Volume Title: Annals of Economic and Social Measurement, Volume 6, number 4

Volume Author/Editor: NBER

Volume Publisher: NBER

Volume URL: http://www.nber.org/books/aesm77-4

Publication Date: October 1977

Chapter Title: Identifying Identical Distributed Lag Structures by the Use of Prior Sum Constraints

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Chapter URL: http://www.nber.org/chapters/c10526

Chapter pages in book: (p. 429 - 444)

Annals of Economic and Social Measurement, 6/4, 1977

IDENTIFYING IDENTICAL DISTRIBUTED LAG STRUCTURES BY THE USE OF PRIOR SUM CONSTRAINTS

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This paper derives an estimation procedure which, when the same distributed lag appears twice in an equation to be estimated by least-squares regression, identifies all of the relevant coefficients and lag weights and also constrains the two sets of individual lag weights to be identical. The procedure for solving this identification-constraint problem involves prior imposition of a restriction on the lag weight sum — i.e., it is necessary to impose the sum restriction before *estimating the equation. A further useful feature of the derived procedure is that it facilitates conveniently imposing the sum restriction on all of the weights in a distributed lag even if the leading weight is independent of a polynomial restriction imposed on the others.*

It is well known that, if an independent variable in an equation to be estimated by least-squares regression is itself a distributed lag, it is necessary to impose some restriction in order to identify both the independent variable's coefficient in the equation and the weights defining the distributed lag. If the proxy variable for "expected permanent income" in a consumption function is defined as a distributed lag on past observations of income, for example, a restriction is necessary to identify both the marginal propensity to consume out of expected permanent income and the weights defining the autoregressive expectation. A familiar practice under such circumstances is to impose the restriction that the weights in the distributed lag must have a prespecified sum, so that the estimated coefficient of the independent variable in the equation is simply the sum of the unrestricted lag weight estimates divided by the prespecified weight sum. This sum restriction, which is easy enough to impose after estimation of the equation, need not represent any complication for the estimation process itself even if the relevant independent variable is a nonlinear term such as the product of the distributed lag and another variable.

But what if the equation to be estimated includes *two* nonlinear independent variables, each defined as the product of the *same* distributed lag and one other variable? Simply estimating the equation and then applying the same prespecified sum restriction to both appearances of the distributed lag is sufficient to identify all of the lag weights as well as the coefficients of both independent variables, but the two sets of estimated lag weight patterns will in general be different. Imposing the usual sum

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restriction after estimation of the equation is not sufficient to constrain the two sets of individual lag weights to be identical.

The object of this paper is to derive a procedure which not only identifies all of the relevant coefficients and lag weights, when the same distributed lag appears twice in an equation to be estimated, but also constrains the two sets of individual lag weights to be identical. In particular, the procedure for solving this identification-constraint problem involves *prior* imposition of the restriction on the lag weight sum — i.e., it is necessary to impose the sum constraint *before* estimating the equation. An additional useful feature of this procedure is that it facilitates readily imposing the sum constraint on all of the lag weights even if, following Sims [14], the leading lag weight is independent of a polynomial constraint imposed on the remaining lag weights.

Section I states in precise terms the nature of the identification problem. Section II, using the direct method of polynomial distributed lag estimation, derives the prior sum constraint procedure. Section III illustrates the use of this procedure with an example drawn from an analysis by one of the authors of corporate financing behavior. Section IV briefly summarizes the paper's principal conclusions.

I. The Problem

Consider the problem of estimating by ordinary least squares the expression

$$y_t = \alpha + \beta(p_t x_t) + u_t \tag{1.1}$$

where

$$x_t \equiv \sum_{\tau=0}^{T+1} \delta_\tau z_{t-\tau}, \tag{1.2}$$

α, β and δ_τ, $\tau = 0, \ldots, T+1$, are the parameters to be estimated, and T is an integer defining the lag length in (1.2). Simply estimating (1.1) with (1.2) substituted for x_t yields a set of estimates $(\widehat{\beta \cdot \delta_\tau})$, $\tau = 0, \ldots, T+1$, thereby still leaving β and δ_τ, $\tau = 0, \ldots, T+1$, unidentified. A commonplace way to identify these parameters is to impose a sum constraint

$$\sum_{\tau=0}^{T+1} \delta_\tau = \bar{\delta} \tag{1.3}$$

for prespecified $\bar{\delta}$,[1] thereby facilitating the solution for $\hat{\beta}$ and $\hat{\delta}_\tau$, $\tau = 0, \ldots$

[1] The most familiar such constraint in expectational models is $\bar{\delta} = 1$, which implies that the autoregressive expectation defined by (1.2) is formed on the assumption that the process generating z_t is borderline stationary/nonstationary — i.e., any level of z which has persisted for $T+1$ time periods is expected to persist indefinitely. For criticisms of the use of a unit sum constraint, see Lucas [10] and Sargent [12].

$T + 1$, as

$$(1.4) \qquad \hat{\beta} = \frac{\sum_{\tau=0}^{T+1} (\widehat{\beta \cdot \delta_\tau})}{\bar{\delta}}$$

$$(1.5) \qquad \hat{\delta}_\tau = \frac{\bar{\delta} \cdot (\widehat{\beta \cdot \delta_\tau})}{\sum_{\tau=0}^{T+1} (\widehat{\beta \cdot \delta_\tau})}, \qquad \tau = 0, \ldots, T + 1.$$

This simple restriction, imposed after estimation of $(\widehat{\beta \cdot \delta_\tau})$, $\tau = 0, \ldots, T + 1$, is sufficient to identify the equation's parameters regardless of additional polynomial constraints on δ_τ, $\tau = 0, \ldots, T + 1$, with or without further zero restrictions, etc.

