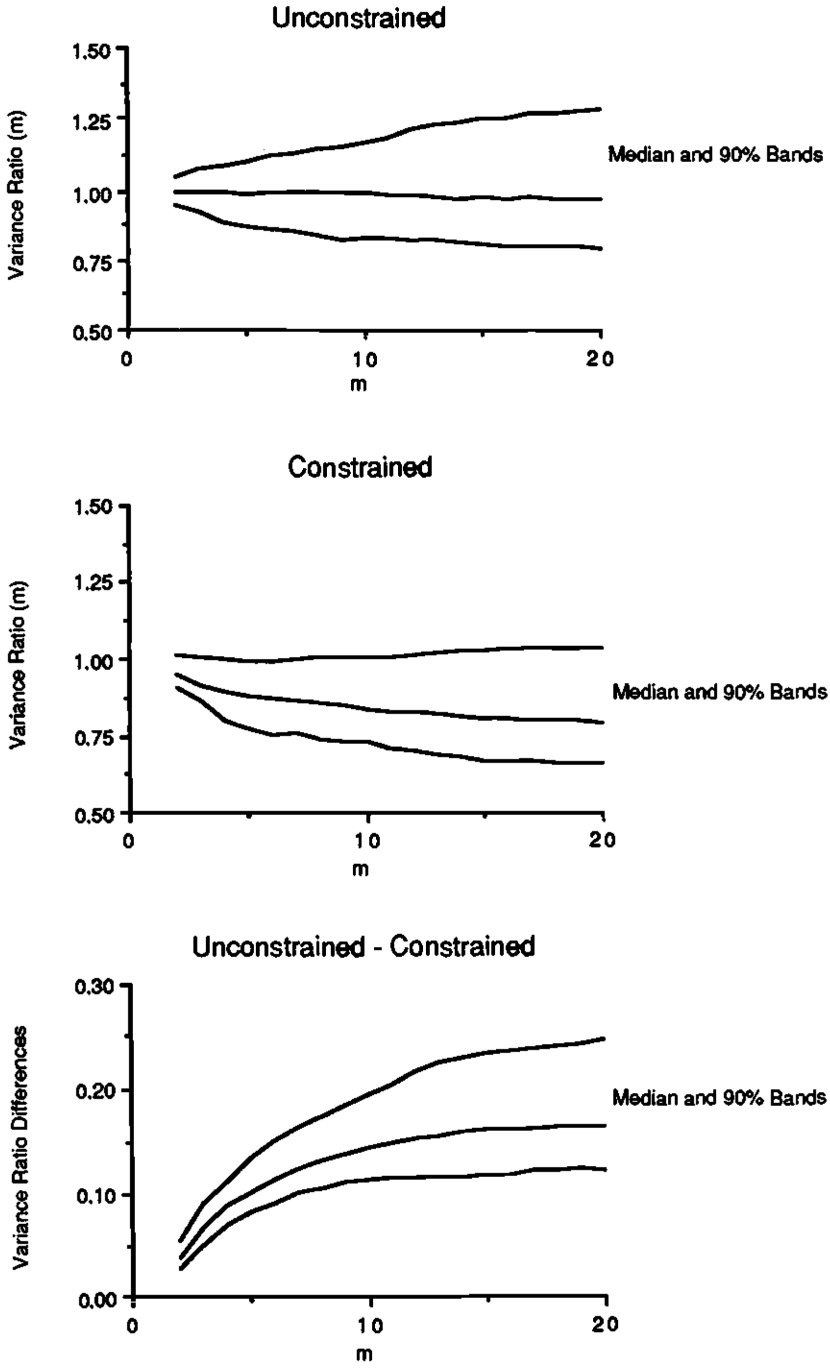

Fig. 9.4 Simulated variance ratios: $\gamma = 0, b = 0.9, d = 0.4, \delta = 0.2, \bar{c} = 0.4$

We do not have any reason to believe that the returns from either generated series are i.i.d. What we are interested in here is comparing the results for the two different types of firms. The top two panels of figure 9.4 show the median and 5% and 95% quantiles for the simulations performed on the unconstrained and constrained firms, respectively. We see that the constrained firms generate more evidence of variance ratios dropping off. This is made clearer in the lower panel where we look at the difference between the variance ratios for each type of firm for a given set of individual shocks. The differences here are significantly positive.

