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SECOND-ORDER APPROXIMATIONS FOR ESTIMATING PRODUCTION FUNCTIONS

BY VITTORIO CORBO*

This paper shows that the CES and VES production functions have the same second order approximation. Furthermore it is shown that in most cases the second order approximation is better for the VES than the CES. Therefore second order approximations should not be used to make inferences with respect to parameters of a CES function without strong independent evidence that the "true" production model is indeed CES.

INTRODUCTION

In the estimation of production functions the usual hypothesis is that the function is one of a restricted class which satisfies some *a priori* restrictions in technology. The production fuctions most frequently used are the Cobb–Douglas, CES and VES, in that order. If relevant data on factor inputs and output are available, these data can be used, in principle, to identify the relevant production function, using quality of fit as a criterion.

The CES and VES production functions are non-linear in the parameters; therefore, direct estimation of these functions requires non-linear estimation procedures. To avoid complications arising from a non-linear estimation procedure[1] Kmenta (1967a) proposed to approximate the CES function with a Taylor-series expansion. Since then, this procedure has been widely used (e.g., Griliches (1967), Zarembka (1970), Griliches and Ringstad (1971)).

G. S. Maddala and J. B. Kadane (1967) have shown, using Monte Carlo techniques, that for samples built using a CES production function, Kmenta's procedure does not give reliable estimates of the elasticity of substitution, although it gives reliable estimates of the returns to scale parameter. Further, in the Kmenta approximation to the CES, only the scale parameter is free of units of measurement in the output and factor inputs.

Further, in a direct non-linear estimation, only scale and substitution parameters are free of units of measurement in the output and factor inputs.

Griliches (1967) and Griliches and Ringstad (1971) have also used Kmenta's approximation, not to estimate the CES production function, but to test for departures from the Cobb–Douglas function. The power of such a test depends on the particular alternative hypothesis being used; in the strict sense, Griliches is testing the null hypothesis that the production function is Cobb–Douglas against

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[1] Direct use of non-linear estimation procedures have led to problems such as: slow convergence, obtaining of a local maximum but without information about the presence of other maxima, important cancellation errors in the computation of derivates, use of substantial amounts of computer time, etc. On this see S. M. Goldfield and R. E. Quandt (1972, 26–27).

the alternative hypothesis that the *production function* itself is of the Kmenta form. However, this type of hypothesis is not of common interest. Usually, we wish to choose specifically between a Cobb–Douglas and a CES production function, and this objective is not accomplished by the Griliches procedure.

More generally, the purpose of this paper is to show that when we wish to use the data to test the hypothesis that the production function is a CES by using Kmenta's approximation (as a matter of fact, only the scale parameter is free of the units of measurements), then the problem becomes more fundamental. Another well-known production function—the variable elasticity of substitution (VES), of which the CES is a special case, first used by G. H. Hildebrand and T. C. Liu (1965) and developed by M. Bruno (see also Y. Lu and L. B. Fletcher (1968), R. Sato and R. Hoffman (1968), Lovell (1973))—has the same form as Kmenta's approximation of the CES function when second-order approximation of it is developed.

Further, for a person willing to test the null hypothesis that the production function is CES using Kmenta's approximation, the crucial point has been summarised by Kmenta (1967b, p. 193): "An inevitable implication of using a function f_1 as an approximation to another function f_2 is that f_1 is also an approximation to functions other than f_2. This is obvious and hardly relevant; what is relevant is how well f_1 approximates f_2 within some range of practical importance." But Kmenta's approximation to the VES also meets the above requirement. It is shown here that almost always, Kmenta's approximation is a better approximation to a VES than to a CES production function.

Therefore, Kmenta's approximation should not be used to make inferences with respect to parameters of a CES function, without strong independent evidence that the "true" production model is indeed a CES. Although in most studies the data is used to identify the type of production function, in this case Kmenta's approximation cannot be used for this purpose. As a matter of fact, it cannot be used to make inferences with respect to parameters of a VES function either, because in that case all the parameters are under-identified.

The organization of the rest of this paper is as follows. In Section 1, the second-order approximations to the CES and VES production functions are examined. Next, in Section 2 the "goodness" of the approximation is studied. In the Appendix a derivation of the error behaviour in the approximation is presented.

