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RISKY R & D WITH RIVALRY

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A firm's R & D planning problem is modelled and the optimal spending plan over the development period characterized. Both technical uncertainty, through lack of full information about requirements for success in the R & D, and market uncertainty, through unknown actions of potential rivals, are taken into account. The analysis may also be pertinent to a variety of problems involving resource allocation over time. Likewise, the methodology developed may be useful for modelling and solving other stochastic optimal control problems.

INTRODUCTION

A firm contemplating an R & D project faces uncertainties from within and from without. The effort required to complete the R & D, the magnitude of the invention obtained and its value are all uncertain at inception. If there are rivals seeking the same goal, the firm obtains the rewards if it is the innovator, but may be pre-empted and get less or nothing if there is prior claim by a rival. Finally there is uncertainty about the magnitude of potential demand for the innovation.

The R & D manager must find a strategy for pursuing development in the face of these uncertainties. We focus upon two particular sources of uncertainty. First, the magnitude of R & D effort required for successful project development is not known. Second, the firm does not know the research plans of other firms regarding projects closely related to its own. We shall refer to the first type of partial ignorance as technological uncertainty and to the second as market uncertainty, a distinction also employed by Hirshleifer [3], among others.

Optimal behavior under each of these two types of uncertainty has been discussed separately in earlier works. In [4] we derived the time pattern of optimal planned R & D expenditure under technological uncertainty, supposing no rivalry. In [5, 6] we considered the speed of development under the threat of rivalry, with no technological uncertainty. That is, the cost of successful completion by any given date was assumed known, with more rapid development more costly. Some of this work is reviewed by Gittens [2].

Our objective in this paper is to combine features of our previous efforts to obtain a characterization of a firm's R & D expenditure plan in the face of both technological and market uncertainty. The stochastic features of the underlying technology are modelled through an assumed probability function over the amount of cumulative R & D effort required for success. That is, while the total effort required to complete the research satisfactorily is not known, the probability of project completion by any date is a nondecreasing function of cumulated research effort to that date. Effort is accumulated through expenditure of money. Funds alloted the project are spent in the most efficient manner. We suppose there are decreasing returns to the compression of the development period.

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Upon the successful completion of the R & D, a reward stream is collected. The value of the stream may be stochastic, but it is assumed independent of the calendar date of completion. However, if a rival completes its R & D and introduces essentially the same product or process our firm is seeking to develop, our firm can obtain nothing. Thus development expenditures are made only until some firm succeeds at development, and rewards are collected by our firm only if it is the successful firm. Potential rivals are recognized through a single subjective probability distribution over the introduction date of any competing product or process. The assumed objective is maximum present expected value of the project.

In the next section we present the formal model of the firm engaged in R & D activity in the posited environment as an optimal control problem and derive necessary conditions for optimality. In the following section we characterize the behavior of the firm's optimal expenditure plan through time and find a necessary condition for the firm to undertake the R & D project. The impact of market uncertainty on the optimal expenditure policy is examined in the subsequent section. We summarize our results, pose some unresolved questions, and indicate how technological uncertainty might generate market uncertainty in the final section.

The Model

Our model formulation is similar to that of [4]. We let $m(t)$ and $z(t)$ denote the rate of dollar spending on the project and cumulative "effort," respectively, at time t. Cumulative effort grows with dollar spending in accordance with a bounded, strictly concave, monotone increasing function $g(m)$ with properties summarized by

$$(1)\qquad g(0) = 0, \quad g'(0) < \infty \quad g'(m) > 0, \quad g''(m) < 0, \quad g(m) < B$$
$$\text{for all } m \text{ and some constant } B < \infty$$

where prime denotes differentiation. Thus, by hypothesis,

$$(2)\qquad z'(t) = g(m(t)), \qquad z(0) = 0.$$

According to (1) and (2) there are decreasing returns to more rapid spending on the project, a supposition supported by Scherer's empirical studies; see [9] and references therein. There is no initial cumulated effort (but relevant learning from previous projects will be reflected in the function F introduced in the next paragraph). No growth of cumulative effort occurs without spending.

The level of cumulative effort needed for successful completion of the project is not known. However the firm's beliefs about required efforts are manifest through a cumulative probability distribution $F(z)$, twice continuously differentiable with properties

$$(3)\qquad F(0) = 0, \quad F'(0) = 0, \quad F'(z) \geq 0, \quad \lim_{z\to\infty} F(z) = 1.$$

The function F may reflect any relevant experience gained from previous R & D efforts. Conditions (3) express the assumptions that positive effort on the current

R & D project is required for completion, and that completion is possible with devotion of enough resources.