Table 9.2 presents some statistics on the simulated series. The first four columns present the mean, standard deviation, skewness, and kurtosis. For these simulations compare the first two rows. The numbers in parenthesis are the 5% and 95% quantiles. This table shows very little difference between the two series. The constrained series does have a "significantly" larger standard deviation, but it is no where near the order of magnitude difference in the actual data (about a factor of 2). The other moments show few patterns that line up with the data. The hope here was not to line up all moments, but to see that the simulated data generated roughly the same type of results as the data. The last column gives the BDS statistic (Brock et al. 1988, Hsieh and LeBaron 1988). This is a nonparametric test for any kind of dependence in the data. Under the null hypothesis of i.i.d., the BDS statistic is distributed normal with mean zero and unit variance. In table 9.1, the BDS statistic is applied to the monthly series. It is not used for the annual series since this series has too few points. In table 9.2 we see that the BDS statistic generates slightly larger values for the constrained firms. This is consistent with the presence of more dependence of any form in the constrained samples.

To test robustness of these results they are first tested against other preferences. They are run for $\gamma = -0.5$ and $\gamma = -1.5$. Figures 9.5 and 9.6 present variance ratios and differences for these preferences. They repeat the results obtained with logarithmic preferences. In table 9.2 we see that the results for the logarithmic case are repeated for the other two preferences.

The next experiment is to check the importance of $\bar{c}$, the constrained payout amount. We will repeat the first three simulations for $\bar{c} = 0$. In figure 9.7 the variance ratios for logarithmic preferences are given. The differences here are much smaller. Very little difference can be seen in the first two figures. The differences show that the variance ratio are still smaller for the constrained firms, but the magnitude of the spread is now much smaller. Table 9.2 shows very little difference in the statistics from the unconstrained and constrained firms. These results are repeated for the other preference parameters $\gamma = -0.5$, and $\gamma = -1.5$ in figures 9.8 and 9.9 and in table 9.2.

From these results we see that the value of $\bar{c}$ is crucial for obtaining mean reversion for our parameter values. It is necessary to keep the constrained firm up against the constraint for a large enough fraction of time to generate different numbers. So far, our most efficient way to do this is to increase $\bar{c}$. At the

Table 9.2 **Simulated Moments and BDS results**

	γ	Mean	Std.	Skewness	Kurtosis	BDS(2)
Experiments 1–3:[a]						
Unconstrained	.0	.13	.11	−.15	−.51	.31
		(.12, .13)	(.10, .11)	(−.20, −.11)	(−.65, −.36)	(−1.15, 1.97)
Constrained	.0	.13	.12	−.30	−.35	1.78
		(.12, .13)	(.12, .12)	(−.36, −.24)	(−.49, −.18)	(.01, 3.95)
Unconstrained	−.5	.13	.11	−.12	−.47	.99
		(.13, .14)	(.11, .12)	(−.18, −.06)	(−.62, −.31)	(−.49, 2.74)
Constrained	−.5	.13	.13	−.25	−.34	1.76
		(.13, .14)	(.12, .13)	(−.32, −.18)	(−.50, −.15)	(.14, 3.46)
Unconstrained	−1.5	.13	.11	−.14	−.50	.58
		(.12, .13)	(.11, .11)	(−.19, −.08)	(−.64, −.33)	(−1.25, 2.23)
Constrained	−1.5	.13	.12	−.29	−.34	1.91
		(.12, .14)	(.12, .13)	(−.36, −.22)	(−.47, −.15)	(.29, 3.66)
Experiments 4–6:[b]						
Unconstrained	.0	.13	.11	−.15	−.52	.38
		(.12, .13)	(.10, .11)	(−.21, −.09)	(−.64, −.36)	(−1.08, 2.02)
Constrained	.0	.13	.11	−.12	−.50	.61
		(.12, .13)	(.10, .11)	(−.18, −.07)	(−.63, −.35)	(−.64, 2.01)
Unconstrained	−.5	.13	.11	−.12	−.48	.98
		(.12, .13)	(.11, .11)	(−.19, −.06)	(−.60, −.32)	(−.52, 2.50)
Constrained	−.5	.13	.11	−.09	−.47	1.38
		(.12, .13)	(.11, .11)	(−.16, −.02)	(−.57, −.31)	(−.31, 2.98)
Unconstrained	−1.5	.12	.11	−.05	−.53	.38
		(.12, .13)	(.10, .11)	(−.20, −.09)	(−.67, −.38)	(−1.13, 2.16)
Constrained	−1.5	.12	.11	−.12	−.52	.63
		(.12, .13)	(.10, .11)	(−.17, −.07)	(−.66, −.36)	(−.93, 2.29)
Experiment 7:[c]						
Unconstrained	.0	.07	.07	.06	−.42	.49
		(.07, .07)	(.06, .07)	(.01, .11)	(−.54, −.27)	(−1.05, 2.26)
Constrained	.0	.07	.07	−.03	−.36	1.05
		(.07, .07)	(.07, .07)	(−.08, .03)	(−.50, −.15)	(−.84, 3.41)

Note: Results for seven experiments performed on varying parameters. The parameter values ($\bar{c}$, b, d, δ, γ) are given in notes. Each experiment is run for one firm subject to the $\bar{c}$ constraint (constrained) and one firm not subject to the constraint (unconstrained). Numbers in parentheses are 5% and 95% quantiles from 100 replications of length 1,000.

