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Chapter Author: A. C. Harvey

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DISCRIMINATION BETWEEN CES AND VES PRODUCTION FUNCTIONS

BY A. C. HARVEY

Under certain assumptions direct estimation of the parameters of production functions is possible. For the CES function, estimates of the parameters may be obtained via the linear approximation suggested by Kmenta, rather than by using a nonlinear estimation procedure. In this paper it is shown that a similar approximation exists for the Sato-Hoffman/Revankar VES function. However, unlike the approximation for the Bruno VES function, this has a different form to the Kmenta approximation. Discrimination between the two functions is therefore possible by the comprehensive F test and the non-nested procedure developed by Pesaran. Some empirical results are presented which show that the Pesaran test is, on the whole, more effective than the F test.

1. INTRODUCTION

In a recent article, Corbo (1976) has shown that the approximation suggested by Kmenta (1967) for the CES production function also provides a good approximation to Bruno's VES function. However the Kmenta approximation cannot be justified for all VES production functions[1] and, in fact, the more widely known VES form developed by Revankar (1971) and Sato and Hoffman (1968) has an entirely different Taylor series expansion associated with it. This suggests that it is possibly to discriminate between this particular VES function and the CES function on the basis of their Taylor series approximations. In this paper it is shown how this may be done statistically by means of the Pesaran test and the comprehensive classical F test: see Pesaran (1974). The probabilities of arriving at correct decisions by these procedures are then computed for a particular set of data on capital and labour, and a comparison between the tests is made.

2. APPROXIMATIONS TO CES AND VES PRODUCTION FUNCTIONS

The CES production function is:

$$Q = \gamma[(1 - \delta)L^{-\eta} + \delta K^{-\eta}]^{-\nu/\eta} \tag{1}$$

where Q is output, K is capital, L is labour and γ, δ, η and ν are parameters. On dividing through by L and taking logarithms the rather more convenient formulation

[1]Corbo (1976, p. 66) appears to imply that all VES functions have similar properties.

$$\text{(2)} \qquad \log q = \log \gamma + (\nu - 1)\log L - \nu\eta^{-1}\log(1 - \delta + \delta k^{-\eta})$$

in which $q = Q/L$ and $k = K/L$ is obtained.

The approximation suggested by Kmenta involves carrying out a Taylor series expansion around $\eta = 0$ to yield

$$\text{(3)} \quad \log q \sim \log \gamma + (\nu - 1)\log L + \nu\delta \log k - 0.5\,\nu\delta\eta(1 - \delta)[\log k]^2.$$

The Bruno production function is:

$$\text{(4)} \qquad Q = \gamma[\delta K^{-\eta} + (1 - \delta)K^{-m\eta}L^{\eta(m-1)}]^{-\nu/\eta}.$$

As Corbo shows a Taylor series expansion for (4) yields an expression having exactly the same form as the Kmenta approximation. However this is perhaps not too surprising since the CES function is a special case of the Bruno function. It is obtained from the Bruno function by setting $m = 0$ and hence (1) may be regarded as being 'nested' within (4).[2] On the other hand the CES function is not nested within the Sato-Hoffman/Revankar VES function.

$$\text{(5)} \qquad Q = \gamma K^{\nu(1-\delta\rho)}[L + (\rho - 1)K]^{\nu\delta\rho}.$$

This function only reduces to the CES form when the parameter, η, in the CES function takes certain specific values. For example if $\eta = 0$ in equation (1) the Cobb-Douglas function is obtained, and (5) reduces to the Cobb-Douglas form when $\rho = 1$.

Dividing the Sato-Hoffman/Revankar function through by L and taking logarithms gives the equation:

$$\text{(6)} \quad \log q = \log\gamma + (\nu - 1)\log L + \nu(1 - \delta\rho)\log k + \nu\delta\rho\log[1 + (\rho - 1)k].$$

A Taylor series expansion around $\rho = 1$ then yields

$$\text{(7)} \quad \log q \simeq \log\gamma + (\nu - 1)\log L + \nu(1 - \delta\rho)\log k + \nu\delta\rho(\rho - 1)k.$$

Thus a term in k replaces the term in $[\log k]^2$ in the Kmenta approximation. If data on K and L are available a regression[3] run on the basis of equation (7) will give indirect estimates of all four parameters, ν, δ, γ and ρ, in the VES function. Furthermore the hypothesis that the appropriate functional form is Cobb-Douglas may be tested by a test of significance on the regression coefficient of k. This is analogous to the test based on the

[2] Mizon (1974) gives a general discussion of the concept of nested hypotheses in the context of production functions.

