Appendix 2: The continuous time version of the New-Keynesian model

A representative infinite-lived, competitive consumer maximises for all $t \geq 1$ the utility functional given in (A2.1) subject to his instantaneous flow budget identity (A2.2), his solvency constraint (A2.3) and his initial financial wealth, $F(0) \equiv M(0) + B(0)$.

$$\int_t^\infty c(v)^\alpha m(v)^{1-\alpha} \exp\{-\rho(v-t)\}\,dv \qquad \textbf{(A2.1)}$$

$$0 < \alpha < 1;\ \rho > 0$$

$$\dot{M} + \dot{B} \equiv P(y - \tau - c) + iB + i^M M \qquad \textbf{(A2.2)}$$

$$\lim_{v \to \infty} [M(v) + B(v)] \exp\{-\int_t^v i(u)\,du\} \geq 0 \qquad \textbf{(A2.3)}$$

By definition, $f \equiv m + b$, and the household budget identity (A2.2) can be rewritten as

$$\dot{f} \equiv rf + y - \tau - c + (i^M - i)m \qquad \textbf{(A2.4)}$$

where r, the instantaneous real rate of interest on non-monetary assets, is defined by $r \equiv i - \pi$ and $\pi \equiv \dot{P}/P$ is the instantaneous rate of inflation.

The household solvency constraint can be rewritten as

$$\lim_{v \to \infty} f(v) \exp\{-\int_t^v r(u)du\} \geq 0 \qquad \textbf{(A2.5)}$$

and the intertemporal budget constraint for the household sector can be rewritten as:

$$\int_t^\infty \left[c(v) + \tau(v) + [i(v) - i^M(v)]m(v) - y(v)\right]\exp\{-\int_t^v r(u)du\}dv \leq f(t) \qquad \textbf{(A2.6)}$$

The first-order conditions for an optimum imply that the solvency constraint will hold with equality. Also,

$$\dot{c} = (r - \rho)c \qquad \textbf{(A2.7)}$$

and for $i \geq i^M$,

$$m = (1-\alpha)\alpha^{-1}(i-i^M)^{-1}c \qquad \textbf{(A2.8)}$$

There is a continuum of identical consumers whose aggregate measure is normalised to *1*. The individual relationships derived in this section therefore also characterise the aggregate behaviour of the consumers. The consumption function is

$$c(t) = \alpha\rho([M(t)+B(t)]P(t)^{-1} + \int_t^\infty [y(v) - \tau(v)]\exp\{-\int_t^v r(u)du\}dv) \qquad \textbf{(A2.9)}$$

The budget identity of the consolidated General Government and Central Bank is given in (A2.13).

$$\dot{M} + \dot{B} \equiv iB + i^{M}M + P(g - \tau) \quad \textbf{(A2.10)}$$

This budget identity can be rewritten as

$$\dot{f} = rf + g - \tau + (i^{M} - i)m \quad \textbf{(A2.11)}$$

or as

$$\dot{b} = rb + g - \tau - (\dot{M} - i^{M}M)P^{-1} \quad \textbf{(A2.12)}$$

The government solvency constraint is

$$\lim_{v \to \infty} b(v)\exp\{-\int_{t}^{v} r(u)du\} \leq 0 \quad \textbf{(A2.13)}$$

Equations (A2.15) and (A2.16) imply the following equivalent intertemporal government budget constraints:

$$f(t) = \int_{t}^{\infty}\left[\tau(v) - g(v) + [i(v) - i^{M}(v)]m(v)\right]\exp\{-\int_{t}^{v} r(u)du\}dv + \lim_{v \to \infty} m(v)\exp\{-\int_{t}^{v} r(u)du\} \quad \textbf{(A2.14)}$$

$$b(t) = \int_{t}^{\infty}\left(\tau(v) - g(v) + [\dot{M}(v) - i^{M}(v)M(v)]P(v)^{-1}\right)\exp\{-\int_{t}^{v} r(u)du\}dv \quad \textbf{(A2.15)}$$

Substituting (A2.15) into the consumption function (A2.9) yields

$$c(t) = \alpha\rho\left[M(t)P(t)^{-1} + \int_{t}^{\infty}\left(y(v) - g(v) + [\dot{M}(v) - i^{M}(v)M(v)]P(v)^{-1}\right)\exp\{-\int_{t}^{v} r(u)du\}dv\right] \quad \textbf{(A2.16)}$$

