Volume Title: Annals of Economic and Social Measurement, Volume 5, number 3

Volume Author/Editor: Sanford V. Berg, editor

Volume Publisher: NBER

Volume URL: http://www.nber.org/books/aesm76-3

Publication Date: July 1976

Chapter Title: Competitive Analysis of the Arms Race

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Chapter URL: http://www.nber.org/chapters/c10480

Chapter pages in book: (p. 283 - 295)

Annals of Economic and Social Measurement, 5/3, 1976

COMPETITIVE ANALYSIS OF THE ARMS RACE

BY D. D. ŠILJAK

A new nonlinear and nonstationary model is proposed for the arms race, which is a modification of the competitive equilibrium model used in economics to describe multiple markets of gross substitute commodities or services. By recognizing the fact that such models are described by Kamke's functions, we will combine the powerful mathematical machinery of the comparison principle from the theory of differential inequalities, with the strong results obtained in the analysis of competitive equilibrium, to come up with new qualitative results concerning the arms race. By using the framework of the connective stability concept, we will resolve the central problem of alliances and neutrality in the arms race. We will show that formation of alliances or neutral countries cannot destabilize the arms race, but is likely to act as a stabilizing factor in the armament processes of hostile nations involved in the arms race.

1. INTRODUCTION

Once it was recognized in [1] that the armament matrix in Richardson's model [2] of the arms race is a Metzler matrix, a whole host of strong results obtained in economic studies [3, 4] was made available for qualitative analysis of the arms race. Positivity of the armament process is a direct consequence of this fact. Another important result is that the classical Hicks conditions [4] can be applied to show positivity and stability of the armament equilibrium, as they were used to establish the same properties of the equilibrium price on multiple markets of gross substitute commodities or services. Recently, the concept of connective stability [5–8] was introduced in the study of competitive equilibrium under structural perturbations [7]. When applied to the arms race, the concept provides an answer to the central question of how formations of alliances and neutral countries affect the armaments of the countries involved in the arms race. We will show that alliances improve stability of the arms race and, at the same time, decrease the level of armaments at the equilibrium.

By carrying a step further the analogy between the armaments and prices, we propose a new nonlinear and nonstationary model for the arms race, which was introduced recently [9] as a nonstationary generalization of the nonlinear model studied extensively in the general competitive analysis in economics [10, 11]. As expected, it is not possible for the new model to duplicate all the results obtained in the rich Hicks–Metzler algebraic setting [4] for linear constant models. However, the new model is more appealing in bringing closer the mathematical representation to the nonlinear and nonstationary reality underlying the arms race problems. We will be able to show again under reasonable conditions, that the formation of alliances cannot destabilize a stable armament process. If the arms race is stable, it is also connectively stable. Pretty much the same conditions assure the existence of a unique equilibrium ray and its attractivity in the first quadrant of the armament space. Therefore, the proposed model has rich and meaningful properties to motivate future applications of the competitive analysis [7–13] to the study of armament processes.

2. Formation of Alliances and Connective Stability

In order to place in proper perspective the gencralizations of arms race models proposed in this work, let us briefly review the original Richardson model [2]. The model is a linear constant differential equation

$$\dot{x} = Ax + b, \tag{1}$$

where $x = \{x_1, x_2, \ldots, x_n\}$ is the armament n vector, $A = (a_{ij})$ is the $n \times n$ armament matrix, and $b = \{b_1, b_2, \ldots, b_n\}$ is the grievance n vector. The elements a_{ij} of A are such that

$$a_{ij}\begin{cases} <0, & i=j \\ \geq 0, & i \neq j. \end{cases} \tag{2}$$

That is, the defense off-diagonal elements a_{ij} are nonnegative so that an increase in armament of one hostile nation causes an increase in armaments of all hostile nations involved in the arms race. Arms build-up in each nation, is opposed by cost and fatigue effects which are reflected in the Richardson model by negativity of the diagonal elements a_{ii} of the armament matrix A. The vector b in (1) represents grievances and ambitions of the countries in the arms race, which cause the arms build-up in each nation even if the threats are absent. Therefore, the vector b is assumed to be nonnegative, that is, its components $b_i \geq 0$, $i = 1, 2, \ldots, n$, which we denote by $b \geq 0$.

Richardson's characterization of an alliance in the arms race is: "When an alliance is formed, the defense coefficients between allies sink to zero" [2]. A suitable framework for consideration of alliances and neutrality described by this statement, is the concept of connective stability [5–8].

Let us define the elements of the armament matrix A as

$$a_{ij} = \begin{cases} -\alpha_i + e_{ii}\alpha_{ii}, & i=j \\ e_{ij}\alpha_{ij}, & i \neq j \end{cases} \tag{3}$$

where $\alpha_i > \alpha_{ii} \geq 0$, $\alpha_{ij} \geq 0$ are real numbers, and e_{ij} are elements of the $n \times n$ constant matrix $E = (e_{ij})$. We need first the following:

Definition 1: *By $\bar{E} = (\bar{e}_{ij})$ we denote the $n \times n$ fundamental interconnection matrix with binary elements*

$$\bar{e}_{ij} = \begin{cases} 1, & \text{the state } x_j \text{ acts on the state } x_i \\ 0, & \text{the state } x_j \text{ does not act on the state } x_i. \end{cases} \tag{4}$$

Then, we recall the following:

Definition 2: *By $E = (e_{ij})$ we denote a constant $n \times n$ interconnection matrix generated from the $n \times n$ fundamental interconnection matrix $\bar{E} = (\bar{e}_{ij})$ by replacing the unit elements $\bar{e}_{ij}$ with the numbers e_{ij} such that*

$$0 \leq e_{ij} \leq 1, \tag{5}$$

and the zero elements $\bar{e}_{ij}$ of the matrix $\bar{E}$ remain the zero elements e_{ij} of the matrix E.