[3] The circumstances under which such regressions on production functions are justified are well known and will not be dealt with here.

Kmenta approximation[4] which has been widely used, for example, by Griliches and Ringstad (1971).

One final point about equation (7) is that it is of the same form as the 'transcendental' production function proposed by Lovell (1968). Hence Lovell's VES function may be viewed as an approximation to the Sato-Hoffman/Revankar form.

3. Discrimination Between CES and VES Functions

Since the linear approximations to the CES and Sato-Hoffman/Revankar VES functions are non-nested, an appropriate statistical technique for discriminating between them is either the Pesaran test or the classical F test. These procedures may be described for the general case as follows.

Suppose we have two possible regression models, which may be written in conventional matrix terms as

$$H_1: y = X\beta + u_1, y_1 \sim N(0, \sigma_1^2 I), \tag{8}$$

$$H_2: y = Z\gamma + u_2, u_2 \sim N(0, \sigma_2^2 I), \tag{9}$$

where X and Z are assumed to be fixed in repeated samples and are not nested within each other, i.e. all columns of X cannot be obtained from those of Z and vice versa. The problem is to obtain a test on the specification of H_1 which has high power against alternatives belonging to H_2.

The classical procedure consists in forming a comprehensive model which includes both H_1 and H_2 as special cases. The hypothesis H_1 is then rejected if the variables which appear in the comprehensive model but not in H_1 are jointly significant according to the F test.

An alternative test procedure has been developed by Pesaran (1974). Let $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ be the estimated variances from H_1 and H_2 respectively; let b denote the OLS estimator of β and let e_{21} be the vector of OLS residuals in the regression of Xb on Z; let e_{121} be the vector of OLS residuals in the regression of e_{21} on X and finally let $\hat{\sigma}_{21}^2 = \hat{\sigma}_1^2 + n^{-1}e_{21}'e_{21}$, where n is the sample size. Then defining $T_1 = (n/2)\log(\hat{\sigma}_2^2/\hat{\sigma}_1^2)$ and $V_1 = (\hat{\sigma}_1^2/\hat{\sigma}_{21}^4)e_{121}'e_{121}$ it can be shown that the statistic

$$N_1 = T_1/V_1^{1/2}, \tag{10}$$

is asymptotically $N(0, 1)$ when H_1 is true. A significant negative value of N_1 implies a rejection of H_1 in favour of H_2.

[4] Despite the different forms of (3) and (7) some computations by the author show that the Kmenta t test may still have a high probability of rejecting the Cobb-Douglas specification when the true model is VES. Conversely the t test in (7) may have a high power when the true model is CES. See Harvey (1976).

The above procedure can be applied to the problem of discriminating between CES and VES functional forms by using the Taylor series approximations, (3) and (7). The comprehensive model is simply

$$y_j = \alpha_1 + \alpha_2 \log L_j + \alpha_3 \log k_j + \alpha_4 (\log k_j)^2 + \alpha_5 k_j + u_j, j = 1, \ldots, n.$$

and the F test reduces to a t test, irrespective of whether the CES or VES form is taken as the null hypothesis. Thus the CES form is rejected if $\hat{a}_5$, the OLS estimate of α_5, is significantly different from zero, while VES is rejected if $\hat{a}_4$ is significant.

Two N statistics are calculated; one taking CES as the null hypothesis and the other taking VES as the null hypothesis. As with the F test there are four possible outcomes to the procedure: both specifications may be rejected, neither may be rejected or one may be rejected while the other is not.

4. Empirical Results

The relative performances of the two tests described in the previous section were evaluated for a particular data set. The data, obtained from Pyatt and Stone (1964) consisted of observations on capital and labour for 22 British industries in the year 1960, and although it is perhaps of limited value to fit production functions across industries, it was felt that these figures provided a reasonably good reflection of the kind of data sets frequently encountered in production function studies. The same data were, in fact, used by Mizon (1974) in his study except that he took 24 industries. However, we preferred to omit two industries ('Coke Ovens' and 'Mineral, Oil and Refining') since these both had very high capital/labour ratios compared with the other industries and it was felt that the 22 observation set, having a higher degree of multicollinearity, was probably more 'typical'.