[a] $\bar{c} = .4; b = .9; d = .4; \delta = .2.$

[b] $\bar{c} = 0; b = .9; d = .4; \delta = .2.$

[c] $\bar{c} = .6; b = .95; d = .4; \delta = .1.$

current values of $\bar{c}$ the constrained firm is at the constraint about one-third of the time.

The last simulation changes b and δ to more common real business cycle values. The first parameters were chosen to generate roughly the same levels of volatility and consumption and returns as in the data. These are chosen as being a little closer to those used in other simulations. (King, Plosser, and Rebelo 1988 and Cecchetti, Lam, and Mark 1988). Here we run $b = 0.95$, $d = 0.4, \delta = 0.1, \gamma = 0$. For these values we needed to increase $\bar{c}$ to a larger

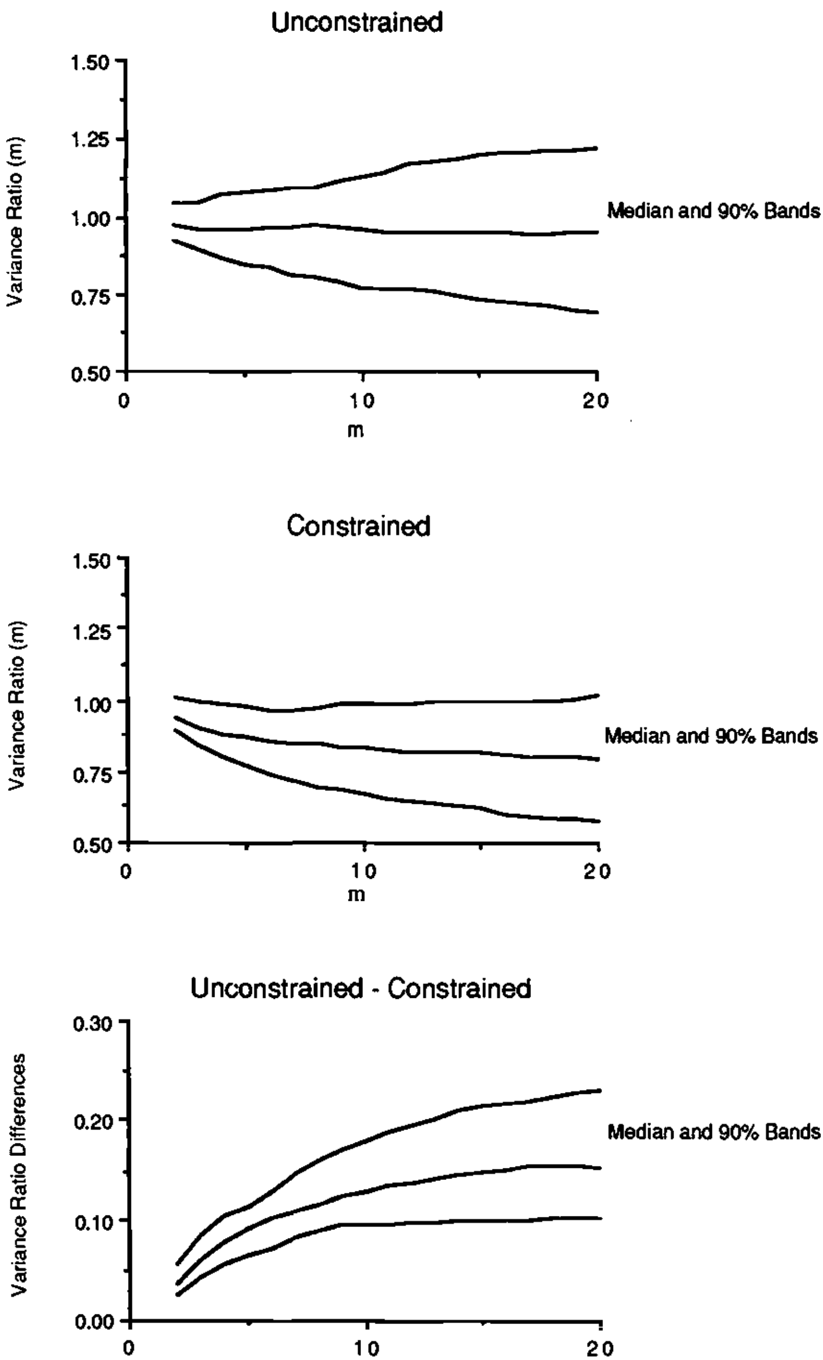

Fig. 9.5 Simulated variance ratios: $\gamma = -0.5, b = 0.9, d = 0.4, \delta = 0.2, \bar{c} = 0.4$