We also need the following:

Definition 3: *By $\mathscr{E}$ we denote the class of all interconnection matrices E generated from a fundamental interconnection matrix $\bar{E}$.*

We immediately note that according to Definitions 1–3, the matrix $\bar{E}$ is also a member of the class $\mathscr{E}$. The effect of these Definitions 1–3 is that for each interconnection matrix $E=(e_{ij})$ we have a different system (1). The underlying idea of the concept of connective stability is a possibility to establish stability of a class of systems (1) corresponding to matrices $E \in \mathscr{E}$ by proving stability of one member of that class corresponding to $\bar{E} \in \mathscr{E}$.

The equilibria of the arms race are constant solutions of equation (1) determined by the algebraic equation

$$Ax + b = 0. \tag{6}$$

If $\det A \neq 0$, the equilibrium x^e is a constant vector

$$x^e = -A^{-1}b, \tag{7}$$

which is the unique solution of (6). We notice from (3) and (7) that for each E, the system (1) has a distinct equilibrium x^e. By $\mathscr{S}_{(1)}$ we denote the system described by equation (1) and the interconnection matrices of class $\mathscr{E}$, and formulate the following:

Definition 4: *The system $\mathscr{S}_{(1)}$ is said to be connectively stable if and only if the equilibrium x^e of equation (1) is stable in the sense of Liapunov for all $E \in \mathscr{E}$.*

To establish conditions for stability of system (1) expressed by Definition 4, let us recall McKenzie's definition of a quasidominant diagonal matrix [14]: An $n \times n$ matrix $A=(a_{ij})$ is called quasidominant diagonal if there exist positive numbers d_j such that

$$d_j|a_{jj}| > \sum_{\substack{i=1 \\ i\neq j}}^{n} d_i|a_{ij}|, \qquad j=1,2,\ldots,n. \tag{8}$$

Since A is a Metzler matrix [3], (8) is necessary and sufficient [14–16] for stability of A. To establish connective stability of $S_{(1)}$, we denote by $\bar{A}=(\bar{a}_{ij})$ the matrix A which corresponds to the fundamental interconnection matrix $\bar{E}$, and prove the following:

Theorem 1. The system $\mathscr{S}_{(1)}$ is connectively asymptotically stable in the large if and only if the matrix $\bar{A}$ is quasidominant diagonal.

Proof. Since $\bar{A}$ is a Metzler matrix [3], the quasidominant property (8) is necessary and sufficient [14–16] for stability of (1) for $E=\bar{E}$. This establishes the "only if" part of the theorem. To prove the "if" part, we note that A is a Metzler matrix for all $E \in \mathscr{E}$ and that

$$A \leq \bar{A}, \qquad \forall E \in \mathscr{E} \tag{9}$$

holds element-by-element (that is, $A-\bar{A} \leq 0$). Thus, if (8) holds for $\bar{E}$, it holds for all E and $\mathscr{S}_{(1)}$ is a connectively stable system. This proves Theorem 1.

As for interpretations of Theorem 1 in the context of the arms race, several Remarks are in order:

Remark 1. From Theorem 1, we conclude immediately that if an arms race is stable, then "sinking to zero" of any number of interconnection elements, which describes formation of alliances in the Richardson sense, cannot destabilize the arms race. Furthermore, such effect of alliances would only strengthen inequalities (8) and, therefore, make the arms race more stable than it was before the alliance was formed. It may be argued against Richardson's description of alliance formation, that the interconnection elements in the alliance are not zero but stay on some positive values. This is, of course, included in the connective stability since we merely use the inequality (9), and require that the interconnection elements among countries involved in an alliance decrease after it is formed, which is reasonable to expect. It may be further argued, however, that after an alliance is formed the countries not involved in the alliance would increase the interaction coefficients corresponding to the allied countries. This effect is not included in connective stability, but the quasidominant condition (8) is still a good measure of how much of the increases can be tolerated by stability.

Remark 2. From connective stability, it follows than an equilibrium x^e exists for all $E \in \mathscr{E}$. Furthermore, since A is a Metzler matrix for all $E \in \mathscr{E}$ and $b \geq 0$, we can use [3] and conclude that the solutions $x(t; t_0, x_0)$ of (1) are nonnegative, that is,

$$x(t; t_0, x_0) \geq 0, \qquad t \geq t_0 \tag{10}$$

for all t_0 and $x_0 \geq 0$. In other words, the armaments are always nonnegative if they "start" nonnegative regardless of the alliance formations. From connective stability of $S_{(1)}$ we conclude that $\lim_{t \to +\infty} x(t; t_0, x_0) = x^e$ for all t_0, x_0, and $E \in \mathscr{E}$. Therefore $x^e \geq 0$, for all $E \in \mathscr{E}$. If the grievance vector is positive ($b > 0$) then we can further show that so is the corresponding equilibrium. As shown in [3], if a Metzler matrix A satisfies (8), then (and only then) A^{-1} is nonpositive. However, A^{-1} cannot have a row of zeros since it satisfies (8) and, thus, it is nonsingular. Therefore, from positivity of b and (7), follows positivity of the arms race equilibrium x^e for all $E \in \mathscr{E}$.