1. The CES and VES Functions and Their Second-Order Approximations

The CES production function allowing for non-constant returns to scale is given by:

$$(1)\quad V=\gamma[\delta K^{-\rho}+(1-\delta)L^{-\rho}]^{-\nu/\rho}\quad \text{with } 0\leq\rho\leq 1,\quad \rho>-1,\quad \gamma>0,\quad \nu>0$$

where:

V = Output
L = Input of labor services
K = Input of capital services

Kmenta approximated it with a Taylor series expansion of the first- and second-order terms around $\rho=0$ to obtain:

$$(1')\quad \ln V = \ln \gamma + \nu\delta \ln K + \nu(1-\delta)\ln L - \tfrac{1}{2}\nu\rho\delta(1-\delta)(\ln K - \ln L)^2$$

Nerlove (1967) presents a VES function with constant returns to scale which he attributes to Bruno. The same type of function has been presented recently also by Lu and Fletcher (1968). The Nerlove nomenclature is followed here.

The Bruno production function allowing for non-constant returns to scale can be written as:

$$(2)\qquad V = \gamma[\delta K^{-\rho} + (1-\delta)K^{-m\rho}L^{\rho(m-1)}]^{-\nu/\rho}, \qquad \text{with } \nu>0$$

$$V = K^{\nu}\gamma[\delta + (1-\delta)k^{\rho(1-m)}]^{-\nu/\rho}, \qquad \text{where } k = K/L > 0.$$

To have a real valued function with positive output the following restrictions are imposed:

$$\gamma>0 \text{ and } \delta+(1-\delta)k^{\rho(1-m)}>0.$$

For any positive ν, a positive marginal product of labor requires:

$$\frac{(1-m)(1-\delta)k^{\rho(1-m)}}{\delta+(1-\delta)k^{\rho(1-m)}}>0,$$

and a positive marginal product of capital requires:[2]

$$\frac{\delta+(1-\delta)mk^{\rho(1-m)}}{\delta+(1-\delta)k^{\rho(1-m)}}>0$$

Strict quasi-concavity of the production function requires:

$$\rho\delta(1-m)+\delta+m(1-\delta)k^{\rho(1-m)}>0$$

(This condition and the positive marginal product conditions imply that the elasticity of substitution is greater than zero.)

Function (2) is homogeneous of degree ν and has a variable elasticity of substitution given by:

$$\sigma = \frac{1}{1+\rho-\dfrac{m\rho\nu}{\alpha_K}},$$

where α_K is the partial elasticity of output with respect to capital.

From the above constraints the following inequalities can be derived:

(i) $\nu>0, \quad \gamma>0, \quad k>0$

(ii) $\delta+(1-\delta)k^{\rho(1-m)}>0$

(iii) $(1-m)(1-\delta)>0$

(iv) $\delta+m(1-\delta)k^{\rho(1-m)}>0$

(v) $\rho\delta(1-m)+\delta+m(1-\delta)k^{\rho(1-m)}>0$

[2] Within a range of K and L this function has diminishing marginal returns to each factor. This range depends on ν, δ, ρ, and m.

Let us impose the additional restrictions:[3]

(vi) $0<\delta<1$

(vii) $1+\rho>0$

Inequalities (iii) and (vi) imply $1-m>0$. If a Taylor-series expansion of (2) is taken around $\rho=0$, and if only the first- and second-order terms are considered, the following is obtained:

$$\text{(3)}\quad \ln V = \ln \gamma + \nu[\delta + m(1-\delta)] \ln K - \nu(m-1)(1-\delta) \ln L - \frac{\nu\rho}{2}(m-1)^2\delta(1-\delta)[\ln K - \ln L]^2$$

This equation is under-identified, its estimation is not of interest. Rather, the important point is that (3) is of the same form as (1′), and therefore (1′) cannot be used to estimate the coefficients of a CES function, without further *a priori* information that the CES is indeed the true model.

