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Age, Luck, and Inheritance

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Abstract

We present a mechanism to analytically generate a double Pareto distribution of wealth in a continuous time OLG model with optimizing agents who have bequest motives, are subject to stochastic returns on capital and have uncertain lifespans. We disentangle, roughly, the contribution of inheritance, age and stochastic rates of capital return to wealth inequality, in particular to the Gini coefficient. We investigate the role of the fiscal and redistributive policies for wealth inequality and social welfare.

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1. Introduction

Age, luck and inheritance all play a significant role in the accumulation of wealth. Households accumulate wealth as they age.[1] Rates of return vary across households and over time depending on the luck of the draw. And some agents start their economic life with large inheritances.[2] We develop a dynamic model of wealth distribution with utility-optimizing agents to understand the well-known features of empirical wealth distributions: skewness to the right and heavy tails. We provide analytic solutions and then calibrate to match the U.S. distribution. We then decompose the Gini coefficient to isolate the effects of age, luck and inheritance. Finally we study the effects of capital and estate taxes on steady state inequality and welfare.

Our analysis is based on Benhabib and Bisin (2006) who also investigate the impact of intergenerational transmission and redistributive policies on the wealth inequality. In a model with one riskless asset Benhabib and Bisin (2006) find that wealth inequality induced by inheritance accounts for just a little less than a third of the size of the Gini coefficient in the U.S. in 1992. We introduce a risky asset into the model to better identify the contributions of the stochastic rate of return, of age, and of inheritance to the inequality of wealth.

Wealth distribution displays right skewness and a heavy upper tail in different countries and times. Atkinson and Harrison (1978) document the heavy upper tail of the wealth distribution in Britain during 1923-1972. Wolff (1995) presents the percentage share of net wealth held by the richest 1% of wealth holders in U.S. during 1922-1989, which ranges from 19.9% to 36.7%. Using the data of Survey of Consumer Finances in the U.S. in 2001, Wolff (2004) computes the Gini coefficient of wealth, 0.826. The top 1% of population holds 33.4% of the wealth in the U.S. Using the richest sample of the U.S., the Forbes 400, during 1988-2003 Klass et. al. (2006) find that the top end of the wealth distribution obeys a Pareto law with an average exponent of 1.49. Dragulescu and Yakovenko (2001) present the 1996 data of the personal total net wealth in the U.K. and find

[1] At least until 65, or retirement; see Rodriguez, Diaz-Gimenez, Quadrini and Rios-Rull (2002), chart 10.

[2] Intergenerational transfers also play an important role in the aggregate capital accumulation. Kotlikoff and Summers (1981) find that intergenerational transfers account for sustaining the vast majority, up to 70%, of the aggregate U.S. capital formation. See also Gale and Scholtz (1994) for more moderate findings on this topic. For an account of the the role of inhertance on the Forbes 400 see Elwood et al. (1997) and Burris (2000).

that the high-end tail follows a power law with exponent of 1.9. Hegyi, Neda and Santos (2007) show that the Pareto index in a rank/frequency plot is 0.92 for the top wealth family of Hungarian in 1550.[3] Sinha (2006) investigates the data of higher-end tail of the wealth distribution in India between the years 2002-2004 and finds that the resulting rank distribution seems to imply a power-law tail for the wealth distribution, with a Pareto exponent between 0.81 and 0.92.

A standard mechanism to generate right-skewed stationary distributions is to construct a stochastic process with negative drift and a lower reflecting barrier.[4] Champernowne (1953) was the first to employ a multiplicative stochastic process of income dynamics with independent proportionate changes that have negative expected value and a reflective lower barrier to derive a power law distribution for income.[5] Wold and Whittle (1957) introduced a birth and death process with exogenous exponential wealth accumulation and bequests to generate a stable stationary wealth distribution. More recently Gabaix (1999) used this mechanism to study the distribution of city sizes, and Levy (2003) used it to study conditions that ensure convergence to the Pareto wealth distribution. In Benhabib and Bisin (2006) the wealth accumulation process is based on optimizing behavior of agents and has positive deterministic growth. But as in Wold and Whittle (1957), the geometrically distributed death and inheritance processes, together with estate taxes, result in a stationary distribution. In our model we will dispense with the need for a lower reflecting barrier to the stochastic process: bad luck will in fact drive wealth down.

Our model is a continuous time overlapping generations model with a continuum of agents as in Yaari (1965) and Blanchard (1985), with optimizing agents. There are three kinds of financial assets: a risk-free asset, a risky asset and life insurance or annuities. Life insurance or annuity companies operate competitively

[3] In a rank/frequency plot, the Pareto index is the reciprocal of the Pareto exponent. For a derivation of this relationship, see Levy and Solomon (1997).

[4] Kesten (1973) studies the limit distribution of the solution Y_n of the difference equation $Y_n = M_n Y_{n-1} + Q_n$, $n \geq 1$, where M_n is an i.i.d. random $d \times d$ matrices, and Q_n is an i.i.d. random d-vector and Y_n is also a d-vector. Takayasu, Sato, and Takayasu (1997) clarify necessary and sufficient conditions for a quantity described by $X(t+1) = b(t)x(t) + f(t)$ to follow a power law distribution with divergent moment. Sornette and Cont (1997) show that the multiplicative process with the reflective barrier and the Kesten variable are deeply related: the additive term in Kesten processes plays the role of effective repulsion from the origin. Sornette (1998) presents a review of applications, highlights the common physical mechanism and summerize the main known results of the stochastic processes with multiplicative noise.

[5] See also Simon (1955) for a related mechanism to generate the Power Law distribution.

and make zero profits. For each agent the return to the risky asset is stochastic and follows a Geometric Brownian Motion. There are two heterogenous groups of agents: one group has a bequest motive and the other group does not. Under optimal consumption and investment behavior, the wealth of agents also follows a Geometric Brownian Motion. The geometrically distributed death rate, the Geometric Brownian Motion of wealth, and the inheritance of bequests results in a stationary distribution of wealth that follows a Pareto law in both tails (double Pareto distribution). While newborn agents are introduced into the economy at some arbitrary minimum level of wealth through transfers determined by a redistributive welfare policy, unlike the previous models in the literature described above, this level does not constitute a reflecting barrier: low realizations of the return on the risky asset can draw down wealth below this birth minimum, and so we end up with a double Pareto distribution.[6]

We try to disentangle the contributions of stochastic rate of return, of the age profile, and of inheritance to wealth inequality. From our rough calibration and simulation exercises, we find that luck captured by the stochastic rate of return contributes about 31% to wealth inequality in terms of the Gini coefficient while life-cycle accumulation or age contributes about 37%. We show that surprisingly, bequests and inheritance can decrease wealth inequality because they have an impact on the growth rate of average wealth, dampening the dissipative effect of luck and age on the relative growth of individual wealth. We also show that government redistributive policies have important consequences for wealth inequality through their effects on the growth rates of wealth, on the size of government subsidies, and on bequests. Finally we show that fiscal policies can have an impact on social welfare defined as the sum of the discounted utility streams of those who are alive.

[6] Reed (2001) proposes a double Pareto distribution to explain the power law behaviour in the upper tail of income and of city size distributions. The income distribution within each cohort is lognormal and the age is an exponential random variable. Provided all income earners have the same starting income (no inheritance), the current distribution of incomes for the economy mixing all cohorts is a double Pareto distribution. The simpler techniques used by Reed that avoid PDEs however cannot be applied directly to our model with inheritance. The plots of Reed (2001) reveal power law behaviour in both the upper tail and lower tail of 1998 U.S. male earnings distributions and 1998 U.S. settlement sizes distribution.

1.0.1. Related literature

A large literature of incomplete markets such as Aiyagari (1994) and Huggett (1993) study the stationary distribution of wealth in models with heterogenous agents. Agents face uncertain labor income and a constant interest rate, and hold precautionary savings against uninsurable labor earnings. As pointed out by Schechtman and Escudero (1977) the constant or bounded relative risk aversion utility functions employed in these models mean that the stationary distribution of wealth has bounded support.[7] To generate skewness and fat tails, we use a model with idiosyncratic stochastic rates of return to capital as well as bequests and inheritance, but we abstract away from modelling labor earnings or their distribution. We obtain a stationary distribution of wealth with unbounded support, which displays fat-tails and upper skewness.

A number of authors have recently introduced new features to the basic incomplete market models to simulate the U.S. wealth distribution. Huggett (1996) calibrates life-cycle economies to match features of the U.S. earnings distribution and then examines the wealth distribution implications of his model. His model produces less than half the fraction of wealth held by the top 1% of U.S. households. Krusell and Smith (1998) study incomplete market economies with aggregate uncertainty. They introduce preference heterogeneity into the economy in the form of random discount factors to match the dispersion and the key features of the U.S. wealth distribution. Quadrini (2000) generates a concentration of wealth similar to the one observed in the U.S. economy by introducing entrepreneurship into his model. Castaneda, Diaz-Gimenez and Rios-Rull (2003) incorporate life cycle features, a social security system, progressive income and estate taxes and intergenerational transmission of stochastic earnings ability into their model, and find through simulations that the labor efficiency shock helps to account for the U.S. distribution of earnings and wealth almost exactly. De Nardi (2004) constructs an OLG model in which parents and children are linked by accidental and voluntary bequests and by earnings ability. Cagetti and De Nardi (2005) summarize some key facts about the U.S. wealth distribution and of equi-

[7] The mechanism that generates a non-degenarate stationary distribution for such additive shock (stochastic labor income) models requires the gross interest rate to be no greater than the reciprocal of the time discount rate. This is also the feature used to generate a stationary distribution in the calibrated models, for example of Aiyagari (1994), Huggett (1993), Castaneda, Diaz-Gimenez and Rios-Rull (2003). In our model the product of the interest rate and the time discount rate exceeds unity so we have growth.

librium models with incomplete markets. More recently, Wang (2006) investigates the equilibrium wealth distribution in an economy endogenous time preferences that can differ across infinitely-lived agents. With endogenous time preference, the stronger incentives to consume for the rich agents narrow the wealth dispersion and generate a stationary distribution. This mechanism differs from of our model where the spread of the wealth process is checked by death, annuity markets, and estate taxes.

While the agents have uncertain lifetimes in our model, in the presence of the perfect life insurance markets there are no accidental bequests. This feature is in contrast to some of the literature on precautionary saving, such as Abel (1985) and Fuster (2000), who investigate how the lack of annuity markets affects saving behavior and the intergenerational transfer of wealth under uncertain lifetimes.

The rest of this paper is organized as follows. In section 2, we present the basic structure of our continuous time OLG economy. We investigate the cross-sectional wealth distribution of the economy in section 3. Section 4 contains the analysis of the effect of redistributive policy on wealth inequality and social welfare. We present an alternative economy with across the board lump-sum subsidies in section 5 and conclude with a discussion in section 6. We leave the proofs to the Appendix.

2. An OLG economy

There is a continuum of agents in the economy who invest their wealth in a riskless asset and a risky asset. There is continuum of risky assets in the economy. Every agent invests their wealth in their own risky asset. The stochastic processes for the risky assets held by the agents are independent, but they follow the same Geometric Brownian Motion process

$$dS(t) = \alpha S(t)dt + \sigma S(t)dB(t)$$

where $B(t)$ is the standard Brownian motion, α is the instantaneous conditional expected percentage change in value per unit time and σ is the instantaneous conditional standard deviation per unit time. The Geometric Brownian Motion process implies that the value of the risky asset is log-normally distributed and the rate of return of the risky asset does not depend on the level of the risky asset value.

The value of the riskless asset follows

$$dQ(t) = Q(t)rdt$$

where r is the rate of return of the riskless asset and $r < \alpha$. The rate of return of the riskless asset is identical for all agents in the economy.

The agent allocates individual wealth among current consumption, investment in a risky asset, a riskless asset and the purchase of life insurance. Negative life insurance purchases, allowed in our model, corresponds to the purchase of annuities. $Z(s,t)$ denotes the bequest that the agent born at time s leaves at time t if the agent dies. The bequest consists of two parts: the agent's wealth invested in riskless asset and risky asset, and the life insurance/annuities that the agent purchases. The price of the life insurance is μ. The agent pays $P(s,t)dt$ to buy the life insurance. Should the agent die in the next short period dt, the life insurance company pays $\frac{P(s,t)}{\mu}$. Let $W(s,t)$ be wealth at time t of an agent born at time s. The relation between $Z(s,t)$, the bequest, $W(s,t)$, wealth, and the payment from the insurance company, $\frac{P(s,t)}{\mu}$, is given by

$$Z(s,t) = W(s,t) + \frac{P(s,t)}{\mu}$$

If $P(s,t) < 0$, the life insurance company is an annuity company, paying $P(s,t)dt$ to those alive and receiving $\frac{P(s,t)}{\mu}$ from the estate of the agents who die.

There is an uncertainty about the duration of the agent's life: it follows an exponential distribution with rate parameter of p. Each agent will die at a time $t \in [0, +\infty)$ by a probability density function $\pi(t) = pe^{-pt}$. In a small time Δt, the agent has probability of $p\Delta t$ to die, conditioning on the event that the agent is sill alive. When the agent dies, the agent's child is born. Each agent has one child.

Life insurance or annuity companies earn zero profits and effectively act as clearing houses. In a short period of length Δt payments and disbursements are equal: $p\frac{P(s,t)}{\mu}\Delta t = P(s,t)\Delta t$, so that $\mu = p$.[8]

We assume that the bequest motive takes the form of "the joy of giving": The bequest enters parents' utility function but parents do not care about children's utility directly.(See however the Pure Altruism section 8.14 at the end of the Appendix for how, following Abel and Warshawsky (1988), we can parametrize

[8] In fact collections and disbursements occur every instant in continuous time as $\Delta t \to 0$.

the bequest function so that it reduces to the standard infinitely-lived dynastic utility Ramsey model) Utility from bequests is given by $\chi\phi((1-\zeta)Z(s,t))$ where ζ is the estate tax rate, χ represents the strength of the bequest motive, and $\phi(\cdot)$ is the bequest utility function. We choose CRRA functions for both the consumption and the bequest utilities.

We assume that there are two groups of agents in the economy. A fraction $\frac{q}{p} < 1$ of the people have a bequest motive,with bequest motive parameter $\chi > 0$, and a fraction $1 - \frac{q}{p}$ of people do not have a bequest motive with bequest motive parameter $\chi = 0$.

Let $J(W(s,t))$ be the optimal value function of agents. The agent's utility maximization problem is

$$J(W(s,t)) = \max_{C,\omega,P} E_t \int_t^{+\infty} e^{-(\theta+p)(v-t)}[\frac{C^{1-\gamma}(s,v)}{1-\gamma} + p\chi\frac{((1-\zeta)Z(s,v))^{1-\gamma}}{1-\gamma}]dv \quad (1)$$

subject to

$$dW(s,t) = [(r-\tau)W(s,t) + (\alpha-r)\omega(s,t)W(s,t) - C(s,t) - P(s,t)]dt + \sigma\omega(s,t)W(s,t)dB(s,t) \quad (2)$$

where θ is the time discount rate. $C(s,t)$ is the consumption at time t of an agent born at time s and $\omega(s,t)$ is the share of wealth the agent invests in risky asset. τ is the capital tax on wealth[9]. The transversality condition for the agent's problem is[10]

$$\lim_{t\to+\infty} Ee^{-(\theta+p)(t-s)}J(W(s,t)) = 0$$

The set-up of the agent's problem is that of Richard (1975). We add a capital tax and an estate tax to Richard's (1975) model. The agent's optimal policy is same as that of Richard (1975), except that we have to take into account the influence of taxes.

[9] Of course τ can be redefined so it is a tax on capital income. We also use the term of capital income tax for τ in this paper.

[10] For the transversality condition of the continuous-time stochastic dynamic programming problem, see Merton (1992).

Proposition 1. *The agent's optimal policies are characterized by*

$$
\begin{aligned}
C(s,t) &= A^{-\frac{1}{\gamma}} W(s,t),\ \omega(s,t) = \frac{\alpha - r}{\gamma\sigma^2}, \\
Z(s,t) &= \left(\frac{p\chi}{A\mu}\right)^{\frac{1}{\gamma}} (1-\zeta)^{\frac{1-\gamma}{\gamma}} W(s,t)
\end{aligned}
$$

with $A = \left(\frac{\theta + p - (1-\gamma)(r - \tau + \mu + \frac{(\alpha - r)^2}{2\gamma\sigma^2})}{\gamma(1 + (p\chi)^{\frac{1}{\gamma}} \mu^{\frac{\gamma-1}{\gamma}} (1-\zeta)^{\frac{1-\gamma}{\gamma}})} \right)^{-\gamma}$ *and*

$$dW(s,t) = gW(s,t)dt + \kappa W(s,t)dB(s,t).$$

with $g = \frac{r - \tau + \mu - \theta - p}{\gamma} + \frac{1+\gamma}{2\gamma} \frac{(\alpha - r)^2}{\gamma\sigma^2}$ *and* $\kappa = \frac{\alpha - r}{\gamma\sigma}$.