We define the completion rate

$$h(z) \equiv F'(z)/(1 - F(z)) \tag{4}$$

as the instantaneous probability of successful completion given completion has not yet occurred. Roughly, $h(z)\,dz$ is the probability of project completion with the increment of effort dz given that total effort z did not bring the project to fruition. Definition (4) can also be expressed as

$$F(z) = 1 - \exp \int_0^z -h(x)\,dx$$

on recollecting (3) for evaluation of the constant of integration. The stipulation that $F(z)$ is a proper distribution implies that the integral on the right side be divergent. This will be true if

$$h'(z) \geq 0 \qquad \text{for } z \geq 0. \tag{5}$$

We assume (5) holds. The intuitive meaning of (5) is that the firm's expectation of successful completion with incremental effort rises as effort is accumulated.

The firm's uncertain beliefs about when a rival might successfully complete a similar project are presented by the cumulative probability distribution $P(t)$, the assumed probability of rival introduction by time t. We assume P is twice continuously differentiable with

$$P(0) = 0, \qquad P'(t) \geq 0, \qquad \lim_{t \to \infty} P(t) \leq 1. \tag{6}$$

We define, analogously to (4),

$$p(t) \equiv P'(t)/(1 - P(t)) \tag{7}$$

as the instantaneous conditional probability of rival preemption, or entry rate, and suppose that

$$p'(t) \geq 0. \tag{8}$$

Thus the probability of rival introduction at any time, given that it has not occurred to date, does not diminish through time.

Unlike technological uncertainty, market uncertainty is exogenous to the firm. In this model, the first firm to complete its R & D project captures the entire market for which the rivals were competing. We have considered in [5] the possibility that latecomers can also profit. In that situation, the first firm to enter the market may attempt to retard further entry through its pricing policy; then market uncertainty becomes partially endogenous. The more restrictive assumptions of this model facilitate the present analysis.

We suppose a finite time horizon T for the firm's planning process. The firm seeks maximum discounted expected value of all cash flows associated with the R & D project. We now examine these expected cash flows. The expected reward to the innovating firm upon success of the R & D, discounted to completion time, is R. This value R of the expected discounted stream of future benefits is independent of calendar time. The probability of receiving R at t, i.e. of project completion at time t, is $dF(z)/dt = F'(z)z'(t)$ while the probability that the reward is still available at time t is $(1 - P(t))$. Prior to completion or preemption, the firm spends at the rate $m(t)$. Thus the firms seeks to

$$\text{(9)} \qquad \underset{m(t) \geq 0}{\text{maximize}} \int_0^T e^{-rt}(1 - P(t))[RF'(z(t))z'(t) - m(t)(1 - F(z(t))] \, dt$$

subject to (2), where r represents a constant discount rate.

It is important to note that this is a *planning* model. That is, the firm seeks a contingency development plan. The planned spending $m(t)$ that maximizes (9), subject to (2), will actually be expended only so long as neither of the random events "project completion" or "rival preemption" have occurred. Further, any change in data would necessitate a recomputation of the optimal policy, employing the then current situation as initial conditions.

It will be convenient to rephrase the problem slightly. We choose to view $g(m)$ as the control variable. Define

$$\text{(10)} \qquad u \equiv g(m).$$

Since $g(m)$ is strictly monotone, the inverse g^{-1} exists. Calling this inverse f,

$$\text{(11)} \qquad m = g^{-1}(u) \equiv f(u).$$

The properties of $g(m)$ summarized in (1) imply that

$$\text{(12)} \qquad f(0) = 0, \qquad f'(u) > 0, \qquad f''(u) > 0.$$

The optimal control problem can be restated, therefore, on substituting from (2), (10) and (11) as

$$\text{(13)} \qquad \text{maximize} \int_0^T e^{-rt}(1 - P(t))[RF'(z)u - f(u)\ (1 - F(z))] \, dt$$

$$\text{(14)} \qquad \text{subject to } z'(t) = u(t), u \geq 0$$

where $u(t)$ is the control variable and the cumulative effort $z(t)$ is the state.

Because of the monotonicity of g, or f, with $m = 0$ if and only if $u = 0$, there is a direct relation between the behavior of the control $u(t)$ and that of planned spending $m(t)$. Consequently, we shall henceforth deal with problem (13)–(14) only and refer to behavior of $u(t)$ as reflecting the planned spending pattern.

