Volume Title: Annals of Economic and Social Measurement, Volume 6, number 2

Volume Author/Editor: NBER

Volume Publisher: NBER

Volume URL: http://www.nber.org/books/aesm77-2

Publication Date: April 1977

Chapter Title: The Allocation of a Natural Resource When the Cost of a Substitute Is Uncertain

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Chapter URL: http://www.nber.org/chapters/c10516

Chapter pages in book: (p. 189 - 201)

Annals of Economic and Social Measurement, 6/2, 1977

THE ALLOCATION OF A NATURAL RESOURCE WHEN THE COST OF A SUBSTITUTE IS UNCERTAIN*

BY DONALD A. HANSON

The mix of capital and resource assets which should be saved for the future depends on whether there is uncertainty in the cost of a substitute for the resource. For linear homogeneous production functions and classes of strictly concave production functions and with risk neutrality, future welfare is improved with more capital and less resource when the cost is uncertain. That is, more resource should be used initially. It is also shown that the effect of risk aversion is in the opposite direction. For sufficiently strong risk aversion, the direction of shift in desired future assets can be reversed.

I. INTRODUCTION

Suppose there is a homogeneous stock of an exhaustible natural resource which is a primary factor of production. Further, suppose that a substitute for the resource can be produced, but the cost is initially not known with certainty. Consider the problem of how much resource to use initially. In this paper the problem will be viewed as one of efficiency: Once initial consumption is determined, what mix of reproducible capital and remaining resource stock will in some sense maximize future welfare?

The paper begins by postulating a production possibility frontier (p.p.f.) between capital and resources available for the future. The idea is that more (less) capital will be accumulated for the future if more (less) resource is used initially in production (with initial consumption fixed). In a world of certainty, the efficient mix of assets is determined by the Hotelling condition; that is, the ratio of the net marginal product of the resource between any two periods must equal the marginal rate of transformation of capital between those periods.[1] If the cost of the substitute is uncertain, should the mix of future assets contain more or less resource? It is shown that with risk neutrality and a modified class of CES production functions, future welfare is improved if more capital and less resource is saved under uncertainty. The affect of risk aversion is a force in the opposite direction.

*I would like to thank Richard Gilbert, Michael Hoel, Daniel Newlon, Robert Solow and an anonymous referee for useful suggestions. Support by National Science Foundation grant GK-42098 is gratefully acknowledged.

[1] In the conventional theory net marginal product is the scarcity rent of the resource, i.e., its market price less extraction cost. The Hotelling result is usually stated in terms of growth rates: The growth rate of the scarcity rent must equal the rate of return on investment (see [1-3]).

2. The Model

Let output in period i be given by $F(K_i, Z_i)$ where K_i is reproducible capital and Z_i is total resource flow consisting of the natural resource and a perfect substitute. It is assumed that $F(K_i, Z_i)$ is sufficiently smooth, increasing in its arguments, and strictly concave. Strict concavity implies

$$F_{KK}(i)F_{ZZ}(i) - F_{KZ}^2(i) > 0 \tag{1}$$

(Here, subscripts denote partial derivatives and i denotes the period.) If $F_{KZ} > 0$, it will be said that K and Z are technical complements. That is, the return on investment F_K increases if more resource is utilized. It will be assumed that $F_{KZ} \geq 0$.

Consider a two period model. The p.p.f. between capital K_2 and resources R_2 available for period 2 is described by

$$K_2 = F(K_1, \hat{R} - R_2) - C_1 \tag{2}$$

where initial consumption C_1, initial capital K_1, and total resources R are fixed. The p.p.f. is a decreasing, concave function with $dK_2/dR_2 = -F_Z(1)$ and $d^2K_2/dR_2^2 = F_{ZZ}(1)$ (see Figure 1).