Note that $P(s,t)$ may be positive or negative depending on the sign of $Z(s,t) - W(s,t)$, and determines whether the agent buys annuities or life insurance.[11] The mean growth rate of the agent's wealth is $g = \frac{r-\tau+\mu-\theta-p}{\gamma} + \frac{1+\gamma}{2\gamma}\frac{(\alpha-r)^2}{\gamma\sigma^2}$. The mean growth rate is independent of the bequest parameter χ. This is due to the specific form of the utility function and the completeness of the life insurance market. The growth rate, g, depends on the income tax rate τ, but not the estate tax rate ζ.

The share of risky asset, $\omega(s,t) = \frac{\alpha - r}{\gamma\sigma^2}$, is only influenced by the risk premium of the risky asset, the degree of risk aversion, and the volatility of the return of the risky asset. This is the same result as that of Merton (1971). The share of wealth invested in the risky asset, $\omega(s,t)$, does not depend on "joy of giving" parameter, χ, or the government tax policies. The volatility of the growth rate of the agent's wealth, $\kappa = \frac{\alpha - r}{\gamma\sigma}$, does not depend on the bequest motive parameter, χ. Note that κ is negatively related to the standard deviation of the price of the risky asset, σ, even though κ is positively related to $\sigma\omega(s,t)$. This is because $\omega(s,t)$ is negatively related to σ^2. The aversion to the risk causes the agent to overreact to risk such that the volatility of wealth is negatively related to the volatility of the risky asset. Government policy has no impact on κ.

The agent's wealth evolves as a Geometric Brownian Motion

$$dW(s,t) = gW(s,t)dt + \kappa W(s,t)dB(s,t) \tag{3}$$

This equation means that even though there are heterogeneous bequest motives

[11] If $\mu = p$, then the sign of $Z(s,t) - W(s,t)$ is determined by $(\frac{\chi}{A})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}} - 1$.

in our economy, the agents follow the same wealth accumulation process during their life time. Equation (3) also implies that wealth growth displays Gibrat's Law. The growth rate of the wealth is independent of the level of wealth. In many of the mechanisms generating a power law distribution, Gibrat's Law plays a fundamental role, and this is also true in our model.

2.1. The aggregate economy

The age cohorts are large enough such that the law of large numbers holds whenever we try to use it. This assumption implies: 1) Even though each agent faces uncertainty about the duration of life, the size of the cohort born at s is $pe^{-p(t-s)}$ at time t. The size of the population at any time t is $\int_{-\infty}^{t} pe^{p(s-t)}ds = 1$. 2) Although different agents within a cohort have different wealth levels, the aggregate wealth level of a cohort depends on the age of the cohort, but not on the wealth distribution within the cohort. At time t, conditional on the event that the agents born at time s are still alive, the mean wealth of the cohort is denoted by $E_sW(s,t)$. Let $E_sW(s,s)$ be the mean starting wealth of the agents born at time s. Then

$$E_sW(s,t) = E_sW(s,s)e^{g(t-s)} \tag{4}$$

In a small time interval Δt, a fraction $p\Delta t$ of people die. Given the heterogeneity of the bequest motive, a fraction $q\Delta t$ of people leave bequests. A fraction $(p-q)\Delta t$ of people leave no bequests when they die, since $\chi = 0$ in their utility function.

Following Benhabib and Bisin (2006), we derive the aggregate wealth growth rate. Let $W(t)$ be the aggregate wealth of the economy. Integrating the mean wealth of all age cohorts with respect to the stationary population distribution, we obtain the aggregate wealth

$$W(t) = \int_{-\infty}^{t} E_sW(s,t)pe^{p(s-t)}ds \tag{5}$$

Plugging formula (4) into formula (5), we have

$$W(t) = \int_{-\infty}^{t} E_sW(s,s)pe^{(g-p)(t-s)}ds \tag{6}$$

Differentiating equation with respect to t, we have the aggregate wealth growth equation

$$\frac{dW(t)}{dt} = pE_tW(t,t) + (g-p)W(t) \tag{7}$$

We need to compute $pE_tW(t,t)$ in formula (7). Since $E_tW(t,t)$ is the mean starting wealth of the agents born at time t, $pE_tW(t,t)$ represents the aggregate starting wealth of the newborns at time t. The aggregate starting wealth $pE_tW(t,t)$ consists of two parts: the private bequest and the public subsidy.

The newborn whose parents have a bequest motive receives an inheritance. By the bequest function in Proposition 1, we find that the aggregate inheritance which the newborns receive from their parents after estate tax is $q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}(1-\zeta)W(t)$.

The public subsidy is determined by the government's budget. The government collects capital and estate taxes, finances government expenditure which is proportional to the aggregate wealth, and provides subsidies to qualifying newborns so that the government budget is balanced at any time.

Government collects estate taxes when agents leave bequests to their children. From the expression of bequest function in Proposition 1 we obtain government's revenue from the estate tax, $q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}\zeta W(t)$. Government's capital tax is $\tau W(t)$. Let η denote the ratio of government expenditure to aggregate wealth. Deducting the government expenditure from the total revenue of the government, we obtain the subsidies to the newborns, $q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}\zeta W(t)+\tau W(t)-\eta W(t)$. Note that the total subsidies to newborns do not depend on how they subsidies are distributed. We assume that the government tax revenue is greater than government expenditure.

The aggregate subsidy is $\left(q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}\zeta+\tau-\eta\right)W(t)$ and the aggregate inheritance is $q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}(1-\zeta)W(t)$. Suming these two, we have the aggregate starting wealth of newborns:

$$pE_tW(t,t) = (q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}} + \tau - \eta)W(t) \tag{8}$$

Substituting equation (8) into equation (7), we obtain:

$$\frac{dW(t)}{dt} = (q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}} + g - p + \tau - \eta)W(t) \tag{9}$$

Then formula (9) plus the initial aggregate wealth, $W(0)$, determines the evolution of the aggregate wealth of the economy. Let $\tilde{g}$ denote the growth rate of the aggregate wealth. From equation (9), we know that $\tilde{g} = q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}} + g - p + \tau - \eta$. Thus the relative growth rate of individual wealth to aggregate wealth is

$$g - \tilde{g} = p - q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}} - (\tau - \eta) \tag{10}$$

3. Wealth distribution and inequality

We now investigate the cross-sectional wealth distribution. Since the aggregate wealth level is growing, one way to study the cross-sectional wealth distribution is to investigate the distribution of the ratio of individual wealth to aggregate wealth.[12] Following Gabaix (1999), we define $X(s,t)$ as the the ratio of the individual wealth to the aggregate wealth.

$$X(s,t) = \frac{W(s,t)}{W(0)e^{\tilde{g}t}} \tag{11}$$

By equations (3) and (11), $X(s,t)$ also generates Geometric Brownian Motion.

$$dX(s,t) = (g - \tilde{g})X(s,t)dt + \kappa X(s,t)dB(s,t) \tag{12}$$

Then $X(s,t)$ is lognormally distributed, and

$$X(s,t) = X(s,s)\exp[(g - \tilde{g} - \frac{1}{2}\kappa^2)(t-s) + \kappa(B(s,t) - B(s,s))]$$

where we assume that $g - \tilde{g} - \frac{1}{2}\kappa^2 \geq 0$.[13]

To investigate the cross-sectional distribution of $X(s,t)$, we need to know not only the evolution function of $X(s,t)$ during an agent's the lifetime, but also the change of $X(s,t)$ between two consecutive generations. The evolution of wealth during an agent's the lifetime reflects the impact of age and of the stochastic rates of return on capital. The change of $X(s,t)$ between two consecutive generations reflects the role of inheritance and government subsidies.

[12] This is equivalent to discounting the individual wealth level by the aggregate wealth growth rate, $\tilde{g}$, plus the normalization, $W(0) = 1$, as in Benhabib and Bisin (2006).

[13] This is a technical assumption. Otherwise $X(\cdot,t)$ converges to 0 almost surely. Note that this assumption implies that $g - \tilde{g} \geq \frac{1}{2}\kappa^2 > 0$.

Government subsidies are distributed by the following rule. Government only subsidizes the newborns. If a newborn's inheritance is lower than a threshold level that is proportional to the aggregate wealth, the government gives the newborn a subsidy that brings their starting wealth to the threshold level.[14] If the newborn's inheritance is higher than the threshold level, the newborn does not receive a wealth subsidy from the government. The newborn whose parents do not have a bequest motive receives a wealth subsidy which is equal to the threshold level, and starts life at the threshold level of wealth.[15] Since we assume that the government tax revenue is greater than government expenditure, governement can always guarantee a positive threshold level. The redistributive policy then implies that the newborn without inheritance has positive starting wealth. This specific wealth level is endogenous and will be determined below.

Let $x^*W(t)$ be the threshold level of wealth below which newborns qualify for the government wealth subsidy. Suppose that a parent with wealth $W(e,s)$ leaves bequest $Z(e,s)$ to his child.[16] If $(1-\zeta)Z(e,s) \geq x^*W(s)$, the child's starting wealth is determined by $W(s,s) = (1-\zeta)Z(e,s)$. Since, by the optimal policy, $Z(e,s) = (\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}W(e,s)$, we have

$$W(s,s) = (1-\zeta)Z(e,s) = (\frac{p\chi(1-\zeta)}{A\mu})^{\frac{1}{\gamma}}W(e,s) \tag{13}$$

Dividing both sides of equation (13) by $W(0)e^{\tilde{g}s}$, and applying the definition of

[14]We may think of part of these subsidies as the discounted value of lifetime transfers.. or consider initial wealth at birth to be the discounted value of lifetime earnings. See also section 6 below.

[15]Benhabib and Bisin (2006) also use a welfare policy where the subsidy is designed to top up all bequests to newborns, including zero bequests, that fall short of a minimum wealth level that grows at the rate of growth of the economy. Both in the Benhabib-Bisin model and in our model, if the parents cared not about the gross bequest, but the bequest net of estate taxes, the optimal bequests would be different due to the induced non-convexity of the agent's optimization problem. Using standard smooth pasting arguments Benhabib and Bisin (2006) show in their appendix that if parents cared about net bequests, that the optimal net bequest would be zero until a threshold level that exceeds the minimum wealth for the newborn, and then revert exactly to the level prescribed by the model. As the wealth subsidy threshold to newborns goes to zero, the optimal bequest and consumption functions of the two models converge. In section 6 below we also consider a policy where all newborns, irrespective of inheritance, receive the same subsidy, which may also be interpreted, after adjusting the fiscal policies, as the discounted value of lifetime labor earnings.

[16]By our notation, e in $W(e,s)$ and $Z(e,s)$ means that the parent is born at time e and $e \leq s$.

$X(s,t)$, equation (11), we have $X(s,s) = (\frac{p\chi(1-\zeta)}{A\mu})^{\frac{1}{\gamma}} X(e,s)$. Let

$$\rho = (\frac{p\chi(1-\zeta)}{A\mu})^{\frac{1}{\gamma}} \tag{14}$$

The transfer wealth ratio between two consecutive generations when the inherited wealth of the newborn is above the threshold for a government subsidy then is $X(s,s) = \rho X(e,s)$.

If on the other hand the parents have a bequest motive but their wealth level $W(e,s) < \frac{x^*}{\rho} W(s)$, or if the parents do not have a bequest motive, then the government subsidizes their children.The children start their lives at the wealth level of $x^* W(s)$.

We now characterize the cross-sectional distribution of $X(\cdot,t)$ at time t, $f(x,t)$. In section 8.2 of the Appendix we derive the forward Kolmogorov equation of $f(x,t)$

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\kappa^2 x^2 f(x,t)) - \frac{\partial}{\partial x}((g-\tilde{g})xf(x,t)) - pf(x,t) + qf(\frac{x}{\rho},t)\frac{1}{\rho}, \quad x > x^* \tag{15}$$

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\kappa^2 x^2 f(x,t)) - \frac{\partial}{\partial x}((g-\tilde{g})xf(x,t)) - pf(x,t), \qquad x < x^* \tag{16}$$

The partial differential equations do not hold at $x = x^*$.[17] By the definition of $X(s,t)$, equation (11), we know that

$$\int_0^{+\infty} xf(x,t)dx = 1, \forall t \geq 0 \tag{17}$$

It is difficult to solve the partial differential equations with the initial distribution. Instead, we investigate the behavior of the equations in the long run, the stationary distribution of the wealth.[18]

Proposition 2. *The stochastic process, $X(\cdot,t)$, is ergodic.*

[17] For this point, we greatly benefited from the discussion with Matthias Kredler and Henry P. McKean.

[18] Benhabib and Bisin (2006) do study the transition dynamics and convergence of the PDE above, but their setting is simpler because it does not involve stochastic returns.

Proposition 2 guarantees the existence and uniqueness of the stationary distribution of stochastic process, $X(\cdot,t)$: starting from any initial distribution, the stochastic process converges to the unique stationary distribution.

In the stationary distribution, we have $\frac{\partial f(x,t)}{\partial t}=0$. We deduce, from the partial differential equations, the stationary distribution $f(x)$ which satisfies the following ordinary differential equations:

$$\frac{1}{2}\kappa^2x^2f''(x)+(2\kappa^2-(g-\tilde{g}))xf'(x)+(\kappa^2-(g-\tilde{g})-p)f(x)+qf(\frac{x}{\rho})\frac{1}{\rho}=0, x>x^* \tag{18}$$

and

$$\frac{1}{2}\kappa^2x^2f''(x)+(2\kappa^2-(g-\tilde{g}))xf'(x)+(\kappa^2-(g-\tilde{g})-p)f(x)=0, x<x^* \tag{19}$$

Proposition 2 guarantees the existence and uniqueness of the solution of equations (18) and (19) with the boundary condtions

$$\int_0^{+\infty} f(x)dx=1 \tag{20}$$

and

$$\int_0^{+\infty} xf(x)dx=1 \tag{21}$$

Condition (20) follows from the normalization of population size. Condition (21), which is the mean preservation condition, is from equation (17).

The endogenous value x^* is determined by government's subsidy policy:

$$(p-q)x^*+q\int_0^{\frac{x^*}{\rho}}(x^*-\rho x)f(x)dx=q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}\zeta+\tau-\eta. \tag{22}$$

In the left hand side of equation (22), the term $(p-q)x^*$ is the government subsidy to newborns whose parents do not have a bequest motive, and the term $q\int_0^{\frac{x^*}{\rho}}(x^*-\rho x)f(x)dx$ is the government subsidy to the newborns who receive inheritance lower than x^*. The term on the right hand side of equation (22), $q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}\zeta+\tau-\eta$, is the subsidy to all the newborns from the government's budget.

3.1. Pareto distribution

In this subsection, we first discuss the special case of no inheritance and then proceed to the general case of inheritance. In both cases the stationary distribution turns out to be a double Pareto distribution.

3.1.1. No inheritance

If agents do not have bequest motive, they leave no bequest to their children. The starting wealth of the newborn is the government subsidy. This closes one of the channels for the intergenerational transmission of inequality in wealth distribution, and corresponds to the special case of $q = 0$ in the general model. From equations (18) and (19), we know that the density function of the stationary distribution, $f(x)$, solves

$$\frac{1}{2}\kappa^2 x^2 f''(x) + (2\kappa^2 - (g - \tilde{g}))xf'(x) + (\kappa^2 - (g - \tilde{g}) - p)f(x) = 0, x \neq x^*$$

All of the newborns are injected into the economy through the discounted wealth level $x^* = \frac{\tau - \eta}{p} > 0$.

Proposition 3. *The stationary distribution in the no inheritance case has the following kernel*

$$f(x) = \begin{cases} C_1 x^{-\beta_1} & \text{when } x \leq x^* \\ C_2 x^{-\beta_2} & \text{when } x \geq x^* \end{cases}$$

where β_1 and β_2 are the two roots of

$$\frac{\kappa^2}{2}\beta^2 - (\frac{3}{2}\kappa^2 - (g - \tilde{g}))\beta + \kappa^2 - p - (g - \tilde{g}) = 0.$$

Then $\beta_1 = \frac{\frac{3}{2}\kappa^2 - (g-\tilde{g}) - \sqrt{(\frac{1}{2}\kappa^2 - (g-\tilde{g}))^2 + 2\kappa^2 p}}{\kappa^2}$ and $\beta_2 = \frac{\frac{3}{2}\kappa^2 - (g-\tilde{g}) + \sqrt{(\frac{1}{2}\kappa^2 - (g-\tilde{g}))^2 + 2\kappa^2 p}}{\kappa^2}$.

When people die, there is a shift of the wealth level.[19] By assumption, the individual wealth growth rate is higher than the aggregate growth rate, $g - \tilde{g} = p - (\tau - \eta) > 0$. The following proposition shows that $f(x)$ is integrable on $(0, +\infty)$

[19] For the dying people whose wealth levels are higher than $x^* = \frac{\tau - \eta}{p}$, the wealth levels of heirs shift down. For the dying people whose wealth levels are lower than $x^* = \frac{\tau - \eta}{p}$, the wealth levels of heirs shift up.

and therefore is a distribution function. Furthermore the proposition shows that $xf(x)$ is also integrable on $(0, +\infty)$.