It further can be concluded that if $\bar{A}$ satisfies (8), then from (9) follows $\bar{A}^{-1} \leq A^{-1}$ [15]. Therefore, applying (7), we have $\bar{x}^e \geq x^e$, where $\bar{x}^e = \bar{A}^{-1}b$ is the armament equilibrium for $E = \bar{E}$. The inequality $\bar{x}^e \geq x^e$ establishes the fact that a formation of alliances is likely to decrease the armaments levels at the equilibrium for all countries involved in the arms race.

Remark 3. Since connective stability involves $E = 0$, we conclude that

$$a_{ii} = -\alpha_i + e_{ii}\alpha_{ii} < 0, \qquad i = 1, 2, \ldots, n \tag{11}$$

for all $E \in \mathscr{E}$. That is, each country involved in the arms race must exhibit the "expense and fatigue" effect for the arms race to be stable.

Once the armament matrix is recognized as a Metzler matrix, another possibility is open for further generalizations of the arms race models. That is, we can use the powerful analysis of competitive equilibrium [10] in nonlinear multiple market models coupled with the comparison principle from the theory of differential inequalities [17] to come up with new important results. This route was made available only recently by a study of nonstationary competitive processes [9], and will be shown in the following development to be suitable for establishing the qualitative characteristics of the arms race expressed by the above Remarks, in a much more general setting, but at the cost of a more refined analysis.

Another nonlinear generalization of the Richardson model was proposed and analyzed by Sandberg [18] using a different mathematical framework. The major difference between Sandberg's model and the one presented here, is that we follow Caspary's critique [19] of the Richardson model, and assume that relative rather than absolute armament levels motivate countries to arm. The armaments in the arms race can be treated as prices of commodities (or services) on multiple markets, and the competitive analysis becomes an ideal setting for studying qualitative characteristics of armament processes.

3. A General Model

For a model of the arms race involving n hostile countries, we propose a differential equation

$$\dot{x} = h(t, x) + g(t), \tag{12}$$

where $x(t) \in \mathcal{R}^n$ is the armament vector; $h\colon \mathcal{T} \times \mathcal{R}^n_+ \to \mathcal{R}^n$ is the function describing the interaction of armament levels among the countries; and $g\colon \mathcal{T} \to \mathcal{R}^n$ is the function representing the grievances that motivate the countries to arm regardless of the armament levels. We assume that $h(t, x) \in C^{(0,0)}(\mathcal{T} \times \mathcal{R}^n_+)$ and $g(t) \in C^0(\mathcal{T})$, where $\mathcal{T} = (\tau, +\infty)$ and τ is a number or the symbol $-\infty$, $\mathcal{R}^n$ is the real n-dimensional Euclidean vector space, and $\mathcal{R}^n_+ = \{x \in \mathcal{R}^n : x \geq 0\}$.

As in the classical Richardson model (1), we assume that an increase in the armament level of one country causes an increase in the armament level of the other countries involved in the arms race. Therefore, we say that the function $h(t, x)$ belongs to the class of functions

$$\begin{aligned} \mathcal{K}\colon h_i(t, a) < h_i(t, b), \qquad & \forall (t, a), (t, b) \in \mathcal{T} \times \mathcal{R}^n_+ \\ a_i = b_i,\ a_j < b_j; \qquad & i \in M, j \in N - M \end{aligned} \tag{13}$$

where N is the set of indices $\{1, 2, \ldots, n\}$, and M is a nonvoid subset of N.

If (12) is used to represent multiple markets of commodities or services, then the fact that the excess demand function $h(t, x) \in \mathcal{K}$ means that (12) describes the time-dependent gross substitute case introduced in [8]. More importantly, in [8, 9], the class of functions $\mathcal{K}$ was recognized as Kamke's functions, and powerful methods of comparison principle [17] were made available for analysis of nonstationary competitive processes.

By following Caspary's critique [19] of the Richardson model, we assume that a nation's security is dependent upon the relative rather than the absolute size of its own and its opponent's forces. This amounts to assuming that

$$h(t, \lambda x) = h(t, x), \qquad \forall \lambda > 0, \tag{14}$$

which is a time-dependent version of the usual "positive" homogeneity condition of degree zero, well-known in the traditional microeconomics analysis.

Finally, we assume that there exists a positive equilibrium x^e as a solution of the equation

$$h(t, x)+g(t)=0, \tag{15}$$

such that $x^e \in \mathscr{C}$, where $\mathscr{C}=\{x \in \mathscr{R}_+^n : x>0\}$ is an open cone in $\mathscr{R}_+^n$. It should be noted here that for (15) to have a constant solution $x^e \in \mathscr{C}$, it is more realistic to assume that g is a constant vector. Our results however, do not depend on constancy of g.

Let us summarize the properties of the function $h(t, x)$ as the following hypotheses:

$$(H_1) h(t, x) \in \mathscr{K};$$
$$(H_2) h(t, \lambda x)=h(t, x), \qquad \forall \lambda>0$$
$$(H_3) \exists x^e>0 : h(t, x^e)+g(t)=0, \qquad \forall t \in \mathscr{T}.$$

Our immediate interest is the existence of solutions $x(t; t_0, x_0)$ of equation (12) on the time interval $\mathscr{T}_0=[t_0, +\infty)$. A preliminary result to the existence question is the following:

Lemma 1. If the function $h(t, x)$ satisfies the hypotheses (H_1), (H_2), and (H_3), then there exists a unique equilibrium ray $\mathscr{X}=\{x^e \in \mathscr{C} : x^e=\lambda e\}$ of equation (12), where $e=\{1, 1, \dots, 1\}$ is an n vector and λ is a positive number.