Suppose that a perfect substitute S for the natural resource is developed in time for the second period production. Let the cost per unit be "b." Then

$$C_2 = F(K_2, R_2 + S) - bS \tag{3}$$

S must be chosen efficiently to maximize C_2 given K_2, R_2 and b. Specifically,

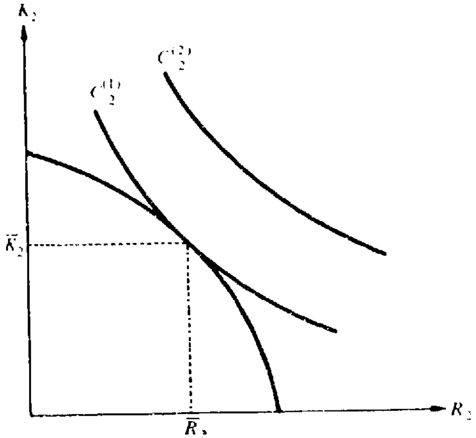

Figure 1

$$(4)\qquad S = \begin{cases} 0 & \text{if } F_Z(K_2, R_2) \leq b \\ \text{solution to } F_Z(K_2, R_2 + S) = b & \text{if } F_Z(K_2, R_2) > b \end{cases}$$

3. Substitute Cost "b" Known

Suppose for the moment that b is known with certainty in period 1. It is easily shown that isoquants (with C_2 fixed) are decreasing, convex functions with slope $dK_2/dR_2 = -F_Z(2)/F_K(2)$ (see Figure 1). Then there exists a unique point $(\bar{K}_2, \bar{R}_2)$ on the p.p.f. which maximizes C_2. This point must satisfy the tangency condition

$$(5)\qquad F_Z(1) = \frac{F_Z(2)}{F_K(2)}$$

which is the Hotelling result. Proposition 1 follows.

PROPOSITION 1. For any fixed initial consumption C_1, initial capital K_1, resource stock $\hat{R}$ and substitute cost $\bar{b}$, there exists a unique solution for the efficient mix of future assets $(\bar{K}_2, \bar{R}_2)$ and substitute production $\bar{S}$ given by the solution to (2), (4) and (5).

Note that if $S = 0$, $dR_2/db = 0$. If $S > 0$

$$\text{Sign } \frac{dR_2}{db} = \text{sign}\,[F_Z(1)F_{KZ}(2) - F_{ZZ}(2)]$$

which is obtained by differentiating (5) subject to (2) and (4). Hence $F_{KZ}(2) \geq 0$ implies $dR_2/db > 0$ and Proposition 2 follows.

PROPOSITION 2. Suppose the cost is known to be greater than $\bar{b}$ and $S(\bar{b}) > 0$. Then it is efficient to save more resource than $\bar{R}_2$ and less capital than $\bar{K}_2$. Hence less resource is used initially.

4. Substitute Cost "b" Random

Now suppose that b is a random variable with mean $\bar{b}$. Note that b is random only in the first period. In the second period the outcome is revealed and then S is chosen optimally according to (4)

For any fixed point (K_2, R_2) on the p.p.f. consider how the outcome of C_2 in (3) depends on the outcome of b with S given by (4). Note that

$$(6)\qquad \frac{\partial C_2}{\partial b} = -S < 0$$

$$(7)\qquad \frac{\partial^2 C_2}{\partial b^2} = -\frac{\partial S}{\partial b} = -\frac{1}{F_{ZZ}(2)} > 0$$

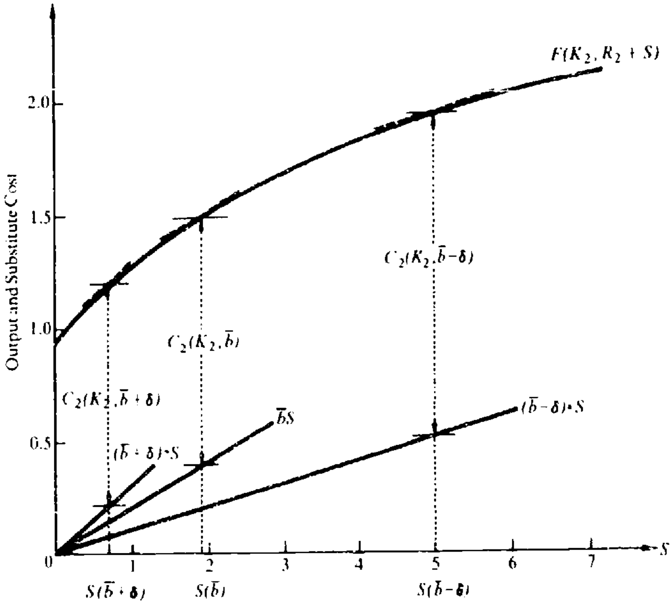

Figure 2

Therefore C_2 is a decreasing, convex function of b. The determination of C_2 as a function of b is shown in Figure 2. The linear rays are total cost functions for producing the substitute. The optimal S is the point where gross output $F(K_2, R_2 + S)$ is parallel to the total cost function. Suppose that the high cost outcome $b + \delta$ is realized. Gross output drops fast as S is reduced to $S(\bar{b} + \delta)$ but substitute production cost drops even faster. Convexity with respect to b implies

