**Econ 110** **Skillman**
**Intro Econ** **Fall 2002**

## Math Review 1: Derivatives

The *derivative* of a real scalar-valued function *f* evaluated at point *x* (written $f'(x)$, $df/dx$, or $dy/dx$ when *y* is understood to represent possible values of the function) is defined as

$$\lim_{t\to x}\frac{f(t)-f(x)}{t-x},$$

assuming this limit exists. If it does, the function is said to be *differentiable* at point *x*. If the derivative exists for all *x* in [*a*, *b*], $a < b$, then the function is said to be differentiable on the set [*a*, *b*]. Note that differentiability implies continuity (but not vice-versa) and that the derivative, if it exists, is unique. The derivative $f'(x)$ is represented visually as the slope of the tangent line to the function *f* at point *x*.

*Example* Let $f(x)=x^2$. Then $f'(x)=\lim_{t\to x}\frac{t^2-x^2}{t-x}=\lim_{t\to x}\frac{(t+x)(t-x)}{t-x}=2x$.

Application of this definition can be shown to yield the following differentiation "rules":

1) *Addition rule*

Let $f(x)=g(x)+h(x)$. Then $f'(x)=g'(x)+h'(x)$.

2) *Scalar multiplication rule*

Let $f(x)=ag(x)$, where *a* is a scalar. Then $f'(x)=ag'(x)$.

3) *Product rule*

Let $f(x)=g(x)\cdot h(x)$. Then $f'(x)=g'(x)\cdot h(x)+g(x)\cdot h'(x)$.

4) *Quotient rule*

Let $f(x)=g(x)/h(x),\ h(x)\neq 0$. Then $f'(x)=\frac{h(x)g'(x)-g(x)h'(x)}{\left(h(x)\right)^2}$.

(Exercise: derive the quotient rule directly from the product rule by substitution.)

5) *Power rule*

Let $f(x)=\left(g(x)\right)^n$, *n* a scalar. Then $f'(x)=n\left(g(x)\right)^{n-1}\cdot g'(x)$.

6) *Natural-logarithmic rule*

Let $f(x) = \ln g(x)$, $g(x) > 0$. Then $f'(x) = g'(x)/g(x)$

(Generalization to other logarithmic bases can achieved by noting that $\log_b g(x) = \ln g(x)/\ln b$, $b \neq 1$. Thus if $f(x) = \log_b g(x)$, $b \neq 1$, then $f'(x) = g'(x)/\left(g(x) \cdot \ln b\right)$.

7) *Exponential rule*

Let $f(x) = a^{g(x)}$, $a > 0$ a scalar. Then $f'(x) = \ln a \cdot g'(x) \cdot a^{g(x)}$.

(Note that this implies $f'(x) \equiv f(x)$ if $a = e$ and $g(x) = x$.)
(Exercise: derive the exponential rule from the natural-logarithmic rule.)

8) *Chain rule*

Let $f(x) = g(h(x))$. Then $f'(x) = g'(z) \cdot h'(x)$, where $z = h(x)$.

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## Math Review 2: Integrals (Anti-Derivatives)

Just as the natural log function can be thought of as the inverse of the exponential function, *integration* can be thought of as the inverse of the process of *differentiation.* Thus we can define the *indefinite integral (anti-derivative)* of a function $f(x)$ as

$$\int f(x)dx = F(x) + c \text{, where } F'(x) = f(x) \text{,}$$

assuming the derivative exists, and $c$ is an arbitrary constant. From the general properties of derivatives, one can infer two properties of indefinite integrals:

*Constant multiple rule*

$$\int af(x)dx = a\int f(x)dx \text{, where } a \text{ is a real-valued constant.}$$

Example: $\int 5e^x dx = 5\int e^x dx = 5e^x + c$

*Summation rule*

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

The *definite integral* of a function $f(x)$ on the domain $x \in (a,b)$ is defined as

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a) \text{, where } F'(x) = f(x) \text{ for all } x \in (a,b) \text{,}$$

assuming the derivative exists. It turns out that the derivative always does exist, and thus the definite integral is well defined, if $f$ is a *continuous* function. In that case the definite integral can be written as

$$\int_a^b f(x)dx = \lim_{\Delta x_i \to 0} \sum_{i=0}^{n-1} f(\xi_i)\Delta x_i \text{ ,}$$

where the set of intervals $\Delta x_i \Big|_{i=1}^{n-1}$ are chosen so as to exhaustively partition the domain [$a$,$b$], and $\xi_i$ is arbitrarily chosen from within the respective intervals. This expression, the so-called *Riemann integral*, confirms the intuition that integration is a sort of summation for continuously-valued variables. More specifically, it's the sum of vanishingly small strips of area underneath a curve over a given domain, so that the integral of a continuous function over that domain just equals the area between the function and the $x$-axis over that domain.

In addition to the properties mentioned above, the definite integral satisfies the following properties:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^a f(x)dx = 0$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad \text{given } c \in (a,b)$$

More specific rules for indefinite or definite integration can be generated by running certain differentiation rules "backwards." Thus, for example,

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + c\,,\ n \neq -1$$

$$\int dx/x = \ln x + c$$

$$\int ae^{f(x)}dx = \frac{ae^{f(x)}}{f'(x)} + c\text{, given } f'(x) \neq 0$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx \quad \text{("Integration by parts")}$$