Using Lemma 1, one can show the following:

Theorem 2. If the function $h(t, x)$ satisfies the hypotheses (H_1), (H_2), and (H_3), then there exists a solution $x(t; t_0, x_0)$ of equation (12) for any $(t_0, x_0) \in \mathscr{T} \times \mathscr{C}$ and for all $t \in \mathscr{T}_0$.

Both Lemma 1 and Theorem 2 are proved in Appendix A following references [9] as slightly stronger Lemma A.1 and Theorem A.1.

From the proof of Theorem A.1, one concludes directly that under the hypotheses (H_1)–(H_3), all solutions are bounded on $\mathscr{T}_0$ for $(t_0, x_0) \in \mathscr{T} \times \mathscr{C}$. Furthermore, the solutions are positive, that is, they have the following property:

$$(P_1)(t_0, x_0) \in \mathscr{T} \times \mathscr{C} \Rightarrow x(t; t_0, x_0) \in \mathscr{C}, \qquad \forall t \in \mathscr{T}_0$$

and the open cone $\mathscr{C}$ is an invariant set.

The property (P_1) is important in the context of the arms race in that the armaments during the adjustment process never become negative. It is a pleasing fact to conclude that nonnegativity of the armament process (10) established for the simple Richardson model, carries over to the generalized model (12). As in the Richardson model, positivity of the armaments is essential for demonstrating the structural properties of the model (12) in the context of connectivity.

As shown in [9], hypotheses (H_1)–(H_3) imply not only that the cone $\mathscr{C}$ is an invariant set, but that it is also a region of attraction of the equilibrium ray $\mathscr{X} \subset \mathscr{C}$. That is, we demonstrate the following property of the solution $x(t; t_0, x_0)$:

$$(P_2)(t_0, x_0) \in \mathscr{T} \times \mathscr{C} \Rightarrow \lim_{t \to +\infty} d[x(t; t_0, x_0), \mathscr{X}]=0$$

where $d(x, \mathscr{X}) = \inf_{x^e \in \mathscr{X}} \{\|x - x^e\|_N\}$, $\|x\|_N = \sup_{i \in N} \{\|x_i\|\}$. That is, in the Appendix, we prove Theorem A.2 which is slightly stronger than the following:

Theorem 3. If the function $h(t, x)$ satisfies the hypotheses (H_1), (H_2), and (H_3), then the solutions $x(t; t_0, x_0)$ of equation (12) have the property (P_2).

Theorem 3 can be widened to include a connective version of the property (P_2). For this purpose, we specify the components $h_i(t, x)$ of the function $h(t, x)$ as

$$h_i(t, x) = \tilde{h}_i(t, x_i, e_{i1}x_1, \ldots, e_{ii}x_i, \ldots, e_{in}x_n), \qquad i \in N \tag{16}$$

where e_{ij} are elements of the $n \times n$ interconnection matrix E.

We introduce the "nonlinear" analog to Definition 5 as the following:

Definition 5. The system $\mathscr{S}_{(12)}$ is said to be connectively attractive if and only if the equilibrium ray $\mathscr{X}$ of equation (12) has the property (P_2) for all $E \in \mathscr{E}$.

To be able to establish the connective version of (P_2), as specified by Definition 5, we require the interconnection matrices E to be indecomposable [4]. That is, we consider the class of matrices $\tilde{\mathscr{E}}$ such that $E \in \tilde{\mathscr{E}}$ implies that E cannot be permuted into a matrix of the form

$$E = \begin{bmatrix} E_{11} & 0 \\ E_{21} & E_{22} \end{bmatrix} \tag{17}$$

where E_{11}, E_{22} are square submatrices of E and 0 is a rectangular block of zeros. Equation (12) and the class of interconnection matrices $\mathscr{E}$ define the $\tilde{\mathscr{S}}_{(12)}$.

By $\bar{h}(t, x)$ we denote the function $h(t, x)$ in (16) which corresponds to the fundamental interconnection matrix $\bar{E} \in \mathscr{E}$. We need the following hypotheses:

$$(\bar{H}_1)\bar{h}(t, x) \in \mathscr{K};$$
$$(\bar{H}_2)\bar{h}(t, \lambda x) = \bar{h}(t, x), \qquad \forall \lambda > 0.$$

Now, we prove the following:

Theorem 4. If the function $\bar{h}(t, x)$ satisfies the hypotheses $(\bar{H}_1)$ and $(\bar{H}_2)$, and the function $h(t, x)$ satisfies the hypothesis (H_3) for all $E \in \tilde{\mathscr{E}}$, then $\tilde{\mathscr{S}}_{(12)}$ is connectively attractive.

Proof. From (16), indecomposability of $E \in \tilde{\mathscr{E}}$, and the property (P_1), it follows that $\bar{h}(t, x) \in \mathscr{K}$ implies $h(t, x) \in \mathscr{K}$ for all $E \in \tilde{\mathscr{E}}$. That is, if $\bar{h}(t, x)$ satisfies $(\bar{H}_1)$, then $h(t, x)$ satisfies (H_1) for all $E \in \tilde{\mathscr{E}}$. Furthermore, if $\bar{h}(t, x)$ satisfies $(\bar{H}_2)$, then $h(t, x)$ satisfies (H_2) for all $E \in \tilde{\mathscr{E}}$. Therefore, under the conditions of the Theorem, $h(t, x)$ satisfies the hypotheses (H_1)-(H_3) for all $E \in \tilde{\mathscr{E}}$, and by Theorem 3 system $\tilde{\mathscr{S}}_{(12)}$ is connectively attractive. This proves Theorem 4.

