Volume Title: Annals of Economic and Social Measurement, Volume 6, number 5

Volume Author/Editor: NBER

Volume Publisher: NBER

Volume URL: http://www.nber.org/books/aesm77-5

Publication Date: December 1977

Chapter Title: Effect of Uncertainties on the Control Performance of Linear Systems with Unknown Parameters and Trajectory Confidence Tubes

Chapter Author: Yaakov Bar-Shalom

Chapter URL: http://www.nber.org/chapters/c10546

Chapter pages in book: (p. 599 - 611)

Annals of Economic and Social Measurement, 6/5, 1978

EFFECT OF UNCERTAINTIES ON THE CONTROL PERFORMANCE OF LINEAR SYSTEMS WITH UNKNOWN PARAMETERS AND TRAJECTORY CONFIDENCE TUBES*

BY Y. BAR-SHALOM

This investigation deals with the assessment of the effects of the errors in the parameters' estimates on the quadratic performance index used for controlling linear econometric models. An algorithm based upon the open-loop feedback concept is used to evaluate the increase in the cost due to parameter errors. Furthermore, the open-loop feedback concept is used to derive confidence regions for the various state components of interest at given future times. These confidence regions combined together define the confidence tube for the trajectory the system will follow in the state space. The technique is illustrated on two macroeconometric models where the "tightness" of the control, i.e., the extent to which the tube is narrow is of major interest.

1. INTRODUCTION

The unavoidable uncertainties associated with econometric models have been long since recognized and much effort has been spent in improving the quality of models. More recently work has been done in using stochastic control techniques for econometric models [C1, A1]. These methods account for the existing uncertainties in the model and the decision variables are modified from the deterministic case. Assuming the performance index is a cost function to be minimized, the uncertainty in the model will lead to an increase in this cost. While the additive disturbances are the effects that cannot be explained by the model, the parameter uncertainty is more of a basic "imperfectness" of the model. The latter might be reduced by more sophisticated estimation techniques, by longer data records or by using the control's "dual effect" [F2, B1].

The present investigation addresses first the question of how much is the penalty in terms of increased cost due to the parameter uncertainties. The goal is to obtain a simple, non Monte Carlo, evaluation technique that can be used to calculate the cost increase due to the parameter uncertainties. Such a technique based upon a stochastic control method is presented in Section 2. The approach takes the Bayesian point of view, i.e., that the true parameters are random variables.

*Research supported in part by NSF under Grants GS 32271 and ENG 77-08177. Presented at the NBER Conference on Stochastic Control in Economics, New Haven, Connecticut, May 1977. Stimulating Discussions with David Kendrick and Kent Wall are gratefully acknowledged.

The second question relates to the ability of controlling the variables of interest in an uncertain econometric model. The true criterion in controlling any system is in general a vector-valued objective function. For reasons of tractability a scalar criterion is usually set up that represents a compromise between the various objectives. While this scalar criterion becomes the design tool, it is of interest to examine separately the behavior of several variables. There are two aspects one can consider here:

(i) predicted values

(ii) confidence regions

The sequence of predicted values for a certain variable is the trajectory that the variable is (most) likely to follow. However, the goodness, or reliability, of the control derived using a given model should also be evaluated using a measure of how close the actual values of the variables of interest will be to the predicted ones. This leads to the concept of a "confidence tube" for a trajectory, made up from a sequence of confidence regions. While most studies concentrated on examining predicted trajectories little attention has been apparently paid to the region in which the realizations of these trajectories are likely to be. A simple technique that evaluates the confidence tube's width is presented in Section 3.

The application of these methods to two macroeconometric models is carried out in Section 4.