$$| C_2(\bar{b} - \delta) - C_2(\bar{b}) | > | C_2(\bar{b} + \delta) - C_2(\bar{b}) |$$

That is, the gain associated with a low cost substitute exceeds the loss associated with a high cost substitute. Hence $E\{C_2\} > C_2(\bar{b})$ where $E\{\cdot\}$ denotes expected value.

Now consider C_2 as a function not only of b but also the point on the p.p.f. The notation will be $C_2(K_2, b)$ where K_2 denotes the point on the p.p.f. with R_2 and S calculated from (2) and (4) respectively. In Figure 3, $C_2(\bar{K}_2, b)$ is shown. If $b = \bar{b}$ with certainty in period 1, then $C_2(\bar{K}_2, b)$ is the maximum of $C_2(K_2, \bar{b})$ over all points on the p.p.f. Now suppose that the random variable b takes on values only in a small range about $\bar{b}$. Further, suppose that $S > 0$ for these values of b. How should the point on the p.p.f. be shifted from the point $(\bar{K}_2, \bar{R}_2)$?

Note that from (6) and (7)

$$(8) \qquad \frac{\partial}{\partial K_2} C(\bar{K}_2, \bar{b}) = 0$$

$$(9) \qquad \frac{\partial}{\partial K_2} \frac{\partial C(\bar{K}_2, \bar{b})}{\partial b} = - \frac{\partial S(\bar{K}_2, \bar{b})}{\partial K_2}$$

$$(10) \qquad \frac{\partial}{\partial K_2} \frac{\partial^2 C(\bar{K}_2, \bar{b})}{\partial b^2} = \frac{1}{F_{ZZ}^2(2)} \frac{\partial F_{ZZ}(K_2, Z_2)}{\partial K_2} \Bigg|_{\bar{K}_2, \bar{R}_2, \bar{b}}$$

Hence the following are needed:

$$(11) \qquad \frac{\partial Z_2}{\partial K_2} = - \frac{F_{KZ}(2)}{F_{ZZ}(2)} > 0$$

$$(12) \qquad \frac{\partial S}{\partial K_2} = \frac{\partial Z_2}{\partial K_2} - \frac{dR_2}{dK_2} > 0$$

where (11) follows from (4). The critical term will turn out to be $\partial F_{ZZ}/\partial K_2$ holding b fixed.

$$(13) \qquad \frac{\partial F_{ZZ}(2)}{\partial K_2} = F_{KZZ}(2) + F_{ZZZ}(2) \frac{\partial Z_2}{\partial K_2}$$

$$= F_{KZZ}(2) - \frac{F_{KZ}(2)}{F_{ZZ}(2)} F_{ZZZ}(2)$$

Suppose that $\partial F_{ZZ}(2)/\partial K_2$ is positive, which will be discussed in a moment. Consider the effect of using slightly more resource initially and hence increasing K_2. Since (10) is positive $C_2(\bar{K}_2 + \Delta, b)$ will be more con-

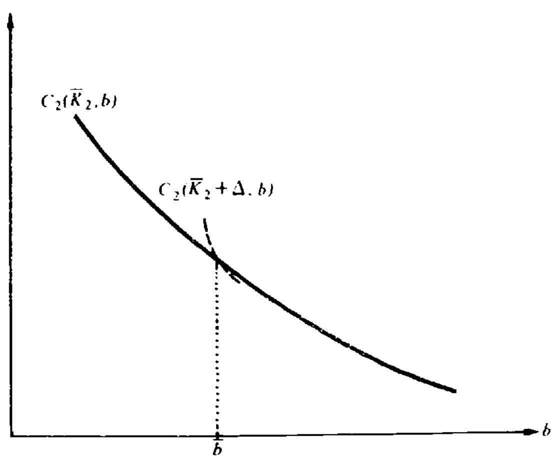

Figure 3

vex in b than $C_2(\overline{K}_2, b)$. Hence

$$E\{C_2(\overline{K}_2 + \Delta, b)\} > E\{C_2(\overline{K}_2, b)\} \tag{14}$$

Therefore, if society is risk neutral and welfare is measured by $E\{C_2\}$, welfare is improved by using more resource initially, accumulating more capital than $\overline{K}_2$ but saving less resource than $\overline{R}_2$ when b is uncertain.

