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A MODEL OF A PROJECT ACTIVITY*

BY EDWARD A. STOHR

This paper presents a simple model of a project activity in which the objective is to complete a given task at minimum cost. The problem is formulated as a decision problem with an uncertain number of stages. The optimal solution is found for the time-invariant case and the implications for the design of activity control systems are discussed.

1. INTRODUCTION

This paper discusses a simple model of a project activity. It will be assumed that a given task must be completed at minimum cost. The time taken to complete the task is not specified beforehand and will not be known exactly until after the task has been finished. The expected duration of the activities may not be long enough to allow them to be described by stochastic processes which have achieved a steady state. Thus the activities have a "project" rather than a "process" orientation. When the task has been completed, the organization which performed the work either disbands or goes on to perform another task. In the mathematical model the system will have to move from an initial state to a final state; attainment of the latter will represent completion of the task. The problem will be stated as a single-person multi-stage decision problem under uncertainty. It will be assumed that the state, x_t, of the system at time t is a scalar variable representing the amount of work remaining to be completed. The decision at time t, $a_t \in R^m_+$, specifies the levels of m different resources which are to be used at time t.

Examples of economic activities which might be modelled in this way are: (i) (simple) construction projects in which the total amount of work involved can be aggregated and represented by a scalar quantity, (ii) a single activity from a PERT or CPM network, or (iii) a single production run from a job shop (see [8] for further details). The objective of the paper is to study the design of management control systems for this type of activity.

The project activity model described here has several unusual features. In the first place, the objective is to design a control system for a single activity rather than for a network of activities as in PERT or CPM. In network models the problem of controlling individual activities is not explicitly considered and each activity is described either in terms of a given probability distribution of finishing times as in PERT or by a given cost-time trade-off curve as in "CPM cost" [9]. One possible use of the type of model developed in this paper would be to provide a rational method of developing data concerning the characteristics of individual activities for inclusion in these network models. In addition, the

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model in this paper involves an uncertain time horizon, while the management science literature concerning production activities has usually assumed either finite time horizons or infinite time horizons. A typical example is the aggregate production planning and smoothing model of Holt *et al.* [4].

In Section 2, the activity control system design problem is described. Section 3 analyzes a version of the activity model in which it is assumed that the actions can be adjusted continuously over time, while Section 4 is concerned with the case where the decision stages are discrete. The optimal solution for the continuous case has a very simple and convenient form. The results for the discrete case approximate those for the continuous case for activities of long expected duration. Section 5 states solutions of the activity control problem for some commonly used cost and production functions. Section 6 uses the results of the previous sections to discuss the general problem of designing control systems for activities of random duration.

2. The Activity Control System Problem

Both continuous and discrete-time versions of the activity model will be discussed in this paper. However, organizational decisions cannot usually be adjusted continuously over time, and it seems more appropriate to introduce the activity control system model initially as a problem with discrete decision stages.

Knowledge of the technology of the activity will be described by a sequence of cost functions, $c_t(a_t)$ and production functions, $f_t(a_t)$, $t = 0, 1, 2, \ldots$. In general, uncertainty will exist concerning these functions. For example, uncertainty about future factor prices will prevent exact specification of the function, c_t, and uncertainty with respect to such factors as the quality of the work force, quality of material inputs, and future weather conditions will prevent exact specification of the production function, f_t. These uncertainties are modelled by including additive random disturbance terms, γ_t and ξ_t, in the cost and production functions as shown in (1) below. The functions c_t and f_t are themselves assumed to be deterministic and continuous. Uncertainty will also exist with respect to the total quantity of work, x^0, involved in the task. In a construction context this uncertainty occurs for example, because estimates of the quantity of work involved are obtained from blueprints which may be based on only approximate data concerning actual topological and geological conditions. In a production setting, x^0 might represent the total orders outstanding for a product at the beginning of the production run. Uncertainty here might be due to inaccuracies or delays in the information system. Although in general, the states x_t cannot be observed exactly, an assumption of perfect observation will be made throughout this paper. It will be shown that this assumption is not of great importance in that the expected value of perfect information will usually be small for the problems analyzed.

The objective of the activity manager is to choose actions, $a_t \in A_t$, $t = 0, 1, 2, \ldots$, which will minimize the expected cost of the activity. The action possibility set, $A_t \subseteq R^m_+$, defines a constraint on the actions available at time t. It is assumed that $f_t(a_t) + \xi_t$, $a_t \in A_t$, is always non-negative, or in other words, that the amount of work remaining to be completed decreases monotonically over time. Information, concerning the current level of x_t, becomes available at time t and an action,

$a_t \in A_t$, is selected according to a decision rule, α_t. The decision rules can be functions of the history of prior observations, $x^t = (x_0, x_1, \ldots, x_t)$ and actions, $a^{t-1} = (a_0, a_1, \ldots, a_{t-1})$. Thus the period t action is given, in general, by $a_t = \alpha_t(x^t, a^{t-1})$. A policy, α, is a collection of decision rules, $(\alpha_0, \alpha_1, \alpha_2, \ldots)$. The state of the system is a random variable with a probability distribution which depends on the policy, α, chosen. Sometimes this dependence on α will be recognized explicitly by denoting the state at time t by x_t^α. The activity model can now be stated as follows. Find the policy, $\hat{\alpha}$, which solves

$$(1) \qquad V(\hat{\alpha}) = \min_{\alpha} E\left[\sum_{t=0}^{T-2} (c_t(a_t) + \gamma_t) + \frac{(c_{T-1}(a_{T-1}) + \gamma_{T-1})x_{T-1}}{f(a_{T-1}) + \xi_{T-1}}\right].$$