Suppose, however, that the equation to be estimated is not (1.1) but

$$(1.6) \qquad y_t = \alpha + \beta(p_t x_t) + \gamma(q_t x_t) + u_t$$

where x_t is again the distributed lag defined in (1.2) and γ is an additional parameter to be estimated. Repetition of the procedure described above for equation (1.1), now with the addition of

$$(1.7) \qquad \hat{\gamma} = \frac{\sum_{\tau=0}^{T+1} (\widehat{\gamma \cdot \delta_\tau})}{\bar{\delta}}$$

$$(1.8) \qquad \hat{\delta}_\tau = \frac{\bar{\delta} \cdot (\widehat{\gamma \cdot \delta_\tau})}{\sum_{\tau=0}^{T+1} (\widehat{\gamma \cdot \delta_\tau})}, \qquad \tau = 0, \ldots, T + 1.$$

results in two different values of each $\hat{\delta}_\tau$, $\tau = 0, \ldots, T + 1$ — one from (1.5) and one from (1.8). By contrast, the economic logic of (1.6), in which the two independent variables involve the same distributed lag, clearly indicates that the $\hat{\delta}_\tau$ relevant to $(p_t x_{t-\tau})$ should be identical to the $\hat{\delta}_\tau$ relevant to $(q_t x_{t-\tau})$, $\tau = 0, \ldots, T + 1$.

Hence unrestricted estimation of (1.6), with subsequent imposition of the sum restriction (1.3) via (1.4, 1.5) and (1.7, 1.8), oversolves the problem of identifying the parameters of (1.6). Section II derives a procedure for solving this problem which uses (1.3) to yield estimates $\hat{\beta}$ and $\hat{\gamma}$ and identical sets of estimates $\hat{\delta}_\tau$, $\tau = 0, \ldots, T + 1$.

II. The Prior Sum Constraint Procedure

Direct Estimation of Polynomial Distributed Lags. Constraining distributed lag weights such as δ_τ, $\tau = 0, \ldots, T + 1$, in (1.2) to depend on the corresponding lag τ according to some polynomial expression is a familiar procedure, intended to reduce the number of independent parameters to be estimated as well as to enforce a priori beliefs about smoothness.[2] The most common method of imposing polynomial distributed lag constraints is due to Almon [1]. In the context of prior imposition of a sum constraint, however, it is more convenient to work from what Cooper [3] has called the "direct" method. Cooper demonstrated that, since the two methods differ only by a nonsingular transformation, the corresponding sets of estimated lag weights are identical, so that the reason for using the direct method here is merely a matter of computational convenience. The Appendix to this paper derives procedures, based on the Almon method, which are equivalent to the procedures derived in this section using the direct method.

For a generalized distributed lag term like (1.2), the direct approach to imposing polynomial constraints on the lag weights δ_τ, $\tau = 0, \ldots, T+1$, represents these coefficients in the form

$$\delta_\tau = \sum_{j=0}^{Q+1} \lambda_j \tau^j, \qquad \tau = 0, \ldots, T+1, \tag{2.1}$$

where $Q+1$ is the degree of the polynomial, and the λ_j, $j = 0, \ldots, Q+1$, are the fixed parameters to be estimated. Substituting (2.1) into (1.2) yields

$$x_t = \sum_{j=0}^{Q+1} \lambda_j Z_{jt} \tag{2.2}$$

where

$$Z_{jt} \equiv \sum_{\tau=0}^{T+1} \tau^j z_{t-\tau}, \qquad j = 0, \ldots, Q+1.$$

In the simplest polynomial distributed lag models, variable x_t in (1.2) is observable, and the problem is to estimate (1.2) directly, constrained only by the polynomial pattern of the lag weights. Ordinary least-squares regression, with x_t as the dependent variable and the distributed lag in the form (2.2), yields an estimate $\hat{\lambda}_j$ for each λ_j, $j = 0, \ldots, Q+1$, together with the respective variances and covariances of these estimates. Corresponding estimates of the distributed lag weights themselves follow directly from (2.1) as

$$\hat{\delta}_\tau = \sum_{j=0}^{Q+1} \hat{\lambda}_j \tau^j, \qquad \tau = 0, \ldots, T+1. \tag{2.3}$$

The variances and covariances of the distributed lag weight estimates follow as

[2] For additional reference, see Jorgenson [9] and Griliches [8]. Shiller's [13] procedure meets these two objectives in a somewhat different way. Beliefs about smoothness are especially prevalent in the context of lags representing autoregressive expectations.

$$(2.4)\quad \operatorname{cov}(\hat{\delta}_\tau, \hat{\delta}_{\tau'}) = \sum_{j=0}^{Q+1} \sum_{j'=0}^{Q+1} \tau^j \tau'^{j'} \operatorname{cov}(\hat{\lambda}_j, \hat{\lambda}_{j'}), \qquad \tau, \tau' = 0, \ldots, T+1.$$