2. The Cost Increase Due to Uncertainty

$$x_{t+1} = Ax_t + Bu_t + c + v_t \qquad t = 1, \ldots, N-1 \tag{2.1}$$

where x_t is the state vector at time t. The decision variable u_t is obtained at time t with the knowledge of x_i and u_{i-1}, $i \leq t$. The additive noise v_t is assumed zero-mean, white and with covariance matrix V. The system matrices A, B and c contain some unknown parameters. Following the Bayesian point of view, which is needed in order to define a stochastic control problem for systems with parameter uncertainties, the uncertain parameters can be modelled as:

(i) a single realization from a distribution which remains fixed over the control horizon [C4] ("random variables that do not change in time")

(ii) the result of independent drawings from a fixed distribution [C1, C2, C3] ("multiplicative white noise" [A1]).

The cost function to be minimized is taken as quadratic about a desired trajectory $\overline{x}_k$, $k = 0, \ldots, N$, and desired controls $\overline{u}_k$, $k = 0, \ldots, N-1$ for a certain horizon N. The cost is thus the expected value of a sum of quadratic forms of these deviations

(2.2)

$$J = \frac{1}{2} E[(x_N - \bar{x}_N)'Q_N(x_N - \bar{x}_N) + \sum_{k=0}^{N-1} (x_k - \bar{x}_k)'Q_k(x_k - \bar{x}_k)$$

$$+ (u_k - \bar{u}_k)'R_k(u_k - \bar{u}_k) + 2(x_k - \bar{x}_k)'T_k(u_k - \bar{u}_k)]$$

where Q, R and T are matrices of appropriate dimensions.

Note that in the above expression the expectation is over all the random variables, i.e., all the noises as well as the ensemble of parameters. This is a consequence of the Bayesian framework underlying all stochastic control problems.

Denote by J^{HCE} the minimum value of the cost J if all the uncertainties are ignored and each random variable is replaced by its mean, (estimates for the parameters and zero for the noises) i.e., applying heuristically the certainty equivalence principle (HCE). This is obtained by well-known recursions [A1].

To assess the effect of the parameter uncertainty alone, we shall consider system (2.1) without the additive noise and evaluate the cost (2.2) with a control of the open-loop feedback (OLF) type [F1, C1]. The basic assumption in this control policy (also called stochastic control without learning in [C1], Chapter 10) are:

a) future state feedback will be available
b) the parameter statistics will not be updated during the control period (in practice only the first decision is retained and the entire solution is recomputed at every period)

Applications of this technique to econometric models have been reported in [B3, C1–C4, S1]. Note that this policy is different from the open-loop optimal feedback (OLOF) [B2, T1] because the latter ignores future state feedback.

A brief review of the equations pertinent to this OLF policy is given next. The assumed optimal cost-to-go is, starting from time $t + 1$,

$$J^{OLF}_{N-t-1} = \tfrac{1}{2}x'_{t+1}K_{t+1}x_{t+1} + p'_{t+1}x_{t+1} + g_{t+1} \tag{2.3}$$

Inserting this into the stochastic dynamic programming equation and using (2.1) without the additive noise yields

(2.4)

$$J^{OLF}_{N-t} = \min_{u_t} E[\tfrac{1}{2}(x_t - \bar{x}_t)'Q_t(x_t - \bar{x}_t) + \tfrac{1}{2}(u_t - \bar{u}_t)'R_t(u_t - \bar{u}_t)$$

$$+ (x_t - \bar{x}_t)'T_t(u_t - \bar{u}_t) + \tfrac{1}{2}(Ax_t + Bu_t + c)' \cdot$$

$$K_{t+1}(Ax_t + Bu_t + c)$$

$$+ p'_{t+1}(Ax_t + Bu_t + c) + g_{t+1} \mid I^t]$$

where I^t stands for the cumulated information at time t (all the states through t).

Since x_t is not a random variable when I^t is given, the expectation in (2.4) is taken as follows on a generic term:

$$E[x_t'A'K_{t+1}Ax_t \mid I^t] = x_t'E[A'K_{t+1}A \mid I^t]x_t \tag{2.5}$$

In view of assumption (b) one has

$$E[A'K_tA \mid I^t] = E[A'K_tA] \triangleq \overline{A'K_tA} \tag{2.6}$$

Assumption (b) states that the posterior distribution of the parameters is replaced by the prior. This simplifying assumption allows one to obtain the solution to this stochastic control problem with the resulting algorithm being only slightly more complex than in the deterministic case. However, there is a less obvious implication of (2.6): by using the prior distribution of the parameters in (2.6) the dependence of the expectation on x_t is ignored.