Now suppose instead that society is extremely risk averse and associates welfare with the value of C_2 resulting from the worst outcome for b. Eqs. (9) and (12) imply that the slope of $C_2(\overline{K}_2 + \Delta, b)$ with respect to b is less (greater in magnitude) than the slope of $C_2(\overline{K}_2, b)$ (see Figure 3). Therefore with respect to this extremely risk averse welfare criterion, welfare is improved by using less resource initially, accumulating less capital than $\overline{K}_2$ but saving more resource than $\overline{R}_2$ when b is uncertain.

In general, the direction of shift along the p.p.f. from $(\overline{K}_2, \overline{R}_2)$ when b is uncertain depends on the relative magnitudes of the consumption convexity effect and the risk aversion effect.

Now consider the meaning of the term $\partial F_{ZZ}(2)/\partial K_2$ holding b fixed. For any given capital K_2, the resource demand function relates the resource's marginal product $F_Z(2)$ to the total flow Z_2. In a competitive market $F_Z(2)$ is the price which clears the market. Note that the slope of the demand function is $F_{ZZ}(2)$ which is negative. Provided K and Z are technical complements ($F_{KZ} > 0$), increasing K_2 shifts the demand function to the right. $\partial F_{ZZ}(2)/\partial K_2$ is positive if the slope of the demand curve increases (decreases in magnitude) as the demand curve is shifted to the right (i.e. as K_2 increases). The change in slope is evaluated along a horizontal constant price line, since b is held fixed and $F_Z(2) = b$ (see Figure 4).

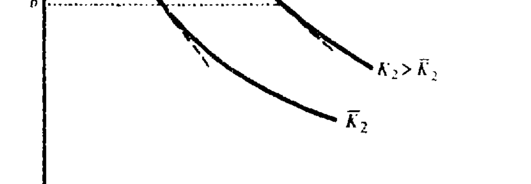

Figure 4

Society can choose among a set of demand functions which it will face in the second period by choosing a point (K_2, R_2) on the p.p.f. However, where society will be on the demand function is random. That is, the cost $b = F_Z$ is a random variable. Let $\bar{D}$ be the demand function associated with $(\bar{K}_2, \bar{R}_2)$. If society is risk neutral, uncertainty in b implies that the demand function should be shifted in the direction in which its slope is increasing. (I.e. the demand curve becomes flatter.) Therefore, for a fixed variance of b, the demand curve is shifted from $\bar{D}$ in a direction to increase the variance of Z. Society puts itself into a position to be more responsive (in the level of substitute production) to the outcome of b.

The demand curves in Figure 4 are drawn so that they become flatter for shifts to the right. It will be shown in the next section that large classes of production functions have this property. In this case the $E\{C_2\}$ is increased when capital is increased above $\bar{K}_2$ even though resources drop below $\bar{R}_2$. It is interesting to note that although R_2 decreases, for any given b Z_2 increases (see (11)). The difference $Z_2 - R_2$ is made up with substitute production S.

5. The Risk Neutral Case

Let the random variable b have a discrete probability distribution $\Pi(b_j), j = 1, J$, with mean $\bar{b}$, minimum b_1 and maximum b_J. In this section it is no longer necessary to assume that the variance of b is small. It will be assumed that not all outcomes of b lie in the range where $S = 0$. That is, Prob $\{S > 0\} > 0$.