Imposing zero constraints on particular elements of the polynomial distributed lag (typically δ_{-1} or δ_{T+2}, or both) is also common and is straightforward. For example, the constraint

$$(2.5)\qquad \delta_{T+2} = 0$$

implies from (2.1)

$$(2.6)\qquad \sum_{j=0}^{Q+1} \lambda_j (T+2)^j = 0.$$

To impose this constraint, it is necessary to solve (2.6) for any one of the $\lambda_j, j = 0, \ldots Q+1$. For λ_0, for example, the solution of (2.6) yields simply

$$(2.7)\qquad \lambda_0 = -\sum_{j=1}^{Q+1} \lambda_j (T+2)^j.$$

Substituting (2.7) into (2.2) yields

$$(2.8)\qquad x_t = \sum_{j=1}^{Q+1} \lambda_j Z'_{jt}$$

where

$$Z'_{jt} \equiv Z_{jt} - (T+2)^j Z_{0t}.$$

Ordinary least-squares regression, with x_t as the dependent variable and the distributed lag in the form (2.8), yields estimates $\hat{\lambda}_j, j = 1, \ldots, Q+1$, together with their respective variances and covariances, and the estimate of λ_0 follows from (2.7) as

$$\hat{\lambda}_0 = \sum_{j=1}^{Q+1} \hat{\lambda}_j (T+2)^j.$$

The distributed lag weight estimates $\hat{\delta}_\tau$, $\tau = 0, \ldots, T+1$, again follow from (2.3). The variances and covariances of these estimates again follow from (2.4), where

$$\operatorname{var}(\hat{\lambda}_0) = \sum_{j=1}^{Q+1} \sum_{j'=1}^{Q+1} (T+2)^{j+j'} \operatorname{cov}(\hat{\lambda}_j, \hat{\lambda}_{j'}).$$

$$\operatorname{cov}(\hat{\lambda}_0, \hat{\lambda}_j) = \sum_{j'=1}^{Q+1} (T+2)^{j'} \operatorname{cov}(\hat{\lambda}_j, \hat{\lambda}_{j'}).$$

Imposing the Prior Sum Constraint. As Section I explains, when the equation to be estimated is (1.1) instead of (1.2)—for example, if x_t is unobservable it is useful to impose, in addition to the polynomial constraint (2.1) and the zero constraint (2.5), the sum constraint (1.3). Furthermore, following Sims' [14] suggestion, in many circumstances it is appropriate to exclude the leading lag weight δ_0 from the polynomial constraint, which then becomes

$$(2.1') \qquad \delta_{\tau+1} = \sum_{j=0}^{Q+1} \lambda_j \tau^j, \qquad \tau = 0, \ldots, T,$$

while still including δ_0 within the sum constraint (1.3).[3]

Substituting (1.3) into (2.1′) yields

$$(2.9) \qquad \delta_0 + (T+1)\lambda_0 + \phi_1\lambda_1 + \sum_{j=2}^{Q+1} \phi_j\lambda_j = \bar{\delta},$$

where

$$\phi_j \equiv \sum_{\tau=0}^{T} \tau^j, \qquad j = 1, \ldots, Q+1,$$

and substituting (2.5) into (2.1′) yields

$$(2.6') \qquad \sum_{j=0}^{Q+1} \lambda_j (T+1)^j = 0.$$

To impose jointly the full set of constraints, it is sufficient to solve (2.9) and (2.6′) for any two of the λ_j, $j = 0, \ldots, Q+1$. For λ_0 and λ_1, for example, the solution of (2.9) and (2.6′) yields

$$(2.10) \qquad \lambda_0 = -\eta_1\bar{\delta} + \eta_1\delta_0 + \sum_{j=2}^{Q+1} \eta_j\lambda_j$$

$$(2.11) \qquad \lambda_1 = -\eta_1'\bar{\delta} + \eta_1'\delta_0 + \sum_{j=2}^{Q+1} \eta_j'\lambda_j$$

where

$$\eta_1 = \frac{T+1}{\phi_1 - (T+1)^2}$$

$$\eta_j = \left[\frac{\phi_j(T+1)}{\phi_1} - (T+1)^j\right]\left[\frac{\phi_1}{\phi_1 - (T+1)^2}\right], \qquad j = 2, \ldots, Q+1$$

$$\eta_1' = -\frac{[1 + \eta_1(T+1)]}{\phi_1}$$

[3] Freeing the leading lag weight from the polynomial constraint is computationally trivial in the absence of the sum constraint.

$$\eta_j' = -\frac{[\phi_j + \eta_j(T+1)]}{\phi_1}, \qquad j = 2, \ldots, Q+1.$$

Substituting (2.1′) into (1.2) yields

$$x_t = \delta_0 z_t + \sum_{j=0}^{Q+1} \lambda_j Z'_{jt} \tag{2.12}$$

where

$$Z'_{jt} \equiv \sum_{\tau=0}^{T} \tau^j z_{t-\tau-1}, \qquad j = 0, \ldots, Q+1,$$

and substituting (2.10) and (2.11) into (2.12) yields

$$x_t = \delta_0 z_t + (\bar{\delta} - \delta_0) Z''_{1t} + \sum_{j=2}^{Q+1} \lambda_j Z''_{jt} \tag{2.13}$$

where

$$Z''_{1t} \equiv -\eta_1 Z_{0t} - \eta_1' Z_{1t}$$

$$Z''_{jt} \equiv \eta_j Z_{0t} + \eta_j' Z_{1t} + Z_{jt}, \qquad j = 2, \ldots, Q+1.$$