Subject to:

(a) Initial condition: $x_0 = x^0$

(b) Dynamics: $x_{t+1} = x_t - (f_t(a_t) + \xi_t), \qquad t = 0, 1, 2, \ldots$

(c) Final condition: $0 \leq x_{T-1} \leq f_{T-1}(a_{T-1}) + \xi_{T-1}$

(d) Admissible actions: $a_t = \alpha_t(x^t, a^{t-1}), \qquad t = 0, 1, 2, \ldots.$

In (1) the expectation is taken with respect to $x^0, \xi_0, \xi_1, \ldots, \gamma_0, \gamma_1, \ldots$. The time of the last decision, $T - 1$, is a random variable. It is assumed that the output, $f_t(a_t) + \xi_t$, and cost, $c_t(a_t) + \gamma_t$, occur uniformly over time. The random variable defined by the ratio, $x_{T-1}/[f_{T-1}(a_{T-1}) + \xi_{T-1}]$, in the objective function is therefore the fraction of the last time period in which work takes place and the term, $(c_{T-1}(a_{T-1}) + \gamma_{T-1})/[f_{T-1}(a_{T-1}) + \xi_{T-1}]x_{T-1}$, in (1) is the cost incurred in the last time period. In the following discussion the activity model (1) will be specialized to the time-invariant case where $c_t = c$, $f_t = f$, $A_t = A$, $t = 0, 1, 2, \ldots$ and $\{\gamma_t, t = 0, 1, 2, \ldots\}$ and $\{\xi_t, t = 0, 1, 2, \ldots\}$ are each assumed to be identically distributed sequences of random variables. It is also assumed that $x^0, \xi_0, \xi_1, \ldots, \gamma_0, \gamma_1, \ldots$ are independent.

As stated above, the possibility of imperfect information concerning the states of the system is not considered in this model. However it is worth noting that the general problem of activity control system design would modify (1) to allow for imperfect observation and would explicitly take into account the cost of generating information concerning the system states. The modified model would then be solved to find the expected cost of completing the activity for each available information system and finally, the optimal information system would be chosen (see [6]).

3. Continuous-Time Random Duration Model

The results for a continuous time version of problem (1) are much simpler and will therefore be presented first. Thus, in this section it will be assumed that the level of resources applied to the task can be adjusted continuously. In other words, the set of possible times at which a decision can be made is the positive real line $R_+^1 = [0, \infty]$. Let $x_t \in R_+^1$ be the amount of work left at time t. A continuous, time-invariant version of the dynamic equation, (1b), is $dx_t = -f(a_t)\,dt -$

$d\xi_t$, where $f: R^m \to R^1_+$ is the production function and ξ_t is a continuous martingale with constant mean, g. Let $\xi_t = g + u_t$, where u_t is a continuous martingale with a zero mean. The system equation becomes:

$$dx_t = -f(a_t)\,dt - g\,dt - du_t. \tag{2}$$

Similarly, a continuous, time-invariant version of the cost equation in (1)

$$dc_t = c(a_t)\,dt + d\gamma_t,$$

where $c: R^m \to R^1_+$ is the cost function and γ_t is a continuous martingale with constant mean h. Let $\gamma_t = h + w_t$, where w_t is a continuous martingale with a zero mean. The instantaneous cost is therefore:

$$dc_t = c(a_t)\,dt + h\,dt + dw_t. \tag{3}$$

The initial condition, x_0, will be a random variable with mean μ_0. The random variables, $x_0, u_t, w_t, t \geq 0$ will be assumed to be independent of one another. For $t \geq 0$, let $\mathcal{F}_t$ be the σ-algebra generated by $\{x_0, u_s, w_s, s \leq t\}$. Let $(\Omega_t, \mathcal{F}_t, p_t)$ be the probability space of the dynamic process defined by (2) at time t. Let $A \subseteq R^m_+$ be a compact set of feasible actions. The admissible decision functions $\alpha_t: \Omega_t \to A$ at time t will be measurable with respect to $\mathcal{F}_t$. The objective of the system will be to finish the task (drive x_t to zero) at minimum cost. Let $T = \inf\{t | x_t = 0\}$. Then the objective is to find the admissible policy, α^*, which solves

$$\begin{aligned} V(\alpha^*) &= \min_{\alpha}\left\{E\left[\int_0^T (c(a_t) + h)\,\mathrm{d}t + \int_0^T dw_t\right]\right\} \\ &= \min_{\alpha}\left\{E\left[\int_0^T (c(a_t) + h)\,dt\right]\right\} \end{aligned} \tag{4}$$

where the second equality follows since w_t is continuous with zero mean.