In general the error in approximating the VES function by (3) is given by:

$$\text{(4)}\quad \ln V_{\text{appr}} - \ln V_{\text{exact}} = -\nu(1-m)(1-\delta)\ln k - \frac{\nu\rho}{2}(1-m)^2\delta(1-\delta)[\ln k]^2 + \frac{\nu}{\rho}\ln[\delta + (1-\delta)k^{\rho(1-m)}]$$

2. Measuring the "Goodness" of the Approximation

To study how well (3) approximates (2), numerical experiments were performed for different values of the parameters. For the first case, let us employ the same parameter values as Kmenta ($\nu=0.9$ and $\delta=4/9$), so that the results will be comparable. However, there is an additional parameter, m, for which values are needed. It is already known (Section 1) that $m<1$. In order to obtain a more restricted range of values for this parameter, the Hildebrand and Liu estimates (presented by Nerlove, (1967)) can be used; these estimates are presented in Table 1. These estimates must be used cautiously because they were derived for the constant-returns-to-scale case. In any event, only those cases within the neighbourhood of constant returns are of interest.

Table 1 shows that in 13 of 17 cases m is a number less than one in absolute value and, in 10 of the 13, m lies between zero and one. Thus, in the experiments the following values were used for m: −1.00, −0.80, −0.60, −0.40, −0.20, 0, 0.20, 0.40, 0.60, 0.80, 1.00.

When $m=0$, (2) reduces to (1) so that the results are equal to those obtained by Kmenta. When $m=1$, (2) reduces to a Leontief production function, and therefore the approximation in (3) becomes an exact one. The ratio of V_{appr} to V_{exact} was calculated for the same range of values of ρ and k used by Kmenta. The numerical experiments indicate that for the most common empirical case of $0<m<1$ (10 out of 17 industries in the Hildebrand and Liu estimates), (3) is

[3] These constraints are consistent with the assumption that the associated CES production function also should be a positive real valued function with positive marginal product of the factors and be strictly quasi-concave (i.e. by substituting $m=0$ in (iii), (iv) and (v)).

TABLE 1

VALUE OF THE PARAMETER m IN DIFFERENT INDUSTRIES

Industry	m
Food and kindred products	0.752
Textile mill products	6.400
Apparel and related products	− 1.366
Lumber and wood products	0.200
Furniture and fixtures	0.597
Pulp, paper, and products	0.539
Chemicals and products	1.763
Petroleum and coal products	0.344
Rubber products	− 0.065
Leather and leather goods	− 0.455
Stone, clay, and glass products	0.640
Primary metal products	0.451
Fabricated metal products	0.297
Machinery except electrical	− 0.327
Electrical machinery	0.397
Transportation equipment	−26.750
Instruments and related products	0.544

Source: Nerlove (1967, p. 78).

almost a better approximation of (2) than of (1) (case $m = 0$). Further, in over 98 percent of the cases considered, the approximation improves monotonically as m increases from zero to one.

Table 2 presents the value of V_{appr}/V_{exact} for the pair of values (4/9, 0.90) for the parameters δ and ν respectively and for several values of the labor-capital ratio and parameters ρ and m.

For experiments performed with the pairs of values (0.44, 1.10), (0.56, 0.90), (0.56, 1.10) for the parameters δ and ν respectively, the conclusions do not change.[4]

TABLE 2

VALUES OF V_{appr}/V_{exact}

	Labor-Capital Ratios					
Control values are: $m = -1.00$, $\delta = 0.44$, $\nu = 0.90$						
ρ	0.10	0.50	1.00	2.00	5.00	10.00
−1.00	2.1651	1.0055	1.0000	1.0242	1.4396	2.8184
−0.50	1.1312	0.9994	1.0000	1.0046	1.0821	1.2913
−0.10	0.9980	0.9999	1.0000	1.0001	1.0018	1.0059
0.10	0.9941	0.9999	1.0000	1.0001	1.0009	1.0020
0.20	0.9704	0.9994	1.0000	1.0003	1.0015	0.9996
0.50	0.7744	0.9954	1.0000	1.0006	0.9761	0.8840
1.00	0.3548	0.9763	1.0000	0.9945	0.7963	0.4619
10.00	0.0000	0.1952	1.0000	0.2198	0.0000	0.0000

[4] These results are available from the author upon request.