To obtain necessary conditions for solution of the optimal control problem (13)–(14), we introduce the multiplier function $\lambda(t)$ and form the Hamiltonian

$$H = e^{-rt}(1 - P(t))[RF'(z)u - f(u)(1 - F(z))] + \lambda u$$

from which we obtain

$$\text{(15)} \qquad \partial H/\partial u = e^{-rt}(1 - P(t))[RF'(z) - f'(u)(1 - F(z))] + \lambda \leq 0$$

$$\text{(16)} \qquad u\partial H/\partial u = 0$$

$$\text{(17)} \qquad \lambda' = -\partial H/\partial z = -e^{-rt}(1 - P(t))[RF''u + fF'], \qquad \lambda(T) = 0$$

Since $f'' > 0$,

$$\partial^2 H/\partial u^2 = -e^{-rt}(1 - P)(1 - F)f'' < 0$$

so a critical point of the Hamiltonian will be a maximum.

Conditions (14)–(17) describe the optimal expenditure plan implicitly. One plan that satisfies these conditions is that of not undertaking the project at all. The value of this plan is zero. The subsequent discussion will focus upon a plan that does involve some positive spending; such a plan is called non-null. The best non-null plan satisfying (14)–(17) will be optimal if its value is positive. Otherwise, the optimal plan is not to undertake the R & D project.

To obtain a more explicit characterization of a non-null plan, we note that throughout a time interval during which u is positive, $\partial H/\partial u = 0$ is constant. Therefore $d(\partial H/\partial u)/dt = 0$ on that interval, i.e. from (15), (17), (14), (4) and (7)

$$\text{(18)} \quad \frac{d}{dt}(\partial H/\partial u) = e^{-rt}(1 - P)(1 - F)[-(r + p)(Rh - f') - f''u' + h(f'u - f)] = 0$$

where all variables and suppressed arguments are evaluated at t, for each t at which the optimal $u(t) > 0$. Planning is needed only for those times t at which the reward may still be available ($P < 1$) and at which the development may not yet be complete ($F < 1$). Therefore subsequent discussion is based on, and relevant for,

$$\text{(19)} \qquad P(t) < 1, \qquad F(z) < 1.$$

Thus over a time interval of positive spending, the bracketed expression in (18) must be zero:

$$\text{(20)} \qquad f''u' = h(f'u - f) - (r + p)(Rh - f') \qquad \text{when } u(t) > 0.$$

This equation will play a central role in establishing the behavior of optimal non-null spending through time.

Characterization of the Solution

We shall now show that if the R & D project is undertaken, then, under the foregoing assumptions, planned spending will be increasing through time. Therefore actual spending will be increasing until completion, preemption, or horizon. This characterization will be formalized as Proposition 1 and proved in a series of steps. Then a necessary condition for the existence of a non-null optimal solution is presented in Proposition 2.

Proposition 1

In an optimal non-null expenditure plan for problem (13), subject to (14), (12), (5), (8), planned spending increases through time.

Proof. All assertions in the proof pertain to a non-null spending plan obeying the necessary conditions for optimality.

Step 1. Evaluate (20) at $t = 0$, recalling that $z(0) = 0$, $h(0) = 0$: $f''u' = (r + p)f' > 0$ at $t = 0$, which, by (12), implies $u'(0) > 0$. Thus spending must be increasing initially.

Step 2. We show that in an optimal non-null program, $u'(t)$ can change sign at most once. Suppose there is a time t_0 at which $u(t_0) > 0$, $u'(t_0) = 0$. Then, from (20),

$$(21) \qquad h(f'u - f) = (r + p)(Rh - f') \qquad \text{at } t_0.$$

The time rate of change of the right side of (20) is

$$(22) \qquad h'u(f'u - f - (r + p)R) - p'(Rh - f') \qquad \text{at } t_0$$

$$= -h'u(r + p)f'/h - p'(Rh - f') \qquad \text{at } t_0, \text{ using (21).}$$

The left side of (21) is positive, since f is strictly convex with $f(0) = 0$ and $u(t_0) > 0$. Therefore, the right side of (21) is also positive, which in turn implies that expression (22) is negative. This means the right side of (20) is decreasing at time t_0. Hence any non-null optimal policy $u(t)$ is either increasing throughout or else is initially increasing, peaks, and then falls.