Now consider a constant policy $\alpha_t \equiv a, t \geq 0$ and let x_0 be given. Then,

$$x_T = 0 = x_0 - \int_0^T (f(a) + g)\,dt - u_T. \tag{5}$$

From the independence assumption, and since $u_t, t \geq 0$ is a zero-mean martingale, $E[u_T | x_0] = E[u_T] = 0$. So, from (4), $E[T|x_0] = x_0/(f(a) + g)$. From (4) the cost of the constant policy is therefore

$$\begin{aligned} V(\alpha, x_0) &= (c(a) + h)E[T|x_0] \\ &= \frac{(c(a) + h)}{f(a) + g} x_0. \end{aligned}$$

Let a^* be a solution of $\min_{a \in A}\{c(a) + h/f(a) + g\}$; $\alpha_t^* \equiv a^*, t \geq 0$ is an optimal constant strategy. Let $c^* = (c(a^*) + h)/(f(a^*) + g)$ and define a function $v: R^1 \to R^1$ by

$$v(x) = \frac{(c(a^*) + h)}{(f(a^*) + g)} x = c^* x, \qquad x \geq 0. \tag{6}$$

Theorem 1:[1]

Given x_0, the optimal strategy for the continuous-time problem is the constant policy $\alpha_t^* \equiv a^*, t \geq 0$.

Proof. Let β be any other admissible strategy, b_t the action taken at time t, T_β the random-activity completion time using policy β, and x_t^β, $t \geq 0$, the corresponding trajectory.

By the Itô differential rule, [10],

$$\begin{aligned} dv(x_t^\beta) &= v_x\, dx^\beta + \tfrac{1}{2} v_{xx}\, dR_t \\ &= c^*[-f(b_t) - g]\, dt - c^*\, du_t, \end{aligned}$$

where dR_t is the incremental covariance of ξ_t and the second equality follows from (2) and (6), since $v_{xx} = 0$. Taking the stochastic integral of the last equation and then taking expectations gives

$$-v(x_0) + E[v(x_{T_\beta}^\beta)] = -E\left[c^* \int_0^{T_\beta} (f(b_t) + g)\, dt\right], \tag{7}$$

where, again, use has been made of the fact that u_t is a zero-mean martingale so that $E[\int_0^{T_\beta} u_t\, dt] = 0$. Now, $v(x_{T_\beta}^\beta) = 0$ by definition of T_β since $x_{T\beta} = 0$ a.e. and by definition of c^*, $c(b_t) + h \geq c^*(f(b_t) + g), t \geq 0$. So, using these facts in (6):

$$\begin{aligned} v(x_0) &= c^* x_0 \\ &\leq E\left[\int_0^{T_\beta} (c(b_t) + h)\, dt\right] = V(\beta, x_0). \end{aligned} \tag{8}$$

On the other hand, if $b_t \equiv a^*$, $t \geq 0$, equality is obtained in (7), since $v(x_0) = V(\alpha^*, x_0)$.

Corollary. The optimal policy for the continuous-time problem, defined by (2) to (4) is the constant action: $\alpha_t^* \equiv a^*$, $t \geq 0$, where a^* is a solution of $\min_{a \in A} (c(a) + h)/(f(a) + g)$.

The minimum expected cost is

$$V^*(\alpha^*) = \frac{(c(a^*) + h)}{f(a^*) + g} \mu_0 = c^* \mu_0.$$

The expected completion time using the optimal policy is given by

$$E[T_{\alpha^*}] = \frac{\mu_0}{f(a^*) + g}.$$

Proof. The proof of the corollary follows immediately from the theorem after taking expectations with respect to x_0.

The optimal solution of the continuous-time problem depends on the distributions of ξ_t and γ_t only through the means h and g. Hence, theorem 1 is an example of a "certainty equivalent" result. Furthermore, the expected cost due to the uncertainty in x^0 and $\xi_t, t \geq 0$ is zero. Since the optimal policy is a constant independent of $x_t, t \geq 0$ there is no advantage to be gained from making observations of the system state.

[1] I am indebted to Professor P. F. Varaiya for his help with the proof of this theorem.

4. Free End Time Problems with Discrete Decision Stages

In this Section the analysis will be concerned with the discrete stages activity model (1). In the discrete case the random elements in the problem complicate the analysis and the simple solution obtained in the previous section no longer holds. To illustrate the effect of the random elements consider first the deterministic time-invariant free-end time problem (9) which is obtained from (1) by omitting the random disturbance terms:

$$V(\hat{\alpha}) = \min_{\alpha}\left\{\sum_{t=0}^{T-2} c(a_t) + \frac{c(a_{T-1})}{f(a_{T-1})}x_{T-1}^{\alpha}\right\} \tag{9}$$

subject to:

(a) Initial condition: $x_0^{\alpha} = x^0 > 0$

(b) Dynamics: $x_{t+1}^{\alpha} = x_t^{\alpha} - f(a_t), \qquad t = 0, 1, 2, \ldots$

(c) Final condition: $0 \le x_{T-1}^{\alpha} \le f(a_{T-1})$

(d) Admissible actions: $a_t = \alpha_t(x_t^{\alpha}) \in A, \qquad t = 0, 1, 2, \ldots.$