Through integration by parts, $\frac{M(t)}{P(t)} + \int_t^{\infty} e^{-\int_t^v r(u)du} \frac{\dot{M}(v)}{P(v)} dv = \int_t^{\infty} e^{-\int_t^v r(u)du} \frac{i(v)M(v)}{P(v)} dv + \lim_{v\to\infty} e^{-\int_t^v r(u)du} \frac{M(v)}{P(v)}$. Using this and the Euler equation (A2.7) in (A2.16) yields:

$$\begin{aligned} c(t) &= \rho\left(\int_t^{\infty} [y(v)-g(v)]\exp\{-\int_t^v r(u)du\}dv + \lim_{v\to\infty} m(v)\exp\{-\int_t^v r(u)du\} \right) \\ &\equiv \rho\left(\int_t^{\infty} [y(v)-g(v)]\exp\{-\int_t^v r(u)du\}dv + P(t)^{-1}\lim_{v\to\infty} M(v)\exp\{-\int_t^v i(u)du\} \right) \end{aligned} \qquad \textbf{(A2.17)}$$

The FFMP of the government is as follows. Lump-sum taxes are continuously adjusted to keep the real stock of public debt (monetary and non-monetary) constant. To achieve $\dot{f} = 0$, taxes are

$$\tau = rf + g + (i^M - i)m \qquad \textbf{(A2.18)}$$

The nominal interest rate on base money is exogenous and constant, $i^M(t) = \bar{i}^M, \; t \geq 0$. The short nominal interest follows a Taylor rule as long as this does not imply a nominal interest rate below the nominal interest rate on base money. Otherwise the short nominal interest rate equals the nominal interest rate on base money and the nominal stock of base money grows as a constant proportional rate.

$$
\begin{aligned}
&\text{If } \rho + \pi^* + \gamma(\pi - \pi^*) \geq \bar{i}^M \\
&\qquad i = \rho + \pi^* + \gamma(\pi - \pi^*) \qquad\qquad \gamma > 1 \\
&\text{If } \rho + \pi^* + \gamma(\pi - \pi^*) < \bar{i}^M \\
&\qquad i = \bar{i}^M \\
&\quad \text{and} \\
&\qquad \dot{M}/M = \bar{\nu} > \bar{i}^M
\end{aligned}
\tag{A2.19}
$$

$$
0 \leq g(t) = \bar{g} \leq y^* \tag{A2.20}
$$

Output equals demand

$$
y = c + g \tag{A2.21}
$$

The continuous time version of Calvo's (1983) model staggered overlapping price setting specifies the price set at time t by firms that have the opportunity to change their price at that instant, $p_i(t) = \ln P_i(t)$, to be a weighted average of future average prices, $p(s) = \ln P(s), \; s \geq t$ and of future excess demands. Weights decline exponentially to reflect the declining probability of a price set at time t still being in effect at increasingly distant future times.

$$
\begin{aligned}
&p_i(t) = \delta \int_t^{\infty} \left(p(s) + \eta[y(s) - y^*]\right) e^{-\delta(s-t)} ds + e^{\delta t} \lim_{s \to \infty} e^{-\delta s} p_i(s) \\
&\delta > 0;\; \eta' > 0;\; \eta(0) = 0;\; y \leq \bar{y} < \infty;\; y^* < \bar{y};\; \lim_{y \uparrow \bar{y}} \eta'(y - y^*) = +\infty
\end{aligned}
\tag{A2.22}
$$

The current contract price $p_i(t)$ is therefore a non-predetermined state variable.

The general price level, p, is a backward-looking, exponentially declining moving average of past contract prices.

$$p(t) = e^{-\delta(t-t_0)}p(t_0) + \delta\int_{t_0}^{t} p_i(s)e^{-\delta(t-s)}ds \tag{A2.23}$$

The (natural logarithm of the) general price level is therefore a predetermined state variable. From (A2.23) and (A2.24) it follows that the rate of inflation of the general price level, $\pi = \dot{p}$, is given by

$$\pi(t) = \delta^2\int_{t}^{\infty}\eta[y(s) - y^*]ds + k \tag{A2.24}$$

where k is an arbitrary constant, which can be given the interpretation of the long-run rate of inflation, that is, $k = \lim_{s\to\infty}\pi(s)$. This implies the following 'quasi-accelerationist' Phillips curve

$$\dot{\pi} = -\delta^2\eta[y(t) - y^*] \tag{A2.25}$$

The rate of inflation, π, is, unlike the price level, p, a non-predetermined or 'forward-looking' state variable. The specific function form for the η function is

$$\begin{aligned}\eta(y - y^*) &= \left(-\eta_0 + \eta_1(y^* - y + \eta_1\eta_0^{-1})^{-1}\right)\delta^{-2}\\ \eta_0,\ \eta_1 &> 0;\ 0 \le y \le \bar{y} = y^* + \eta_1\eta_0^{-1}\end{aligned} \tag{A2.26}$$

The New-Keynesian Phillips curve is therefore given by

$$\text{ft} \; ' \; \eta_0 \; \& \; \eta_1\left(y^{(} \; \& \; y \; \% \; \eta_1\eta_0^{\&1}\right)^{\&1} \qquad \textbf{(A2.27)}$$