Therefore this OLF algorithm is *suboptimal* for model (i) described above (parameters that are random but time invariant) and *optimal* for model (ii) (parameters that are independent from period to period (white)). In practice the situation is probably in between. Nevertheless, the resulting algorithm, due to its simplicity is a useful tool in evaluating the effect of the parameter uncertainties. The confidence tube discussed in the next section is also based on this algorithm.

The resulting control and cost from (2.4) can be found in [C1]. This is also summarized in Appendix A.

The effect of the parameter uncertainty under the above assumptions is

$$\Delta J^P = J^{OLF} - J^{HCE} \tag{2.7}$$

The effect of the additive noise (disturbance) can be easily obtained explicitly by incorporating it into (2.4). The result is

$$\Delta J^D = \tfrac{1}{2}\sum_{t=1}^{N} \operatorname{tr}(K_t V) \tag{2.8}$$

where K_t follows from recursion (A.4) and V is the covariance matrix of v_t.

3. The Confidence Tube for the Trajectory

Assume that the feedback rule (A.1) resulting from (2.4) is used* for system (2.1). Then the predicted trajectory for the system will be

*Any other feedback rule or sequence of controls can be assumed.

$$(3.1)\qquad x^0_{t+1} = (A + BL_t)x^0_t + Bm_t + c$$

$$t = 0, \ldots, N-1;\; x^0_0 = x_0$$

A question of major importance is how close (or far) the actual trajectory will be to what (3.1) yields. This will serve as a measure of reliability of what a proposed control can be expected to do in a system.

Define the mean square value of the deviation of the ξ-th component of x_t from the corresponding predicted value as

$$(3.2)^*\qquad p^2_{t,\xi} = \tfrac{1}{2}E\,[(x_t - x^0_t)'Q^0_{t,\xi}(x_t - x^0_t)]$$

where the elements of the matrix in the above quadratic form are all zero except for

$$(3.3)\qquad Q^0_{t,\xi}(\xi,\xi) = 1$$

Eq. (3.2) is an approximate expression of the variance of the forecast. A discussion of one period ahead forecasts can be found in standard textbooks, e.g., [C-1]. Here we present a recursive algorithm that allows the evaluation of the forecase variance for an arbitrary number of periods. The result will be that the standard deviation of the ξ-th component of the state at time t is (approximately) $\sqrt{2}\,p_{t,\xi}$. The confidence region of width $2p$ corresponds then to a level of approximately 50% and was chosen arbitrarily.

The evaluation of (3.2) is done exactly as the OLF control was obtained in the previous section except that there is no need to minimize w.r.t. the control. Note that (3.2) is a quadratic cost and thus it will be written in the form

$$(3.4)\qquad p^2_{t,\xi} = \tfrac{1}{2}x_0'K_0x_0 + p_0'x_0 + g_0$$

where K, p and g (which are, for simplicity not indexed by t and ξ) are obtained from the following recursions

$$(3.5)\qquad K_i = A'K_{i+1}A + L_i'B'K_{i+1}BL_i + A'\tilde{K}_{i+1}BL_i + L_i'B'K_{i+1}A$$

$$(3.6)\qquad p_i = A'K_{i+1}Bm_i + L_i'B'K_{i+1}Bm_i + L_i'B'\tilde{K}_{i+1}c$$
$$+ A'K_{i+1}c + L_i'B'p_{i+1} + A'p_{i+1}$$

$$(3.7)\qquad g_i = \tfrac{1}{2}m_i'B'K_{i+1}Bm_i + \tfrac{1}{2}c'K_{i+1}c + c'K_{i+1}Bm_i$$
$$+ \tfrac{1}{2}\mathrm{tr}\,(K_{i+1}V) + p'_{i+1}Bm_i + p'_{i+1}c + g_{i+1}$$

for $i = t-1, \ldots, 0$ with initial conditions

$$(3.8)\qquad K_t = Q^0_{t,\xi}$$

* The 1/2 was used for convenience of programming

(3.9) $$\dot{P}_t = -Q^0_{t,f}\, x^0_t$$

(3.10) $$\tilde{g}^t = \tfrac{1}{2} x^{0\prime}_t Q^0_{t,f}\, x^0_t$$

One backward iteration has to be done with Eqs. (3.5)–(3.7), which are linear recursions, for each time t and component f of interest. This will give cross-sections of the trajectory uncertainty tube.