Nonlinear regression, with x_t in the form (2.13) replaced by $(x_t - \bar{\delta} Z''_{1t})$ on the right-hand side of (1.6), yields estimates $\hat{\delta}_0$ and $\hat{\lambda}_j$, $j = 2, \ldots, Q+1$, together with their respective variances and covariances, as well as estimates $\hat{\beta}$ and $\hat{\gamma}$. Estimates $\hat{\lambda}_0$ and $\hat{\lambda}_1$ then follow from (2.10) and (2.11) as

$$\hat{\lambda}_0 = -\eta_1 \bar{\delta} + \eta_1 \hat{\delta}_0 + \sum_{j=2}^{Q+1} \eta_j \hat{\lambda}_j$$

$$\hat{\lambda}_1 = -\eta_1' \bar{\delta} + \eta_1' \hat{\delta}_0 + \sum_{j=2}^{Q+1} \eta_j' \hat{\lambda}_j$$

and estimates of the remaining distributed lag weights follow in turn from (2.1′) as

$$\hat{\delta}_{\tau+1} = \sum_{j=0}^{Q+1} \hat{\lambda}_j \tau^j, \qquad \tau = 0, \ldots, T. \tag{2.14}$$

Hence imposing the sum constraint prior to estimation, in the manner of (2.9)–(2.14), yields only a single set of lag weights for the two appearances of the distributed lag in (1.6). The variances and covariances of the distributed lag weight estimates follow from

$$\operatorname{cov}(\hat{\delta}_{\tau+1}, \hat{\delta}_{\tau'+1}) = \sum_{j=0}^{Q+1} \sum_{j'=0}^{Q+1} \tau^j \tau'^{j'} \operatorname{cov}(\hat{\lambda}_j, \hat{\lambda}_{j'}), \qquad \tau, \tau' = 0, \ldots, T$$

$$\operatorname{cov}(\hat{\delta}_0, \hat{\delta}_{\tau+1}) = \sum_{i=0}^{Q+1} \tau^i \operatorname{cov}(\hat{\delta}_0, \hat{\lambda}_i), \qquad \tau = 0, \ldots, T$$

where

$$\operatorname{var}(\hat{\lambda}_0) = (\eta_1)^2 \cdot \operatorname{var}(\hat{\delta}_0) + 2\eta_1 \cdot \sum_{i=2}^{Q+1} \eta_i \operatorname{cov}(\hat{\delta}_0, \hat{\lambda}_i)$$

$$+ \sum_{i=2}^{Q+1} \sum_{i'=2}^{Q+1} \eta_i \eta_{i'} \operatorname{cov}(\hat{\lambda}_i, \hat{\lambda}_{i'})$$

$$\operatorname{var}(\hat{\lambda}_1) = (\eta_1')^2 \cdot \operatorname{var}(\hat{\delta}_0) + 2\eta_1' \cdot \sum_{i=2}^{Q+1} \eta_i' \operatorname{cov}(\hat{\delta}_0, \hat{\lambda}_i)$$

$$+ \sum_{i=2}^{Q+1} \sum_{i'=2}^{Q+1} \eta_i' \eta_{i'}' \operatorname{cov}(\hat{\lambda}_i, \hat{\lambda}_{i'})$$

$$\operatorname{cov}(\hat{\lambda}_0, \hat{\lambda}_1) = \eta_1 \eta_1' \cdot \operatorname{var}(\hat{\delta}_0) + \sum_{i=2}^{Q+1} (\eta_1 \eta_i' + \eta_1' \eta_i) \operatorname{cov}(\hat{\delta}_0, \hat{\lambda}_i)$$

$$+ \sum_{i=2}^{Q+1} \sum_{i'=2}^{Q+1} \eta_i \eta_{i'}' \operatorname{cov}(\hat{\lambda}_i, \hat{\lambda}_{i'})$$

$$\operatorname{cov}(\hat{\lambda}_0, \hat{\lambda}_j) = \eta_1 \cdot \operatorname{cov}(\hat{\delta}_0, \hat{\lambda}_j) + \sum_{i'=2}^{Q+1} \eta_{i'} \operatorname{cov}(\hat{\lambda}_j, \hat{\lambda}_{i'}), \qquad j = 2, \ldots, Q+1$$

$$\operatorname{cov}(\hat{\lambda}_1, \hat{\lambda}_j) = \eta_1' \cdot \operatorname{cov}(\hat{\delta}_0, \hat{\lambda}_j) + \sum_{i'=2}^{Q+1} \eta_{i'}' \operatorname{cov}(\hat{\lambda}_j, \hat{\lambda}_{i'}), \qquad j = 2, \ldots, Q+1$$

$$\operatorname{cov}(\hat{\delta}_0, \hat{\lambda}_0) = \eta_1 \cdot \operatorname{var}(\hat{\delta}_0) + \sum_{i=2}^{Q+1} \eta_i \operatorname{cov}(\hat{\delta}_0, \hat{\lambda}_i)$$

$$\operatorname{cov}(\hat{\delta}_0, \hat{\lambda}_1) = \eta_1' \cdot \operatorname{var}(\hat{\delta}_0) + \sum_{i=2}^{Q+1} \eta_i' \operatorname{cov}(\hat{\delta}_0, \hat{\lambda}_i).$$

In all cases considered here, it is of course possible to use $\hat{\delta}_0$ and $\operatorname{var}(\hat{\delta}_0)$ to test directly the null hypothesis that the (free) leading weight δ_0 is zero. If $\delta_0 = 0$, the procedure developed above is still valid for the remaining weights δ_τ, $\tau = 1, \ldots, T+1$. All that is necessary is to set $\delta_0 = 0$ in (2.13) and to re-estimate the equation accordingly. All estimates, variances and covariances follow as before, with $\hat{\delta}_0$, $\operatorname{var}(\hat{\delta}_0)$ and all covariances of $\hat{\delta}_0$ with the other estimated parameters simply set equal to zero.

