SOPHISTICATED MONETARY POLICIES*

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May 2008

ABSTRACT

The Ramsey approach to policy analysis finds the best competitive equilibrium given available instruments but is silent about how to get there uniquely. Many ways of specifying monetary policy lead to indeterminacy. *Sophisticated policies* do not. They depend on the history of past actions and exogenous events, differ on and off the equilibrium path, and can uniquely produce any desired competitive equilibrium. This result holds in two standard monetary economies and is robust to trembles and imperfect monitoring. The result implies that adherence to the Taylor principle is unnecessary. We also show that such adherence is inefficient.

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Frank Ramsey's now-classic approach to policy analysis under commitment specifies the set of instruments available to policymakers and describes their basic problem: to use those instruments to find the competitive equilibrium that maximizes social welfare. The Ramsey approach has been extended (by Barro (1979) and Lucas and Stokey (1983), among others) to situations with uncertainty by specifying the instruments as functions of exogenous events. This extension has made the approach useful in addressing macroeconomic policy questions.[1] The Ramsey approach still does not, however, describe how to structure policy so that its outcome coincides with the desired equilibrium and only that equilibrium. The approach ignores, that is, the issue of *unique implementation.*

Here we suggest a new approach to policy analysis which fills that void: *sophisticated policies.* The approach allows policies to depend on the history of private agents' actions as well as on the histories of policies and exogenous events. It also allows policies to differ on and off the equilibrium path. In two standard monetary economies, we show that an appropriately chosen sophisticated monetary policy ensures unique implementation of any desired competitive equilibrium.

We demonstrate the use of sophisticated policies in monetary economies because this approach is of particular significance for monetary policy. The simplest way to apply the Ramsey approach to policymaking has been to find the desired competitive equilibrium by specifying policy as a function of exogenous events and then to use that function for policy. While this way to structure policies has often led to a unique equilibrium in fiscal policy problems, it leads to indeterminacy in many monetary policy environments, when interest rates are used as the principal policy instrument. (This indeterminacy problem has been known since at least the 1975 work of Sargent and Wallace.) Interest rate policies could lead to the best equilibrium, but they could also lead to undesirable outcomes, including hyperinflation and outcomes with excessive volatility due to sunspot-like fluctuations. In this sense, policies which lead to indeterminacy are risky and researchers generally agree that such policies should be avoided.

The concern with the risks arising from indeterminacy has led to a substantial literature aimed at finding policy rules that eliminate it. (For example, see McCallum 1981 and Woodford 2003.) The most popular recent solution is to follow interest rate rules that are consistent with

the *Taylor principle:* the interest rate should rise more than one-for-one with a rise in inflation rates relative to target inflation. Obeying this principle, the literature argues, leads to unique outcomes and violating it leads to indeterminacy. A related literature argues that the failure to obey this principle explains the U.S. economy's undesirable inflation experiences of the 1970s. (See, for example, the 2000 work of Clarida, Gali, and Gertler.)

Our sophisticated policy approach makes the Taylor principle irrelevant as a device to avoid indeterminacy. Sophisticated policies can produce unique outcomes which violate that principle along the equilibrium path as well as those which do not.

Moreover, our findings imply that historical evidence that policy violated the Taylor principle in some periods does not necessarily imply that such policy was unwise.

Our conclusions differ from those of the recent literature because of differences in the way we specify policy. Both ways allow policy to depend on the history of past choices. We allow policy to be different on and off the equilibrium path while the recent literature considers only *restricted policies,* which are required to be the same on and off the equilibrium path. This restriction leads to the critical role of the Taylor principle in ensuring determinacy. Since we impose no such restriction, the Taylor Principle becomes irrelevant.

We also show that in our economies rules that obey the Taylor principle are inefficient in the sense that some of the outcomes associated with a rule that violates the Taylor principle have higher welfare than any of the outcomes associated with rules that obey the Taylor principle. In this sense, obeying the Taylor Principle is not just unnecessary, it is harmful in our economies.