The above algorithm represents a convenient implementation of the concept of the "variance of a forecast."

4. Simulation Results

The techniques presented in Sections 2 and 3 were applied to two macroeconometric models with endogenous variables total private consumption C_t, total gross investment I_t and GNP (less net exports) Y_t and exogenous variable the government expenditure

(4.1) $$G_t \triangleq u_{t-1}$$

First a 3 state model identified in reduced form with OLS in [K1] is considered. For the purpose of this study, which was to illustrate the technique of Section 3, out of the 15 parameters of this model, only the 5 entering the consumption equation were considered random with covariance matrix as yielded by the identification procedure.* Additive noise was assumed to enter in each equation.

This model is characterized by the following equations (the notations from (2.1) are used)

(4.2) $$x_t' = [C_t I_t Y_t]$$

(4.3) $$A = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 \\ -.328 & .425 & .403 \\ .527 & .301 & .557 \end{bmatrix}$$

(4.4) $$b' = [\theta_5 \quad -.499 \quad .345]$$

(4.5) $$c' = [\theta_4 \quad .008 \quad -1.512]$$

The estimate of the unknown parameter vector was:

(4.6) $$\hat{\theta}' = [-.852 \quad -.125 \quad .156 \quad -1.482 \quad -.158]$$

with the associated variance-covariance matrix

*This was the extent to which data were available.

$$(4.7)\qquad S = \begin{bmatrix} .02914 & .02176 & -.02767 & .02983 & .02720 \\ .02176 & .01973 & -.02139 & -.00618 & .02122 \\ -.02767 & -.02139 & .02652 & -.03398 & -.02642 \\ .02983 & -.00618 & -.03398 & 4.122 & .08987 \\ .02720 & -.02122 & -.02642 & .08987 & .02832 \end{bmatrix}$$

The covariance matrix of the process noise (random shocks) was

$$(4.8)\qquad V = \text{diag}(9.61,\ 18.92,\ 28.94)$$

The second model is the one from [W1]. This is a structural form identified with FIML by the ERSF algorithm [W2] for increments rather than levels. The same data base was used in both cases. The state space form has 11 states and only 3 unknown parameters (due to the model specification).

$$(4.9)\qquad x_t' = [\Delta Y_t \Delta \tilde{C}_t \Delta \tilde{I}_t \Delta \tilde{I}_{t-1} \Delta \tilde{I}_{t-2} e_{t-1}^1 e_{t-2}^1 Y_t C_t I_t u_{t-1}]$$

where Δ stands for increment (first difference): the quantities with the "tilde" sign had their mean variations subtracted: the appearance of lagged noises is a consequence of the prewhitening procedure carried out in the course of the identification (see [W1] for details).

The non-zero elements of the system matrices were

$$(4.10)\qquad \begin{aligned} a_{1,1} &= a_{2,1} = a_{8,1} = a_{9,1} = \theta_1 \\ a_{1,2} &= a_{3,2} = a_{8,2} = a_{10,2} = \theta_2 \\ a_{1,5} &= a_{2,5} = a_{8,5} = a_{9,5} = \theta_3 \\ a_{1,6} &= a_{2,6} = a_{8,6} = a_{9,6} = -.211 \\ a_{1,7} &= a_{2,7} = a_{8,7} = a_{9,7} = .357 \\ a_{4,3} &= a_{5,4} = a_{7,6} = a_{8,8} = a_{9,9} = a_{10,10} = -a_{1,11} = -a_{8,11} = 1 \\ b_1 &= b_8 = b_{11} = 1 \\ c_1 &= c_8 = -1.32 \\ c_9 &= 2.974 \\ c_{10} &= .658 \end{aligned}$$

where θ are the unknown parameters with estimates

$$(4.11)\qquad \hat{\theta}' = [.227 \quad .703 \quad -.1399]$$

with covariance matrix

$$(4.12)\qquad S = \begin{bmatrix} .00114 & .00094 & .00365 \\ .00094 & .00449 & .00152 \\ .00365 & .00152 & .02020 \end{bmatrix}$$