Proposition 4. $\beta_1 < 1$ *and* $\beta_2 > 2$.

The normalization condition, equation (20), gives us $\int_0^{x^*} C_1 x^{-\beta_1} dx + \int_{x^*}^{+\infty} C_2 x^{-\beta_2} dx = 1$, and the mean preservation condition, equation (21), gives us $\int_0^{x^*} C_1 x^{1-\beta_1} dx + \int_{x^*}^{+\infty} C_2 x^{1-\beta_2} dx = 1$. Combining these two conditions, we can determine C_1 and C_2.[20] Reed (2001) uses moment generating functions to derive the stationary distribution of a Geometric Brownian Motion process. The no inheritance case treated here is an alternative way to derive the same distribution that Reed (2001) derives.

3.1.2. The general case

Now we set $p > q > 0$ so that some agents have a bequest motive.

Proposition 5. *The stationary distribution has the following kernel*

$$f(x) = \begin{cases} C_1 x^{-\beta_1} & \text{when } x \leq x^* \\ C_2 x^{-\beta_2} & \text{when } x \geq x^* \end{cases}$$

where β_1 *is the smaller root of the characteristic equation*

$$\frac{\kappa^2}{2}\beta^2 - (\frac{3}{2}\kappa^2 - (g - \tilde{g}))\beta + \kappa^2 - p - (g - \tilde{g}) = 0 \tag{23}$$

and β_2 *is the larger solution of the characteristic equation*

$$\frac{\kappa^2}{2}\beta^2 - (\frac{3}{2}\kappa^2 - (g - \tilde{g}))\beta + \kappa^2 - p - (g - \tilde{g}) + q\rho^{\beta-1} = 0. \tag{24}$$

The characteristic equations represent the forces that influence wealth inequality in the economy. Note that $g - \tilde{g} = p - q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}} - (\tau - \eta)$ reflects the relative growth rate of individual wealth to the aggregate wealth. Through

[20] For an even simpler model that still generates the double Pareto distribution without inheritance and without government taxes and transfers, and where all agents are born with the same positive initial wealth, see Appendix 8.10.

this term the parameter of preference for bequests, χ, and the capital tax rate, τ, influence the Pareto coefficients β_1 and β_2. The volatility of the price of the risky asset is reflected by the term $\kappa = \frac{\alpha - r}{\gamma\sigma}$. The capital tax rate, τ, is one of the two fiscal policy tools of the government. The other policy tool is the estate tax rate, ζ, which is reflected in the intergenerational transmission term, $\rho = (\frac{p\chi(1-\zeta)}{A\mu})^{\frac{1}{\gamma}}$. The strength of bequest motive χ also influences ρ. Note that while the volatility of risky asset, σ, influences the individual wealth growth rate, it does not influence the relative growth rate of wealth. Therefore σ does not influence the Pareto coefficient through the relative growth rate, $g - \tilde{g}$. On the other hand, the capital tax rate, τ, has an impact on the relative growth rate, but has no impact on the volatility of wealth growth, κ.

The following proposition characterizes the two solutions of the characteristic equations (23) and (24).

Proposition 6. $\beta_1 < 1$ *and* $\beta_2 > 2$.

The proposition implies the integrability of $f(x)$ and $xf(x)$ on $(0, +\infty)$. The integrability of $f(x)$assures that it is a distribution function. But this does not imply that the variance necessarily exists.

From the normalization condition, $\int_0^{x^*} C_1 x^{-\beta_1} dx + \int_{x^*}^{+\infty} C_2 x^{-\beta_2} dx = 1$, and the mean preservation condition, $\int_0^{x^*} C_1 x^{1-\beta_1} dx + \int_{x^*}^{+\infty} C_2 x^{1-\beta_2} dx = 1$, we can determine the terms C_1 and C_2 of the stationary distribution density function, $C_1 = (1 - \frac{1-\beta_2}{2-\beta_2} x^*)(x^*)^{\beta_1 - 2} \frac{(2-\beta_1)(2-\beta_2)(1-\beta_1)}{\beta_2 - \beta_1}$and $C_2 = (1 - \frac{1-\beta_1}{2-\beta_1} x^*)(x^*)^{\beta_2 - 2} \frac{(2-\beta_1)(2-\beta_2)(1-\beta_2)}{\beta_2 - \beta_1}$.

With the explicit form of $f(x)$, we can find the endogenous x^* by equation (22).

$$x^* = \frac{q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}\zeta + \tau - \eta + \rho q - \frac{\rho^{\beta_2 - 1}(2-\beta_1)}{\beta_2 - \beta_1} q}{p - \frac{\rho^{\beta_2 - 1}(1-\beta_1)}{\beta_2 - \beta_1} q}$$

Both in cases of no inheritance and inheritance, the relative wealth ratios follow double Pareto distributions. In the Appendix 8.8 we derive the Lorenz curve and the Gini coefficient of the double Pareto distribution.

4. The calibrated economy

We calibrate parameters to simulate our highly stylized and abstract model economy. We explore the numerical relationship between the Gini coefficient and the fundamental parameters. We choose the annual time discount rate, $\theta = 0.03$, the preference parameter for bequests, $\chi = 15$, the coefficient of relative risk aversion, $\gamma = 3$, the annual risk-free interest rate, $r = 1.8\%$, the annual average return on the risky asset, $\alpha = 8.8\%$, which implies that the risk premium of the risky asset is $\alpha - r = 7\%$, and the volatility of return of the risky asset, $\sigma = 0.26$.[21] As in Benhabib and Bisin (2006), we pick $p = 0.016$, which implies that agents have an expected working life of $\frac{1}{p} = 62.5$ years. As noted earlier, under the fair insurance market we set the life insurance price $\mu = p = 0.016$. Kopczuk and Lupton (2006) find that roughly 3/4 of the elderly single population has a bequest motive. Setting $\frac{q}{p} = 0.75$ implies that $q = 0.012$. Following Friedman and Carlitz (2005) we calibrate the effective estate tax rate at $\zeta = 0.19$. Since only net government expenditures affect results and play a role in the analysis, we set government expenditures $\eta = 0$. This leaves the calibration of the capital tax on wealth at τ. Since we have $\eta = 0$ we have to consider τ as net of capital and income taxes that are collected for purposes other than redistribution. We have to set τ such that, together with estate taxes, it will generate the revenue to subsidize transfer payments. These transfers, in discounted value, correspond to the expected government wealth transfer to the young. At about $9 - 10\%$ of GDP in the US, transfers amount to about a trillion dollars or about $\$9,000$ per household.[22] Discounted over working life at an interest rate of 6.5%, this corresponds to an initial wealth of about $\$130,000$. Thus we set $\tau = 0.004$ so that together with estate taxes, the capital taxes can finance the redistributive transfers.[23] Section 5.1 and the tables in section 8.13 of the Appendix provide sensitivity results for alternative calibrations of capital and estate taxes and other parameters.

[21] Our calibration means that the volatility of the return of the risky asset is 3 times of the mean of the return of the risky asset. This ratio is slightly higher than that of S&P 500 Index during the perod of 1952-1999, which is 1.7794 in Campbell and Viceira (2002) to capture the variance in returns of less diversified portfolios including private business initiatives and private assets.

[22] If we add state subsidies to public education, transfers would be even higher.

[23] From the U.S. 2004 Survey of Consumer Finances average household wealth is about \$448,000, so that total household wealth is about 50 trillion. At the calibrated estate tax of 19% our model would produce a fraction $q * \mu * \zeta$ of household wealth in estate taxes, amounting to 86 billion, about 2.5-3 times the actual collection, but still a small fraction of government revenues. As shown in the last table of the appendix, lowering ζ to match the actual collections has little effect on the results of our calibration. Note that in our model, relative to capital taxes collected through τ to finance transfers for new households, estate taxes are quite small. In the US economy they are insignificant.

The following table reports the numerical results of the calibrated economy:

	A	ω	ρ	g	$\tilde{g}$	$g-\tilde{g}$	κ
Results	13988.6	0.345168	0.0954114	0.0107745	0.000187993	0.0105865	0.0897436

From the simulation results, the portfolio share that the agent invests in the risky asset is $\omega = 0.345168$. The bequest function is[24] $Z(s,t) = (\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}W(s,t) = 0.117792W(s,t)$ so that purchased life insurance is given by $P(s,t) = p(Z(s,t) - W(s,t)) = -0.0141153W(s,t)$. Negative life insurance corresponds precisely to annuities. Given, τ, and ω the average return on wealth, including annuity receipts, is 5.18% and the individual consumption function is $C(s,t) = A^{-\frac{1}{\gamma}}W(s,t) = 0.0415026W(s,t)$.

The growth rate of the wealth of agents is $g = 0.0107745$,while the growth rate of the aggregate economy, $\tilde{g}$, is almost zero. The reason for the low growth rates is simple and can be explained to a rough approximation as follows. We calibrate the riskless rate of return at 1.8% and the mean of the risky rate at 8.8%,to match the 7% premium for risky asset. In the optimal portfolio 34% of wealth is held in the risky asset, so that together with annuity receipts agents receive a return on wealth of 5.18%. The Euler equation relates the consumption growth rate to the difference between the mean return on wealth and the discount rate, multiplied by the intertemporal elasticity of substitution $\sigma^{-1} = \frac{1}{3}$. Thus the small difference between the mean return on wealth and the discount rate necessarily results in low growth rates. In section 4.5 we provide a sensitivity analysis by raising both the riskless and the mean risky returns while maintaining the 7% risk premium, and we obtain higher growth rates of wealth.

We can now plot the stationary distribution of wealth for this calibration, with mean wealth normalized to 1:

In the simulated economy, $x^* = 0.313596$,corresponding to $\$129,000$ or 31% of mean household wealth of $\$448,000$, normalized to unity in our model. This is very close to the discounted value of lifetime transfers of roughly $\$130,000$ that we used in order to calibrate τ above. When $x < x^*$, the density of the distribution is governed by C_1 and β_1 (the increasing part of the density in Figure 4.1). When

[24] The bequest function, abstracting from inter-vivos transfers, is only for the agents who do leave bequests. Therefore for the pre-tax bequest flow we must multiply the right side by q, so bequest flows are $0.0014W$.

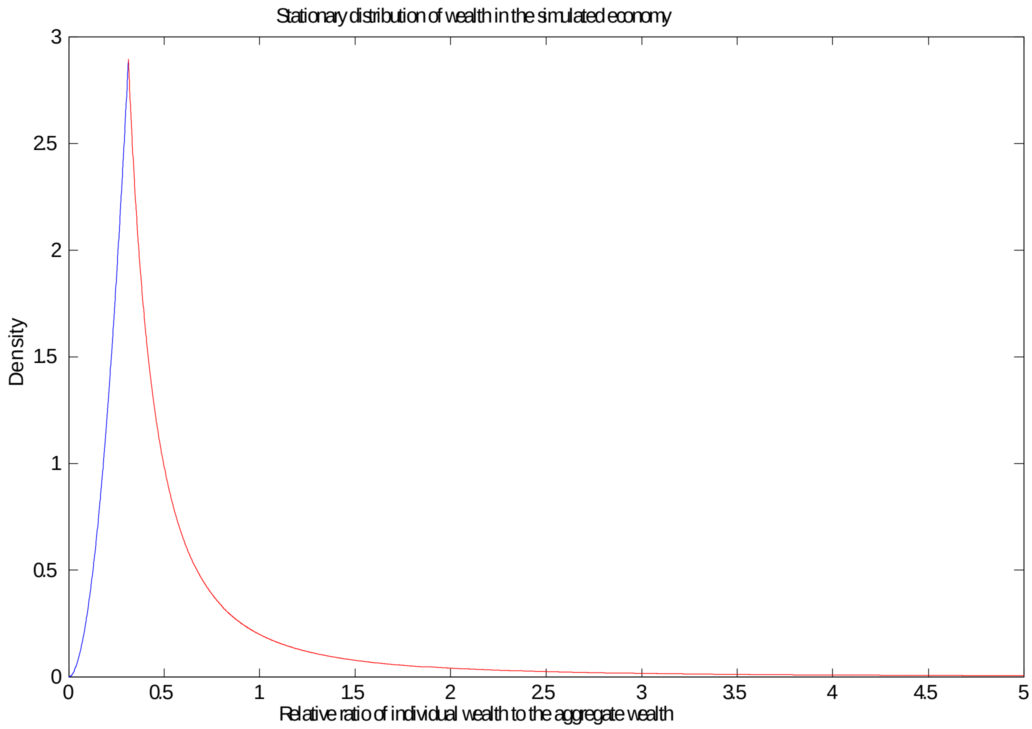

Figure 4.1: Model Data

$x > x^*$, the density is governed by C_2 and β_2 (the decreasing part of the density in Figure 4.1). Let F^* denote the percentage of the population whose wealth level is lower than x^*. The following table lists the parameters of the distribution:

	x^*	F^*	β_1	C_1	β_2	C_2
Distribution	0.313596	0.30558	-1.96772	28.3256	2.30647	0.199409

Plugging the numbers of C_1, β_1, C_2, β_2, and x^* into the density function

$$f(x) = \begin{cases} C_1 x^{-\beta_1} & \text{when } x \leq x^* \\ C_2 x^{-\beta_2} & \text{when } x \geq x^* \end{cases}$$

we find that

$$f\_(x^*) - f_+(x^*) = -0.0011666 \tag{25}$$

where $f\_(x^*)$ and $f_+(x^*)$ are the left and right limit of the density function $f(x)$ at x^*. We expect that $f(x)$ will be continuous at $x = x^*$. We rigorously prove this result by finding two boundary condtions of the stationary distribuion $f(x)$ at $x = x^*$:

$$f\_(x^*) - f_+(x^*) = 0 \tag{26}$$

and

$$\frac{1}{2}\kappa^2(x^*)^2[f'\_(x^*) - f'_+(x^*)] = p\int_0^{\frac{x^*}{\rho}} f(x)dx + (p-q)\int_{\frac{x^*}{\rho}}^{+\infty} f(x)dx \tag{27}$$

where $f'\_(x^*)$ and $f'_+(x^*)$ are the left and right limit of $f'(x)$ at x^*. Equation (26) means that the density function $f(x)$ is continuous at $x = x^*$. Note that the right hand side of equation (27) is exactly the injection of the newborns at $x = x^*$. Thus equation (27) relates the injection and the difference between the left derivative and the right derivative of $f(x)$ at $x = x^*$. For the two boundary conditions, equations (26) and (27), we can not verify them explicitly in the general case, since the equation (24) has no explicit solution. However, we can explicitly verify that equations (26) and (27) are satisfied in the no inheritance case. For the general case, we employ numerical methods to check whether the boundary conditions are satisfied. Equation (25) and equation (28) below in the simulated results show that both boundary conditions are satisfied in our calibrated economy.

$$\frac{1}{2}\kappa^2(x^*)^2[f'\_(x^*)-f'_+(x^*)]-p\int_0^{\frac{x^*}{\rho}} f(x)dx-(p-q)\int_{\frac{x^*}{\rho}}^{+\infty} f(x)dx = -4.74491\times 10^{-7} \tag{28}$$

The empirical wealth distribution from the U.S. 2004 Survey of Consumer Finances displays (see Figure 4.2) the two power-law-like tails, albeit with some jagged wiggles. Even though in our model we cannot generate the zero and negative wealth levels held by 8.9% of the population in the data, our model replicates the "double Pareto" distribution. The mode of the empirical distribution is around the zero wealth level, since unlike our model, it excludes the discounted value of government transfers to households.

The prominent features of wealth distribution are the fat-tail and upper skewness. Using quintiles and the Gini coefficient we compare the wealth distribution

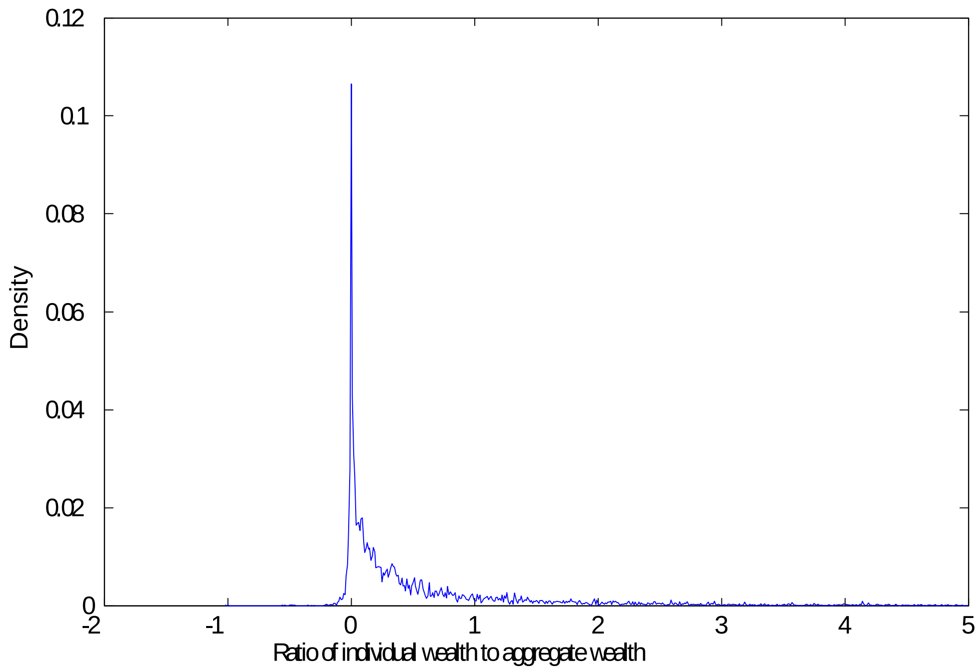

Figure 4.2: U.S. Data

of our model economy with the U.S. wealth distribution data.[25]

Economy	*Gini*	*First*	*Second*	*Third*	*Fourth*	*Fifth*
United States	0.78	−0.39	1.74	5.72	13.43	79.49
Model	0.64	4.07	6.21	8.16	12.24	69.32

We further disaggregate the top groups, and compare the percentiles of the wealth distribution for the United States and the benchmark model economies.