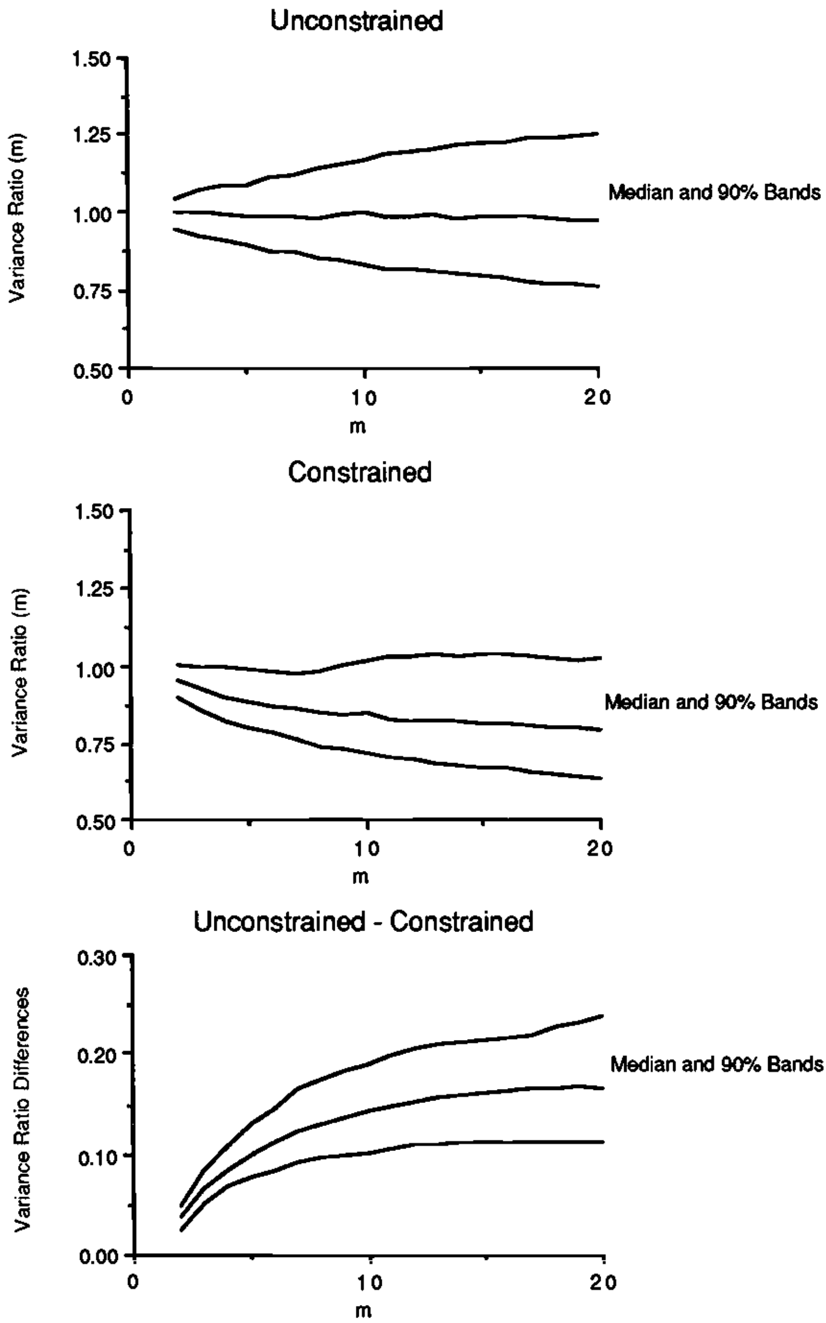

Fig. 9.6 Simulated variance ratios: $\gamma = -1.5, b = 0.9, d = 0.4, \delta = 0.2, \bar{c} = 0.4$