Note that the time, $T - 1$, of the last decision is determined implicitly by the chosen policy and the constraint (9c). Define the fraction of the last time period in which work takes place under policy α by

$$m(\alpha) = \frac{x_{T(\alpha)-1}^{\alpha}}{f(\alpha_{T(\alpha)-1}(x_{T(\alpha)-1}^{\alpha}))} \tag{10}$$

where the dependence of T and m on α has been made explicit. Let α be the constant policy, $\alpha_t \equiv a$, $t = 0, 1, \ldots$. From (9c) and (10) and the assumption that work is completed at a uniform pace during each time period: $T(\alpha) - 1 + m(\alpha) = x_0/f(a)$. Hence:

$$V(\alpha) = \left(\sum_{t=0}^{T(\alpha)-2} c(a)\right) + m(\alpha)c(a)$$

$$= (T(\alpha) - 1 + m(\alpha))c(a) = \frac{c(a)}{f(a)}x_0.$$

Let $\hat{a} \in R^m$ be a solution to $\hat{c} = c(\hat{a})/f(\hat{a}) = \min_{a\in A} c(a)/f(a)$. The cost of the optimal constant policy, $\hat{\alpha}_t \equiv \hat{a}$; $t \ge 0$, is given by $V(\hat{a}) = \hat{c}x_0$. Let β be any other admissible policy, b_t the action taken at time t, and $T(\beta) - 1 + m(\beta)$ the activity duration. The cost of policy, β, is

$$V(\beta) = \sum_{t=0}^{T(\beta)-2} c(b_t) + m(\beta)c(b_{T(\beta)-1}).$$

Now, $c(b_t) \ge c(\hat{a})/f(\hat{a})f(b_t)$, $t \ge 0$, so

$$V(\beta) \ge \hat{c}\left(\sum_{t=0}^{T(\beta)-2} f(b_t) + m(\beta)f(b_{T(\beta)-1})\right)$$

$$= \hat{c}x_0 = V(\hat{\alpha}).$$

Hence $\hat{\alpha}$ is the optimal policy, the minimum cost of completing the activity is $V(\hat{\alpha}) = \hat{c}x^0$ and the optimal completion is $\hat{T} = x^0/f(\hat{a})$. These results are similar to those which were obtained in Section 2 for the stochastic continuous-time problem.

The remainder of this section is concerned with the "time invariant" version of problem (1) in which all of the parameters of the problem remain constant over time. The analysis of the stochastic discrete stages problem is complicated by the fact that the time, $T - 1$, at which the last decision is made is a random variable with a probability distribution which depends on the chosen policy. To begin the analysis note that the final condition, (1c), is equivalent to the definition of the last decision stage:

$$T = \min\left\{ s \geq 1 \,\middle|\, \sum_{t=0}^{s-1} (f(a_t) + \xi_t) \geq x^0 \right\}.$$

Now a_t is a function of $x^0, \xi_0, \xi_1, \ldots, \xi_{t-1}$ and the event $\{T \leq i\}$ is equivalent to the event $\{\sum_{t=0}^{i-1} (f(a_t) + \xi_t) \geq x^0\}$. Hence T is a stopping time for the dynamic process defined by (1b). It follows from the Wald identity [7, p. 38] that:

$$E\left[\sum_{t=0}^{T-1} \xi_t\right] = E[T] \cdot E[\xi_0]. \tag{11}$$

Before proceeding with a more general analysis it will be useful to consider the class of policies involving a constant action in each time-period. Let α be any such constant policy, $\alpha_t \equiv a, t = 0, 1, 2, \ldots$. By definition:

$$\sum_{t=0}^{T-1} (f(a) + \xi_t) - (f(a) + \xi_{T-1}) + x_{T-1} = x^0.$$

Taking expectations and using (11):

$$E[T](f(a) + E[\xi_0]) - f(a) - E[\xi_{T-1}] + E[x_{T-1}] = E[x^0],$$

or

$$E[T - 1](f(a) + E[\xi_0]) + f(a) + E[\xi_0] - f(a) - E[\xi_{T-1}] + E[x_{T-1}] = E(x^0].$$

Rearranging, and defining $\mu_0 = E[x^0]$:

$$E[T - 1] = \frac{\mu_0 - E[x_{T-1}] - E[\xi_0] + E[\xi_{T-1}]}{f(a) + E[\xi_0]}. \tag{12}$$

From (1) and the time invariance assumption, the expected cost of the constant policy, α, is given by

$$V(\alpha) = E\left[\sum_{t=0}^{T-2} (c(a) + \gamma_t) + \frac{(c(a) + \gamma_{T-1})x_{T-1}}{f(a) + \xi_{T-1}}\right].$$