The covariance of the process noise was

$$(4.13)\qquad Q = \Gamma \begin{bmatrix} 7.498 & .561 \\ .561 & 17.27 \end{bmatrix} \Gamma'$$

where Γ is an 11×12 matrix with units in locations (1,1), (1,2), (2,1), (3,2), (6,1), (8,1), (8,2), (9,1), (10,2) and zeros elsewhere. Further details on this model as well as on the way it was obtained can be found in [W1].

Both models were identified from the same data base: constant (1958) dollar time series from the NBER data bank for the period 1947-1 to 1968-4.

The performance index was chosen quadratic with unity weights for the deviations of C, I, Y, G from desired levels. These levels were chosen as increasing 0.75% per quarter from the initial condition of 69-1 until 70-4.

Table 1 shows the cost defined in Section 2 for these two models. It appears that the 11 state model suffers a relatively small penalty due to the uncertain parameters. This seems to be due to the fact that its specification in increments captured quite accurately the behavior. On the other hand

TABLE 1

COSTS FOR THE TWO MODELS

Model	Deterministic Cost J^{DET}	Cost with Uncertain Parameters J^{UP}	Increase Due to Uncertain Parameters ΔJ^{P}	Increase Due to Additive Disturbances ΔJ^{D}
4 state	415.6	972.2	556.6	616.[illegible]
11 state	644.5	655.5	11.0	[illegible]