Unfortunately, by Theorem 4 we are not providing the entire "nonlinear" analog to Theorem 1 since we are not able to express our conditions only in terms of the function $\bar{h}(t, x)$. That comes from our inability to show that if $\bar{h}(t, x)$ satisfies (H_3) so does $h(t, x)$ for all $E \in \tilde{\mathscr{E}}$.

As in Remark 1, we conclude that under somewhat restricted conditions ($E \in \tilde{\mathscr{E}}$ instead of $E \in \mathscr{E}$), formation of alliances in the general model (12), cannot ruin stability of the arms race. Remark 2 has not its counterpart in the general model (12). As for the Remark 3, we can show that there is a "nonlinear" counterpart to the necessary condition (11)), which is expressed by

$$h_i(t, a) \leq h_i(t, b),\ a_i > b_i,\ a_j = b_j,\ i \neq j;\ i, j \in N. \tag{18}$$

Condition (18) is a necessary condition for system (12) to have the property (P_2), and represents the "expense and fatigue" effect established in the linear model (1) by the condition (11). A proof of necessity of (18) is provided by Theorem A.3 of the Appendix.

4. Conclusion

By exploring the analogy between the competitive equilibrium in economics and the arms race models, we were able to show a number of results concerning the armament processes involving hostile countries. The most important conclusion reached by the foregoing analysis is that the formation of alliances or neutral countries, cannot destroy stability, but is likely to stabilize the arms race. It would be interesting to show that the same conclusion can be made for the stochastic arms race model [1] on the basis of results obtained for model ecosystems in randomly varying environment [20] and stochastic large-scale systems [21].

5. Acknowledgment

The author is grateful to Professor G. S. Ladde, Department of Mathematics, State University of New York at Potsdam, Potsdam, New York, for his help in proving the theorems in the Appendix.

Appendix

In this section, we will establish the existence result that is slightly stronger than that expressed by Theorem 2.

For simplicity, let us rewrite the equation (29) as

$$\dot{x} = f(t, x), \tag{A.1}$$

where $f(t, x) = h(t, x) + g(t)$ and, therefore, $f(t, x) \in C^{(0,0)}(\mathcal{T} \times \mathcal{R}^n_+)$ with $t \in \mathcal{T}$, $x \in \mathcal{R}^n_+$.

We see that the function $f(t, x)$ satisfies the same hypotheses (H_1)–(H_3) as $h(t, x)$:

$$(H_1')\, f(t, x) \in \mathcal{H};$$

$$(H_2')\, f(t, \lambda x) = f(t, x), \qquad \forall \lambda > 0;$$

$$(H_3')\, \exists x^e > 0 : f(t, x^e) = 0, \qquad \forall t \in \mathcal{T}.$$

To prove the existence result for (A.1), we first establish the following [9]:

Lemma A.1. If the function $f(t, x)$ satisfies the hypotheses (H_1'), (H_2') and (H_3'), then for any two vectors $u > 0$, $v > 0$, $u \neq v$, there exist indices $k, l \in N$, $k \neq l$ such that

$$f_k(t, u) < f_k(t, v),\ f_l(t, u) > f_l(t, v) \tag{A.2}$$

for each fixed $t \in \mathcal{T}$ and all $u, v \in \mathcal{R}^n_+$.

Proof. Define $\xi_k = \max_{i\in N}\{u_i/v_i\}$, $n_l = \min_{i\in N}\{u_i/v_i\}$ for any pair of vectors $u, v > 0$. With each pair (u, v), we associate the pair (u_*, u^*) given as $u_* = \xi_k^{-1}u$, $u^* = \eta_l^{-1}u$, so that $u_* \le v$ and $u^* \ge v$. That is, $u_{*i} \le v_i$, $i \ne k$, $u_{*l} = v_k$ and likewise, $u_i^* \ge v_i$, $i \ne l$, $u_l^* = v_l$, and since $u \ne v$, at least for some i we have $u_{*i} < v_i$, $u_i^* > v_i$, $i \ne k$, $i \ne l$.

From (H_1') and (H_2'), we have

$$\text{(A.3)} \qquad f_k(t, u) = f_k(t, u_*) < f_k(t, v), f_l(t, u) = f_l(t, u^*) > f_l(t, v)$$

for each $t \in \mathcal{T}$, which proves Lemma A.1.

We recall that the equilibrium ray of (A.1) is $\mathcal{X} = \{x^e \in \mathcal{C}, x^e = \lambda e\}$, where $e = \{1, 1, \dots, 1\}$ such that $e \in \mathcal{R}_+^n$ and $\lambda > 0$. We provide the following:

Proof of Lemma 1. Uniqueness of $\mathcal{X}$ means that for any pair of equilibrium values $x', x'' \in \mathcal{C}$, $x' \ne x''$,

$$\text{(A.4)} \qquad f(t, x'') = f(t, x') = 0 \Rightarrow x'' = \lambda x'$$

for all $t \in \mathcal{T}$ and some $\lambda > 0$.