[25] The data of the U.S. economy in the following two tables are from Castaneda, Diaz-Gimenez and Rios-Rull (2003).

$Economy$	$90th - 95th$	$95th - 99th$	$99th - 100th$
$United\ States$	12.62	23.95	29.55
$Model$	8.84	15.75	34.33

Our model overpredicts the wealth share of the upper 1% wealth group. Our prediction is 34.33%, while in the data the number is 29.55%. Our model underpredicts the wealth share of the 90% − 95% group and 95% − 99% group. The predicted shares are, respectively, 8.84% and 15.75% while in the data the corresponding numbers are 12.62% and 23.95%. For the group of top 20%, the predicted number is 69.32% which is lower than the number of data, 79.49%. The model overpredicts the wealth shares held by the first, second and third quintile group. Our model predicts a Gini coefficient lower than that of data: Our prediction is 0.64, while in the data Gini is 0.78. Much of the wealth inequality in our model is due to the heavy upper tail. Our model predicts the heavy tail, but underpredicts the Gini of the wealth distribution as a whole. This in part is because we do not capture the inequalities in wealth induced by disparities in labor earnings.

4.1. Wealth distribution conditional on age

Within the same age group, even though the agents have the same age, they have the different starting wealth levels and the realizations of the stochastic rate of return. These two factors generate wealth inequality within an age cohort. The wealth distribution within the age 0 group stems from the initial wealth inequality of newborns. It reflects the heterogeneity of their inheritance. This distribution has a mass point, x^*, which has a positive probability. All the newborns, including those who do not receive inheritance (since their parents do not have bequest motive) and those whose inheritance is lower than the threshold level x^* (even though their parents have bequest motive), have a starting wealth x^*. For older cohorts, the distribution of wealth also reflects the element of luck coming from stochastic returns. For a fixed starting wealth level, and for luck driven by Brownian motion, the distribution of wealth conditional on age t is lognormal. In Appendix 8.8 we derive the distribution of wealth within age groups. We can plot the wealth distribution conditional on age in our simulated model. The wealth distribution within age groups also display upper skewness and fat tails.

To investigate the relationship between wealth inequality within the cohort and the age, we plot the Gini coefficient of the wealth distribution within the age

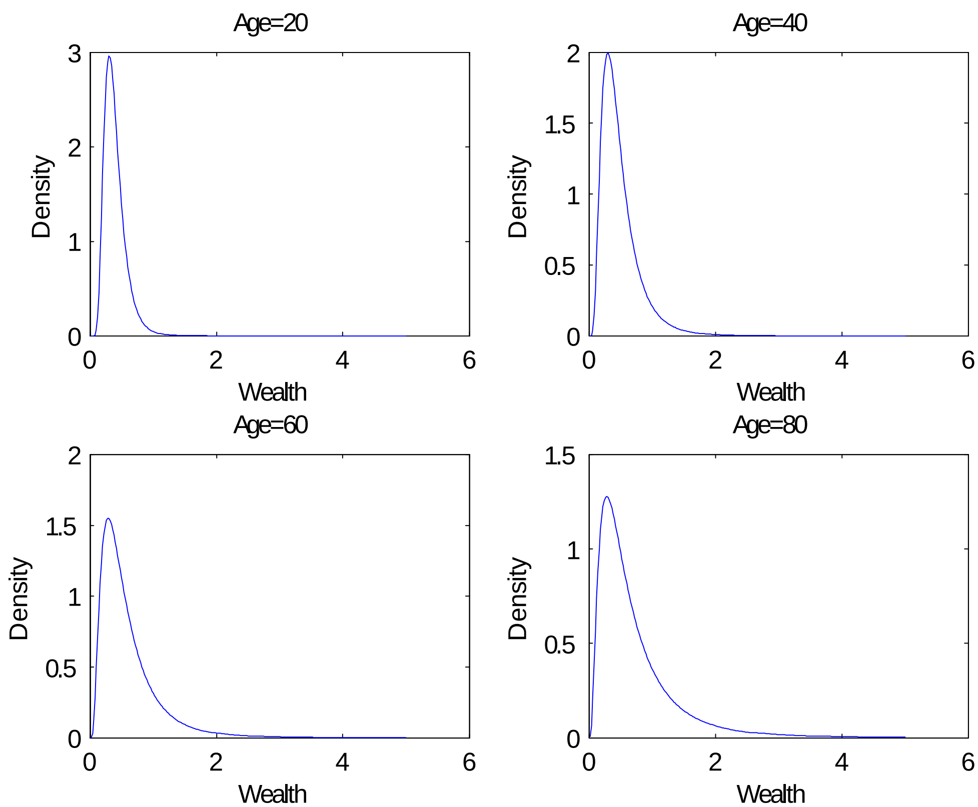

Figure 4.3: Wealth Distribution by Age Cohorts

group for ages 1 to age 80.[26] We find that the wealth inequality decreases as age goes up.[27]

[26]We do not plot the Gini coefficient of wealth distribution within age 0 (newborn) cohort, even though it is an interesting distribution.

[27]Huggett (1996) also studies the wealth inequality within age groups and notes a U shape: the Gini coeffienct declines to about agee 50 and then picks up again. See also Hendricks (2007) for empirical findings that the Gini coefficient declines with age of cohorts.

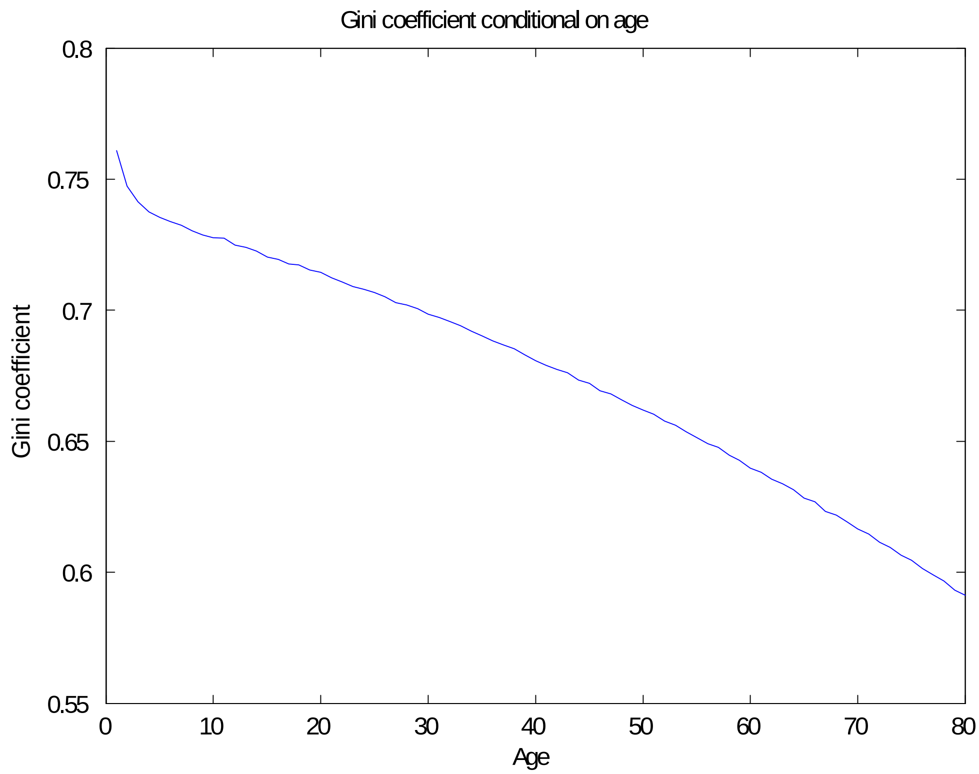

Figure 4.4: Gini by Age Cohorts

4.2. Inequality and bequests

Wealth inequality decreases as the parameter of bequest motive, χ, increases. On the one hand, when people have stronger bequest motives and leave higher bequests, the wealth process becomes more persistent across generations. More wealth inequality is inherited. On the other hand, if people purchase more life insurance or buy fewer annuities, the aggregate growth rate of wealth increases and the relative growth rate of individual wealth decreases. The lower relative growth rate of wealth causes the wealth distribution to become more equal. The simulated results below show that the relative growth effect dominates the inheritance effect. Therefore a higher bequest motive, χ, implies a lower Gini coefficient. We cut a slice of $\sigma = 0.26$ in Figure 4.5 to highlight this relationship:

χ	β_1	β_2	$Gini$
14	-1.97299	2.30493	0.637806
15	-1.96772	2.30647	0.636731
16	-1.9627	2.30795	0.635706
17	-1.95789	2.30936	0.634729

4.3. Inequality and the volatility of the risky asset

Wealth inequality decreases as the volatility of the risky asset increases. This counter-intuitive result is mainly due to the endogenous choice of the risky asset. When the volatility of risky asset increases, people hold a smaller share of risky asset, and in effect the volatility of the overall portfolio declines. The net effect of the portfolio reallocation and lower volatility of the wealth portfolio in turn lowers inequality. This first channel is the direct volatility effect on inequality. From the expression for A in Proposition 1, A increases as the volatility of the risky asset, σ, increases. By the formula (10) we also know that the higher is A, the higher is the relative growth rate. The second channel through which the volatility of the risky asset influences inequality is through the relative growth rate of the individual wealth: higher volatility implies a higher relative growth rate and therefore more inequality. The third channel works through the influence of the volatility of the risky asset on the bequests. By the formula (14), the higher is A, the lower is ρ. This means that the wealth process across generations becomes less persistent as the volatility of the risky asset, σ, increases. This channel reduces wealth inequality when the volatility of the risky asset increases. In our simulation, the direct portfolio effect of volatility and the bequest effect dominate the growth effect, so inequality declines with volatility. Setting $\chi = 15$, we can show the relationship between inequality and the volatility of the risky asset:

σ	β_1	β_2	$Gini$
0.2	-0.820429	2.25666	0.690649
0.21	-0.992948	2.26628	0.67958
0.22	-1.17291	2.27534	0.669451
0.23	-1.36038	2.28385	0.66018
0.24	-1.55528	2.29185	0.651688
0.25	-1.75772	2.29942	0.643865
0.26	-1.96772	2.30647	0.636731

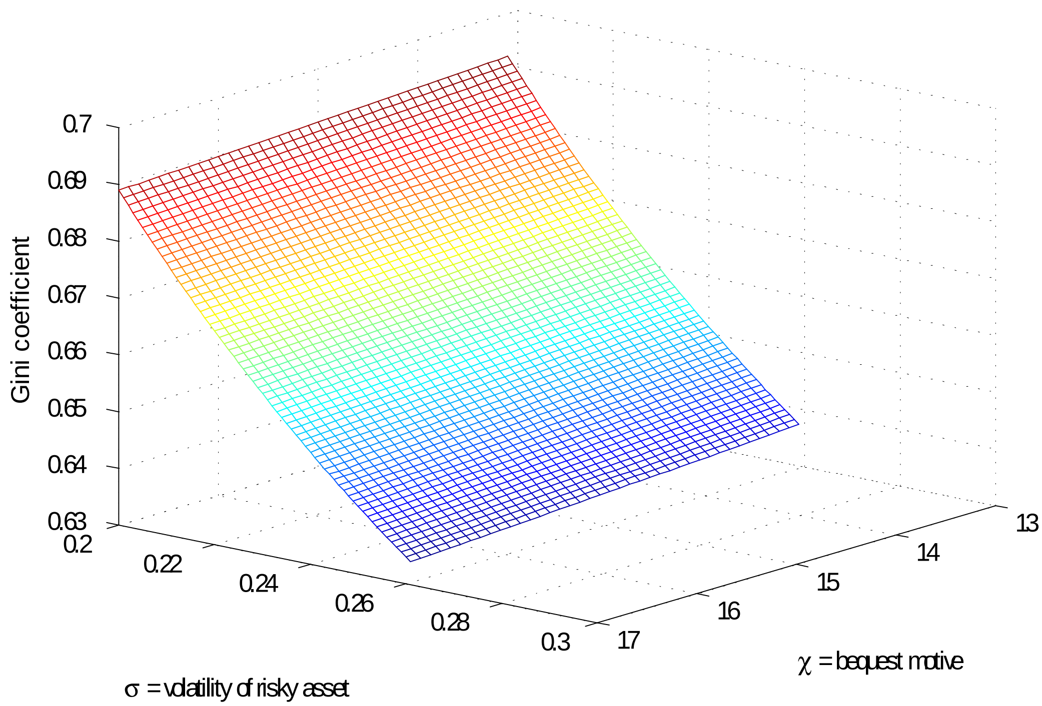

Figure 4.5: Gini by return volatility and bequest motive

We plot the relationship between the Gini coefficient, the preference for bequests, χ, and the volatility of the risky asset, σ, in Figure 4.5. For the simulated numbers, see the first table in section 8.13 of the Appendix.

Inequality and risk aversion

Risk aversion affects wealth inequality through all the three channels: growth, volatility and inheritance. We set $\chi = 15$ and $\sigma = 0.26$ to simulate the economy for $\gamma = 2$, $\gamma = 2.5$ and $\gamma = 3$. The Gini coefficient decreases with the increase of the coefficient of relative risk aversion.

γ	β_1	β_2	$Gini$
2	-0.366113	2.24326	0.714494
2.5	-1.09162	2.28013	0.667223
3	-1.96772	2.30647	0.636731

4.4. Inheritance, stochastic return and the age effect

We now explore the roles of inheritance, age, and stochastic rates of the capital return on wealth inequality. We investigate special cases to isolate the effect of each of these factors. To identify the effect of these three factors we construct two schemes. In scheme I, we first eliminate the investment opportunity of agents in the risky asset. Our model reduces to that of Benhabib and Bisin (2006). After we close the channel of stochastic returns, we find that the Gini coefficient of the economy decreases. Comparing the Gini coefficient of this special economy with the general case, we isolate the effect of the luck on wealth inequality. We then eliminate the bequest motive by setting $\chi = 0$, and study an economy without luck or inheritance. In scheme II, we first limit the age effect by setting the wealth growth rate of the agent relative to the growth rate of the economy to be as low as possible. Comparing the Gini coefficient of this special economy with that of the general case, we estimate the age effect. We then close the inheritance channel while keeping the relative growth rate as low as possible to identify the effect of inheritance on wealth inequality.

4.4.1. Scheme I

Stochastic rates of capital return We disentangle the contribution of stochastic rates of capital return to wealth inequality by shutting down the investment opportunity in the risky asset. In the economy without risky asset, the agent's discounted wealth can not be lower than the threshold level, x^*. The stationary distribution of wealth is a Pareto distribution:

$$f(x) = C_2 x^{-\beta_2} \quad x \geq x^*$$

where $C_2 = (\beta_2 - 1)(x^*)^{\beta_2 - 1}$ and β_2 satisfies the characteristic function

$$(g - \tilde{g})\beta - p - (g - \tilde{g}) + q\rho^{\beta - 1} = 0.$$

From the balance in the government budget, we have

$$(p - q)x^* + q\int_{x^*}^{\frac{x^*}{\rho}} (x^* - \rho x) C_2 x^{-\beta_2} dx = q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1 - \zeta)^{\frac{1-\gamma}{\gamma}}\zeta + \tau - \eta$$

We can determine x^*as

$$x^* = \frac{q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}\zeta + \tau - \eta}{p + \frac{1}{\beta_2 - 2}q\rho^{\beta_2 - 1} - \frac{\beta_2 - 1}{\beta_2 - 2}q\rho}$$

In the stationary distribution, the Gini coefficient is[28]

$$G = \frac{1}{2\beta_2 - 3}$$

For the standard calibration of our model, the Gini coefficient without risky asset is 0.439633. Comparing this Gini coefficient with that of the economy with risky asset, 0.636731, we find that the Gini coefficient decreases by about 31% when we close the investment opportunity for the risky asset. We can view this number as the contribution of luck to wealth inequality.