CES and VES functions of the form (2) and (6), respectively, were considered with additive disturbance terms, independently and normally distributed with mean zero and constant variance, σ^2. Suitable values of σ^2 were chosen as follows. Denoting the $n \times 1$ vector of expected values of the dependent variable, in deviation from the mean form, by y_E, we may define the quantity

$$R_*^2 = 1 - \sigma^2 (n^{-1} y_E' y_E + \sigma^2)^{-1}. \tag{11}$$

Although R_*^2 is not the expectation of R^2 it may still be regarded as an indication of an "average" value of R^2 since, provided $n^{-1} y_E' y_E$ is bounded as $n \to \infty$, it may be shown that plim $R^2 = R_*^2$; see Koerts and

TABLE 1

PROBABILITIES THAT COMPREHENSIVE (F) AND PESARAN (N) TESTS REJECT CES AND VES WHEN TRUE MODEL IS CES

($\delta = 0.27, \nu = 1.0, R_*^2 = 0.99$)

ϕ		0.83	0.67	0.50	2.00
Prob. of rejecting VES	F	.092	.316	.665	.175
	N	.245	.610	.890	.428
Prob. of rejecting CES	F	.050	.055	.060	.076
	N	.073	.068	.048	.215

TABLE 2

PROBABILITIES THAT COMPREHENSIVE (F) AND PESARAN (N) TESTS REJECT CES AND VES WHEN TRUE MODEL IS VES

($\delta\rho = 0.27, \nu = 1.0, R_*^2 = 0.99$)

$\bar{\phi}$		0.83	0.67	0.50	2.00
Prob. of rejecting CES	F	.086	.341	.927	.104
	N	.203	.595	.990	.288
Prob. of rejecting VES	F	.050	.055	.103	.063
	N	.103	.058	.033	.180

Abrahamse (1969, p. 135 6). The value of R_*^2 was then set equal to 0.99, and the appropriate value of σ^2 was obtained by solving (11). Suitable values of the parameters (required for calculating y_E) were obtained from a regression on the original data, but in all cases constant returns to scale were assumed. i.e. $\nu = 1$.

Table 1 presents results for the N and F tests when the true model is CES, while Table 2 gives the corresponding results for a VES model. Although there are four outcomes to the test procedure, only the probabilities of rejecting each of the two models are given. Very little is lost by doing this (c.f. the presentation in Pesaran, 1974), since the probabilities of rejecting both models are very small in all cases; on the other hand a certain amount is gained in clarity of presentation.

The F test probabilities were computed exactly by the method of Imhof as set out in the Appendix. The N test probabilities[5] were esti-

[5]A one-sided test was assumed. This is in contrast to Pesaran (1974), who in his empirical results used a two-sided N test "in order to make the two tests comparable." The rationale behind this is somewhat unclear

mated by Monte Carlo methods. Four hundred independent replications were used in each case. Thus the 95% confidence interval for an estimated probability of 0.50 is approximately 0.50 ± 0.05, while for a probability of 0.10 it is 0.10 ± 0.03.

The results in the Tables are given for four different values of the elasticity of substitution, ϕ. In the VES case ϕ depends on k as

$$\phi = 1 + (1 - \delta\rho)^{-1}(\rho - 1)k. \tag{12}$$

However, by setting k equal to the average value over all observations, and treating $\delta\rho$ as a fixed parameter (equal to $1 - \delta$ in the CES function), an average elasticity of substitution, $\bar{\phi}$, was defined together with a corresponding value of ρ.

The results in both tables indicate that when $\phi = 0.67$ or 0.50, the N test is clearly superior to the F test in that it gives a much higher probability of rejecting the incorrect model while having a probability of rejecting the true model which is not significantly larger (and for $\phi = 0.50$ it appears to be smaller) that that of the F test. However as ϕ increases the greater 'power' of the N test is only achieved at the expense of a high probability of rejecting the true model. Nevertheless its performance is still better if the criterion adopted is the proportion of correct decisions, i.e. incorrect model rejected and true model accepted. As previously indicated this proportion is, in all cases, only marginally below the estimated probability of rejecting the incorrect model.

Overall the results indicate that statistical discrimination between the CES and Sato-Hoffman/Revankar VES functions is possible. However the tests are unlikely to be effective unless the variance of the disturbance has a relatively small value. The figures presented were obtained with $R_*^2 = 0.99$ and corresponding calculations for $R_*^2 = 0.95$ gave considerably lower 'powers': see the results for the F test set out in Table 3.