Define $\mu = \min_{i\in N}\{x_i'/x_i''\}$, where x_i', x_i'' are the i-th components of the two equilibria x', x'', and $x''' = \mu x''$. Then, we have $x''' \le x'$, that is, $x_i''' \le x_i'$, $i \ne l$, $x_l''' = x_l'$ and at least for some $i \ne l$, $x_i''' < x_i'$. Assume that the statement (A.4) is false. That is $x'' \ne \lambda x'$, for all $\lambda > 0$. By (H_1')–(H_3') and $f(t, x'') = f(t, x') = 0$, we have $0 = f_l(t, x'') = f_l(t, x''') < f_l(t, x') = 0$, which is absurd. The proof of Lemma 1 is complete.

Remark 4. If we take any pair of vectors $x^e, x \in \mathcal{C}$ such that $x^e \in \mathcal{X}$, $x \notin \mathcal{X}$, and use Lemma 1 and inequalities (A.2), we conclude that $f_k(t, x) < 0$ and $f_l(t, x) > 0$ for all $t \in \mathcal{T}$ and some indices $k, l \in N$.

To establish the existence result for equation (A.1), we can replace the hypotheses (H_1')–(H_3') by the following weaker hypothesis: (H_4') $f(t, x^e) = 0 \Leftrightarrow x^e \in \mathcal{X}$, and for any $x \in \mathcal{C}$ and $x \notin \mathcal{X}$ and any $x^e \in \mathcal{X}$, there exists a pair of indices $k, l \in N$, $k \ne l$, such that

$$\text{(A.5)} \qquad x_k = \max_{i\in N}\{x_i\} \Rightarrow f_k(t, x) < 0, \qquad x_l = \min_{i\in N}\{x_i\} \Rightarrow f_l(t, x) > 0$$

for all $t \in \mathcal{T}$.

In view of Remark 4, (H_1')–(H_3') imply (H_4') but not vice versa.

Now, we can prove a slightly stronger result than that of Theorem 2, which we state as the following [9]:

Theorem A.1. If the function $f(t, x)$ satisfies the hypothesis (H_4'), then there exists a solution $x(t) = x(t; t_0, x_0)$ for any $(t_0, x_0) \in \mathcal{T} \times \mathcal{C}$ and for all $t \in \mathcal{T}_0$.

Proof. Consider $x_0 \notin \mathcal{X}$, and $\alpha = x_{k0} = \max_{i\in N}\{x_{i0}\}$, $\beta = x_{l0} = \min_{i\in N}\{x_{i0}\}$, $\alpha > \beta > 0$. Define

$$\text{(A.6)} \qquad \begin{aligned} \mathcal{B}' &= \{x \in \mathcal{R}_+^n : \beta \le x_i \le \alpha, \forall i \in N\} \\ \mathcal{B}'' &= \{x \in \mathcal{R}_+^n : \tfrac{1}{2}\beta \le x_i \le \alpha + \tfrac{1}{2}\beta, \forall i \in N\} \end{aligned}$$

and note $\mathcal{B}' \subset \mathcal{B}''$. For any $\tau > 0$, we define the time interval $\mathcal{T}_1 = [t_0, t_0 + \tau]$ and the rectangle $\mathcal{T}_1 \times \mathcal{B}''$. By continuity of $f(t, x)$ we can find a number $\mu > 0$ such that $|f_i(t, x)| \le \mu_1$, for all $(t, x) \in \mathcal{T}_1 \times \mathcal{B}''$ and all $i \in N$. By Peano's existence Theorem

[22], there exists at least one solution $x(t)$ for all $t \in [t_0, t_0+\varepsilon]$, where $\varepsilon_1 = \min\{\tau, \alpha/\mu_1\}$. Now, either $x(t_1) \in \mathscr{X}$ for some $t_1 \in (t_0, t_0+\varepsilon_1]$, or $x(t) \notin \mathscr{X}$ for all $t \in (t_0, t_0+\varepsilon_1]$. In the first case, the solution $x(t)$ exists for all $t \in \mathscr{T}_0$. If we have the second case, then we extend successively the solution $x(t)$ beyond the time interval $(t_0, t_0+\varepsilon_1]$.

Note that for all $t \in [t_0, t_0+\varepsilon_1]$, $x(t) \in \mathscr{B}''$. In fact, we can show that $x(t) \in \mathscr{B}'$ for all $t \in [t_0, t_0+\varepsilon_1]$. Since $x_0 \notin \mathscr{X}$, by using (H_4'), we have (A.5) with $x = x_0$ which implies for $t = t_0$,

$$f_k(t_0, x_0) < 0, \quad f_l(t_0, x_0) > 0. \tag{A.7}$$

By continuity of $f(t, x)$ and $x(t)$, we can find a $\delta_1 > 0$ such that

$$f_k[t, x(t)] < 0, \quad f_l[t, x(t)] > 0, \quad \forall t \in [t_0, t_0+\delta_1] \tag{A.8}$$

which by integration yields

$$x_k(t) \leq x_{k0}, \quad x_l(t) \geq x_{l0}, \quad \forall t \in [t_0, t_0+\delta_1] \tag{A.9}$$

which in turn, implies that $x(t) \in \mathscr{B}'$ for all $t \in [t_0, t_0+\delta_1]$. Since $x(t_0+\delta_1) \in \mathscr{B}''$ and also $x(t_0+\delta_1) \notin \mathscr{X}$, by using (H_4'), we have

$$f_k[t_0+\delta_1, x(t_0+\delta_1)] < 0, \quad f_l[t_0+\delta_1, x(t_0+\delta_1)] > 0 \tag{A.10}$$

and conclude that there exists a $\delta_2 > 0$ such that

$$f_k[t, x(t)] < 0, \quad f_l[t, x(t)] > 0, \quad \forall t \in [t_0+\delta_1, t_0+\delta_1+\delta_2] \tag{A.11}$$

and

$$x_k(t) \leq x_k(t_0+\delta_1), \quad x_l(t) \geq x_l(t_0+\delta_1), \quad \forall t \in [t_0+\delta_1, t_0+\delta_1+\delta_2]. \tag{A.12}$$

Therefore, (A.8) and (A.12) imply that $x(t) \in \mathscr{B}'$ for all $t \in [t_0, t_0+\delta_1+\delta_2]$. By continuing this process, one arrives in finite number of steps to the conclusion that $x(t) \in \mathscr{B}'$ for all $t \in [t_0, t_0+\varepsilon_1]$.