Intergenerational transmission and age effect After we close the channel for stochastic returns our model reduces to that of Benhabib and Bisin (2006). In order to close the intergenerational transmission channel, we set $\chi = 0$. Note that in the standard calibration of section 4, $\chi = 15$. The wealth process is more persistent across generations with inheritance than without. When people have bequest motives and leave higher bequests, more of the wealth inequality is inherited. On the other hand if people leave bequests, they receive smaller annuities and consume less. The growth rate of the aggregate economy increases because the initial wealth of agents at birth is higher due to higher bequests. As shown in Proposition 1 however, for our CRRA preferences the individual agent's growth rate is not influenced by the bequest motive parameter χ. When the aggregate economy grows faster, the lucky agents who earn high returns relative to the economy will not break away as easily and leave others behind by as much. The lower relative wealth growth rate, $g - \tilde{g}$, therefore causes the wealth distribution to become more equal. The simulated results below show that in fact the growth effect dominates the inheritance effect. Therefore we can have a higher Gini coefficient after we close the intergenerational transmission channel. Surprisingly, a stronger bequest motive and higher inheritance rates may decrease wealth inequality.

[28]See, for example, Chipman (1974). Note that in standard terminology the Pareto exponent corresponds to $\beta_2 - 1$ in our model.

χ	g	$\tilde{g}$	$g-\tilde{g}$	β_2	x^*	$Gini$
0	0.018	0.006	0.012	2.33333	0.25	0.6
15	0.018	0.00867146	0.00932854	2.63731	0.389242	0.439633

The empirical literature initiated by Kotlikoff and Summers (1981) emphasizes the role of bequests on the wealth accumulation. They show that life cycle savings without intergenerational transfers and bequests cannot account for the level of the U.S. capital stock. Updating the work of Kotlikoff and Summers, Gale and Sholtz (1994) show that inheritances (even excluding accidental bequests) plus various inter-vivos transfers account for at least 50% of the accumulation and transmission of the U.S. capital stock.

If we completely shut down the age effect on the other hand, for example by raising the tax rate τ to set the relative growth rate $g - \tilde{g} = 0$, the long-run stationary distribution becomes degenerate and its support consists of only x^*. Without luck or stochastic returns, inheritance alone will not generate or amplify inequality in the stationary distribution.

4.4.2. Scheme II

Age effect In this experiment we allow the stochastic return to remain and we pick $\tau = 0.0107$ so that the relative growth rate is $g - \tilde{g} \approx \frac{1}{2}\kappa^2$. Note that with stochastic returns $\frac{1}{2}\kappa^2$ is the lowest bound for $g - \tilde{g}$ that yields a non-degenerate stationary distribution. In this economy, the Gini coefficient is 0.402667 whereas in our benchmark economy the Gini coefficient is 0.636731. After we close the age effect, the Gini coefficient decreases by about 37%. We can view this number as the lower bound for the contribution of the age effect to wealth inequality.

Inheritance We now pick $\chi = 0$ in order to close the inheritance or intergenerational transmission channel. We set $\tau = 0.011973$ so that the economy-wide relative growth rate is $g - \tilde{g} = \frac{1}{2}\kappa^2$. In this economy the Gini coefficient becomes 0.401526. Relative to the case with inheritance and no age effect, the Gini is marginally lower.

4.5. Sensitivity analysis for rates of return

Here we explore increasing rates of return so as to obtain higher growth rates for the economy. To keep the risk premium at 7%, we adjust upwards both the return of the riskless asset and the return on the risky asset. We show the effects of raising the rates of return on the consumption function, the bequest function, the individual wealth growth rate and aggregate wealth growth rate. The higher are the rates of return, the higher is aggregate growth rate of the economy, $\tilde{g}$.

	A	ω	ρ	g	$\tilde{g}$	$g-\tilde{g}$	κ
$r=1.8\%, \alpha=8.8\%$	13988.6	0.345168	0.095411	0.010774	0.000187	0.010586	0.089743
$r=2\%, \alpha=9\%$	12774.4	0.345168	0.098343	0.011441	0.000898	0.010543	0.089743
$r=3\%, \alpha=10\%$	8419.67	0.345168	0.113004	0.014774	0.004448	0.010325	0.089743
$r=4\%, \alpha=11\%$	5839.38	0.345168	0.127664	0.018107	0.007999	0.010108	0.089743
$r=5\%, \alpha=12\%$	4214.38	0.345168	0.142325	0.021441	0.011549	0.009891	0.089743
$r=6\%, \alpha=13\%$	3140.5	0.345168	0.156985	0.024774	0.015100	0.009674	0.089743
$r=7\%, \alpha=14\%$	2402.58	0.345168	0.171646	0.028107	0.018650	0.009457	0.089743
$r=8\%, \alpha=15\%$	1878.8	0.345168	0.186306	0.031441	0.022201	0.009239	0.089743

The different wealth accumulation processes in the different economies result in the different stationary distributions of wealth. The effects of increasing the returns on capital are shown in the following table.

	x^*	F^*	β_1	C_1	β_2	C_2	$Gini$
$r=1.8\%, \alpha=8.8\%$	0.313596	0.30558	-1.96772	28.3256	2.30647	0.199409	0.636731
$r=2\%, \alpha=9\%$	0.315502	0.30647	-1.96029	27.5945	2.30866	0.200566	0.635217
$r=3\%, \alpha=10\%$	0.325052	0.31095	-1.92333	24.2825	2.31964	0.206376	0.627737
$r=4\%, \alpha=11\%$	0.334644	0.315486	-1.88666	21.4657	2.33078	0.212233	0.620396
$r=5\%, \alpha=12\%$	0.344287	0.320079	-1.85029	19.0571	2.34207	0.218147	0.613184
$r=6\%, \alpha=13\%$	0.353988	0.324729	-1.81422	16.9874	2.35353	0.224124	0.606098
$r=7\%, \alpha=14\%$	0.363749	0.329437	-1.77846	15.2009	2.36516	0.230169	0.599133
$r=8\%, \alpha=15\%$	0.373576	0.334203	-1.74301	13.6523	2.37698	0.236289	0.592287

5. Redistributive Policies

We now investigate the impact of redistributive government policy on wealth inequality and aggregate welfare in our calibrated model. Government fiscal policies affect wealth inequality through their effect on the relative growth rate, bequests and the redistributive subsidy. Note that government policy has no impact on the portfolio choice of the risky asset, and does not influence the volatility of individual wealth.

5.1. Tax and wealth inequality

Capital and estate taxes influence wealth inequality through the relative growth rate of wealth, $g - \tilde{g}$, through their effect on the intergenerational transmission of wealth and through their redistributive effects. Neither of the government policy tools has an impact on the volatility of wealth. In Figure 5.1 we plot the Gini coefficient as a function of the capital tax rate, τ, and the estate tax rate, ζ.[29] For the simulated numbers, see the last table in section 8.13 of the Appendix.

We calculate the Gini coefficients for combinations of the capital tax, τ and the estate tax, ζ, in a parameter region such that $g > 0$, $\tilde{g} > 0$ and $g - \tilde{g} - \frac{1}{2}\kappa^2 \geq 0$. The minimum value of the Gini coefficient is 0.4020, obtained for $\zeta = 0.95$ and $\tau = 0.0047$ where both ζ and τ are on the boundary of the parameter region and cannot be further increased.

The higher is the capital tax, τ, the lower is the relative growth rate, $g - \tilde{g}$. A lower $g - \tilde{g}$ implies that the wealth distribution is more equal. The higher is the capital tax, τ, the lower is the intergenerational transmission parameter, ρ. Lower ρ implies that the wealth process between two consecutive generations becomes less persistent and the wealth distribution becomes more equal. Furthermore when τ increases, x^* increases. In our calibrated economy therefore, a higher τ implies a lower Gini coefficient.

The higher the estate tax, ζ, the lower is the relative growth rate, $g - \tilde{g}$. At the same time the higher the estate tax, ζ, the lower is the intergenerational transmission parameter, ρ. The effect of ζ on $g - \tilde{g}$ and ρ can be obtained by analyzing equations (10) and (14). Furthermore, the higher is the estate tax, ζ, the higher is the bequest that the agent leaves to his children to partly offset the higher estate taxes. Thus a higher estate tax rate implies a lower consumption propensity and a larger bequest. The aggregate growth rate of the economy increases because

[29] In Figure (5.1), the range of τ is $0.004 - 0.0099$ and the range of ζ is $0 - 0.6$.

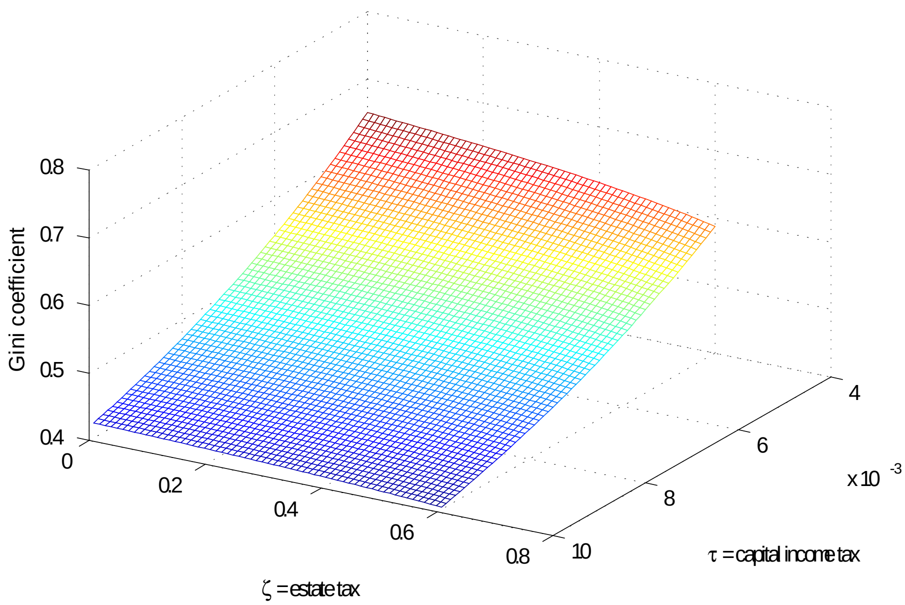

Figure 5.1: Gini by taxes

of the higher bequest levels. This in turn decreases the difference between the individual wealth growth rate and the aggregate growth rate since the estate tax ζ has no impact on the individual wealth growth rate. The wealth distribution is more equal as $g - \tilde{g}$ decreases. The wealth process between two consecutive generations becomes less persistent when ρ decreases. Furthermore x^* increases as ζ increases. The overall effect of a higher ζ therefore is a lower the Gini coefficient.

These results on the effects of taxes are consistent with our intuition: redistributive policies tend to reduce wealth inequality.

5.2. Taxes and welfare

Government policies influence both the individual utilities and the wealth distribution in the economy. We take aggregate welfare to be the integral of the individual utilities with respect to the cross-sectional wealth distribution. Thus we simply add the utilities of those currently alive. We compute aggregate welfare of the economy and find the optimal government fiscal policies.

There are two kinds of people in the economy. People with a bequest motive account for $\frac{q}{p}$ in the population. And people without a bequest motive account for $1 - \frac{q}{p}$ in the population. In section 8.11 of the Appendix we the derive the value function of people with bequest motives:

$$U(w) = \frac{1}{1-\gamma} \left(\frac{\theta + p - (1-\gamma)(r - \tau + \mu + \frac{(\alpha - r)^2}{2\gamma\sigma^2})}{\gamma(1 + (p\chi)^{\frac{1}{\gamma}} \mu^{\frac{\gamma-1}{\gamma}} (1-\zeta)^{\frac{1-\gamma}{\gamma}})} \right)^{-\gamma} w^{1-\gamma}$$

and the value function of people with no bequest motive:

$$U_0(w) = \frac{1}{1-\gamma} \left(\frac{\theta + p - (1-\gamma)(r - \tau + \mu + \frac{(\alpha - r)^2}{2\gamma\sigma^2})}{\gamma} \right)^{-\gamma} w^{1-\gamma}$$

The aggregate welfare of the economy is the weighted sum of the individual utilities with weights according to the cross-sectional wealth distribution of the two groups of agents.

$$\Omega(\tau, \zeta) = \frac{q}{p} \int_0^{+\infty} U(w) f(w) dw + \frac{p-q}{p} \int_0^{+\infty} U_0(w) f(w) dw$$

In section 8.11 of the Appendix we derive the aggregate welfare function:

$$\begin{aligned}\Omega(\tau,\zeta) &= [\frac{q}{p}\frac{1}{1-\gamma}\left(\frac{\theta+p-(1-\gamma)(r-\tau+\mu+\frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma(1+(p\chi)^{\frac{1}{\gamma}}\mu^{\frac{\gamma-1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}})}\right)^{-\gamma} \\ &\quad +\frac{p-q}{p}\frac{1}{1-\gamma}\left(\frac{\theta+p-(1-\gamma)(r-\tau+\mu+\frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma}\right)^{-\gamma}]\times \\ &\quad [\frac{C_1}{2-\gamma-\beta_1}(x^*)^{2-\gamma-\beta_1}-\frac{C_2}{2-\gamma-\beta_2}(x^*)^{2-\gamma-\beta_2}]\end{aligned}$$

Note that β_1 and β_2 are functions of capital tax rate, τ, and estate tax rate, ζ. Note also that β_2 has a non-linear relationship with τ, and ζ. In section 8.11 of the Appendix, we show that the aggregate welfare function is well-defined when $\beta_1 < -1$. In Figure (5.2) we plot welfare as a function of the capital tax rate, τ, and the estate tax rate, ζ.[30]

For the set of combinations of the capital tax, τ and the estate tax, ζ, such that $g > 0$, $\tilde{g} > 0$, $g-\tilde{g}-\frac{1}{2}\kappa^2 \geq 0$ and $\beta_1 < -1$, we calculate the aggregate welfare for our calibrated economy. The maximum of the welfare function is obtained for $\zeta = 0.18$ and $\tau = 0.0063$. The estate tax is very close to our calibration in the benchmark economy, but maximizing social welfare requires a high capital tax because our social welfare function, weighting only generations currently alive, puts a high emphasis on equality. Setting $\tau = 0.0063$ decreases the relative growth rate so that the Gini coefficient for the tax rates maximizing social welfare is now 0.5260. Of course a different welfare specification that puts more weight on future generations by including the utilities of those not yet born would put a higher weight on growth, and shift the optimal taxes from ζ that does not affect individual growth rates to τ that does.

6. A lump-sum subsidy policy

Previously we discussed a welfare policy for which only those newborns whose inheritance is lower than x^* receive a subsidy. Here we discuss the alternative policy where all the newborns receive a subsidy.[31] Note that this subsidy, after

[30] In Figure (5.2), the range of τ is $0.004-0.0099$ and the range of ζ is $0-0.5$.

[31] See Huggett (1996) for a calibrated model where accidental bequests are distributed equally to everyone, not just newborns. DeNardi (2004) also has some specifications of calibrated models

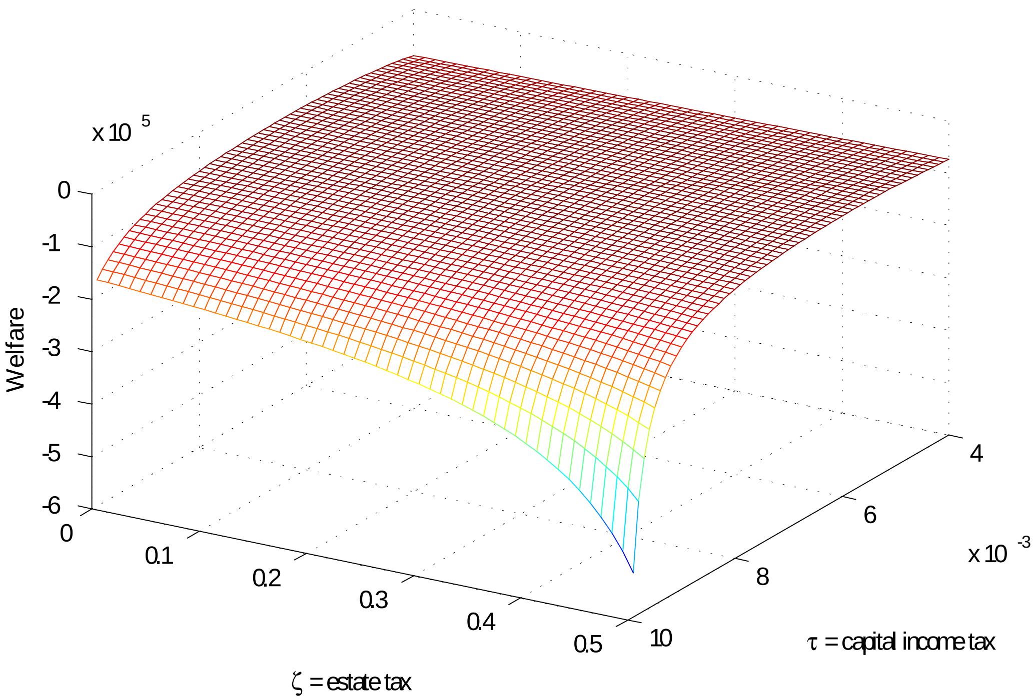

Figure 5.2: Welfare by taxes

adjusting the redistributive taxes for a balanced government budget, can also be interpreted as the discounted value of lifetime labor earnings received by all newborns. As we point out in section 2, the total subsidies to the newborns do not depend on how the subsidies are distributed to the newborns. Equation (8) gives us the total subsidies to the newborns. Under the lump-sum subsidy policy, each newborn with or without inheritance receive the same subsidy, which we denote by $b(t)$.

$$b(t) = \frac{q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}\zeta + \tau - \eta}{p} W(t) \tag{29}$$

Change of the subsidy policy from that in section 3 to the lump-sum subsidy policy only influences the change of wealth between two consecutive generations, while the individual wealth accumulation equation and the aggregate wealth growth rate are the same as those in section 2.