TABLE 2 (Continued)

	Labor-Capital Ratios					
ρ	0.10	0.50	1.00	2.00	5.00	10.00
Control values are: $m = -0.80$, $\delta = 0.44$, $\nu = 0.90$						
−1.00	1.7307	1.0029	1.0000	1.0169	1.3019	2.1565
−0.50	1.0818	0.9994	1.0000	1.0032	1.0570	1.1993
−0.10	0.9984	0.9999	1.0000	1.0001	1.0013	1.0042
0.10	0.9958	0.9999	1.0000	1.0001	1.0007	1.0016
0.20	0.9792	0.9996	1.0000	1.0002	1.0014	1.0009
0.50	0.8338	0.9968	1.0000	1.0006	0.9861	0.9244
1.00	0.4637	0.9834	1.0000	0.9971	0.8574	0.5778
10.00	0.0000	0.2771	1.0000	0.3077	0.0004	0.0000
Control values are: $m = -0.60$, $\delta = 0.44$, $\nu = 0.90$						
−1.00	1.4449	1.0013	1.0000	1.0113	1.2003	1.7255
−0.50	1.0474	0.9995	1.0000	1.0022	1.0380	1.1315
−0.10	0.9988	1.0000	1.0000	1.0001	1.0009	1.0028
0.10	0.9972	0.9999	1.0000	1.0000	1.0005	1.0012
0.20	0.9859	0.9997	1.0000	1.0002	1.0012	1.0015
0.50	0.8838	0.9978	1.0000	1.0005	0.9929	0.9547
1.00	0.5795	0.9888	1.0000	0.9987	0.9073	0.6921
10.00	0.0000	0.3769	1.0000	0.4127	0.0021	0.0000
Control values are: $m = -0.40$, $\delta = 0.44$, $\nu = 0.90$						
−1.00	1.2573	1.0003	1.0000	1.0072	1.1266	1.4419
−0.50	1.0247	0.9996	1.0000	1.0014	1.0241	1.0825
−0.10	0.9991	1.0000	1.0000	1.0000	1.0006	1.0018
0.10	0.9982	1.0000	1.0000	1.0000	1.0004	1.0009
0.20	0.9910	0.9998	1.0000	1.0001	1.0009	1.0015
0.50	0.9237	0.9986	1.0000	1.0004	0.9972	0.9759
1.00	0.6935	0.9928	1.0000	0.9997	0.9453	0.7954
10.00	0.0000	0.4911	1.0000	0.5304	0.0102	0.0000
Control values are: $m = -0.20$, $\delta = 0.44$, $\nu = 0.90$						
−1.00	1.1369	0.9998	1.0000	1.0043	1.0748	1.2558
−0.50	1.0108	0.9997	1.0000	1.0008	1.0142	1.0485
−0.10	0.9994	1.0000	1.0000	1.0000	1.0004	1.0011
0.10	0.9989	1.0000	1.0000	1.0000	1.0002	1.0006
0.20	0.9946	0.9999	1.0000	1.0001	1.0007	1.0013
0.50	0.9538	0.9992	1.0000	1.0003	0.9995	0.9893
1.00	0.7963	0.9957	1.0000	1.0002	0.9717	0.8796
10.00	0.0006	0.6133	1.0000	0.6533	0.0387	0.0008

TABLE 2 (Continued)

ρ	Labor-Capital Ratios 0.10	0.50	1.00	2.00	5.00	10.00
Control values are: $m = 0.00$, $\delta = 0.44$, $\nu = 0.90$						
−1.00	1.0636	0.9997	1.0000	1.0023	1.0403	1.1363
−0.50	1.0034	0.9998	1.0000	1.0005	1.0077	1.0259
−0.10	0.9996	1.0000	1.0000	1.0000	1.0002	1.0006
0.10	0.9994	1.0000	1.0000	1.0000	1.0001	1.0004
0.20	0.9970	0.9999	1.0000	1.0000	1.0004	1.0010
0.50	0.9747	0.9995	1.0000	1.0002	1.0004	0.9967
1.00	0.8800	0.9977	1.0000	1.0003	0.9880	0.9402
10.00	0.0066	0.7339	1.0000	0.7709	0.1169	0.0081
Control values are: $m = 0.20$, $\delta = 0.44$, $\nu = 0.90$						
−1.00	1.0234	0.9997	1.0000	1.0011	1.0188	1.0637
−0.50	1.0002	0.9999	1.0000	1.0002	1.0036	1.0121
−0.10	0.9998	1.0000	1.0000	1.0000	1.0001	1.0003
0.10	0.9997	1.0000	1.0000	1.0000	1.0001	1.0002
0.20	0.9986	1.0000	1.0000	1.0000	1.0003	1.0006
0.50	0.9880	0.9998	1.0000	1.0001	1.0006	0.9998
1.00	0.9401	0.9989	1.0000	1.0003	0.9965	0.9771
10.00	0.0457	0.8416	1.0000	0.8720	0.2805	0.0538
Control values are: $m = 0.40$, $\delta = 0.44$, $\nu = 0.90$						
−1.00	1.0054	0.9998	1.0000	1.0004	1.0071	1.0239
−0.50	0.9994	0.9999	1.0000	1.0001	1.0014	1.0046
−0.10	0.9999	1.0000	1.0000	1.0000	1.0000	1.0001
0.10	0.9999	1.0000	1.0000	1.0000	1.0000	1.0001
0.20	0.9994	1.0000	1.0000	1.0000	1.0001	1.0003
0.50	0.9954	0.9999	1.0000	1.0001	1.0004	1.0006
1.00	0.9766	0.9996	1.0000	1.0002	0.9997	0.9946
10.00	0.1977	0.9253	1.0000	0.9460	0.5346	0.2225