Step 3. We show that either planned spending stops by time T or else it is increasing at T. If $u(T) > 0$, then $\partial H(T)/\partial u = 0$ and $P(T) < 1$, $F(z(T)) < 1$. Thus, from (15) and (17) ($\lambda(T) = 0$),

$$(23) \qquad Rh(z(T)) - f'(u(T)) = 0 \qquad \text{if } u(T) > 0.$$

Use (23) in evaluating (20) at T:

$$f''u' = h(f'u - f) > 0 \qquad \text{at } T \text{ if } u(T) > 0.$$

Thus $u'(T) > 0$. Hence $u(T) > 0$ implies $u'(T) > 0$ as claimed.

Step 4. We show that if planned spending ceases prior to T, i.e. if there is a $0 < \mathrm{t} < T$ such that $u(\mathrm{t}) = 0$, then

(a) $u'(\mathrm{t}) = 0$
(b) $Rh(z(\mathrm{t})) = f'(0)$
(c) $u(t) = 0$ for $\mathrm{t} \le t \le T$.

To demonstrate these assertions, suppose there are t, t^* such that

$$(24) \qquad \begin{cases} u(t) > 0 & \text{for } 0 \le t < \mathrm{t} \\ u(t) = 0 & \text{for } \mathrm{t} \le t \le t^* \le T \qquad \text{where } \mathrm{t} < t^*. \end{cases}$$

Evaluate (20) at this t, recalling $f(0) = 0$:

$$(25) \qquad f''(0)u'(\mathrm{t}) = -(r + p(\mathrm{t}))[Rh(z(\mathrm{t})) - f'(0)].$$

Since $u \ge 0$ is required, the left-hand derivative $u'(\mathrm{t}) \le 0$, which implies from (25)

$$(26) \qquad Rh(z(\mathrm{t})) \ge f'(0).$$

Claim (b) will follow if strict inequality is eliminated. To do this, we assume strict inequality in (26), reach a contradiction, and thus establish (b). Evaluate the left equality in (18)

$$(27) \qquad \frac{d}{dt}\left(\frac{\partial H}{\partial u}\right) = -e^{-rt}(1 - P)(1 - F)[Rh(z(\mathrm{t})) - f'(0)], \qquad \mathrm{t} \le t \le t^*$$

in view of (14), (12). But if strict inequality holds in (26) then from (27),

$$\frac{d}{dt}\left(\frac{\partial H}{\partial u}\right) < 0$$

which in turn implies, since $\mathrm{t} > 0$, that $\partial H/\partial u < 0$ for $t \geq \mathrm{t}$ so $t^* = T$ and $u(t) = 0$, $\mathrm{t} \leq t \leq T$. Hence $\lambda'(t) = 0$, $\mathrm{t} \leq t \leq T$. But since, from (17), $\lambda(T) = 0$, we conclude that

$$\lambda(t) = 0, \qquad \mathrm{t} \leq t \leq T.$$

Now we can evaluate (15)

$$\partial H/\partial u = e^{-rt}(1 - P)(1 - F)[Rh(z(\mathrm{t})) - f'(0)], \qquad \mathrm{t} \leq t \leq T. \tag{28}$$

This is positive by supposition (26) but must be nonpositive by necessary condition (15). This contradiction indicates that supposition of strict inequality in (26) is in error, and result (b) obtains.

Verification of (a), given that (b) has been shown, follows on evaluating (20) at $\mathrm{t}: f''u'(\mathrm{t}) = 0$ so $u'(t) = 0$.

Finally to establish (c), we observe that (28) and (b) together imply $\partial H/\partial u = 0$ at $u(t) = 0$, $\mathrm{t} \leq t \leq T$. Since H is strictly concave in u, it follows that the Hamiltonian attains its maximum at $u = 0$, $\mathrm{t} \leq t \leq T$.

Step 5. We can now show that $u'(t) > 0$ for $0 \leq t \leq T$. It follows from the conclusions of Steps 1–3 that in a non-null policy, either $u'(t) > 0$, $0 \leq t \leq T$ or else $u(T) = 0$. We now show that $u(T) = 0$ cannot happen under our assumptions, thus completing the proof of the Proposition.

If $u(T) = 0$ in a non-null policy, then (since $u'(0) > 0$ by Step 1) $u(t)$ must attain its maximum at some time t_0 and then decline to zero. At t_0, (21) holds. The left side of (21) is positive so the right side is also

$$Rh(z(t_0)) - f'(u(t_0)) > 0. \tag{29}$$

On the other hand, since z and h are nondecreasing functions of their arguments and since f' is strictly increasing in u,

$$Rh(z(t_0)) - f'(u(t_0)) < Rh(z(T)) - f'(u(T)) = 0 \tag{30}$$

where the right hand equality follows from Step 4. Since (29) and (30) are inconsistent, the supposition of $u(T) = 0$ is erroneous. We must have $u'(t) > 0$ for $0 \leq t \leq T$ in a non-null optimal policy. q.e.d.