Now let

$$x^* = \frac{q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}\zeta + \tau - \eta}{p}$$

Note also that by equation (29) newborns without any inheritance are injected into the economy with the wealth level of $x^*W(t)$. Furthermore x^* here is different from that in equation (22).

We still investigate the stochastic process of the ratio of individual wealth to aggregate wealth. Even though the change of wealth between two consecutive generations here is different from that in section 3, the stochastic process during the agent's life time is the same as that in section 3. For large x, we can write the equation to characterize the evolution of the cross-sectional wealth distribution which is similar to equation (15) in section 3. An argument similar to Proposition 2 guarantees the existence and uniqueness of the stationary distribution. The stationary distribution converges to the unique stationary distribution from any initial distribution. The stationary distribution $f(x)$ now satisfies

$$\frac{1}{2}\kappa^2 x^2 f''(x) + (2\kappa^2 - (g-\tilde{g}))xf'(x) + (\kappa^2 - (g-\tilde{g}) - p)f(x) + qf(\frac{x-x^*}{\rho})\frac{1}{\rho} = 0 \quad \text{when } x > \frac{x^*}{1-\rho} \tag{30}$$

Note that equation (30) differs from equation (18) only by x^* in the last term: $qf(\frac{x-x^*}{\rho})\frac{1}{\rho}$ as opposed to $qf(\frac{x}{\rho})\frac{1}{\rho}$. For large x, the influence of the shift term, x^*,

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where accidental bequests are equally distributed to the population.

can be ignored in $qf(\frac{x-x^*}{\rho})\frac{1}{\rho}$. The solution of equation (30), $f(x)$, is approximated by the solution of equations (18) and (19). Therefore the stationary wealth distribution under the lump-sum subsidy policy has an approximately Pareto upper tail, and is approximated by the stationary distribution in Proposition 5 for large x.

7. Conclusions

There are three basic forces in our model that cause wealth inequality: stochastic rates of return, age, and inheritance. The effects of age and stochastic returns are captured by the mean relative growth rate and the volatility of the Geometric Brownian Motion. The role of inheritance is represented by the bequest motive which generates further jumps and reshuffling in the stochastic process for wealth accumulation. Geometric Brownian Motion coupled with the exponential death rate, despite the complications introduced by inheritance, generates a double Pareto distribution as the stationary distribution of wealth.

The dispersion of the age distribution causes wealth inequality because those who remain alive have an individual growth rate of wealth higher than the aggregate growth rate of the economy. The stochastic return of the wealth itself is a source of wealth inequality. And inheritance plays a role by perpetuating wealth inequality across generations while at the same time limiting dispersion by increasing the relative aggregate growth rate. The heterogeneity of the bequest motive, the estate taxes, as well as the annuities assure the existence of stationary distribution of wealth: together they restrict the spread of the Geometric Brownian Motion. Inheritance and luck are also responsible for generating a skewed distribution of wealth conditional upon age, that is for every age cohort.

Our rough calibration and simulation exercises disentangle some of the main sources of wealth inequality: stochastic returns, age and inheritance. Luck, or the stochastic rate of capital return contributes about 31% to wealth inequality in terms of the Gini coefficient and the age effect contributes about 37%. We show that surprisingly, inheritance can decrease wealth inequality because it increases the growth rate of aggregate wealth relative to individual wealth.

Finally, we show that government redistributive policies have important consequences for wealth inequality and welfare through their effects on the relative growth rates of wealth, on bequests, and on the size of government subsidies.

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8. Appendix

8.1. Proof of Proposition 1.

Proof: Let $J(W(s,t))$ be the optimal value of the agent with wealth $W(s,t)$. Following Merton (1992) and Kamien and Schwartz (1991), we set up the Hamilton-Jacobi-Bellman equation of the maximization problem

$$\begin{aligned} & (\theta+p)J(W(s,t)) \\ = & \max_{C,\omega,P}\{\frac{C(s,t)^{1-\gamma}}{1-\gamma}+p\chi\frac{((1-\zeta)Z(s,t))^{1-\gamma}}{1-\gamma} \\ & +J_W(W(s,t))[(r-\tau)W(s,t)+(\alpha-r)\omega(s,t)W(s,t)-C(s,t)-P(s,t)] \\ & +\frac{1}{2}J_{WW}(W(s,t))\sigma^2\omega^2(s,t)W^2(s,t)\} \end{aligned}$$

Using the relationship

$$Z(s,t)=W(s,t)+\frac{P(s,t)}{\mu}$$

we find the first order conditions:

$$C(s,t)^{-\gamma}=J_W$$

$$p\chi(1-\zeta)^{1-\gamma}Z(s,t)^{-\gamma}\frac{1}{\mu} = J_W$$
$$(\alpha - r)J_W W(s,t) = -J_{WW}\sigma^2\omega(s,t)W^2(s,t)$$

We guess the value function

$$J(W(s,t)) = \frac{A}{1-\gamma}W(s,t)^{1-\gamma}$$

where A is the undetermined constant. Then we find the expressions of $C(s,t)$, $Z(s,t)$, $P(s,t)$, and $\omega(s,t)$ from the first order conditions

$$C(s,t) = A^{-\frac{1}{\gamma}}W(s,t)$$
$$Z(s,t) = (\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}W(s,t)$$
$$P(s,t) = (\mu^{1-\frac{1}{\gamma}}(\frac{p\chi}{A})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}} - \mu)W(s,t)$$
$$\omega(s,t) = \frac{\alpha - r}{\gamma\sigma^2}$$

Plugging these equations into the Hamilton-Jacobi-Bellman equation, we can determine the constant A:

$$A = \left(\frac{\theta + p - (1-\gamma)(r - \tau + \mu + \frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma(1+(p\chi)^{\frac{1}{\gamma}}\mu^{\frac{\gamma-1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}})}\right)^{-\gamma}$$

From the budget constraint we obtain the wealth accumulation equation

$$dW(s,t) = [\frac{r-\tau+\mu-\theta-p}{\gamma} + \frac{1+\gamma}{2\gamma}\frac{(\alpha-r)^2}{\gamma\sigma^2}]W(s,t)dt + \frac{\alpha-r}{\gamma\sigma}W(s,t)dB(s,t).$$

8.2. Derivation of the forward Kolmogorov equation

Following Ross (1996), we heuristically derive the forward Kolmogorov equations (14) and (15).

Let $f(x,t;y)$ be the probability density of $X(t)$, given $X(0) = y$. By the Markovian property of the process

$$\Pr\{X(t) = x | X(0) = y, X(t-\Delta t) = a\} = \Pr\{X(\Delta t) = x | X(0) = a\} = \Pr\{DB = \log x - \log a\}$$

where DB is a normal distribution with mean of $(g - \tilde{g} - \frac{1}{2}\kappa^2)\Delta t$, and variance of $\kappa^2 \Delta t$. Let $f_{DB}(\cdot)$ be the density function of the normal distribution with mean of $(g - \tilde{g} - \frac{1}{2}\kappa^2)\Delta t$, and variance of $\kappa^2 \Delta t$.

When $x > x^*$, we have

$$\begin{aligned}
f(x,t;y) &= (1-p\Delta t)\int_0^{+\infty} f(a,t-\Delta t;y) f_{DB}(\log x - \log a)\frac{1}{x}da + q\Delta t \cdot f(\frac{x}{\rho}, t-\Delta t; y)\frac{1}{\rho} \\
&= (1-p\Delta t)\int_0^{+\infty} [f(x,t;y) + (a-x)\frac{\partial}{\partial x}f(x,t;y) - \Delta t\frac{\partial}{\partial t}f(x,t;y) \\
&\quad + \frac{(a-x)^2}{2}\frac{\partial^2}{\partial x^2}f(x,t;y)] f_{DB}(\log x - \log a)\frac{a}{x}d\log a \\
&\quad + q\Delta t \cdot f(\frac{x}{\rho}, t-\Delta t; y)\frac{1}{\rho} + o(\Delta t) \\
&= (1-p\Delta t)(1-(g-\tilde{g})\Delta t + \kappa^2\Delta t)f(x,t;y) + (1-p\Delta t)(2\kappa^2 - (g-\tilde{g}))\Delta t x\frac{\partial}{\partial x}f(x,t;y) \\
&\quad -(1-p\Delta t)\Delta t\frac{\partial}{\partial t}f(x,t;y) + (1-p\Delta t)\frac{\kappa^2}{2}x^2\Delta t\frac{\partial^2}{\partial x^2}f(x,t;y) + q\Delta t \cdot f(\frac{x}{\rho}, t-\Delta t; y)\frac{1}{\rho} \\
&\quad + o(\Delta t)
\end{aligned}$$

where we use the Taylor expansion in the second and third equality. Divide by Δt on both sides and let $\Delta t \to 0$

$$\begin{aligned}
\frac{\partial}{\partial t}f(x,t;y) &= (\kappa^2 - p - (g-\tilde{g}))f(x,t;y) + (2\kappa^2 - (g-\tilde{g}))x\frac{\partial}{\partial x}f(x,t;y) \\
&\quad + \frac{1}{2}\kappa^2 x^2\frac{\partial^2}{\partial x^2}f(x,t;y) + qf(\frac{x}{\rho}, t; y)\frac{1}{\rho}, \quad x > x^*
\end{aligned}$$

Then

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\kappa^2 x^2 f(x,t)) - \frac{\partial}{\partial x}((g-\tilde{g})xf(x,t)) - pf(x,t) + qf(\frac{x}{\rho}, t)\frac{1}{\rho}, \ x > x^*.$$

Similarly, we have

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\kappa^2 x^2 f(x,t)) - \frac{\partial}{\partial x}((g-\tilde{g})xf(x,t)) - pf(x,t), x < x^*$$

8.3. Proof of Proposition 2

Proof: Following Benhabib and Bisin (2006), we use the Embedded Markov Chain method to establish the ergodicity of the wealth distribution of newborns, which then implies the ergodicity of the wealth distribution of the whole economy.

As in Karlin and Taylor (1981), we construct the embedded Markov Chain from the continuous time process, $X(\cdot, t)$. Let t_1, t_2, t_3, $\cdots$, denote the birthday of the generation 1, generation 2, generation 3,$\cdots$. By our notation, their starting wealth is $X(t_1, t_1)$, $X(t_2, t_2)$, $X(t_3, t_3)$, $\cdots$.

Let

$$\Phi_0 = X(\cdot, 0), \quad \Phi_n = X(t_n, t_n), \quad n = 1, 2, 3, \cdots$$

Thus Φ_n is the newborns's starting wealth. Note that the state space for Φ_n is $S = [x^*, +\infty)$ by the subsidy policy of the government. The stochastic process Φ_n is a Markov Chain. Note that the duration of the life follows an exponential distribution with parameter p. When the agent is alive, his wealth follows a Geometrical Brownian Motion as in equation (12). Given the government subsidy policy for the newborns, the transition probability of Φ_n is

$$\begin{aligned} P(\Phi_n &= x^* \mid \Phi_n = x) = \frac{p-q}{p} \\ &+ \int_0^{x^*} \int_0^{+\infty} q e^{-pt} \frac{1}{y} \frac{1}{\sqrt{2\pi t \kappa^2}} \exp[-\frac{(\log(\frac{y}{\rho}) - \log(x) - (g - \tilde{g} - \frac{1}{2}\kappa^2)t)^2}{2t\kappa^2}] dt dy \end{aligned}$$

and

$$\begin{aligned} P(\Phi_n &= y \mid \Phi_n = x) \\ &= \int_0^{+\infty} q e^{-pt} \frac{1}{y} \frac{1}{\sqrt{2\pi t \kappa^2}} \exp[-\frac{(\log(\frac{y}{\rho}) - \log(x) - (g - \tilde{g} - \frac{1}{2}\kappa^2)t)^2}{2t\kappa^2}] dt \quad \text{for} \quad y > x^* \end{aligned}$$

By Theorem 16.0.2 of Meyn and Tweedie (1993), $\{\Phi_n\}_{n=0}^{\infty}$ will be uniformly ergodic whenever the state space $S = [x^*, +\infty)$ is v_m-small for some m.

Definition 1. *Small sets*

A set $C \in \mathbf{B}(S)$ is called a small set if there exists an $m > 0$, and a non-trivial measure v_m on $\mathbf{B}(S)$, such that for all $s \in C$, $B \in \mathbf{B}(S)$, $P^m(s, B) \geq v_m(B)$.[32]

[32] $\mathbf{B}(S)$ *is the Borel σ-algebra on S.*

To see that the state space S is a small set for $m = 1$, we just need to define a non-trivial measure v_1 on $\mathbf{B}(S)$ by, for $\forall B \in \mathbf{B}(S)$

$$v_1(B) = \begin{cases} \frac{p-q}{p} & \text{if } \{x^*\} \in B \\ 0 & \text{otherwise} \end{cases}.$$

8.4. Proof of Proposition 3

Proof: Plugging $f(x) = Cx^{-\beta}$ into the ordinary differential equation

$$\frac{1}{2}\kappa^2 x^2 f''(x) + (2\kappa^2 - (g - \tilde{g}))xf'(x) + (\kappa^2 - (g - \tilde{g}) - p)f(x) = 0$$

we have the characteristic equation

$$\frac{\kappa^2}{2}\beta^2 - (\frac{3}{2}\kappa^2 - (g - \tilde{g}))\beta + \kappa^2 - p - (g - \tilde{g}) = 0$$

This quadratic equation has two roots

$$\beta_1 = \frac{\frac{3}{2}\kappa^2 - (g - \tilde{g}) - \sqrt{(\frac{1}{2}\kappa^2 - (g - \tilde{g}))^2 + 2\kappa^2 p}}{\kappa^2}$$

and

$$\beta_2 = \frac{\frac{3}{2}\kappa^2 - (g - \tilde{g}) + \sqrt{(\frac{1}{2}\kappa^2 - (g - \tilde{g}))^2 + 2\kappa^2 p}}{\kappa^2}.$$

8.5. Proof of Proposition 4

Proof: From Proposition 3, we have

$$\begin{aligned} \beta_1 &< 1 \Leftrightarrow \frac{\frac{3}{2}\kappa^2 - (g - \tilde{g}) - \sqrt{(\frac{1}{2}\kappa^2 - (g - \tilde{g}))^2 + 2\kappa^2 p}}{\kappa^2} < 1 \\ &\Leftrightarrow \frac{1}{2}\kappa^2 - (g - \tilde{g}) < \sqrt{(\frac{1}{2}\kappa^2 - (g - \tilde{g}))^2 + 2\kappa^2 p} \end{aligned}$$

The last inequality obviously holds. Then $\beta_1 < 1$. Similarly,

$$\begin{aligned}\beta_2 &> 2 \Leftrightarrow \frac{\frac{3}{2}\kappa^2 - (g-\tilde{g}) + \sqrt{(\frac{1}{2}\kappa^2 - (g-\tilde{g}))^2 + 2\kappa^2 p}}{\kappa^2} > 2 \\ &\Leftrightarrow \sqrt{(\frac{1}{2}\kappa^2 - (g-\tilde{g}))^2 + 2\kappa^2 p} > \frac{1}{2}\kappa^2 + (g-\tilde{g}) \\ &\Leftrightarrow p > g - \tilde{g}\end{aligned}$$

The last inequality holds since our assumption that government revenue is greater than the government expenditure, implies that $g - \tilde{g} = p - (\tau - \eta) < p$. Then $\beta_2 > 2$.