APPENDIX

The Behavior of the Error of Approximation

Let us write:

$$E(m) = \ln V_{\text{appr}} - \ln V_{\text{exact}} = \ln \frac{V_{\text{appr}}}{V_{\text{exact}}}$$

We are interested in studying the behavior of the absolute value of $H(m)$, where $H(m)$ is defined as:

$$H(m)=\frac{V_{\text{appr}}}{V_{\text{exact}}}-1$$

but we can rewrite $H(m)$ as:

$$H(m)=e^{E(m)}-1$$

Now $H(m)$ is a strictly increasing monotonic function of $E(m)$. Further:

$$E(m)>0 \Leftrightarrow H(m)>0$$
$$E(m)<0 \Leftrightarrow H(m)<0$$

This suggests that instead of working with $|H(m)|$ we could work with $|E(m)|$. But

$$|E(m)|=\sqrt{(E(m))^2}$$

Therefore

$$\frac{\partial|E(m)|}{\partial m}=\frac{1}{\sqrt{(E(m))^2}}E(m)\frac{\partial E(m)}{\partial m}$$

So, sign of $\left\{\frac{\partial|E(m)|}{\partial m}\right\}=\text{sign of}\left\{E(m)\frac{\partial E(m)}{\partial m}\right\}$

We have:

$$E(m)=\nu(1-m)(1-\delta)\ln k-\frac{\nu\rho}{2}(1-m)^2\delta(1-\delta)[\ln k]^2 + \nu/\rho\ln[\delta+(1-\delta)k^{\rho(1-m)}]$$

From here we obtain:

$$\frac{\partial E(m)}{\partial m}=\frac{\nu\delta(1-\delta)}{\delta+(1-\delta)k}\delta(1-m)\{(1-k^{\rho(1-m)})\ln k+\rho(1-m) \times(\delta+(1-\delta)k^{\rho(1-m)})(\ln k)^2\}$$

Therefore:

$$\begin{aligned}E(m)\frac{\partial E(m)}{\partial m}=\frac{\nu\delta(1-\delta)}{\delta+(1-\delta)k}\rho(1-m)\Big\{&(1-k^{\rho(1-m)})\frac{1}{\rho}\ln[\delta+(1-\delta)k^{\rho(1-m)}]\ln k\\
&+(\ln k)^2[-(1-m)(1-\delta)+(1-m)(1-\delta)k^{\rho(1-m)}\\
&+(1-m)[\delta+(1-\delta)k^{\rho(1-m)}]\ln(\delta+(1-\delta)k^{\rho(1-m)})]\\
&+(\ln k)^3\Big[-\frac{\rho}{2}\delta(1-\delta)(1-m)^2+\frac{\rho}{2}\delta(1-\delta)(1-m)^2k^{\rho(1-m)}\\
&-\rho(1-\delta)(1-m)^2(\delta+(1-\delta)k^{\rho(1-m)})\Big]\\
&+(\ln k)^4\Big[-\frac{\rho^2}{2}\delta(1-\delta)(1-m)^3(\delta+(1-\delta)k^{\rho(1-m)})\Big]\Big\}\end{aligned}$$

This expression can be used to obtain the ranges of δ, ρ, m and k for which $E(m)\frac{\partial E(m)}{\partial m}<0$; i.e., where the error $E(m)$ decreases in absolute value as m increases, particularly when we move away from the case $m=0$ (CES) to the region $m>0$ (the most common type of VES production function obtained in empirical studies).

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