We note that the assumption of a nondecreasing completion rate $h(z)$ played an important role in the characterization of the non-null policy. If $h(z)$ were to become a decreasing function after some point, reflecting eventually growing pessimism regarding feasibility of success in the R & D effort, then the expenditure in a non-null policy may well fail to increase through time.

Our analysis thus far has been based on the assumed existence of a non-null optimal policy. We now present a necessary condition for such a policy to exist.

Proposition 2

A necessary condition for the existence of non-null optimal policy is

$$\sup_z Rh(z) > f'(0)$$

Proof. Assume the contrary. Then for arbitrary $z, u \geq 0$,

$$Rh(z) \leq \sup_z Rh(z) \leq f'(0) \leq f'(u).$$

It follows that $Rh(z) - f'(u) \leq 0$, which together with (20) implies $u'(t) > 0$, $0 \leq t \leq T$. Hence $u(T) > 0$, so that

$$f'(u(T)) > f'(0) \geq \sup_z Rh(z) \geq Rh(z(T))$$

which contradicts (23). The contradiction establishes the Proposition.

The necessary condition of Proposition 2 is not sufficient for the optimal policy to be non-null. Nevertheless, it is easy to apply and so can provide a coarse screen for proposed projects. The rewards must be sufficiently large, with probability of successful completion high, and cost of cumulating effort low, to warrant further investigation. Lastly, the existence of a solution to problem (13), subject to (14), (12), (5) and (8) can be established as in [4, Appendix].

Market Uncertainty

An optimal non-null R & D spending plan for a firm facing both market and technological uncertainty, as described in Proposition 1, does not differ qualitatively (in the sense that $u'(t) > 0$ throughout) from that found earlier without market uncertainty ($P = p \equiv 0$; see also [4, Theorem 1a]) or indeed with neither type of uncertainty ($P = p \equiv 0, F = h = 0$ for $z \leq \mathbf{z}$; see also Lucas [7, Lemma 2]). Likewise, the necessary condition of Proposition 2 is independent of market uncertainy, since the function P does not enter. It remains to ascertain what quantitative difference the presence of market uncertainty has on the firm's optimal R & D plan. While we cannot provide a complete answer to this query, a partial response is possible.

Thus, we note that market uncertainty can discourage the firm from undertaking an R & D project that would be pursued in the absence of rival threat. To see this, one need only observe that a positive probability of rival entry P diminishes the expected return associated with any plan u, z; recall (13). More specifically, imagine the following situation. The best non-null plan u^*, z^* for (13) could render (13) nonpositive, so that the null policy (do nothing) would be best. Next imagine modifying the integral (13) by setting $P \equiv 0$ and evaluating the modified integral at the non-null policy u^*, z^* just found. This integral value will be larger; if it is positive, then the optimal value of (13) without rivalry will clearly be positive and the project will be pursued. Thus the market uncertainty can dissuade the firm from undertaking desirable R & D. On the other hand, it follows from Proposition 1, that rivalry will not induce the firm to terminate an accepted R & D project prematurely, i.e. before completion by the firm or some rival.

Before proceeding to Proposition 3, we observe that if $V(z_0, t_0)$ denotes

$$\max \int_{t_0}^{T} e^{-rt}(1 - P)[RF'u - f(1 - F)]\, dt$$

$$\text{s.t. (14)} \quad \text{and} \quad z(t_0) = z_0$$

then $\lambda(t_0) = \partial V/\partial z_0$. That is, $\lambda(t_0)$ measures the marginal contribution of the state z

to the maximum achievable from that time forward; see Arrow [1]. An increment in cumulated effort cannot be detrimental under our hypotheses, so that $\lambda(t) \geq 0$, $0 \leq t \leq T$. Since, from Proposition 1, (15) holds with equality in a non-null policy, it follows that

$$Rh(z) - f'(u) \leq 0, \qquad 0 \leq t \leq T \tag{31}$$

in an optimal non-null plan, with strict inequality prior to T as long as (19) is true.

Proposition 3

Let two firms be identical in every respect except that the second firm anticipates potential rivals ($P \geq 0$) while the first does not ($P \equiv 0$). At time zero, both face problem (13), subject to (14), (12), (5), (8). If both firms have optimal non-null expenditure plans, then their respective paths intersect at most once in (u, z)-space. Further, if an intersection $(\bar{u}, \bar{z})$ exists, it will be reached at a later date by the second firm than by the first.