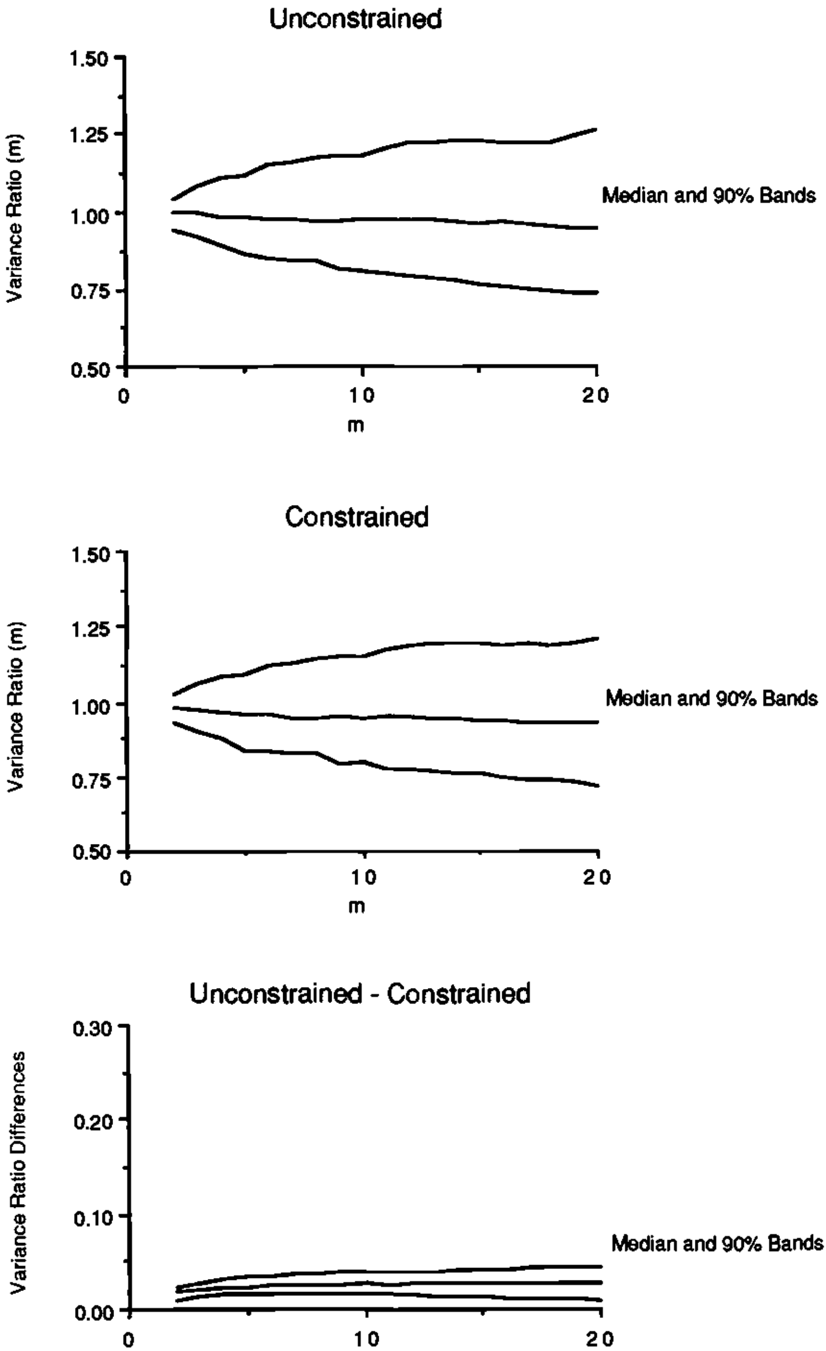

Fig. 9.7 Simulated variance ratios: $\gamma = 0, b = 0.9, d = 0.4, \delta = 0.2, \bar{c} = 0$