Now, $T - 1$ is determined by x^0 and $\xi_0, \xi_1, \ldots, \xi_{T-1}$ and by assumption, $\gamma_0, \gamma_1, \gamma_2, \ldots$ are independent of $x^0, \xi_0, \xi_1, \ldots$. Hence it follows that $T - 2$ is independent of $\gamma_0, \gamma_1, \ldots, \gamma_{T-2}$, that γ_{T-1} is independent of ξ_{T-1} and x_{T-1}, and finally, that $E[\gamma_{T-1}] = E[\gamma_0]$. Therefore from (12):

$$V(\alpha) = E[T-1](c(a) + E[\gamma_0]) + (c(a) + E[\gamma_0])E\left[\frac{x_{T-1}}{f(a) + \xi_{T-1}}\right] \tag{13}$$

$$= \mu_0 \frac{c(a) + E[\gamma_0]}{f(a) + E[\xi_0]}$$

$$+ (c(a) + E[\gamma_0])\left\{E\left[\frac{x_{T-1}}{f(a) + \xi_{T-1}}\right] + \frac{E[\xi_{T-1}]}{f(a) + E[\xi_0]}\right.$$

$$\left. - \frac{E[x_{T-1}]}{f(a) + E[\xi_0]} - \frac{E[\xi_0]}{f(a) + E[\xi_0]}\right\}.$$

Let a^* be the solution to

$$c^* = \frac{c(a^*) + E[\gamma_0]}{f(a^*) + E[\xi_0]} = \min_{a \in A}\left\{\frac{c(a) + E[\gamma_0]}{f(a) + E[\xi_0]}\right\} \tag{14}$$

and $\alpha_t^* \equiv a^*, t = 0, 1, \ldots$. The policy α^* is the "certainty equivalent" policy obtained from the optimal deterministic policy by replacing the random disturbance terms by their expectations. Let α^s be the optimal constant policy, $\alpha_t^s \equiv a^s$, $t \geq 0$. This policy must minimize the value of $V(\alpha)$ given by (13). Because of the last term in (13), a^s depends on the distributions of $x^0, \xi_0, \xi_1, \ldots$ and not just on their mean values.

We turn now to a consideration of the class of all admissible policies and attempt to follow the approach adopted earlier for the deterministic problem. Let β be any admissible policy and $b_t \in R^m$ the action actually taken at time t. Since the action can be any function of the past history of the process, b_t is a random vector. Let $S - 1$ be the random variable denoting the last time at which a decision is made. The duration of the activity under this policy is the random variable, $S - 1 + x_{S-1}/(f(b_{S-1}) + \xi_{S-1})$. The expected cost is given by:

$$V(\beta) = E\left[\sum_{t=0}^{S-2} (c(b_t) + \gamma_t) + x_{S-1}\frac{(c(b_{S-1}) + \gamma_{S-1})}{(f(b_{S-1}) + \xi_{S-1})}\right]$$

$$= E\left[\sum_{t=0}^{S-2} (c(b_t) + E[\gamma_0]) + x_{S-1}\frac{(c(b_{S-1}) + E[\gamma_0])}{f(b_{S-1}) + \xi_{S-1}}\right]$$

where the second line follows since $S - 2$ depends only on $x^0, \xi_0, \xi_1, \ldots \xi_{S-1}$, γ_{S-1} is independent of x_{S-1}, and $E[\gamma_{S-1}] = E[\gamma_0]$. Now for $t = 0, 1, \ldots, c(b_t) + E[\gamma_0] \geq c^*(f(b_t) + E[\xi_0])$, so:

$$V(\beta) \geq c^*E\left[\sum_{t=0}^{S-2} (f(b_t) + E[\xi_0]) + \frac{x_{S-1}(f(b_{S-1}) + E[\xi_0])}{f(b_{S-1}) + \xi_{S-1}}\right] \tag{15}$$

$$= c^*K.$$

where K represents the term under the expectation. By definition,

$$x^0 = \sum_{t=0}^{S-2} (f(b_t) + \xi_t) + x_{S-1}.$$

Taking expectations and using (11) and (15):

$$K - \mu_0 = E[S-1]E[\xi_0] - E\left[\sum_{t=0}^{S-2} \xi_t\right] + E\left[x_{S-1}\frac{(f(b_{S-1}) + E[\xi_0])}{f(b_{S-1}) + \xi_{S-1}} - x_{S-1}\right]$$

$$= E[\xi_{S-1}] - E[\xi_0] + E\left[\frac{x_{S-1}(f(b_{S-1}) + E[\xi_0])}{f(b_{S-1}) + \xi_{S-1}} - x_{S-1}\right]$$

$$= E[\xi_{S-1}] - E[\xi_0] - E\left[\frac{x_{S-1}(\xi_{S-1} - E[\xi_0])}{f(b_{S-1}) + \xi_{S-1}}\right] \geq 0 \text{ with probability 1.}$$

The final inequality above follows since, from equation, (1b), and the definition of S, $0 \leq x_{S-1}/f(b_{S-1}) + \xi_{S-1} \leq 1$ with probability 1 and therefore

$$\left|E\left[\frac{x_{S-1}(\xi_{S-1} - E[\xi_0])}{f(b_{S-1}) + \xi_{S-1}}\right]\right| \leq |E[\xi_{S-1} - E[\xi_0]|$$

with probability 1. It follows that $V(\beta) > c^*\mu_0$ for any admissible policy β. Evidently, the certainty equivalent policy, α^*, the optimal constant policy, α^s, and the optimal admissible policy $\hat{\alpha}$, satisfy:

(16) $$V(\alpha^*) \geq V(\alpha^s) \geq V(\hat{\alpha}) \geq c^*\mu_0.$$

Temporarily, let x^0 be a known constant. For a constant action the dynamics of the time invariant random duration control problem with the stated independence assumptions define a renewal process (in the "amount of work completed" rather than in "time" as in the usual interpretation of renewal processes). In fact, the problem reduces to the usual definition of a "renewal reward process," [7], except for the terminating condition (1c) and the assumption that costs are incurred, and progress of work is achieved, uniformly over time. From (13):

$$V(\alpha^*) = E[T-1](c(a^*) + E[\gamma_0]) + (c(a^*) + E[\gamma_0])E\left[\frac{x_{T-1}}{f(a) + \xi_{T-1}}\right],$$

hence using (14):

(17) $$V(\alpha^*) - c^*x^0 = \left\{E[T-1] - \frac{x^0}{f(a^*) + E[\xi_0]} + E\frac{x_{T-1}}{f(a) + \xi_{T-1}}\right\} \times (c(a^*) + E[\gamma_0]).$$

Now $E[T-1]$ can be regarded as a function of x^0 (the "renewal function") and has the following property [2, p. 366]:

$$E[T-1] - \frac{x^0}{f(a^*) + E[\xi_0]} \to \frac{E[(f(a^*) + \xi_0)^2]}{2(f(a^*) + E[\xi_0])^2} - 1 \quad \text{as } x^0 \to \infty.$$

Using the "Key Renewal Theorem" [7, p. 42] it can be shown that

$$E\left[\frac{x_{T-1}}{f(a^*) + \xi_{T-1}}\right] \to \frac{1}{2},$$

as $x^0 \to \infty$. Hence substituting in (17):

$$(V(\alpha^*) - c^*x^0) \to L = \frac{\operatorname{var}[\xi_0](c(a^*) + E[\gamma_0])}{2(f(a) + E[\xi_0])^2} \quad \text{as } x^0 \to \infty. \tag{18}$$

From (16) and (18) it is clear that $0 < V(\alpha^*) - V(\hat{\alpha}) < L$ if μ_0 is suitably large. This gives some measure of the expected loss incurred by following the "certainty equivalent" policy, α^*, rather than the true optimal policy $\hat{\alpha}$.

It can be seen from (13) that a^* is the optimal constant action if the contribution of the final term in the objective of (1) is neglected. Furthermore, $a^s \to a^*$ as $\mu_0 \to \infty$ since the last terms in the expression for $V(\alpha)$ have finite limits. In order to compare α^* with the optimal policy, $\hat{\alpha}$, it will be necessary to introduce some more terminology. For simplicity it will be temporarily assumed that the random variables, $\xi_t, t \geq 0$, may have any non-negative value and that $A = \{a \in R^m | a \geq a_{\min} \geq 0\}$. Define $A(x) = \{a | a_{\min} \leq a; f(a) \geq x\}$ and let $A^c(x)$ be the complement of $A(x)$. $A(x_t)$ is the set of feasible actions which will guarantee completion of the task before time $t + 1$. From the assumption about A, $A(x)$ is non-empty for all $x \geq 0$. If $a \in A^c(x)$, then under the above assumptions the activity may or may not be completed before $t + 1$. Let $v(x_t)$ be the expected cost of completing the project given that the state is x_t at time t. Then:

$$v(x_t) = \min\{g_1(x_t), g_2(x_t)\} \tag{19}$$

where:

$$g_1(x_t) = \min_{a_t \in A^c(x_t)} \left\{ c(a_t) + E[\gamma_0] + E[v(x_t - f(a_t) - \xi_t)]F(x_t - f(a_t)) + x_t E\left[\frac{c(a_t) + \gamma_t}{f(a_t) + \xi_t}\right](1 - F(x_t - f(a_t)) \right\} \tag{20}$$

$$g_2(x_t) = x_t \min_{a_t \in A(x_t)} \left\{ E\left[\frac{c(a_t) + \gamma_t}{f(a_t) + \xi_t}\right]\right\} \tag{21}$$

and F is the probability distribution function for $\xi_0, \xi_1, \ldots$.

From the previous analysis the optimal action, a_t^*, for large values of x_t will approximate the action, a^*, which minimizes $(c(a) + E[\gamma_0])/(f(a) + E[\xi_0])$. However, if $x_t > 0$ is small enough and the decision is made to complete the activity during the next time period then, from (14) and the independence assumption, the optimal action $\hat{a}_t = a'$, where a' solves:

$$E\left[\frac{c(a') + E[\gamma_0]}{f(a') + \xi_0}\right] = \min_{a \in A} \left\{ E\left[\frac{c(a) + E[\gamma_0]}{f(a') + \xi_0}\right]\right\} \tag{22}$$

Evidently if $0 \leq x_t \leq f(a')$ then the optimal policy, $\hat{\alpha}_t(x_t) = a'$. Finally, the following proposition gives a relationship between a^* and a' which provides upper and lower bounds for the optimal actions, $\hat{\alpha}_t(x_t)$, $x_t \geq 0$, $t \geq 0$ under certain conditions.