Let (K_2^*, R_2^*) be the solution to the problem

$$\text{Max}\, E\{C_2\} \tag{15}$$

where (K_2^*, R_2^*) lies on the p.p.f. and S satisfies (4). In this section sufficient conditions and classes of production functions are given which imply

$$\begin{aligned} K_2^* &> \bar{K}_2 \\ R_2^* &< \bar{R}_2 \end{aligned} \tag{16}$$

LEMMA. For any fixed K_2, R_2, let the return on investment $F_K(K_2, R_2 + S)$ be a strictly convex function of b for $S > 0$ where S given by (4) is viewed as a function of b. That is, let

$$F_{KZZ}(2) - \frac{F_{KZ}(2)}{F_{ZZ}(2)} F_{ZZZ}(2) > 0 \tag{17}$$

for all b on $[b_1, b_J]$. Then (16) holds.

The proof is given in an appendix. Note that condition (17) is equivalent to $\partial F_{ZZ}(2)/\partial K_2 > 0$ (see (13)). It can be shown that if F_K is independent of Z, which implies that F_{KZ} is zero, and if $S > 0$ for all b on

$[b_1, b_j]$, then $K_2^* = \overline{K}_2$ and $\overline{R}_2^* = R_2$ (In the proof of the lemma $y(R_2, b)$ is a linear function of b.)

PROPOSITION 3. (a) Let $F(K, Z)$ be linear homogeneous. Then (16) holds. (b) Let $F(K, Z)$ be homogeneous of degree $0 < n < 1$. Then (16) holds whenever

$$\frac{bF_{ZZZ}(2)}{F_{ZZ}^2(2)} < \frac{2 - n}{1 - n} \tag{18}$$

Proof: The following property holds

$$(n - 1)F_Z(2) = (n - 1)b = F_{KZ}(2)K_2 + F_{ZZ}(2)Z_2$$

$$(n - 2)F_{ZZ}(2) = F_{KZZ}(2)K_2 + F_{ZZZ}(2)Z_2$$

Therefore, (17) becomes

$$\frac{(n - 2)F_{ZZ}(2)}{K_2} - \left[\frac{F_{KZ}(2)}{F_{ZZ}(2)} + \frac{Z_2}{K_2}\right] F_{ZZZ}(2) > 0$$

or

$$\frac{(n - 2)F_{ZZ}(2)}{K_2} - \frac{(n - 1)b}{F_{ZZ}(2)K_2} F_{ZZZ}(2) > 0$$

This inequality must hold for $n = 1$. For $n < 1$, F_K strictly convex in b for $S > 0$ is equivalent to the condition (18).

A special class of production functions homogeneous of degree n is the CES class

$$AK^{n(1-\alpha)}Z^{n\alpha} \qquad \rho = 0 \tag{19a}$$

$$A[(1 - \alpha)K^{-\rho} + \alpha Z^{-\rho}]^{-n/\rho} \qquad \rho > 0 \tag{19b}$$

PROPOSITION 4. Let $F(K, Z)$ be in the class of CES production functions (19). Then (16) holds.

Proof. Suppose

$$\frac{bF_{ZZZ}(2)}{F_{ZZ}^2(2)} \geq \frac{2 - n}{1 - n} > 1 \quad \text{for} \quad 0 < n < 1 \tag{20}$$

Then $F(K_2, R_2 + S)$ is a convex function of b with K_2, R_2 fixed and S viewed as a function of b. This is seen by computing the following

$$\frac{\partial^2 F}{\partial b^2} = \frac{1}{F_{ZZ}(2)}\left[1 - \frac{bF_{ZZZ}(2)}{F_{ZZ}^2(2)}\right] > 0$$

Now note that $F_K(K_2, R_2 + S)$ is given by

$$H(F) = \left[\frac{n(1-\alpha)A^{-\rho/n}}{K_2^{\rho+1}}\right]F^{1+\rho/n}$$

for K_2, R_2 fixed. F and F_K are viewed as functions of b only. Note that H is an increasing, convex function. Then $F_K(K_2, R_2 + S)$ must be convex in b, since

$$\frac{\partial^2 F_k}{\partial b^2} = H''(F)\left(\frac{\partial F}{\partial b}\right)^2 + H'(F)\frac{\partial^2 F}{\partial b^2} > 0$$

This leads to a contradiction, for $F_K(K_2, R_2 + S)$ convex in b implies (17) holds which in turn implies that (18) holds (see the proof of Proposition 3). Therefore (20) is false, (18) must be true and the result (16) follows.