TABLE 3

PROBABILITIES THAT COMPREHENSIVE (F) TESTS REJECT CES AND VES WHEN: (a) TRUE MODEL IS CES, (b) TRUE MODEL IS VES ($R_*^2 = 0.95$)

Model	ϕ or $\bar{\phi}$	0.83	0.67	0.50	2.00
(a) (b)	Prob. of rejecting CES	.050 .057	.051 .104	.052 .323	.055 .060
(a) (b)	Prob. of rejecting VES	.058 .050	.100 .051	.182 .060	.073 .052

5. Conclusion

The Sato-Hoffman/Revankar VES production function has a Taylor series expansion which is different to the Kmenta approximation used for the CES function. This essentially involves adding a term in k, rather than $[\log k]^2$, to the Cobb-Douglas equation. Statistical discrimination between the two functions on the basis of these linear approximations is possible by means of the Pesaran test and the comprehensive F test, both of which are designed to deal with discrimination between 'non-nested' hypotheses. The empirical results presented indicate that, although the Pesaran test requires rather more computation than the F test, its performance is better in the sense that it gives a higher proportion of correct decisions.

University of Kent at Canterbury

Appendix

Calculation of the Power of the F Test

Consider a model of the form

$$(A.1) \qquad y = X\beta + X_p\beta_p + u,$$

where X and X_p are $n \times k$ and $n \times p$ matrices respectively, β and β_p are respectively $k \times 1$ and $p \times 1$ vectors of parameters, and u is an $n \times 1$ vector of disturbance terms which are assumed to be normally and independently distributed with mean zero and constant variance. In the classical F test the hypothesis that $\beta_p = 0$ is tested. When $p = 1$ this is simply the conventional t test.

The test statistic, which follows an F-distribution with $(p, n - k - p)$ degrees of freedom under the null hypothesis, may be written

$$(A.2) \qquad \omega = \frac{e'e - e'_{k+p}e_{k+p}}{e'_{k+p}e_{k+p}} \cdot \frac{n - k - p}{p}$$

where e and e_{k+p} are the vectors of OLS residuals obtained from regressing y on X and $[X \vdots X_p]$ respectively.

Now suppose the true model is

$$(A.3) \qquad y = X_r\beta_r + f + v, \qquad v \sim N(0, \sigma^2 I),$$

where β_r is an $r \times 1$ parameter vector and X_r is an $n \times r$ (fixed) matrix, with $0 \leq r \leq k$, the columns of which are contained in X. Each element in the $n \times 1$ vector f is a (possibly nonlinear) function of a set of fixed observations, some of which may be observations in the corresponding row of X_r.

Now

$$e = My = M(X_r\beta + f + \varepsilon) = M(f + \varepsilon) = M\zeta.$$

where $M = I - X(X'X)^{-1}X'$. Defining M_{k+p} in a similar manner we have

$$e_{k+p} = M_{k+p}y = M_{k+p}\zeta.$$

Thus (A.2) becomes

$$\omega = \frac{\zeta'(M - M_{k+p})\zeta}{\zeta' M_{k+p}\zeta} \cdot \frac{n-k-p}{p}.$$

and this is a quadratic form in independent normal variables since $\zeta \sim N(f, \sigma^2 I_n)$.

If q_α is the appropriate significance point for a (one-tailed) F test with a Type I error of size α and $q^*_\alpha = 1 + q_\alpha \cdot p/(n-k-p)$ the power of (A.2) when (A.3) is the true model is

(A.4) $$1 - \text{Prob.}\,[\zeta'(M - q^*_\alpha M_{k+p})\zeta \leq 0].$$

Now let the ith characteristic root of $(M - q^*_\alpha M_{k+p})$ be denoted by λ_i and let P be an orthogonal matrix of corresponding characteristic vectors. Denote the ith element of $\sigma^{-1}P'f$ by τ_i. Expression (A.4) may then be rewritten

$$1 - \text{Prob.}\left[\sum_{i=1}^{n} \lambda_i w_i^2 \leq 0\right].$$

where the w_i's are independent non-central Chi-square variates with one degree of freedom and non-centrality parameters, τ_i^2. This probability may be evaluated by the method of Imhof as described in Koerts and Abrahamse (1969, p. 81–2, 155–60). From the point of view of computation it is important to note that $n - k - p$ of the characteristic roots are equal to $(1 - q^*_\alpha)$, k are zero and the remaining p take a value of unity: c.f. a similar result in Koerts and Abrahamse (1969, p. 141–3).

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