8.6. Proof of Proposition 5

Proof: Plugging $f(x) = Cx^{-\beta}$ into the ordinary differential equation

$$\frac{1}{2}\kappa^2 x^2 f''(x) + (2\kappa^2 - (g-\tilde{g}))xf'(x) + (\kappa^2 - (g-\tilde{g}) - p)f(x) = 0,\ x < x^*$$

we have the characteristic equation

$$\frac{\kappa^2}{2}\beta^2 - (\frac{3}{2}\kappa^2 - (g-\tilde{g}))\beta + \kappa^2 - p - (g-\tilde{g}) = 0.$$

Plugging $f(x) = Cx^{-\beta}$ into the ordinary differential equation

$$\frac{1}{2}\kappa^2 x^2 f''(x) + (2\kappa^2 - (g-\tilde{g}))xf'(x) + (\kappa^2 - (g-\tilde{g}) - p)f(x) + qf(\frac{x}{\rho})\frac{1}{\rho} = 0,\ x > x^*$$

we have the characteristic equation

$$\frac{\kappa^2}{2}\beta^2 - (\frac{3}{2}\kappa^2 - (g-\tilde{g}))\beta + \kappa^2 - p - (g-\tilde{g}) + q\rho^{\beta-1} = 0.$$

8.7. Proof of Proposition 6

Proof[33]: We first prove that $\beta_1 < 1$. From Proposition 5, we know

$$\beta_1 = \frac{\frac{3}{2}\kappa^2 - (g-\tilde{g}) - \sqrt{(\frac{1}{2}\kappa^2 - (g-\tilde{g}))^2 + 2\kappa^2 p}}{\kappa^2}$$

Thus

$$\begin{aligned} \beta_1 &< 1 \Leftrightarrow \frac{\frac{3}{2}\kappa^2 - (g-\tilde{g}) - \sqrt{(\frac{1}{2}\kappa^2 - (g-\tilde{g}))^2 + 2\kappa^2 p}}{\kappa^2} < 1 \\ &\Leftrightarrow \frac{1}{2}\kappa^2 - (g-\tilde{g}) < \sqrt{(\frac{1}{2}\kappa^2 - (g-\tilde{g}))^2 + 2\kappa^2 p} \end{aligned}$$

The last inequality obviously holds. Then $\beta_1 < 1$.

We then prove that $\beta_2 > 2$. Let $\Gamma(\beta) = \frac{\kappa^2}{2}\beta^2 - (\frac{3}{2}\kappa^2 - (g-\tilde{g}))\beta + \kappa^2 - p - (g-\tilde{g}) + q\rho^{\beta-1}$. Since $\frac{\kappa^2}{2} > 0$, we know that

$$\lim_{\beta\to-\infty}\Gamma(\beta) = +\infty \quad \text{and} \quad \lim_{\beta\to+\infty}\Gamma(\beta) = +\infty.$$

Note that $\Gamma(1) = q - p < 0$. And

$$\begin{aligned} \Gamma(2) &= g - \tilde{g} - p + q\rho \\ &= p - q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}} - (\tau - \eta) - p + q(\frac{p\chi(1-\zeta)}{A\mu})^{\frac{1}{\gamma}} \\ &= -(q(\frac{p\chi}{A\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}\zeta + \tau - \eta) < 0 \end{aligned}$$

The last inequality is from the assumption that government revenue is greater than the government expenditure. By the continuity of $\Gamma(\beta)$, we know that there exists $\beta < 1$, such that $\Gamma(\beta) = 0$, and there exists $\beta > 2$ such that $\Gamma(\beta) = 0$. Since the function $\Gamma(\beta)$ is convex, it can have at most two roots. Then the unique $\beta_2 > 2$.

[33] For this proof, we benifit from the discussion with Henry P. McKean.

8.8. Gini coefficient of a double Pareto distribution

Following Nygard and Sandstrom (1981) and Gastwirth (1971), we derive the Lorenz curve, and Gini coefficient of a double Pareto distribution.[34]

The cumulative distribution function (CDF) of a double Pareto distribution is

$$F(x) = \int_0^x C_1 v^{-\beta_1} dv = \frac{C_1}{1-\beta_1} x^{1-\beta_1} \quad \text{When} \ \ x \leq x^*$$

and

$$F(x) = \int_0^{x^*} C_1 v^{-\beta_1} dv + \int_{x^*}^{x} C_2 v^{-\beta_2} dv = 1 - \frac{C_2}{\beta_2 - 1} x^{1-\beta_2} \quad \text{When} \ \ x \geq x^*$$

Let $x(F)$ be the inverse of the CDF.

$$x = (\frac{1-\beta_1}{C_1} F)^{\frac{1}{1-\beta_1}} \quad \text{When} \ \ x \leq x^*$$

and

$$x = [\frac{\beta_2 - 1}{C_2}(1-F)]^{\frac{1}{1-\beta_2}} \quad \text{When} \ \ x \geq x^*$$

The function of Lorenz curve is

$$L(F) = \frac{\int_0^{x(F)} x f(x) dx}{\int_0^{+\infty} x f(x) dx} = \frac{\int_0^F x(F') dF'}{\int_0^1 x(F') dF'} = \int_0^F x(F') dF'$$

where $x(F)$ is the inverse of the CDF.

Let $F^* = F(x^*) = \frac{C_1}{1-\beta_1}(x^*)^{1-\beta_1}$. When $F \leq F^*$

$$L(F) = \int_0^F x(F') dF' = (\frac{1-\beta_1}{C_1})^{\frac{1}{1-\beta_1}} \frac{1-\beta_1}{2-\beta_1} F^{\frac{2-\beta_1}{1-\beta_1}}$$

[34] This is also an extension of the derivation of Lorenz curve and Gini coefficient for a Pareto distribution from Wikipedia (http://en.wikipedia.org/wiki/Pareto_distribution) to a double Pareto distribution.

When $F \geq F^*$

$$\begin{aligned}
L(F) &= \int_0^F x(F')dF' \\
&= \int_0^{F^*} x(F')dF' + \int_{F^*}^F x(F')dF' \\
&= \frac{C_1}{2-\beta_1}(x^*)^{2-\beta_1} + \int_{F^*}^F x(F')dF' \\
&= \frac{C_1}{2-\beta_1}(x^*)^{2-\beta_1} + (\frac{\beta_2 - 1}{C_2})^{\frac{1}{1-\beta_2}} \frac{1-\beta_2}{2-\beta_2}\{[1-F^*]^{\frac{2-\beta_2}{1-\beta_2}} - (1-F)^{\frac{2-\beta_2}{1-\beta_2}}\}
\end{aligned}$$

The Gini coefficient of a double Pareto distribution is

$$\begin{aligned}
G &= 1 - 2\int_0^1 L(F)dF \\
&= 1 - 2\int_0^{F^*} L(F)dF - 2\int_{F^*}^1 L(F)dF \\
&= 1 - 2(\frac{1-\beta_1}{C_1})^{\frac{1}{1-\beta_1}} \frac{1-\beta_1}{2-\beta_1}\frac{1-\beta_1}{3-2\beta_1}(F^*)^{\frac{3-2\beta_1}{1-\beta_1}} - 2\frac{C_1}{2-\beta_1}(x^*)^{2-\beta_1}(1-F^*) \\
&\quad -2(\frac{\beta_2-1}{C_2})^{\frac{1}{1-\beta_2}} \frac{1-\beta_2}{3-2\beta_2}(1-F^*)^{\frac{3-2\beta_2}{1-\beta_2}}.
\end{aligned}$$

8.9. Two boundary conditions of the stationary distribution at $x = x^*$

Proof[35]: First, we see the infinitesimal generator of the stochastic process, $\mathcal{L}$, acting on h in the domain of the operator $\mathcal{L}$, $D(\mathcal{L}) \subset C_0^2[0, +\infty)$.[36]

$$(\mathcal{L}h)(x) := \lim_{t \to 0} \frac{E_x h(X_t) - h(x)}{t} \tag{A.1}$$

for $\forall h \in D(\mathcal{L})$. The domain of the operator $\mathcal{L}$, $D(\mathcal{L})$, is specified as

$$D(\mathcal{L}) = \{h(x) = \int_0^{+\infty} e^{-\alpha t} E_x \phi(X_t) dt : \phi \in C[0, +\infty)\}$$

[35] Without the help from Henry P. McKean, we could not write this rigorous proof of the boundary conditons.

[36] $C_0^2[0, +\infty)$ is the set of functions in $C^2[0, +\infty)$ with compact support.

Here, $\alpha \in R$ and $\alpha > 0$. But the set $D(\mathcal{L})$ is independent of the choosing of positive α.[37](See Yosida 1971, Chapter IX, Analytical Theory of Semi-groups)

Applying Ito formula to equation (A.1), we have

$$(\mathcal{L}h)(x) = [h'(x)(g-\tilde{g})x+\frac{1}{2}h''(x)\kappa^2x^2]+q[h(\rho x)-h(x)]+(p-q)[h(x^*)-h(x)] \quad \text{if} \quad x \geq \frac{x^*}{\rho} \tag{A.2}$$

and

$$(\mathcal{L}h)(x) = [h'(x)(g-\tilde{g})x+\frac{1}{2}h''(x)\kappa^2x^2]+p[h(x^*)-h(x)] \quad \text{if} \quad x < \frac{x^*}{\rho} \tag{A.3}$$

For the density function of the stationary distribution, $f(x)$, by the definition of the infinitesimal generator $\mathcal{L}$, we have

$$\int_0^{+\infty} (\mathcal{L}h)(x)f(x)dx = 0 \tag{A.4}$$

for $\forall h \in D(\mathcal{L})$.

The density function $f(x)$ may not be differentiable at $x = x^*$. Thus we write equation (A.3) as

$$\int_0^{x^*} (\mathcal{L}h)(x)f(x)dx + \int_{x^*}^{\frac{x^*}{\rho}} (\mathcal{L}h)(x)f(x)dx + \int_{\frac{x^*}{\rho}}^{+\infty} (\mathcal{L}h)(x)f(x)dx = 0 \tag{A.5}$$

Plugging equations (A.2) and (A.3) into equation (A.5), we have

$$\begin{aligned}
& \int_0^{x^*} \{[h'(x)(g-\tilde{g})x+\frac{1}{2}h''(x)\kappa^2x^2]+p[h(x^*)-h(x)]\ \}f(x)dx \\
& + \int_{x^*}^{\frac{x^*}{\rho}} \{[h'(x)(g-\tilde{g})x+\frac{1}{2}h''(x)\kappa^2x^2]+p[h(x^*)-h(x)]\ \}f(x)dx \\
& + \int_{\frac{x^*}{\rho}}^{+\infty} \{[h'(x)(g-\tilde{g})x+\frac{1}{2}h''(x)\kappa^2x^2]+q[h(\rho x)-h(x)]+(p-q)[h(x^*)-h(x)]\ \}f(x)dx \\
= \ & 0
\end{aligned} \tag{A.6}$$

[37]It can be proved that $\{f = \int_0^{+\infty} e^{-\alpha t}E_x g(X_t)dt : g \in C[0,+\infty)\}$ $= \{f = \int_0^{+\infty} e^{-\beta t}E_x g(X_t)dt : g \in C[0,+\infty)\}$ for $\forall \alpha, \beta > 0$.

Applying integration by parts to equation (A.6) and taking into account the boundary at $x = x^*$, we obtain

$$\begin{aligned}
& (g-\tilde{g})x^* f_-(x^*)h(x^*) - (g-\tilde{g})x^* f_+(x^*)h(x^*) \\
& -(g-\tilde{g})\int_0^{x^*} h(x)[f(x)+xf'(x)]dx - (g-\tilde{g})\int_{x^*}^{+\infty} h(x)[f(x)+xf'(x)]dx \\
& +\frac{1}{2}\kappa^2 (x^*)^2 f_-(x^*)h'(x^*) - \frac{1}{2}\kappa^2 (x^*)^2 f_+(x^*)h'(x^*) \\
& -\frac{1}{2}\kappa^2[2x^* f_-(x^*) + (x^*)^2 f'_-(x^*)]h(x^*) + \frac{1}{2}\kappa^2[2x^* f_+(x^*) + (x^*)^2 f'_+(x^*)]h(x^*) \\
& +\frac{1}{2}\kappa^2 \int_0^{x^*} h(x)[2f(x)+4xf'(x)+x^2 f''(x)]dx \\
& +\frac{1}{2}\kappa^2 \int_{x^*}^{+\infty} h(x)[2f(x)+4xf'(x)+x^2 f''(x)]dx \\
& +ph(x^*)\int_0^{\frac{x^*}{\rho}} f(x)dx + (p-q)h(x^*)\int_{\frac{x^*}{\rho}}^{+\infty} f(x)dx \\
& +\int_{x^*}^{+\infty} qh(x)f(\frac{x}{\rho})\frac{1}{\rho}dx \\
& -\int_0^{+\infty} ph(x)f(x)dx \\
= \; & 0 \qquad \text{(A.7)}
\end{aligned}$$

where $f_-(x^*)$ and $f_+(x^*)$ are the left and right limit of the density function $f(x)$ at x^*. $f'_-(x^*)$ and $f'_+(x^*)$ are the left and right limit of $f'(x)$ at x^*.[38]

We can pick $h \in D(\mathcal{L})$ where $h(x) = 0$, when $x \geq x^*$. Thus we know $h'(x^*) = 0$, since $h \in C_0^2[0, +\infty)$. By equation (A.7), such $h(x)$ satisfies

$$\int_0^{x^*} h(x)\{\frac{1}{2}\kappa^2[2f(x)+4xf'(x)+x^2f''(x)] - (g-\tilde{g})[f(x)+xf'(x)] - pf(x)\}dx = 0 \qquad \text{(A.8)}$$

[38] This integration by parts technique is also used in the derivation of the Kolmogorov's forward equation in Chapter VIII of Oksendal (1995).

Equation (A.8) holds for $\forall h \in D(\mathcal{L})$ where $h(x) = 0$, when $x \geq x^*$. Thus we have

$$\frac{1}{2}\kappa^2[2f(x) + 4xf'(x) + x^2 f''(x)] - (g - \tilde{g})[f(x) + xf'(x)] - pf(x) = 0 \quad \text{if} \quad x < x^* \tag{A.9}$$

Similarly, we can pick $h \in D(\mathcal{L})$ where $h(x) = 0$, when $x \leq x^*$. Thus we know $h'(x^*) = 0$, since $h \in C_0^2[0, +\infty)$. By equation (A.7), such $h(x)$ satisfies

$$\int_{x^*}^{+\infty} h(x)\{\frac{1}{2}\kappa^2[2f(x)+4xf'(x)+x^2f''(x)]-(g-\tilde{g})[f(x)+xf'(x)]-pf(x)+qf(\frac{x}{\rho})\frac{1}{\rho}\}dx = 0 \tag{A.10}$$

Equation (A.10) holds for $\forall h \in D(\mathcal{L})$ where $h(x) = 0$, when $x \leq x^*$. Thus we have

$$\frac{1}{2}\kappa^2[2f(x)+4xf'(x)+x^2f''(x)]-(g-\tilde{g})[f(x)+xf'(x)]-pf(x)+qf(\frac{x}{\rho})\frac{1}{\rho} = 0 \quad \text{if} \quad x > x^* \tag{A.11}$$

Plugging equations (A.9) and (A.11) into equation (A.7), we have

$$\begin{aligned}
& \{(g - \tilde{g})x^*[f_-(x^*) - f_+(x^*)] - \kappa^2 x^*[f_-(x^*) - f_+(x^*)] - \frac{1}{2}\kappa^2(x^*)^2[f'_-(x^*) - f'_+(x^*)] \\
& +p\int_0^{\frac{x^*}{\rho}} f(x)dx + (p-q)\int_{\frac{x^*}{\rho}}^{+\infty} f(x)dx\}h(x^*) \\
& +\frac{1}{2}\kappa^2\left(x^*\right)^2[f_-(x^*) - f_+(x^*)]h'(x^*) \\
= \; & 0
\end{aligned} \tag{A.12}$$

Equation (A.12) holds for $\forall h \in D(\mathcal{L})$. By the coefficient before $h'(x^*)$, we have one of the two boundary conditions at $x = x^*$.

$$f_-(x^*) - f_+(x^*) = 0 \tag{A.13}$$

By the coefficient before $h(x^*)$ and equation (A.13), we have the other boundary condition at $x = x^*$.

$$\frac{1}{2}\kappa^2(x^*)^2[f'_-(x^*) - f'_+(x^*)] = p\int_0^{\frac{x^*}{\rho}} f(x)dx + (p-q)\int_{\frac{x^*}{\rho}}^{+\infty} f(x)dx \tag{A.14}$$

Equation (A.13) means that the density function $f(x)$ is continuous at $x = x^*$.

Note that the right hand side of equation (A.13) is exactly the injection of the newborns at $x = x^*$. Thus equation (A.14) is the relationship about the injection and the difference between the left derivative and the right derivative of $f(x)$ at $x = x^*$.