We use two models to illustrate the basic idea of our construction of sophisticated policies: a simple model with one-period price stickiness and a New Keynesian model with staggered price-setting. While our main focus is the widely used New Keynesian model, we begin with the other model because of the simplicity of the dynamical system associated with its competitive equilibrium. This simplicity lets us focus on the strategic aspects of sophisticated policies.

The basic idea of policy construction is the same in both models. The aim is to construct central bank policies that uniquely produce a desired competitive equilibrium. To use this approach to do that, along the equilibrium path choose the policies which produce the desired competitive equilibrium. Then structure the polices off the equilibrium path, referred to as *reversion policies,* in such a way that they discourage deviations of private agents from that

2

path. Specifically, if the average action of private agents deviates from that in the desired equilibrium, then choose the reversion policies so that the optimal action, or *best response,* of each individual agent is different from the average deviation choice.

Our construction requires that best responses be controllable, in the sense that policies can be found which ensure that after any deviation, the best response of any individual private agent is different than the average action of the private agents. Controllability implies that reversion policies be constructed so that no deviation is optimal and, hence, that the desired equilibrium is uniquely implemented by the sophisticated policy.

In the simple model, we specify the reversion policy as follows. If the average price set by sticky price producers differs from the desired price, then the central bank switches from an interest rate regime to a money regime for one period. In our model, if the central bank chooses a money regime in a period, then equilibrium outcomes are uniquely pinned down for that period by the period's money growth rates. The central bank can use this uniqueness property of money to choose a money growth rate that generates the same expected level of inflation as in the original, desired equilibrium allocation. If price-setters expect the central bank to act this way after a deviation, then even if all other producers make this deviation, an individual producer does not find the deviation profit-maximizing. That is, we set the policies so that the best responses of individual producers differ from those of others in the deviation. Since the best responses are controllable, such reversion policies ensure that a unique competitive equilibrium results.

In the New Keynesian model, we specify the reversion policies after a deviation in one of two ways. One is to specify an infinite reversion to a money regime; the other, to specify an infinite reversion to an alternative interest rate policy. The second way is of particular interest because many economists today find interest rate rules more practical than money rules. This second way also demonstrates that our implementation result does not require uniqueness of equilibria under money regimes. Indeed, some other monetary models commonly have uniqueness of the real allocations under interest rate rules but indeterminacy (of both nominal and real variables) under money regimes. In such economies, reversion to interest rate rules can support any competitive outcome on the equilibrium path.

For the New Keynesian model, we provide sufficient conditions to ensure that equilibrium

outcomes are uniquely pinned down by the reversion policy. This uniqueness property means that reversion policies can be specified so that the best responses of individual producers differ from those of others in the deviation. Here again, controllability of the best responses ensures that any desired competitive equilibrium can be uniquely implemented.

Our reversion policies have two important properties. One is that they are not extreme in any sense. Indeed, after deviations, they simply bring inflation back to the desired path and do not threaten the private economy with dire outcomes after deviations. Also, the reversion policies are structured so that after a deviation, the continuation is an equilibrium. This construction means that we do not declare deviations to be unacceptable by simply asserting that there are no continuation equilibria.

A simple way to state our general message here is that uniqueness somewhere generates uniqueness everywhere. Our analysis says that sophisticated policies can lead to unique implementation of interest rate policy if best responses are controllable. A sufficient condition for controllability is that reversion policies can be found under which the continuation equilibrium is unique and varies with policy. The latter requirement typically holds, so that if policies can be found under which the continuation equilibrium is unique (somewhere), then we have unique implementation (everywhere).

Some may question our approach because it apparently relies on the ideas that private agents and the central bank never tremble in their actions and that the central bank can detect any deviation, that is, perfectly monitors private agents' actions. Trembles by private agents or the central bank is not a difficult issue; we easily show that even with them, our results are not affected. The other concern requires more analysis to dismiss. We consider an environment in which the central bank can only imperfectly monitor private agents' actions and show that if the central bank can monitor with a sufficiently high probability, then suitably modified sophisticated policies can uniquely produce any outcome.