TABLE 2

FIRST PERIOD CONTROLS

	HCE	OLF
4 state	154.92	158.63
11 state	155.66	155.86

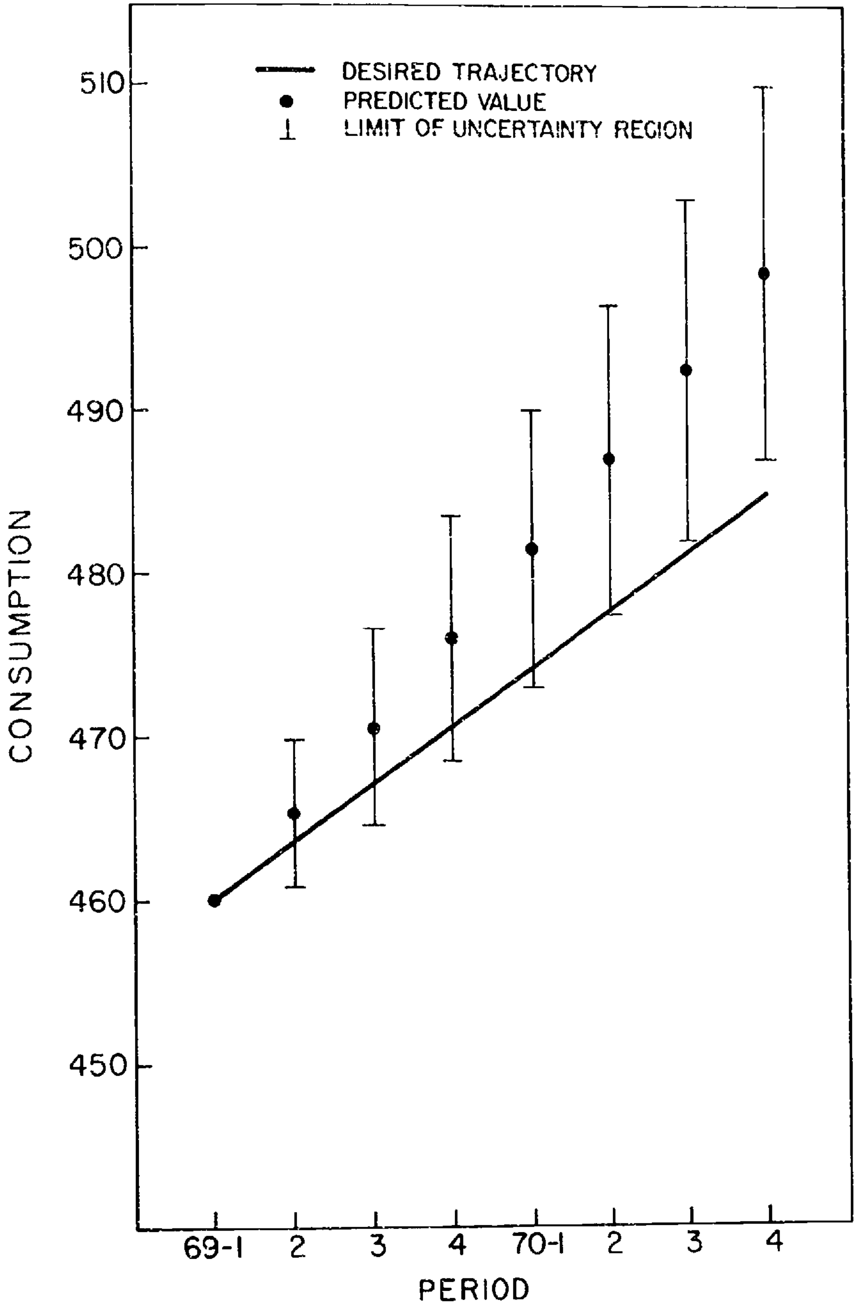

Figure 1 Desired and predicted consumption with uncertainty region for 3-state model.