8.10. Wealth distribution conditional on age

The distribution of the starting wealth consists of two parts, one of which is a mass point. At x^*, the distribution has a positive probability, $\frac{q}{p}(C_1 \int_0^{x^*} x^{-\beta_1} dx + C_2 \int_{x^*}^{\frac{x^*}{\rho}} x^{-\beta_2} dx) + \frac{p-q}{p}$. For wealth levels higher than x^*, the density of the distribution is

$$v(y) = \frac{q}{p} C_2 (\frac{y}{\rho})^{-\beta_2} \frac{1}{\rho}, \qquad y > x^*$$

Conditional on age t, the density of wealth distribution will be the sum of two components: 1). starting from $y > x^*$, using the stationary density $v\,(y)$, we can compute the probability of reaching x, and 2). starting from the mass point x^* we can compute the probability of reaching x:

$$\begin{aligned}
f_t(x) &= \int_{x^*}^{+\infty} \frac{1}{x} \frac{1}{\sqrt{2\pi t\kappa^2}} \exp[-\frac{(\log(x) - \log(y) - (g - \tilde{g} - \frac{1}{2}\kappa^2)t)^2}{2t\kappa^2}] v(y) dy \\
&\quad + \frac{1}{x} \frac{1}{\sqrt{2\pi t\kappa^2}} \exp[-\frac{(\log(x) - \log(x^*) - (g - \tilde{g} - \frac{1}{2}\kappa^2)t)^2}{2t\kappa^2}] \times \\
&\quad [\frac{q}{p}(C_1 \int_0^{x^*} x^{-\beta_1} dx + C_2 \int_{x^*}^{\frac{x^*}{\rho}} x^{-\beta_2} dx) + \frac{p-q}{p}] \\
&= \int_{x^*}^{+\infty} \frac{1}{x} \frac{1}{\sqrt{2\pi t\kappa^2}} \exp[-\frac{(\log(x) - \log(y) - (g - \tilde{g} - \frac{1}{2}\kappa^2)t)^2}{2t\kappa^2}] \frac{q}{p} C_2 (\frac{y}{\rho})^{-\beta_2} \frac{1}{\rho} dy \\
&\quad + \frac{1}{x} \frac{1}{\sqrt{2\pi t\kappa^2}} \exp[-\frac{(\log(x) - \log(x^*) - (g - \tilde{g} - \frac{1}{2}\kappa^2)t)^2}{2t\kappa^2}] \times \\
&\quad [\frac{q}{p}(C_1 \int_0^{x^*} x^{-\beta_1} dx + C_2 \int_{x^*}^{\frac{x^*}{\rho}} x^{-\beta_2} dx) + \frac{p-q}{p}].
\end{aligned}$$

8.11. Derivation of Ω

To derive the aggregate welfare of the economy, we first note that there are two kinds of people in the economy. $\frac{q}{p}$ fraction of the people have a bequest motive and

$1-\frac{q}{p}$ fraction of people do not have a bequest motive. The aggregate welfare of the economy is the weighted sum of the individual utilities with weights according to the cross-sectional wealth distribution of the two groups of agents.

$$\Omega(\tau,\zeta)=\frac{q}{p}\int_0^{+\infty}U(w)f(w)dw+\frac{p-q}{p}\int_0^{+\infty}U_0(w)f(w)dw$$

where $U(w)$ is the optimal value of the people with bequest motives, and $U_0(w)$ is the optimal value of the people without bequest motives. From Proposition 1, we know that people with bequest motives have the following value function:

$$\begin{aligned}U(w) &= \frac{A}{1-\gamma}W(s,t)^{1-\gamma}\\ &= \frac{1}{1-\gamma}\left(\frac{\theta+p-(1-\gamma)(r-\tau+\mu+\frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma(1+(p\chi)^{\frac{1}{\gamma}}\mu^{\frac{\gamma-1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}})}\right)^{-\gamma}w^{1-\gamma}\end{aligned}$$

Similarly, for people with no bequest motive, the value function is:

$$U_0(w)=\frac{1}{1-\gamma}\left(\frac{\theta+p-(1-\gamma)(r-\tau+\mu+\frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma}\right)^{-\gamma}w^{1-\gamma}$$

Thus the aggregate welfare of the economy is

$$\begin{aligned}
\Omega(\tau,\zeta) &= \frac{q}{p}\int_0^{+\infty} U(w)f(w)dw + \frac{p-q}{p}\int_0^{+\infty} U_0(w)f(w)dw \\
&= \frac{q}{p}\frac{1}{1-\gamma}\left(\frac{\theta+p-(1-\gamma)(r-\tau+\mu+\frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma(1+(p\chi)^{\frac{1}{\gamma}}\mu^{\frac{\gamma-1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}})}\right)^{-\gamma}\int_0^{+\infty} w^{1-\gamma}f(w)dw \\
&\quad + \frac{p-q}{p}\frac{1}{1-\gamma}\left(\frac{\theta+p-(1-\gamma)(r-\tau+\mu+\frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma}\right)^{-\gamma}\int_0^{+\infty} w^{1-\gamma}f(w)dw \\
&= [\frac{q}{p}\frac{1}{1-\gamma}\left(\frac{\theta+p-(1-\gamma)(r-\tau+\mu+\frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma(1+(p\chi)^{\frac{1}{\gamma}}\mu^{\frac{\gamma-1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}})}\right)^{-\gamma} \\
&\quad + \frac{p-q}{p}\frac{1}{1-\gamma}\left(\frac{\theta+p-(1-\gamma)(r-\tau+\mu+\frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma}\right)^{-\gamma}]\times \\
&\quad \int_0^{+\infty} w^{1-\gamma}f(w)dw
\end{aligned}$$

We then compute $\int_0^{+\infty} w^{1-\gamma}f(w)dw$

$$\begin{aligned}
&\int_0^{+\infty} w^{1-\gamma}f(w)dw \\
=\ & C_1\int_0^{x^*} x^{1-\gamma}x^{-\beta_1}dx + C_2\int_{x^*}^{+\infty} x^{1-\gamma}x^{-\beta_2}dx \\
=\ & \frac{C_1}{2-\gamma-\beta_1}(x^*)^{2-\gamma-\beta_1} - \frac{C_2}{2-\gamma-\beta_2}(x^*)^{2-\gamma-\beta_2}
\end{aligned}$$

The last step is valid when $\beta_1 < -1$ since $\gamma = 3$ in our calibration. Plugging the above result of the integral, $\int_0^{+\infty} w^{1-\gamma}f(w)dw$, into the formula of $\Omega(\tau,\zeta)$, we have

$$\begin{aligned}
\Omega(\tau,\zeta) &= \frac{q}{p}\int_0^{+\infty} U(z)f(z)dz + \frac{p-q}{p}\int_0^{+\infty} U_0(z)f(z)dz \\
&= [\frac{q}{p}\frac{1}{1-\gamma}\left(\frac{\theta+p-(1-\gamma)(r-\tau+\mu+\frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma(1+(p\chi)^{\frac{1}{\gamma}}\mu^{\frac{\gamma-1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}})}\right)^{-\gamma} \\
&\quad +\frac{p-q}{p}\frac{1}{1-\gamma}\left(\frac{\theta+p-(1-\gamma)(r-\tau+\mu+\frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma}\right)^{-\gamma}]\int_0^{+\infty} w^{1-\gamma}f(w)dw \\
&= [\frac{q}{p}\frac{1}{1-\gamma}\left(\frac{\theta+p-(1-\gamma)(r-\tau+\mu+\frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma(1+(p\chi)^{\frac{1}{\gamma}}\mu^{\frac{\gamma-1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}})}\right)^{-\gamma} \\
&\quad +\frac{p-q}{p}\frac{1}{1-\gamma}\left(\frac{\theta+p-(1-\gamma)(r-\tau+\mu+\frac{(\alpha-r)^2}{2\gamma\sigma^2})}{\gamma}\right)^{-\gamma}]\times \\
&\quad [\frac{C_1}{2-\gamma-\beta_1}(x^*)^{2-\gamma-\beta_1} - \frac{C_2}{2-\gamma-\beta_2}(x^*)^{2-\gamma-\beta_2}].
\end{aligned}$$

8.12. A simple mechanism underlying a double Pareto distribution of wealth without inheritance

Suppose that $X(s,t)$ is a geometric Brownian motion. We have

$$dX(s,t) = gX(s,t)dt + \kappa X(s,t)dB(s,t)$$

and

$$X(s,t) = X(s,s)\exp[(g-\frac{1}{2}\kappa^2)(t-s)+\kappa(B(s,t)-B(s,s))]$$

Normalize $X(s,s)$, $\log(X(s,s))=0$ so initial wealth is fixed (no inheritance). Note that $X(s,t)$ is log-normal

$$\log(X(s,t)) = (g-\frac{1}{2}\kappa^2)(t-s)+\kappa(B(s,t)-B(s,s))$$

Now integrating over the population, we have the density function of the stationary distribution of $X(s,t)$:

$$\begin{aligned} f(x) &= \int_{-\infty}^{t} pe^{-p(t-s)}\frac{1}{x}\frac{1}{\sqrt{2\pi(t-s)\kappa^2}}\exp[-\frac{(\log(x)-(g-\frac{1}{2}\kappa^2)(t-s))^2}{2(t-s)\kappa^2}]ds \\ &= \int_{0}^{+\infty} pe^{-pv}\frac{1}{x}\frac{1}{\sqrt{2\pi v\kappa^2}}\exp[-\frac{(\log(x)-(g-\frac{1}{2}\kappa^2)v)^2}{2v\kappa^2}]dv \end{aligned}$$

Let

$$w(x,v) = \frac{1}{x}\frac{1}{\sqrt{2\pi v\kappa^2}}\exp[-\frac{(\log(x)-(g-\frac{1}{2}\kappa^2)v)^2}{2v\kappa^2}]$$

Thus

$$f(x) = \int_{0}^{+\infty} pe^{-pv}w(x,v)dv$$

$$f'(x) = \int_{0}^{+\infty} pe^{-pv}\frac{\partial w(x,v)}{\partial x}dv$$

$$f''(x) = \int_{0}^{+\infty} pe^{-pv}\frac{\partial^2 w(x,v)}{\partial x^2}dv$$

Note that

$$\frac{\partial w(x,v)}{\partial v} = \frac{1}{2}\kappa^2x^2\frac{\partial^2 w(x,v)}{\partial x^2} + (2\kappa^2-g)x\frac{\partial w(x,v)}{\partial x} + (\kappa^2-g)w(x,v)$$

Then

$$\frac{1}{2}\kappa^2x^2f''(x) + (2\kappa^2-g)xf'(x) + (\kappa^2-g)f(x) = \int_{0}^{+\infty} pe^{-pv}\frac{\partial w(x,v)}{\partial v}dv = pf(x)$$

This gives the characteristic equation

$$\frac{1}{2}\kappa^2x^2f''(x) + (2\kappa^2-g)xf'(x) + (\kappa^2-g-p)f(x) = 0$$

Then $f(x)$ has the functional form

$$f(x) = \begin{cases} C_1x^{-\beta_1} & \text{when } x \leq x^* \\ C_2x^{-\beta_2} & \text{when } x \geq x^* \end{cases}$$

where β_1 and β_2 are the two roots of the characteristic equation

$$\frac{\kappa^2}{2}\beta^2 - (\frac{3}{2}\kappa^2 - g)\beta + \kappa^2 - g - p = 0$$

Solving this equation, we have

$$\beta_1 = \frac{\frac{3}{2}\kappa^2 - g - \sqrt{(\frac{1}{2}\kappa^2 - g)^2 + 2\kappa^2 p}}{\kappa^2}$$

and

$$\beta_2 = \frac{\frac{3}{2}\kappa^2 - g + \sqrt{(\frac{1}{2}\kappa^2 - g)^2 + 2\kappa^2 p}}{\kappa^2}.$$

8.13. Simulation results-Gini coefficient

$\sigma\backslash\chi$	14	15	16	17
0.2	0.691555	0.690649	0.689788	0.688966
0.21	0.680519	0.67958	0.678686	0.677833
0.22	0.670422	0.669451	0.668527	0.667646
0.23	0.66118	0.66018	0.659228	658319
0.24	0.652716	0.651688	0.650711	0.649777
0.25	0.644917	0.643865	0.642864	0.641908
0.26	0.637806	0.636731	0.635706	0.63472

$\zeta\backslash\tau$	0.004	0.005	0.006	0.007	0.008	0.009	0.0099
0	0.650234	0.594185	0.546586	0.506215	0.472115	0.443518	0.421965
0.1	0.643547	0.588788	0.542218	0.502678	0.469257	0.441223	0.420096
0.2	0.635948	0.582625	0.537207	0.498605	0.465956	0.438565	0.417929
0.3	0.627151	0.575452	0.531349	0.493824	0.462069	0.435427	0.415367
0.4	0.616726	0.566905	0.524334	0.488076	0.457382	0.431636	0.412267
0.5	0.603986	0.556398	0.515668	0.480947	0.451551	0.426912	0.408404
0.6	0.587734	0.542913	0.504491	0.471718	0.443984	0.420775	0.408404

8.14. Pure altruism

"Joy of giving" can be viewed as a reduced form of a pure altruistic bequest motive. (See, for example, Abel and Warshawsky (1988).) The parameter of "joy of giving" can be derived from the value of the altruism parameter. In the case of pure altruism, parents care about their offsprings' welfare through the altruism parameter φ. Parents receive utility from their children's utility with a parameter $\varphi \leq 1$. The agent's utility function is

$$V(W(s,t)) = \max_{C,\omega,P} E_t \int_t^{+\infty} e^{-(\theta+p)(v-t)} [\frac{C^{1-\gamma}(s,v)}{1-\gamma} + p\varphi V((1-\zeta)Z(s,v))]dv$$

and the budget constraint is

$$dW(s,t) = [(r-\tau)W(s,t)+(\alpha-r)\omega(s,t)W(s,t)-C(s,t)-P(s,t)]dt+\sigma\omega(s,t)W(s,t)dB(s,t)$$

The optimal policies under pure altruism, as derived below, are

$$C(s,t) = H^{-\frac{1}{\gamma}}W(s,t), \quad Z(s,t) = (\frac{p\varphi}{\mu})^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}}W(s,t), \quad \omega(s,t) = \frac{\alpha-r}{\gamma\sigma^2}.$$

where

$$H = \{\frac{\theta+p}{\gamma} - \mu^{\frac{\gamma-1}{\gamma}}(p\varphi)^{\frac{1}{\gamma}}(1-\zeta)^{\frac{1-\gamma}{\gamma}} - \frac{1-\gamma}{\gamma}[r-\tau+\mu+\frac{(\alpha-r)^2}{2\gamma\sigma^2}]\}^{-\gamma}$$

The individual wealth accumulation equation is

$$dW(s,t) = [\frac{r-\tau+\mu-\theta-p}{\gamma} + \frac{1+\gamma}{2\gamma}\frac{(\alpha-r)^2}{\gamma\sigma^2}]W(s,t)dt + \frac{\alpha-r}{\gamma\sigma}W(s,t)dB(s,t).$$

We can now establish the endogenous formulation of the "joy of giving" from the parameter of pure altruism:

$$\chi = \varphi H$$

Setting $\varphi = 1$ yields the standard infinitely-lived dynastic model. Note that the share of wealth invested in the risky asset does not depend on the parameter of pure altruism and the government policy. The share of wealth allocated to the purchase of life insurance does depend on the estate tax rate.

To derive the results above, we guess the value function

$$V(W(s,t)) = \frac{H}{1-\gamma} W(s,t)^{1-\gamma}$$

where H is the undetermined constant.

Then the Hamilton-Jacobi-Bellman equation is

$$\begin{aligned}
& (\theta + p) \frac{H}{1-\gamma} W(s,t)^{1-\gamma} \\
= \; & \max_{C,\omega,P} \{ \frac{C(s,t)^{1-\gamma}}{1-\gamma} + p\varphi \frac{H}{1-\gamma} ((1-\zeta) Z(s,t))^{1-\gamma} \\
& + HW(s,t)^{-\gamma} [(r-\tau) W(s,t) + (\alpha - r)\omega(s,t) W(s,t) - C(s,t) - P(s,t)] \\
& - \frac{1}{2} H\gamma\sigma^2\omega^2(s,t) W(s,t)^{1-\gamma} \}
\end{aligned}$$

Using the relationship

$$Z(s,t) = W(s,t) + \frac{P(s,t)}{\mu}$$

we find the first order conditions:

$$C(s,t)^{-\gamma} = HW(s,t)^{-\gamma}$$

$$p\varphi H (1-\zeta)^{1-\gamma} Z(s,t)^{-\gamma} \frac{1}{\mu} = HW(s,t)^{-\gamma}$$

$$\omega(s,t) = \frac{\alpha - r}{\gamma\sigma^2}$$

Plugging these equations into the Hamilton-Jacobi-Bellman equation, we can determine the constant H:

$$H = \{ \frac{\theta + p}{\gamma} - \mu^{\frac{\gamma - 1}{\gamma}} (p\varphi)^{\frac{1}{\gamma}} (1-\zeta)^{\frac{1-\gamma}{\gamma}} - \frac{1-\gamma}{\gamma} [r - \tau + \mu + \frac{(\alpha - r)^2}{2\gamma\sigma^2}] \}^{-\gamma}$$

And the optimal policies are

$$C(s,t) = H^{-\frac{1}{\gamma}} W(s,t), \quad Z(s,t) = (\frac{p\varphi}{\mu})^{\frac{1}{\gamma}} (1-\zeta)^{\frac{1-\gamma}{\gamma}} W(s,t), \quad \omega(s,t) = \frac{\alpha - r}{\gamma\sigma^2}.$$