In sum, the estimation procedure based on nonlinear regression using the substituted form (2.13) for the distributed lag variable x_t in (1.2) yields lag weight estimates $\hat{\delta}_\tau$, $\tau = 0, \ldots, T+1$, which satisfy the sum constraint (1.3), the zero constraint (2.5) and the polynomial constraint (2.1) or the equivalent (2.1′) which omits the leading lag weight. In addition, the procedure not only identifies the coefficients β and γ in (1.6) but also constrains the individual lag weights to be identical in both appearances in (1.6) of the distributed lag variable x_t.[4]

III. An Illustration

An example may serve to illustrate the application of the estimation procedure derived in Section II. An analysis of corporate financing behavior by one of the authors [7] modeled nonfinancial business corporations' net new issues of long-term bonds by combining the familiar linear homogeneous model of portfolio allocation, applied to the selection of liabilities to finance externally a given cumulated deficit requirement,[5]

$$\text{(3.1)} \qquad \frac{L^*_{it}}{D_t} = \sum_k^N \beta_{ik} r_{kt} + \sum_h^M \gamma_{ih} q_{ht} + \pi_t, \qquad i = 1, \ldots, N,$$

with the optimal marginal adjustment model of portfolio adjustment out of equilibrium,[6]

$$\text{(3.2)} \qquad \Delta L_{it} = \sum_k^N \theta_{ik}(\lambda^*_{kt} D_{t-1} - L_{k,t-1}) + \lambda^*_{it}\Delta D_t, \qquad i = 1, \ldots, N,$$

where

$$\text{(3.3)} \qquad \lambda^*_{it} \equiv \frac{L^*_{it}}{D_t}, \qquad i = 1, \ldots, N$$

and

$L^*_{it}, i = 1, \ldots, N$ = the borrower's desired equilibrium amount of the i-th liability outstanding at time period t $(\sum_i L^*_{it} = D_t)$

[4] It is clear that this procedure based on a prior sum constraint on the distributed lag weights is not the only way to accomplish these objectives. A prior constraint on the ratio of β and γ in (1.6) for example, would facilitate achieving the same purpose by simply imposing the lag weight sum constraint *after* the nonlinear estimation of (1.6) in the form

$$y_t = \alpha + \beta\left[p_t + \left(\frac{\gamma}{\beta}\right)q_t\right] x_t + u_t$$

with prespecified ratio $(\overline{\gamma/\beta})$. Imposing the lag weight sum constraint *before* the estimation has the advantage, however, of requiring no further restrictions such as a prespecified ratio of β and γ.

[5] See de Leeuw [4] for a discussion of the rationale behind the familiar linear homogenous model of portfolio allocation.

[6] See Friedman [6] for a discussion of the rationale behind the optimal marginal adjustment generalization of the standard stock adjustment model.

D_t = the borrower's total cumulated external deficit at time period t

$r_{kt}, k = 1, \ldots, N$ = the expected "borrowing-period" yield on the k-th liability at time period t

$q_{ht}, h = 1, \ldots, M$ = the values at time period t of additional variables which influence the allocation of the portfolio of outstanding liabilities

$L_{it}, i = 1, \ldots, N$ = the borrower's actual amount of the i-th liability outstanding at time period $t (\sum_i L_{it} = D_t)$

and the β_{ik}, γ_{ih}, π_i and θ_{ik} are parameters satisfying the relevant adding-up constraints specified in Brainard and Tobin [2].

Any r_{kt} or q_{ht} variable which influences the determination of the equilibrium allocation ratios in (3.1) therefore appears twice in (3.2), in nonlinear form both times. Expanding (3.2) after substituting (3.1) for the λ^*_{it}, $i = 1, \ldots, N$, indicates that the coefficient of each resulting $(r_{kt} \Delta D_t)$ or $(q_{ht} \Delta D_t)$ term consists of a single parameter β_{ik} or γ_{ih} which, from (3.1), is presumably of known sign a priori. By contrast, the coefficient of each resulting $(r_{kt} D_{t-1})$ or $(q_{ht} D_{t-1})$ term is a sum of products of parameters from (3.1) and (3.2) and is in general of unknown sign a priori; nevertheless, since these terms do appear in the model specification, it is inappropriate to impose the assumption that their respective coefficients are zero by eliminating them from the estimated equation.