Proposition. If $(c(a) + E[\gamma_0])/(f(a) + E[\xi_0])$ is convex in a for $a \in A$, and $f_i(a) \geq 0$, $1 \leq i \leq m$, then $a_i' \geq a_i^*$, $1 \leq i \leq m$, where a^* and a' are defined by [14] and [22] respectively.

Proof. For any random variable $x \geq 0$, $1/E[x] \leq E[1/x]$ and so:

$$\frac{1}{E[x]}E\left[\frac{1}{x}\right] \leq E\left[\frac{1}{x}\right]^2 = -\operatorname{var}\left[\frac{1}{x}\right] + E\left[\frac{1}{x^2}\right] \leq E\left[\frac{1}{x^2}\right].$$

Let $x = f(a^*) + \xi_0$, then:

$$\frac{1}{f(a^*) + E[\xi_0]}E\left[\frac{1}{f(a^*) + \xi_0}\right] - E\left[\frac{1}{(f(a^*) + \xi_0)^2}\right] \leq 0.$$

From the optimality condition for a^*:

$$\frac{1}{f(a^*) + E[\xi_0]} = \frac{c_i(a^*)}{f_i(a^*)(c(a^*) + E[\gamma_0])}, \qquad 1 \leq i \leq m.$$

Substituting in the previous inequality, and multiplying through by the positive quantity, $f_i(a^*)(c(a^*) + E[\gamma_0])$, it follows that:

$$c_i(a^*)E\left[\frac{1}{f(a^*) + \xi_0}\right] - E\left[\frac{f_i(a^*)(c(a^*) + E[\gamma_0])}{(f(a^*) + \xi_0)^2}\right]$$

$$= \frac{\partial}{\partial a_i}E\left[\frac{c(a) + E[\gamma_0]}{f(a) + \xi_0}\right]\Bigg|_{a=a^*} \leq 0, \qquad 1 \leq i \leq m.$$

Now by definition of a',

$$\frac{\partial}{\partial a_i}E\left[\frac{c(a) + E[\gamma_0]}{f(a) + \xi_0}\right]\Bigg|_{a=a'} = 0, \qquad 1 \leq i \leq m$$

and from the convexity assumption, $E[c(a) + E[\gamma_0]/f(a) + \xi_0]$ is also convex in a, which proves the proposition.

The formulation (19) to (21) provides insight both for the activity problem considered here and for renewal reward processes in general. For large x_t, $v(x_t) = g_1(x_t) \geq c^*x_t$. Also, since $F(x)$ is a distribution function, $\lim_{x\to\infty} xF(x - k) = 0$. Hence the first two terms in (20) will predominate for large values of x_t, so that:

$$g_1(x_t) = \min_{a_t \in A^c(x_t)} \{c(a_t) + E[\gamma_0] + c^*(x_t - f(a_t) - E[\xi_0])\}.$$

This functional equation is obviously solved by a^* as defined in (14). This confirms that the optimizing action for large x_t approximately minimizes the ratio of the expected cost to the expected output in each period (rather than the expectation of the ratio of the cost to output in each period—which seems, at first sight, to be an equally intuitive result).

The introduction of finite bounds on the possible values of the disturbances, ξ_t, and on the possible actions a_t, would complicate the dynamic programming formulation (19) to (21) without adding anything essential to the analysis. The results for the discrete stages time-invariant random duration control problem can be summarized as follows:

Theorem 2:

The constant policy, $\alpha_t^* \equiv a^*, t \geq 0$ defined by (14), the optimal constant policy, α^s, and the optimal policy, $\hat{\alpha}$, satisfy:

$$V(\alpha^*) \geq V(\alpha^s) \geq V(\hat{\alpha}) \geq c^*\mu_0.$$

If μ_0 is large enough, the opportunity cost involved in using a "certainty equivalent" policy, α^*, rather than the true optimal policy, $\hat{\alpha}$, satisfies:

$$0 < V(\alpha^*) - V(\hat{\alpha}) < \frac{\text{var}\,[\xi_0](c(a^*) + E[\gamma_0])}{2(f(a) + E[\xi_0])^2}.$$

As $x_t \to \infty$ the optimal action $\hat{a}_t \to a^*$. Also, if $0 \leq x_t \leq f(a')$ then $\hat{\alpha}_t(x_t) = a'$ where a' is defined by (22). Finally, under the conditions of the proposition, the optimal policy satisfies $a^* \leq \hat{\alpha}_t(x_t) \leq a'$, $x_t \geq 0$, $t = 0, 1, 2, \ldots.$

5. Optimal Solutions for Some Particular Technologies

The optimal (or nearly optimal) action, a^*, for the time-invariant free end time problems discussed in the previous sections is the solution to a problem of the form, $\min_{a \in A} (c(a) + d)/(f(a) + g)$, where d and g are constants representing the means of the additive disturbance terms in the cost and production functions. Since it has been assumed that c and f are continuous and that A is compact this problem always has a solution. Some simple examples are now stated, however the computational task involved in solving this problem is not always trivial.