Suppose $F(K, Z)$ is given by (19a). By direct computation

$$\frac{bF_{ZZZ}(2)}{F_{ZZ}^2(2)} = 1 - \frac{1}{n\alpha - 1} > 1 \tag{21}$$

To see that the upper bound (18) is satisfied, note that (21) is an increasing function of α and for $\alpha = 1$, (21) becomes $(2 - n)/(1 - n)$.

6. The Risk Averse Case

The following proposition assumes an extreme risk averse position.

PROPOSITION 5. Let b_J be the highest cost outcome. Suppose society chooses to maximize the worst consumption outcome $C_2(K_2, b_J)$. The solution (K_2^{**}, R_2^{**}) lies on the p.p.f. Then

$$K_2^{**} < \bar{K}_2 \tag{22}$$

$$R_2^{**} > \bar{R}_2$$

This proposition follows directly from Proposition 2.

The general problem with risk aversion is

$$\text{Max } E\{G(C_2)\} \tag{23}$$

where G is an increasing, concave function, the solution (K_2^{**}, R_2^{**}) lies on the p.p.f., and S is given by (4). It is difficult to relate the solution to the certainty solution $(\bar{K}_2, \bar{R}_2)$. However, one can say that the direction of the effect of risk aversion is to accumulate less capital but save more resource relative to the risk neutral case.

PROPOSITION 6. The solution to the problem with risk aversion relative to the solution to the problem with risk neutrality satisfies

$$K_2^{**} < K_2^* \tag{24}$$

$$R_2^{**} > R_2^*.$$

The proof is in the appendix.

7. Summary and Conclusions

The analysis of the two period model with consumption fixed in period 1 and maximized in period 2 can be summarized as follows: Suppose the cost of the substitute b is known initially. Then the value of the resource p_2 in period 2 is $\bar{b}$. By Hotelling's efficiency result $p_1 = b/F_K(2)$ where $F_K(2)$ is the discount factor. Then the resource is used initially up to the point where its marginal product equals p_1 and the remaining resource is used in period 2. Assets available for period 2 are $(\bar{K}_2, \bar{R}_2)$.

Now suppose at period 1 b is a random variable with mean $\bar{b}$. The outcome for C_2 is a decreasing, convex function of b. That is, the increase in C_2 for b less than $\bar{b}$ is greater than the decrease in C_2 for b greater than $\bar{b}$. One would like to shift the asset mix (K_2, R_2) along the production possibility frontier to achieve two objectives: (1) increase $E\{C_2\}$; (2) reduce the variance of C_2. One can show that a welfare improving shift with respect to (2) is to use less resource initially, produce less capital, but save more resource. This strategy can be justified on the basis of sufficiently strong risk aversion.

Objective (1) is equivalent to increasing the convexity of C_2 with respect to b. It is shown that it is also equivalent to shifting to a period 2 resource demand function which, for a fixed price b, is more flat. In turn the responsiveness (variance) of substitute production to the outcome of b is increased. It seems intuitive that the appropriate shift is to give more capital but less natural resource to period 2 (a shift in the demand function to the right). With more capital K_2, C_2 will increase faster than before as b decreases below $\bar{b}$; and as b increases, at least the capital is there to help offset the increased cost. That is, one expects C_2 to be more convex in b. It was proved that this intuition is right for a class of CES production functions. That is, in order to increase $E\{C_2\}$, the initial price of the resource should be less and more resource should be used initially to produce capital. The assets saved for the future should contain more capital and less natural resource.

In general the two objectives are in conflict and the direction of the shift in natural resource usage and capital accumulation depends on the relative magnitudes of the consumption convexity effect and the risk aversion effect.

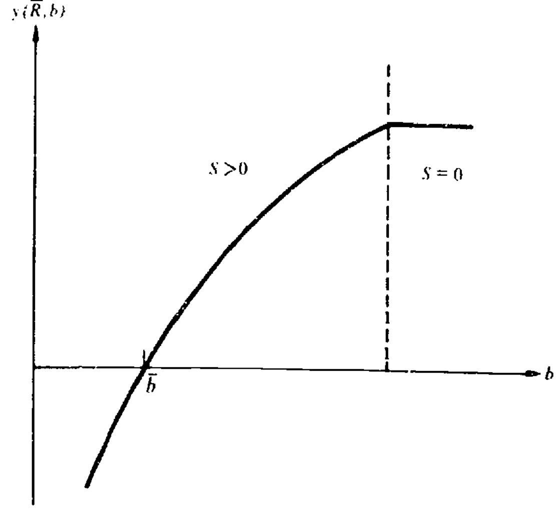

Figure 5

Appendix

Proof of Lemma. Reversing the role of R_2 to now become the independent variable, define