The equation developed in [7] for net new issues of long-term bonds of nonfinancial corporations follows (3.1)–(3.3), introducing three yield variables and four non-yield variables in (3.1). The three yield variables, in particular, are

r_{Bt} = the currently prevailing yield, at time period t, on new issues of corporations' long-term bonds

r^e_{Bt} = corporations' expectation, at time period t, of the average future yield on new issues of their long-term bonds

r^e_{St} = corporations' expectation, at time period t, of the average current and future level of yields on their short-term securities

and the unobservable r^e_{Bt} and r^e_{St} variables are in turn modeled as autoregressive distributed lags as in (1.2). Hence the estimated net bond issues equation is analogous to expression (1.6) in that the distributed lag variables each appear twice, in two separate independent variables. Since the expectation in the $(r^e_{Bt} \Delta D_t)$ term is the same as that in the $(r^e_{Bt} D_{t-1})$ term, it is necessary to use some procedure like that developed in Section II in order to constrain the individual distributed lag weights defining r^e_{Bt} to be identical in the two terms. The same requirement applies to the two appearances of r^e_{St}.

The result of estimating this expression, using quarterly U.S. data for

1960:I–1973:IV, is[7]

$$\Delta B_t = \underset{(4.8)}{1.837\,\Delta D_t} - \underset{(-6.2)}{5.382\,r_{Bt}\Delta D_t} + \underset{(4.5)}{0.04167\,r_{Bt}D_{t-1}}$$

$$+ \underset{(6.0)}{4.732\,r^{e}_{Bt}\Delta D_t} - \underset{(-4.1)}{0.03886\,r^{e}_{Bt}D_{t-1}} + \underset{(3.0)}{0.4046\,r^{e}_{St}\Delta D_t}$$

$$+ \underset{(2.7)}{5.600\,q_{1t}\Delta D_t} - \underset{(-3.0)}{5.331\,q_{2t}\Delta D_t} - \underset{(-1.7)}{0.2579\,q_{3t}\Delta D_t}$$

$$+ \underset{(3.6)}{0.6239\,q_{4t}\Delta D_t} - \underset{(-4.8)}{0.07134\,B_{t-1}} + \underset{(2.6)}{0.07889\,S_t}$$

$$\bar{R}^2 = 0.95 \qquad SE = 303 \qquad H = -1.28$$

where[8]

B_t = corporations' outstanding amount of long-term bonds
q_{1t} = stock of fixed investment
q_{2t} = average retained earnings
q_{3t} = inventory of bond dealers
q_{4t} = equity retirements
S_t = corporations' outstanding amount of short-term liabilities
$\bar{R}^2$ = coefficient of determination, adjusted for degrees of freedom
SE = standard error of estimate (in millions of dollars)
H = Durbin's [5] H-statistic

and the numbers in parentheses are ratios of estimates to standard errors for each coefficient.

All estimated coefficients in the bond issues equation which correspond to single parameters of (3.1) have the signs expected a priori. With two exceptions, the coefficients of the nonlinear terms involving D_{t-1} did not significantly differ from zero, and so these terms are eliminated from the final specification of the equation. In particular, the $(r^{e}_{St}D_{t-1})$ term is eliminated, thereby avoiding the need to constrain the distributed lag weights defining r^{e}_{St} to be identical in two separate terms. Imposition of the sum constraint (1.3) after estimation of the equation is sufficient to identify both the associated $\hat{\beta}_{ik} = 0.4046$ and the set of lag weights.[9]

[7]The equation is estimated using an instrumental variables procedure, because of the joint determination of ΔB_t and r_{Bt}. For a detailed description of the estimation process and an evaluation and interpretation of the results, see Friedman [7].

[8]See Friedman [7] for a more detailed description of the data and variable definitions (especially $q_{1t}, \ldots, q_{4t}$).

[9]The distributed lag defining r^{e}_{St} is

$$r^{e}_{St} = \sum_{\tau=1}^{17} \delta_\tau r_{S,t-\tau} \qquad \sum_{\tau=1}^{17} \delta_\tau = 1.$$

The estimation procedure constrained δ_τ, $\tau = 2, \ldots, 17$, to follow a third-degree polynomial

By contrast, both $(r_{B_t} D_{t-1})$ and $(r^e_{Bt} D_{t-1})$ have coefficients significantly differ from zero, and the presence of $(r^e_{Bt} D_{t-1})$ along with $(r^e_{Bt} \Delta D_t)$ leads to the need for the prior sum constraint procedure developed in Section II. The distributed lag expression for r^e_{Bt}, in both of the appearances of r^e_{Bt} in the estimated equation, is[10]

$$r^e_{Bt} = r_{Bt} + \sum_{\tau=0}^{12} \delta_\tau \Delta r_{B,t-\tau}, \qquad \sum_{\tau=0}^{12} \delta_\tau = 1$$

$$\begin{array}{lll}
\delta_0 = 0.1397 & \delta_5 = 0.1034 & \delta_9 = 0.01474 \\
\delta_1 = 0.1636 & \delta_6 = 0.07992 & \delta_{10} = 0.000124 \\
\delta_2 = 0.1568 & \delta_7 = 0.05624 & \delta_{11} = -0.008271 \\
\delta_3 = 0.1517 & \delta_8 = 0.03398 & \delta_{12} = -0.008846. \\
\delta_4 = 0.1251 & &
\end{array}$$

Following the discussion in Section II, the estimation procedure constrains δ_τ, $\tau = 1, \ldots, 12$, to follow a third-degree polynomial with the implicit $\delta_{13} = 0$, and leaves δ_0 free of the polynomial constraint but still includes it within the sum constraint.[11]