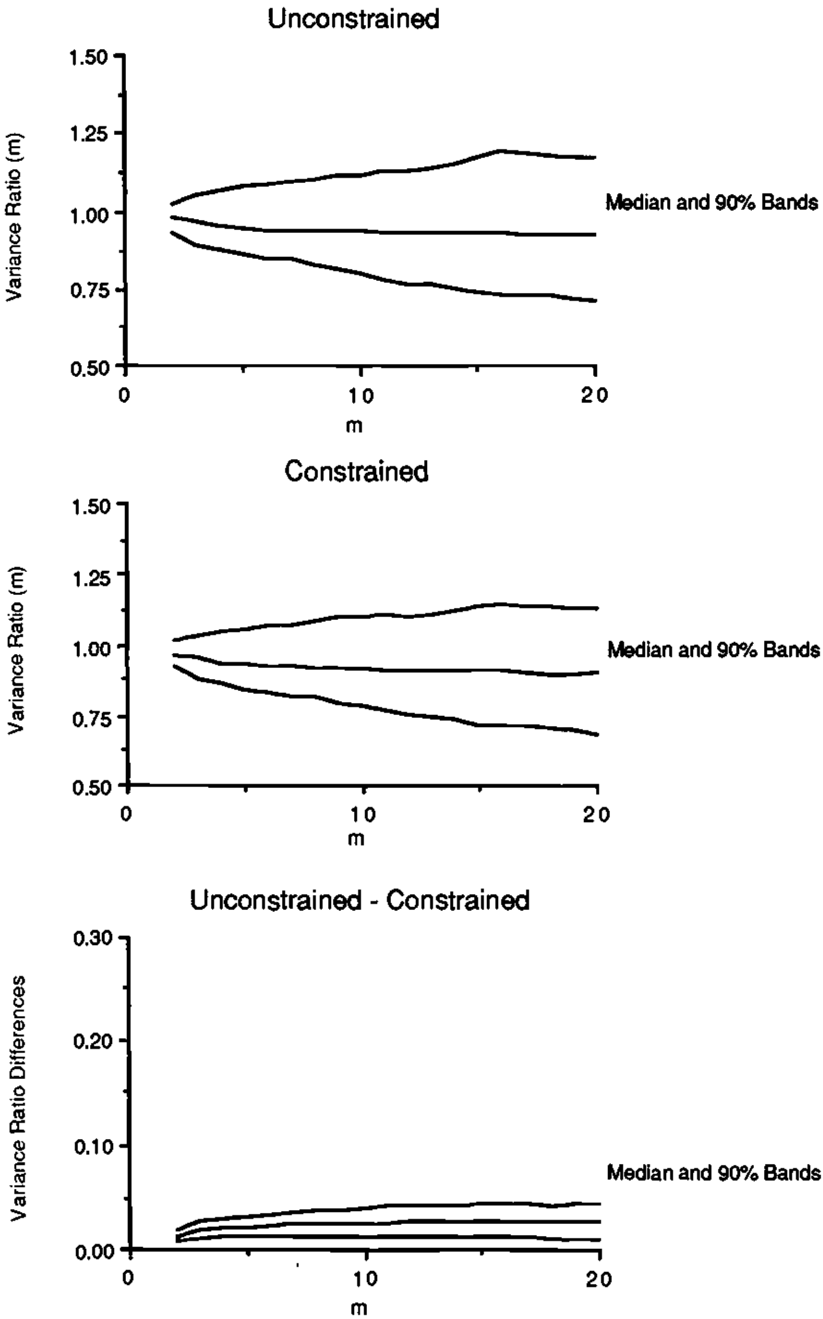

Fig. 9.8 Simulated variance ratios: $\gamma = -0.5, b = 0.9, d = 0.4, \delta = 0.2, \bar{c} = 0$