Let the cost and production functions be given by:

$$c(a_t) + d = c_0 + c_1 a_t + c_2 a_t^2, \qquad t \geq 0$$

$$f(a_t) + g = e a_t, \qquad t \geq 0$$

where $c_0, c_1, c_2 \in R'$ and $c_0, c_2, e > 0$. Then $a^* = \sqrt{c_0/c_2}$ if $a_{\min} \leq \sqrt{c_0/c_2} \leq a_{\max}$ and the optimal solution does not depend on any parameters of the production equation. However, this is a very special case. If a non-zero constant term is present in the production function, an optimal solution to this problem is the solution to a quadratic equation involving parameters from both the cost and production functions. If $c_2 = 0$ in the preceding example the solution would be unbounded except for the constraint on the actions. The optimal action is then:

$$a^* = \begin{cases} a_{\min} & \text{if } c_1 e_0 \geq c_0 e_1 \\ a_{\max} & \text{if } c_1 e_0 \leq c_0 e_1. \end{cases}$$

As another example, let the cost function be linear and the production function be of the Cobb–Douglas type:

$$c(a_t) + d = c_0 + c_1' a_t, \qquad t \geq 0$$

$$f(a_t) + g = b_0 \prod_{i=1}^{m} a_i^{b_i}, \qquad t \geq 0$$

where $c_0, c_1 > 0$, $c_0 \in R^1$, $c_1 \in R^m$, $b_i > 0$, $b_i \in R^1$, $0 \leq i \leq m$ and $\Sigma b_i < 1$. Also let $A = \{a \in R^m | a \geq 0\}$. Then the optimal action is given by:

$$a_i^* = \frac{c_0 b_i}{(1 - \Sigma b_i)c_i}; \qquad 1 \leq i \leq m.$$

If $\Sigma b_i = 1$ the Cobb–Douglas production function gives constant returns to scale and if $\Sigma b_i > 1$ it gives increasing returns to scale. In both of these cases the solution would be unbounded if the action were not constrained.

For more general cases it will be necessary to use numerical approximation or specially devised algorithms in order to solve this problem. For the case of multidimensional linear cost and production functions for example, the algorithm given in [3] might be adopted.

6. The Design of Control Systems for Economic Activities of Random Duration

The significance of the results obtained for the time-invariant, free-end time activity models analyzed in the previous sections will now be discussed. It has been shown that a constant policy, a^*, is optimal for the continuous random duration and deterministic versions of this problem and that a constant policy is approximately optimal for the discrete random duration version. Intuitively, the action, a^*, minimizes the cost per unit output in each time period. This is also the action which minimizes the long-run average cost per unit of time in the corresponding infinite-horizon problem (see [7]).

The fact that the optimal action is approximately constant, independent of the state, x_t, of the system, is quite surprising. This means that the choice of the optimal action is never affected by chance events. In a construction context for example, the action taken after two weeks of heavy rain and poor production should be the same as the action taken after two weeks of fine weather and good production. Of course, this result depends on the time invariance assumption. It is no longer true, for example, when additional penalties are incurred, if the activity is not finished before some given target date. The solutions for free-end time and fixed duration activity models can be very different. For the fixed-duration problem, where the production functions are linear and the cost functions are quadratic, the optimal action is linear and expected cost is quadratic in the state of the system, (see [1]). In the time invariant free-end time problem, however, the optimal (or near optimal) action is a constant and the expected cost is a linear function of the state of the system.

Perhaps the most interesting property of this activity model is that information systems which report the state of the system (amount of work remaining) have little or no value. The additional freedom of a "free-end time" makes the optimal actions less dependent on the state of the system and more a function of the particular technology. The benefits to be derived from the traditional information system which produces periodic "progress reports" may not, therefore, be very significant. Again, this conclusion is dependent on the time invariance assumption. However, it does at least indicate the need for a careful economic analysis of the value and cost of this type of information in real problems.

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References

[1] Åstrom, Karl J., *Introduction to Stochastic Control Theory*, New York: Academic Press, 1970, Chapter 8.

[2] Feller, William, *An Introduction to Probability Theory and Its Applications*, New York: John Wiley and Sons, Inc., 1971, (Volume II).

[3] Geoffrion, A. M., "Solving Bicriterion Mathematical Programs", *Operations Research*, Volume 15, 1967, pp. 915–37.

[4] Holt, C., J. Muth, F. Modigliani and H. Simon, *Planning Production Inventories and Workforce*, New York: Prentice Hall, 1960.

[5] Kushner, Harold, *Introduction to Stochastic Control*, New York: Holt Rinehart and Winston, Inc., 1971.

[6] Marshak, Jacob and Roy Radner, *Economic Theory of Teams*, New Haven and London, Yale University Press, 1972, Chapter 2.

[7] Ross, Sheldon M., *Applied Probability Models with Optimization Applications*, San Francisco, Holden Day, Inc. 1971, Chapter 3.

[8] Stohr, Edward A., "The Design of Information Systems for Economic Activities", *Technical Report No. 22, Collaborative Research on Economic Systems and Organization*, Center for Research in Management Science, University of California, Berkeley, California.

[9] Weist, J. D. and F. K. Levy, *A Management Guide to PERT/CPM*, New York: Prentice Hall, 1969.

[10] Wong, E. E., *Stochastic Processes in Information and Dynamical Systems*, New York: McGraw Hill, 1971, Chapter 4.