$$(25)\quad y(R_2, b) \equiv F_Z(K_2, R_2 + S) - F_Z(K_1, \hat{R} - R_2) F_K(K_2, R_2 + S)$$

where K_2 is given by (2) and S is given by (4). First it is argued that $F_K(2)$ convex with respect to b implies y concave with respect to b. $F_Z(1)$ is independent of b and if $S > 0$, then $F_Z(2) = b$. Therefore any nonlinearity in y arises from $F_K(2)$. Hence the result follows if $S > 0$. For $S = 0$ both y and $F_K(2)$ are horizontal lines as a function of b and the result still holds (see Figure 5).

Note that R_2^* must satisfy

$$(26)\qquad E\left\{\frac{dC_2}{dR_2}\right\} = E\{y(R_2^*, b)\} = 0$$

and from (5) $\bar{R}_2$ must satisfy $y(\bar{R}_2, \bar{b}) = 0$.

Then concavity of y with respect to b implies

$$E\{y(\bar{R}_2, b)\} < 0$$

with strict inequality holding since y is strictly concave when $S > 0$. Suppose that y is a decreasing function of R_2. Then in order to restore equality in (26) it is necessary that $R_2^* < \bar{R}_2$ which yields the result (16). Now note that

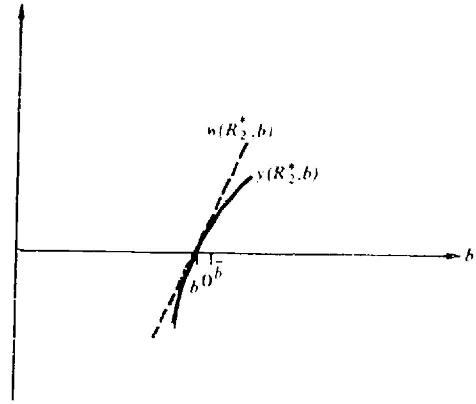

Figure 6

$$\frac{\partial y}{\partial R_2} = \begin{cases} F_{ZZ}(2) - 2F_Z(1)F_{KZ}(2) + F_Z^2(1)F_{KK}(2) + F_{KK}(1) & \text{if } S = 0. \\ F_K(2)F_{ZZ}(1) + \dfrac{F_Z^2(1)}{F_{ZZ}(2)}\,[F_{KK}(2)F_{ZZ}(2) - F_{KZ}^2(2)] & \text{if } S > 0. \end{cases}$$

In both cases (1) implies that y is a strictly decreasing function of R_2 which was needed for the argument above.

Proof of Proposition 6.[2] Define

$$w(R_2, b) \equiv G'(C_2(R_2, b))\, y(R_2, b) \tag{27}$$

where C_2 and y are given by (3) and (25) respectively. Implicitly K_2 and S are given by (2) and (4) respectively. Then it is necessary that R_2^{**} satisfy

$$E\{w(R_2, b)\} = 0$$

Define b^0 so that

$$y(R_2^*, b^0) = 0$$

The solution R_2^{**} must be independent of scaling G by a constant. Therefore, let $G'(C_2(R_2^*, b^0)) = 1$. Then

$$w(R_2^*, b^0) = 0$$

and for $b \neq b^0$, $w(R_2^*, b) > y(R_2^*, b)$ since $G'(C_2(R_2^*, b))$ is an increasing function of b. (See Figure 6). Hence,

$$E\{w(R_2^*, b)\} > E\{y(R_2^*, b)\} = 0 \tag{28}$$

[2] This proof was suggested by Michael Hoel.

Note that

$$\frac{\partial w}{\partial R_2} = G' \frac{\partial y}{\partial R_2} + G'' y^2$$

Hence $\partial y/\partial R_2 < 0$ implies $\partial w/\partial R_2 < 0$. Hence $E\{w(R_2, b)\}$ is a decreasing function of R_2. Therefore, to restore equality in (28) it is necessary that $R_2^{**} > R_2^*$ and the proposition follows.

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