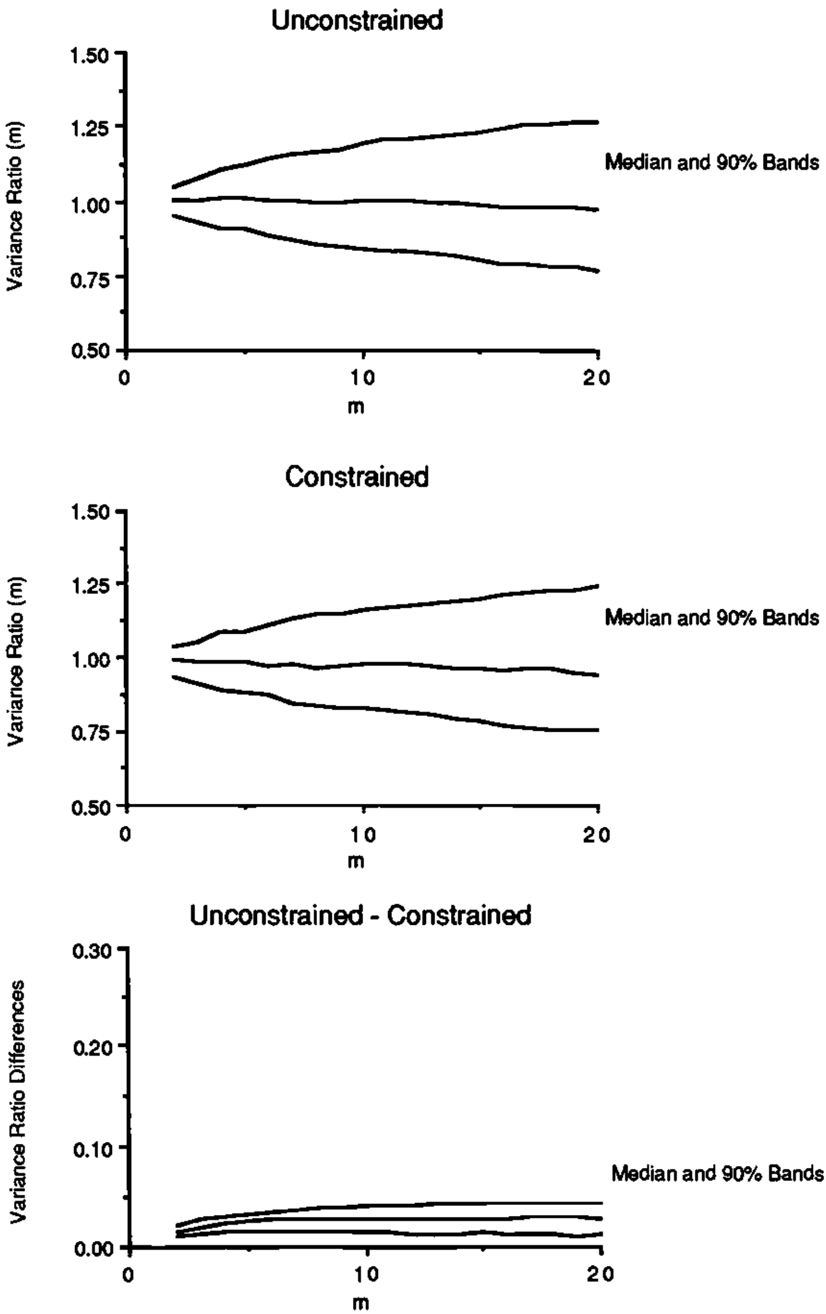

Fig. 9.9 Simulated variance ratios: $\gamma = -1.5, b = 0.9, d = 0.4, \delta = 0.2, \bar{c} = 0$

level, 0.6, to keep the constrained firm against the constraint. These results repeat those of the first three experiments. The plots of variance ratios are in figure 9.10. The differences here appear a little smaller than in the first set of runs, but still detectable. In table 9.2 the results show very little difference between the constrained and unconstrained firms for these parameters.

9.5 Speculations

In this section we make speculations about the possibility of credit constrained asset pricing models along the lines of the above to rationalize stylized facts.

1. De Bondt/Thaler (1985),(1987),(1989) loser/winner effect. Look at firm i whose production function is $a_i(t)r(t)x_i(t-1)^d$ in Section 9.3 above. Imagine conducting De Bondt and Thaler's (1985,1987) study on data generated by our model for $i = 1,2, \ldots, F$ firms where the smaller ones are credit constrained because of scale economies in underwriting loans. Past loser firms would be ones whose borrowing constraint binds tightly because, as was shown in Section 9.3, the value function is smaller for a firm that is constrained. One would expect the value of a firm that had experienced a spell of bad shocks to be low because it is very tightly constrained, but it should be a good bet because there is a good chance in the future that it will be positively shocked. Hence, its value will revert upward. This is what our variance ratio plots suggest.

2. (Capital asset-pricing model) CAPM returns are received only in January. And excess CAPM-adjusted returns appear for small firms in January (DeBondt and Thaler 1989, Haugen and Lakonishok 1988, Lakonishok and Smidt 1987). Barsky and Miron (1989) have documented a strong yearly seasonal in output with a sharp drop from the fourth quarter to the next first quarter. If we put a yearly seasonal in both the systematic shocks and the specific shocks in the above model, then smaller firms may exhibit high returns and high own variance over the turn of the year. Yet the covariance with the market may not move that much. Hence it is possible that there are CAPM-adjusted excess measured returns over the turn of the year for small firms.

3. Low price/earnings ratio firms exhibit "excess" returns (De Bondt and Thaler 1989). In Section three we showed that the value of firm i is low when it is credit constrained and y(i,t)/y(t) is low. In price/earnings (P/E) ratio studies the current price P is compared with some measure of intrinsic earning power. Since that measure is usually some kind of backward or estimated forward average, therefore, the measure of P/E will proxy current $y(i)/y$ in our model. We have shown by simulation that low enough current $y(i)/y$ predicts