IV. Summary

The procedure for distributed lag estimation developed in this paper is useful when two separate independent variables, in an equation to be estimated by least-squares regression, both contain the same distributed lag. The procedure, which involves the prior imposition of a restriction on the sum of the relevant distributed lag weights, serves not only to identify the coefficients of the two nonlinear independent variables but also to constrain the individual distributed lag weights to be identical in the lag's two

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with the implicit $\delta_{18} = 0$, and left δ_1 free of the polynomial constraint but still included it within the sum constraint. (Initial experimentation could not reject the hypothesis $\delta_0 = 0$.) The lag weights (which exhibit a pattern strikingly similar to that reported by Modigliani and Shiller [11] in their reduced-form equation which also includes a distributed lag on past levels of the short-term yield as a proxy for expectations of this yield's future level) are $-.1657$, .06996, .08212, .09451, .09691, .09998, .1005, .09861, .09462, .08873, .08115, .07212, .06186, .05060, .03855, .02596, .01303. The standard error ratio for δ_1 is -2.0, and the F-statistic for the two polynomial variables jointly is 5.7.

[10]Note that, since the first-differences representation of r^e_{Bt} implies the presence of r_{Bt} with unit coefficient, the identification problem of Section I would not arise in this equation if r_{Bt} were not already an argument of the bond issues function. The analysis in [7] exploits this relation to test whether the -5.382 coefficient on $r_{Bt} \Delta D_t$ is significantly different from the 4.732 coefficient on $r^e_{Bt} \Delta D_t$ by re-estimating the equation with r_{Bt} eliminated from the r^e_{Bt} expression; the resulting coefficient or $r_{Bt} \Delta D_t$ (which is then, of course, $-0.650 = -5.382 + 4.732$) does turn out to be significantly different from zero at high confidence levels.

[11]The standard error ratios for δ_0 and the two polynomial variables are, respectively, 6.6., -3.5, and 4.1.

appearances in the estimated equation. In addition, this prior sum constraint procedure is especially convenient in the context of polynomial distributed lags with the leading lag weight left free of the polynomial constraint.

Appendix

Estimation of Polynomial Distributed Lags using the Almon Method. The Almon approach to imposing polynomial constraints on the lag weights δ_τ in (1.2) represents these coefficients in the form

$$\delta_\tau = \sum_{j=0}^{Q+1} \psi_j \Phi_j(\tau), \qquad \tau = 0, \ldots, T+1, \tag{A.1}$$

where $Q + 1$ is the degree of the polynomial as in (2.1); the ψ_j, $j = 0, \ldots, Q+1$, are the fixed parameters to be estimated, and the $\Phi_j(\tau)$ are values of Lagrangian interpolation polynomials given by

$$\Phi_j(\tau) \equiv \frac{(\tau - \tau_0)(\tau - \tau_1)\cdots(\tau - \tau_{j-1})(\tau - \tau_{j+1})\cdots(\tau - \tau_{Q+1})}{(\tau_j - \tau_0)(\tau_j - \tau_1)\cdots(\tau_j - \tau_{j-1})(\tau_j - \tau_{j+1})\cdots(\tau_j - \tau_{Q+1})}$$

where the $\tau_j, j = 0, \ldots, Q+1$, are arbitrary values along the polynomial lag structure.

For $\tau_j = j$, $j = 0, \ldots, Q+1$, the Almon approach reduces to the direct approach of Section II, and, in general,

$$\Phi_j(\tau_j) = 1, \qquad j = 0, \ldots, Q+1, \tag{A.2}$$

$$\Phi_j(\tau_{j'}) = 0, \qquad j \neq j',\ j, j' = 0, \ldots, Q+1. \tag{A.3}$$

Substituting (A.1) into (1.2) yields

$$x_t = \sum_{j=0}^{Q+1} \psi_j W_{jt} \tag{A.4}$$

where

$$W_{jt} \equiv \sum_{\tau=0}^{T+1} \Phi_j(\tau) z_{t-\tau}, \qquad j = 0, \ldots, Q+1.$$

Ordinary least-squares regression, with x_t as the dependent variable and the distributed lag in the form (A.4), yields an estimate $\hat{\psi}_j$ for each ψ_j, $j = 0, \ldots, Q+1$, together with the respective variances and covariances of these estimates. Corresponding estimates of the distributed lag weights themselves follow directly from (A.1) as

$$\hat{\delta}_\tau = \sum_{j=0}^{Q+1} \hat{\psi}_j \Phi_j(\tau), \qquad \tau = 0, \ldots, T+1.$$

The variances and covariances of the distributed lag weight estimates follow as[12]

(A.5) $$\operatorname{cov}(\hat{\delta}_\tau, \hat{\delta}_{\tau'}) = \sum_{j=0}^{Q+1} \sum_{j'=0}^{Q+1} \Phi_j(\tau)\Phi_{j'}(\tau') \operatorname{cov}(\hat{\psi}_j, \hat{\psi}_{j'}),$$

$$\tau, \tau' = 0, \ldots, T+1.$$

From (A.1)–(A.3), it follows that imposing the zero constraint in (2.5) is equivalent to selecting

(A.6) $$\tau_{Q+1} = T + 2$$

(A.7) $$\psi_{Q+1} = 0.$$

Hence it is possible to rewrite the lag coefficients, conditional on (A.6), as

(A.8) $$\delta_\tau = \sum_{j=0}^{Q} \psi_j \Phi_j(\tau), \qquad \tau = 0, \ldots, T+1,$$

thereby deleting all terms involving ψ_{Q+1}.