Since $x(t)$ remains in $\mathscr{B}'$ for the entire interval $[t_0, t_0+\varepsilon_1]$, by using the above arguments, one can show that the solution $x(t)$ can be extended over the interval $[t_0+\varepsilon_1, t_0+2\varepsilon_1]$ and, thus, over the interval $\mathscr{T}_1 = [t_0, t_0+\tau]$. Moreover, $x(t) \in \mathscr{B}'$ for all $t \in \mathscr{T}_1$.

Because, the solution $x(t)$ stays inside $\mathscr{B}'$ for the entire interval $\mathscr{T}_1 = [t_0, t_0+\tau]$, it can be extended over the interval $\mathscr{T}_2 = [t_0+\tau, t_0+2\tau]$ by choosing subintervals of $\mathscr{T}_2$ determined by $\varepsilon_2 = \min\{\tau, \alpha/\mu_2\}$, where μ_2 is defined by the condition $|f_i(t, x)| \leq \mu_2$, for all $(t, x) \in \mathscr{T}_2 \times \mathscr{B}''$ and $i \in N$. Moreover, one shows as before that $x(t) \in \mathscr{B}'$ for all $t \in \mathscr{T}_2$. Therefore, $x(t) \in \mathscr{B}'$ for all $t \in \mathscr{T}_1 \cup \mathscr{T}_2$. In this manner, a solution staying in $\mathscr{B}'$ can be found for all $t \in \mathscr{T}_0$. This proves Theorem A.1.

Now, we can establish the attractivity property of the equilibrium ray $\mathscr{X}$ of equation (A.1) by the following [9]:

Theorem A.2. If the function $f(t, x)$ satisfies the hypothesis (H_4'), then the solutions $x(t; t_0, x_0)$ of equation (A.1) *have the property* (P_2).

Proof. To prove Theorem A.2, we use the Liapunov-like function $V: \mathscr{C} \to \mathscr{R}_+$,

$$V(x) = d(x, \mathscr{X}), \tag{A.13}$$

$V(x) \in C^0(\mathscr{C})$ and $V(x)$ is Lipchitzian, which has

$$D^+V(x) \le -\min_{i\in L} \{|f_i(t, x)|\}, \qquad \forall (t, x) \in \mathscr{T} \times \mathscr{C} - \mathscr{X}. \tag{A.14}$$

where L is a nonvoid subset of N defined by the set of all $i \in N$ such that $|f_i(t, x)| > 0$.

To show (A.14), we note that for each $(t, x) \in \mathscr{T} \times \mathscr{C} - \mathscr{X}$ there exists $\lambda_0 > 0$ such that $V(x)$ can be rewritten as $V(x) = \max_{i\in N} \{|x_i - \lambda_0|\}$. To see this, we recognize the fact that the distance between a point x and the ray $\mathscr{X}$ is equal to the distance between the point x and the foot $x^0 \in \mathscr{X}$ of the normal drawn from the point x to the ray $\mathscr{X}$. Furthermore, there exists an index set L such that $V(x) = |x_i - \lambda_0|$ for all $i \in L$. Now, by hypothesis (H_4'), for $i \in L$ we have either $f_i(t, x) < 0$, or $f_i(t, x) > 0$ and, therefore, $|f_i(t, x)| > 0$, $i \in L$. By continuity of $f(t, x)$ and for $\Delta t > 0$ sufficiently small, we conclude that the index set L remains invariant.

We proceed to compute $D^+V(x)$ as follows:

$$V[x + \Delta t f(t, x)] - V(x) = |x_i + \Delta t f_i(t, x) - \lambda_0| - |x_i - \lambda_0|, \qquad \forall i \in L. \tag{A.15}$$

There are two cases to be considered: (i) $x_i - \lambda_0 > 0$, and (ii) $x_i - \lambda_0 < 0$, for $i \in L$. In either case, (A.15) can be rewritten as

$$V[x + \Delta t f(t, x)] - V(x) = \Delta t |f_i(t, x)|, \qquad \forall i \in L. \tag{A.16}$$

When (i), then $f_i(t, x) < 0$, and when (ii), then $f_i(t, x) > 0$. Hence, from (A.16), we get

$$V[x + \Delta t f(t, x)] - V(x) = -\Delta t |f_i(t, x)|, \qquad \forall i \in L \tag{A.17}$$

and finally, (A.14) is established from (A.17).