Proof. Suppose the i-th firm reaches an assumed intersection $(\bar{u}, \bar{z})$ at time $t_i < T$. Evaluate (20) for the i-th firm at t_i. Subtract the first such equation from the second to obtain

$$f''(\bar{u})[u_2'(t_2) - u_1'(t_1)] = -p(t_2)[Rh(\bar{z}) - f'(\bar{u})] \tag{32}$$

where u_i is the optimal policy for firm i. In view of (31), (32) implies

$$u_2'(t_2) > u_1'(t_1).$$

The slope of the path followed in (u, z) plane is $du/dz = u'/z' = u'/u$. Since $u_i(t_i) = \bar{u}$, the slope du/dz of the second firm's path in the (u, z) plane at $(\bar{u}, \bar{z})$ exceeds the first firm's. Because the relation of the slopes at any intersection would have to be that just shown, the paths can cross at most once.

Eliminating the parameter t, we can view the paths in the (u, z) plane as graphs of functions $U_i(z)$. Then

$$U_2(z) \lesseqgtr U_1(z) \quad \text{as} \quad z \lesseqgtr \bar{z}.$$

Since at values of $z < \bar{z}$, the second firm is spending at a lower rate than the first, it will achieve cumulated effort $\bar{z}$ at a later date than the first firm. q.e.d.

According to Proposition 3, market uncertainty may modify the optimal spending path by retarding spending in early stages of the R & D and accelerating spending in later stages, relative to the expenditure pattern prevailing in its absence. This conclusion is based on the supposition that the two paths do cross, a supposition that we have been unable to verify or deny at this time. The conclusions of Proposition 3 do obtain under somewhat weaker restrictions, for example, that $\max_t p_1(t) \leq \min_t p_2(t)$.

Summary and Further Questions

We have attempted to analyze the optimal expenditure plan for an R & D project by a firm faced with market uncertainty (through actions of potential rivals) and technological uncertainty (through lack of perfect information about the requirements for success in the R & D). Under our assumptions, expenditures on any project undertaken will grow over time, until the project reaches fruition or is preempted by a rival's success. We also concluded that the presence of potential rivals could dissuade a firm from undertaking a potentially profitable R & D project. If the project were undertaken, recognition of potential rivals may retard and accelerate spending at different stages of the project, but will not lead to premature termination of an on-going R & D project. Thus this paper represents another step toward complete analytic characterization of a firm's resource allocation decisions under increasingly realistic assumptions about its stochastic environment.

There are several obvious directions for further work. The effect of increasing market uncertainty upon the optimal expenditure path needs further resolution. The case in which the completion rate is eventually a decreasing function of cumulated effort warrants investigation. Likewise the situation in which the entry of rivals becomes less likely after some point in time might be explored. There may be environmental features other than rivalry that make the rewards from innovation time-dependent. The possibility that latecomers can also collect some reward should be taken into account.

Another, perhaps less obvious, extension is in the direction of making rival behavior endogenous to the model. In at least one important sense, market uncertainty is generated by technological uncertainty. For example, let us posit the existence of n rivals, identical in every respect, each facing problem (13)–(14). If the R & D efforts of these firms can be considered to be statistically independent (at least as an approximation), then the probability that none of our firm's $n - 1$ rivals has successfully completed its R & D by time t is

$$1 - P(t) = [1 - F(z(t))]^{n-1} \tag{33}$$

where $z(t)$ is the common cumulative effort at time t. Moreover, it follows on differentiating with respect to time that

$$p(t) = (n - 1)h(z(t))u(t). \tag{34}$$

Equation (33) or (34) relates the probability of rival preemption by any time to the technological uncertainty experienced by each. Equation (33) or (34) would be an equilibrium condition for solution of the n-firm industry problem, much as is done in Cournot-type analysis; see, for example, Ruff [8].

Finally, we remark that while the model has been developed in the context of R & D, it may be pertinent to a wide variety of situations involving resource allocation over time. In particular, the analysis could apply to planning for any race requiring uncertain effort to reach the goal, and involving rivalry. Examples suggested to us by T. Schelling include the land (gold, oil) rush in the west, the effort to decipher Linear B, the search for the North Pole, the construction of a new building, and the search for a new result in mathematics. Likewise, the methodology employed here might be useful for modelling and solving other stochastic optimal control problems.

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