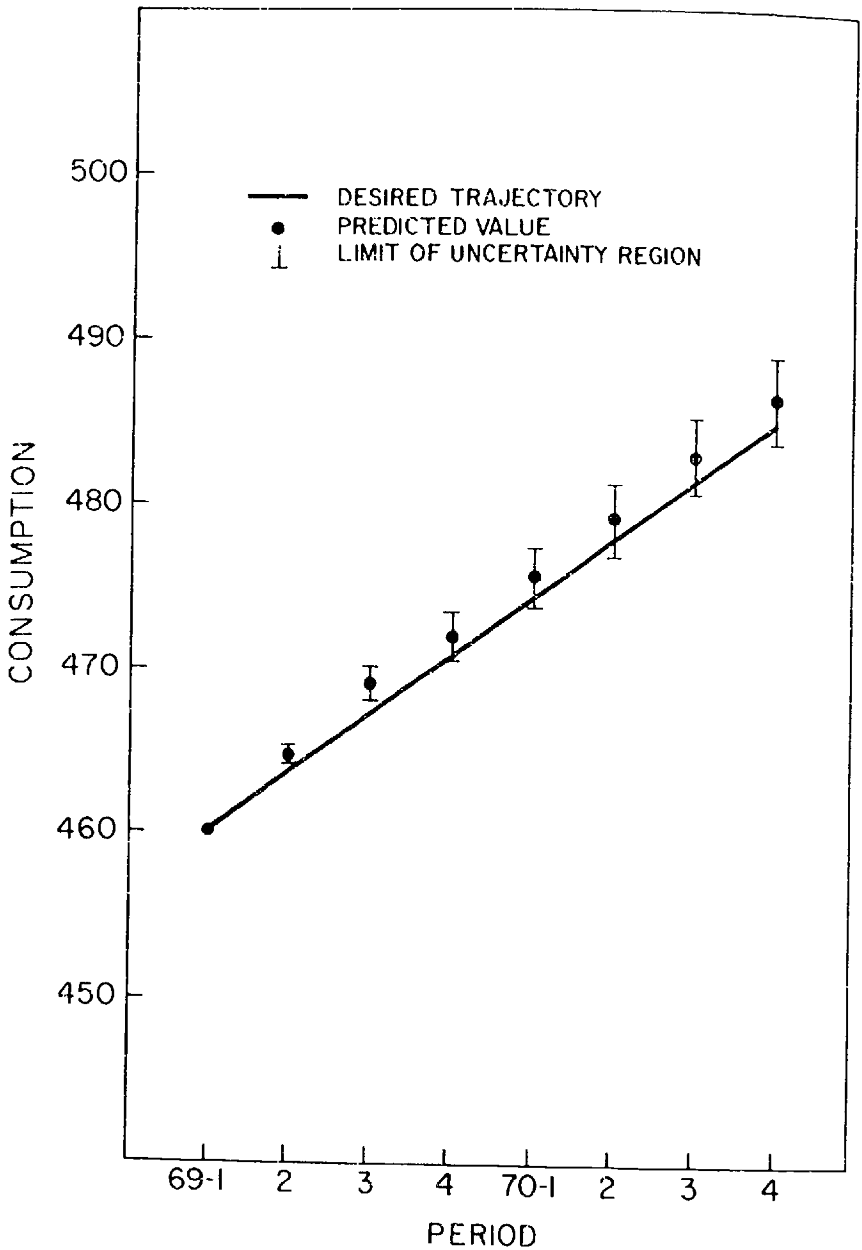

Figure 2 Desired and predicted consumption with uncertainty region for 11-state model.

in the dynamic model identified directly in levels there seems to be a substantial effect of the parameter uncertainty on the cost.

A comparison of the first period control for the two models with both HCE and OLF control strategies is presented in Table 2. The CE control is larger for the second model than for the first one such a discrepancy can be always expected when two different models are obtained. If the parameter uncertainties are accounted for, the first period control in the smaller model decreases by about 1%. In the larger model the parameter uncertainties have a very small effect on the control (about .13%) but, interestingly, in the opposite direction. While this could sound counter-intuitive, it has been pointed out [A2] that the OLF control can be larger than the CE as well as smaller.

Figures 1 and 2 present the desired trajectory for consumption in billions of 1958 dollars (solid line), predicted values (dots) and associated confidence regions as defined in (3.2) for the two models. The predicted trajectory of the 3-state model is substantially farther away from the desired path than for the second model. The first model cannot apparently sustain the uniform growth rate of 0.75% per quarter for each of the variables. Furthermore, the confidence regions associated with the predicted values of the consumption are substantially larger in the first model than in the second. The pattern for the other variables was similar. It is felt that forecasts or recommendations following from models should be judged not only based upon the corresponding values (point estimates) but the associated uncertainty should also be taken into consideration.

5. Conclusion

A method based upon the OLF stochastic control has been developed to assess the effects of model parameter uncertainties on the performance index when controlling an econometric model. An algorithm was presented that calculates the uncertainty tube for the trajectory of an endogenous variable in an econometric model for a given control law or set of values for the control. The potential usefulness of this lies in the following:

1. It can be used to assess the reliability of models and controls derived using those models.
2. It can serve in the comparison of proposed control laws vs. past actual values.

University of Connecticut, Storrs

Appendix A

The OLF Algorithm

The control is given by

$$u_t^* = L_t x_t + m_t \tag{A.1}$$

where

$$L_t \triangleq -(R_t + B'K_{t+1}B)^{-1}(B'K_{t+1}A + T_t') \tag{A.2}$$

$$m_t \triangleq -(R_t + B'K_{t+1}B)^{-1}(B'K_{t+1}c + B'p_{t+1} - R_t\bar{u}_t - T_t'x_t) \tag{A.3}$$

and the backwards recursions for K, p, g are:

$$K_t = Q_t + A'K_{t+1}A + (A'K_{t+1}B + T_t)L_t \tag{A.4}$$

$$p_t = -Q_t x_t - T_t\bar{u}_t + A'K_{t+1}c + A'p_{t+1} + (A'K_{t+1}B + T_t)m_t \tag{A.5}$$