Estimation in this case proceeds as before, upon the substitution of (A.8) into (1.2).

Imposing the Prior Sum Constraint. To impose the constraints in (A.1), (2.5) and (1.3), while leaving the leading lag weight δ_0 free of the polynomial constraint, it is useful to represent the remaining lag weights included in the polynomial lag as

$$\delta_{\tau+1} = \sum_{j=0}^{Q+1} \psi_j \Phi_j(\tau), \qquad \tau = 0, \ldots, T$$

so that imposing the zero constraint (2.5) is then equivalent to selecting

(A.6′) $$\tau_{Q+1} = T + 1$$

in conjunction with (A.7). Hence it is possible to rewrite the lag weights included within the polynomial lag structure, conditional on (A.6′), as

(A.8′) $$\delta_{\tau+1} = \sum_{j=0}^{Q} \psi_j \Phi_j(\tau), \qquad \tau = 0, \ldots, T.$$

Substituting (A.8′) into the sum constraint (1.3) yields

(A.9) $$\delta_0 + \sum_{\tau=0}^{T} \sum_{j=0}^{Q} \psi_j \Phi_j(\tau) = \bar{\delta}.$$

[12] To avoid needless repetition from the body of the paper, the discussion below of the estimation procedure in the presence of the zero and sum constraints does not derive the variances and covariances of the δ_τ, $\tau = 0, \ldots, T+1$; these follow, in each case, from estimating the variances and covariances of ψ_j, $j = 0, \ldots, Q = 1$, and substituting into (A.5).

To impose the sum constraint, it is necessary to solve (A.9) for one of the $\psi_j, j = 0, \ldots, Q$, or for δ_0. For $\delta_0 \neq 0$ the solution to this problem is straightforward and is applicable using most currently available standard polynomial distributed lag estimation programs. For $\delta_0 = 0$, the procedure is computationally more difficult, so that it is most convenient to rely on the direct approach of Section II.

For $\delta_0 \neq 0$, solving (A.9) for δ_0 yields

$$\delta_0 = \bar{\delta} - \sum_{\tau=0}^{T} \sum_{j=0}^{Q} \psi_j \Phi_j(\tau). \tag{A.10}$$

Substituting (A.8') and (A.10) into (1.2) yields

$$x_t = \bar{\delta} z_t + \sum_{j=0}^{Q} \psi_j W'_{jt},$$

where

$$W'_{jt} \equiv \sum_{\tau=0}^{T} \Phi_j(\tau)(z_{t-\tau-1} - z_t).$$

The simplicity of this result is readily apparent. The procedure imposes both zero and sum constraints on a polynomial lag structure, with δ_0 free of the polynomial constraint, simply by representing the equation with

$$x_t = \bar{\delta} z_t + \sum_{\tau=0}^{T} \delta_{\tau+1}(z_{t-\tau-1} - z_t) \tag{A.11}$$

substituted for x_t in the form (1.2), and using a standard polynomial distributed lag estimation procedure to constrain the right-hand tail of the lag structure to zero. The leading lag weight δ_0 is readily computed from the sum of the lag coefficients $\delta_{\tau+1}$, $\tau = 0, \ldots, T$, in (A.11):

$$\hat{\delta}_0 = \bar{\delta} - \sum_{\tau=0}^{T} \hat{\delta}_{\tau+1}. \tag{A.12}$$

and the variance of $\hat{\delta}_0$ follows as

$$\text{var}(\hat{\delta}_0) = \text{var}\left(\sum_{\tau=0}^{T} \hat{\delta}_{\tau+1}\right). \tag{A.13}$$

Hence (A.12) and (A.13) facilitate testing directly the significance of $\hat{\delta}_0$.

If the leading lag weight δ_0 is constrained to equal zero, however, it is necessary to solve (A.9) for some other parameter, thereby complicating the computational aspects of the estimation and rendering the direct approach of Section II substantially easier to implement. Solving (A.9) for ψ_0, for example, yields

$$\psi_0 = \frac{\bar{\delta}}{\Phi} - \frac{\delta_0}{\Phi} - \frac{1}{\Phi}\sum_{j=1}^{Q} \psi_j \sum_{\tau=0}^{I} \Phi_j(\tau). \tag{A.14}$$

where

$$\Phi \equiv \sum_{\tau=0}^{I} \Phi_0(\tau),$$

and imposing the constraint $\delta_0 = 0$ then involves simply deleting the term in δ_0 from (A.14). Substituting (A.8′) and (A.14) into (1.2) yields

$$x_t = \delta_0 z_t + \left(\frac{\bar{\delta}}{\Phi} - \frac{\delta_0}{\Phi}\right) W''_{0t} + \sum_{j=1}^{Q} \psi_j \left(W''_{jt} - \frac{1}{\Phi}\sum_{\tau=0}^{I} \Phi_j(\tau) W''_{0t}\right). \tag{A.15}$$

where

$$W''_{jt} \equiv \sum_{\tau=0}^{I} \Phi_j(\tau) z_{t-\tau-1}, \qquad j = 0, \ldots, Q.$$

The analog of this expression in the direct approach is (2.13). The estimation procedure based on (A.15) is more difficult to implement than that based on (2.13) because of the greater complexity of the $\Phi_j(\tau)$ in (A.8′) in contrast to the τ^j in (2.1′).

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