The second part of the proof consists in showing that the function $V(x)$ defined in (A.13) with (A.14) implies property (P_2). Let $x(t) = x(t; t_0, x_0)$ be any solution of (A.1) for $(t_0, x_0) \in \mathscr{T} \times \mathscr{C}$. Then, $x(t) \in \mathscr{B}'$ for all $t \in \mathscr{T}_0$, where $\mathscr{B}'$ was defined in (A.6). Set

$$\rho(t) = V[x(t)]. \tag{A.18}$$

For sufficiently small $\Delta t > 0$, we have

$$\begin{aligned} \rho(t + \Delta t) - \rho(t) &= V[x(t + \Delta t)] - V[x(t)] \\ &= V[x(t + \Delta t)] - V(x(t) + \Delta t f[t, x(t)]) \\ &\quad + V(x(t) + \Delta t f[t, x(t)]) - V[x(t)]. \end{aligned} \tag{A.19}$$

By using the fact that $V(x)$ is Lipschitzian, from (A.19), we obtain

$$D^+\rho(t) \le \min_{i\in L} \{|f_i[t, x(t)]|\}, \qquad \forall t \in \mathscr{T}_0. \tag{A.20}$$

We proceed to establish the property (P_2) by contradiction. That is, for some $\varepsilon>0$, $(t_0, x_0)\in\mathcal{T}\times\mathscr{C}$, there exists $t_1>t_0$ and a sequence $\{t_k\}$, $t_k>t_1$, $t_k\to+\infty$, $k\to+\infty$, such that $d[x(t_k), \mathscr{X}]=\varepsilon$ and $d[x(t), \mathscr{X}]>\varepsilon$ for $t\in(t_k, t_{k+1})$. Let us denote $\mathscr{B}'''=\{x\in\mathscr{B}': d(x, \mathscr{X})\geq\varepsilon\}$, which is a compact set. For any $t\in\mathcal{T}$ and any fixed $\tilde{x}\in\mathscr{B}'''$, there exists an index subset $L\subset N$ such that the function $\theta(t, \tilde{x})=\min_{i\in L}\{|f_i(t, \tilde{x})|\}>0$. By continuity of $\theta(t, \tilde{x})$, there exists a neighborhood $\mathcal{O}(\tilde{x})$ of $\tilde{x}\in\mathscr{B}'''$ such that $\theta(t, x)>0$, for all $x\in\mathcal{O}(\tilde{x})$. Let $\mathscr{U}=\{\mathcal{O}(\tilde{x}): \tilde{x}\in\mathscr{B}'''\}$ open cover of $\mathscr{B}'''$. Since $\mathscr{B}'''$ is compact, by Heine–Borel Theorem [23], we can extract a finite subcover $\{\mathcal{O}(\tilde{x}_1), \mathcal{O}(\tilde{x}_2), \ldots, \mathcal{O}(\tilde{x}_n)\}$, where to each $\mathcal{O}(\tilde{x}_j)$ there corresponds an index subset L_j and the function $\theta_j(t, x)=\min_{i\in L_j}\{|f_i(t, x)|\}$. We define $\psi(t, x)=\min_j\{\theta_1(t, x), \theta_2(t, x), \ldots, \theta_n(t, x)\}$, and note that $\psi(t, x)\in C^{(0,0)}(\mathcal{T}\times\mathscr{B}''')$ and $\psi(t, x)>0$, for all $(t, x)\in\mathcal{T}\times\mathscr{B}'''$. Therefore, we can take $\inf_{x\in\mathscr{B}'''}\psi(t, x)=\varphi(t)$ and $\varphi(t)\in C^0(\mathcal{T})$ since $\mathscr{B}'''$ is compact.

Now, from inequality (A.20), we can derive

$$D^+\rho(t)\leq-\varphi(t), \tag{A.21}$$

where $x(t)\in\mathscr{B}'''$, $t\in[t_k, t_{k+1}]$. Integrating (A.21) from t_k to t_{k+1}, and using the definitions (A.14) and (A.18) of $V(x)$ and $\rho(t)$ we obtain

$$0=\rho(t_{k+1})-\rho(t_k)\leq-\int_{t_k}^{t_{k+1}}\varphi(\tau)\,d\tau<0 \tag{A.22}$$

which is absurd. Therefore, the proof of Theorem A.2 is complete.

As the final part of the Appendix, let us prove the following:

Theorem A.3. If the function $f(t, x)$ satisfies the hypotheses (H_1'), (H_2'), and (H_3'), and solutions $x(t; t_o, x_0)$ of equation (A.1) *have the property (P_2), then*

$$f_i(t, a)\leq f_i(t, b),\ a_i>b_i,\ a_j=b_j,\ i\neq j;\ i, j\in N. \tag{A.23}$$

Proof. Suppose that (A.23) is false. This would imply that for some $i\in N$,

$$f_i(t, a)>f_i(t, b), \tag{A.24}$$

for all a, b such that $a_i>b_i>0$, $0<a_j=b_j$, $i\neq j$. Then, (A.24) is equivalent to

$$\begin{aligned} &f_i(t, a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n)\\ &\quad>f_i(t, a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_n). \end{aligned} \tag{A.25}$$

This together with (A.23) and the fact that $f(t, x)$ is continuous and belongs to $\mathscr{K}$, implies that

$$\begin{aligned} &f_i(t, a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n)\\ &\quad>f_i(t, a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_n)\\ &\quad>f_i(t, 0, \ldots, 0, b_i, 0, \ldots, 0)\\ &\quad>f_i(t, 0, \ldots, 0, 0, 0, \ldots, 0)>0, \end{aligned}$$

for all $a > 0$. From (A.26) and the fact that under conditions of the Theorem $x(t; t_0, x_0) \in \mathscr{C}$, we conclude that the i-th component $x_i(t; t_0, x_0)$ of the solution $x(t; t_0, x_0)$ is a strictly increasing function for all $t \in \mathscr{T}_0$, which contradicts the property (P_2). This proves Theorem A.2.

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