$$\begin{aligned} g_t = {} & \tfrac{1}{2}x_t'Q_t x_t + \tfrac{1}{2}\bar{u}_t'R_t\bar{u}_t + x_t'T_t\bar{u}_t + \tfrac{1}{2}c'K_{t+1}c + c'p_{t+1} \\ & + \tfrac{1}{2}(B'K_{t+1}C + B'p_{t+1} - R_t\bar{u}_t - T_t'x_t)'m_t + g_{t+1} \end{aligned} \tag{A.6}$$

for $t = N - 1, \ldots, 0$ with initial conditions

$$K_N = Q_N \tag{A.7}$$

$$p_N = -Q_N x_N \tag{A.8}$$

$$g_N = \tfrac{1}{2}x_N'Q_N x_N \tag{A.9}$$

The cost J^{OLF} results then from (2.3) using the above recursions.

The deterministic algorithm is the same as above with the expectations (denoted by overbar) removed.

References

[A1] M. Aoki, Optimal Control and System Theory in Dynamic Economic Analysis, North Holland, 1976.

[A2] M. Athans, R. Ku and S. B. Gershwin, "The Uncertainty Threshold Principle," *IEEE Trans. Auto. Control*, vol. AC-22, pp. 491–495, June 1977.

[B1] Y. Bar-Shalom and E. Tse, "Dual Effect, Certainty Equivalence and Separation in Stochastic Control," *IEEE Trans. Auto. Control*, Vol. AC-19, pp. 494–500, Oct. 1974.

[B2] Y. Bar-Shalom and R. Sivan, "On the Optimal Control of Discrete-Time Linear Systems with Random Parameters," *IEEE Trans. Auto. Control*, vol. AC-14, pp. 3–8, Feb. 1969.

[B3] W. C. Brainard, "Uncertainty and the Effectiveness of Policy," *American Econom. Review*, Vol. 57, pp. 411–425, 1967.

[C1] G. Chow, *Analysis and Control of Dynamic Economic Systems*, Wiley, 1975.

[C2] G. Chow, "Effect of Uncertainty on Optimal Control Policies," *Intn'l Economic Review*, Vol. 14, No. 3, pp. 632–645, Oct. 1973.

[C3] R. Craine and A. Havenner, "Optimal Control in a Linear Macroeconomic Model with Random Coefficients," *Proc. 1973 IEEE Conf. on Decision and Control*, San Diego, Dec. 1973.

[C4] ——, "A Stochastic Optimal Control Technique for Models with Estimated Coefficients," *Econometrica*, Vol. 45, pp. 1013–1021, May 1977.

[F1] J. B. Farison, R. E. Graham and R. C. Shelton, "Identification and Control of Linear Discrete Systems," *IEEE Trans. Auto. Control*, vol. AC-12, pp. 438–442.

[F2] A. A. Feldbaum, *Optimal Control Systems*, Academic, 1965.

[G1] A. Goldberger, *Econometric Theory*, Wiley, 1964.

[K1] D. Kendrick, "Adaptive Control of Macroeconomic Models with Measurement Error," *5th NBER Conf. on Stochastic Control in Economics*, New Haven, Conn., May 1977.

[S1] F. Shupp, "Uncertainty and Optimal Policy Intensity in Fiscal and Income Policies," *Ann. Economic and Social Measurement*, Vol. 5, pp. 225–238, Spring, 1976.

[T1] E. Tse and M. Athans, "Adaptive Stochastic Control for a Class of Linear Systems," *IEEE Trans. Auto. Control*, vol. AC-17, pp. 38–52, Feb. 1972.

[W1] K. D. Wall, Y. Bar-Shalom and D. Kendrick, "Adaptive Control of Macroeconometric Models with Correlated Measurement Errors," *5th NBER Conf. on Stochastic Control in Economics*, New Haven, Conn., May 1977.

[W2] K. D. Wall, "FIML Estimation of Rational Distributed Lag Structural Form Models," *Annals of Economic and Social Measurement*, Vol. 5, No. 1, Winter, 1976.