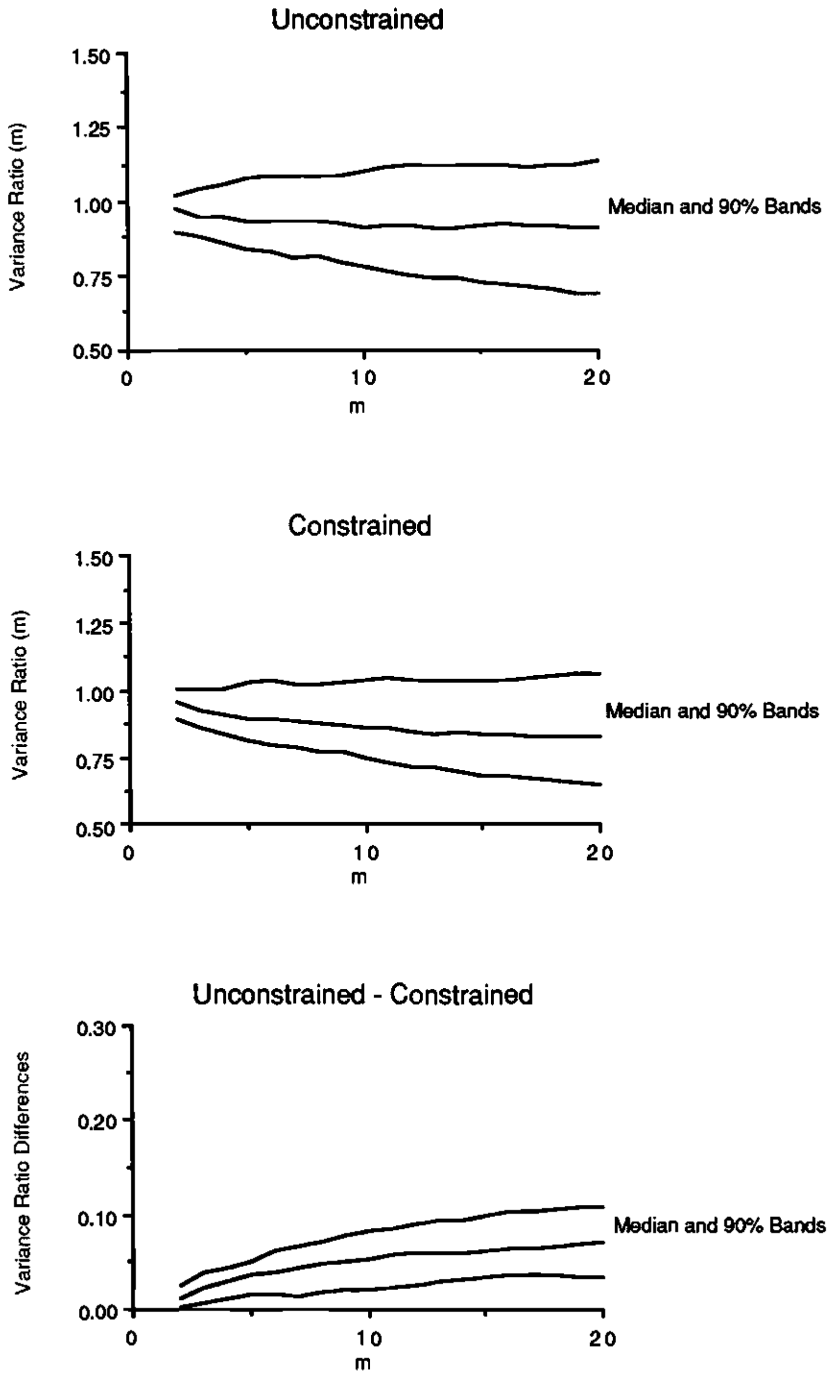

Fig. 9.10 Simulated variance ratios: $\gamma = 0, b = 0.95, d = 0.4, \delta = 0.1, \bar{c} = 0.6$

high returns next period. Hence it seems plausible that P/E ratio studies conducted on simulated data from a many firms version of our model may show "excess" returns to low P/E firms.

This paper set out to explain some stylized facts about small firms using a production-based asset-pricing model. We have been successful in some areas and unsuccessful in others. Simple credit-constraint restrictions, when the constraints are binding, much of the time can account for mean reversion types of effects in return series. There were several other issues that we had hoped to explain, but have been unable to do so with this model. Differences in unconditional moments do not show up in this model with these simple discrete shocks. Also, the lead-lag patterns discovered by Lo and MacKinlay (1988) were not seen here.

Notes

1. See Kim, Nelson, and Startz (1988) and Richardson (1988) for some of this evidence. Bruce Lehmann has chosen the term "factoids" for these stylized facts of questionable significance.
2. Variance ratios are calculated following Poterba and Summers (1987) and Cochrane (1988).

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