1. A Simple Model with One-Period Price Stickiness

The model we use to analyze the unique implementation is a modified version of a basic sticky price model with a New Classical Phillips curve (as in Woodford 2003, Chap. 3, Sec. 1.3). Here, in order to make our results comparable to those in the literature, we describe a simple, linearized version of that model. In Appendix A, we describe the general equilibrium model

that when linearized gives rise to the equilibrium conditions studied here. Our implementation result holds in the nonlinear model as well.

A. The Determinants of Output and Inflation

Here we describe the agents in the simple model, their behavior, and how the economy produces a New Classical Phillips curve.

Consider a monetary economy populated by a large number of identical, infinitely lived consumers, a continuum of producers, and a central bank. Each producer uses labor to produce a differentiated good on the unit interval. Producers $i \in [0, \alpha]$ are *flexible price* producers, and $i \in (\alpha, 1]$ are *sticky price* producers.

The timing within a period is as follows. At the beginning of the period, sticky price producers set their prices, after which the central bank chooses its monetary policy, either by setting interest rates or by choosing the quantity of money. Two shocks η_t and ν_t are then realized. At the end of the period, flexible price producers set their prices, and consumers make their decisions. We interpret the shock η_t as a *flight to quality* shock that affects the attractiveness of government debt relative to private claims and the shock ν_t as a *velocity* shock.

We here develop necessary conditions for an equilibrium in this economy and then, in the next section, formally define an equilibrium. Here and throughout we express all variables in log-deviation form. In particular, this way of expressing variables implies that none of our equations will have constant terms.

Consumer behavior in this model is summarized by an intertemporal Euler equation and a cash-in-advance constraint. We can write the linearized Euler equation as

(1) $\qquad y_t = E_t [y_{t+1}] - \psi (i_t - E_t [\pi_{t+1}]) + \eta_t,$

where y_t is aggregate output, i_t is the nominal interest rate, η_t is an i.i.d. mean zero shock with variance $var(\eta)$, and $\pi_{t+1} = p_{t+1} - p_t$ is the inflation rate from time period t to $t + 1$, where p_t is the aggregate price level. The parameter ψ determines the intertemporal elasticity of substitution in consumption, and E_t denotes the expectations of a representative consumer given that consumer's information in period t, which includes the shock η_t.

The cash-in-advance constraint, when first-differenced, implies that the relationship among

5

inflation π_t, money growth μ_t, and output growth $y_t - y_{t-1}$ is given by a quantity equation of the form

$$(2) \qquad \pi_t = \mu_t - (y_t - y_{t-1}) + \nu_t,$$

where ν_t is an i.i.d. mean zero shock with variance $\text{var}(\nu)$.

We turn now to producer behavior. The aggregate price level p_t is a linear combination of the prices p_{ft} set by the flexible price producers and the prices p_{st} set by the sticky price producers and is given by

$$(3) \qquad p_t = \int_0^\alpha p_{ft}(i) + \int_\alpha^1 p_{st}(i).$$

The optimal price set by an individual flexible price producer i satisfies

$$(4) \qquad p_{ft}(i) = p_t + \gamma y_t,$$

where the parameter γ is the elasticity of the equilibrium real wage with respect to output and is referred to in the literature as *Taylor's* γ. The optimal price set by sticky price producer i satisfies

$$(5) \qquad p_{st}(i) = E_{t-1}\left[p_t + \gamma y_t\right],$$

where E_{t-1} denotes expectations at the beginning of period t before the shock η_t is realized. Using language from game theory, we can think of equations (4) and (5) as akin to the *best responses* of the flexible and sticky price producers given their beliefs about the aggregate price level and aggregate output. Equations (1)–(5) completely describe the simple model.

In this model, the flexible price producers are strategically uninteresting. Their expectations about the future have no influence on their decisions; their prices are set mechanically according to the static considerations reflected in (4). In all that follows, equation (4) will hold on and off the equilibrium path, and we can think of $p_{ft}(i)$ as being residually determined by (4) and substitute out for $p_{ft}(i)$ from these equations. To do so, substitute (4) into (3) and solve for p_t to get

$$(6) \qquad p_t = \kappa y_t + \frac{1}{1-\alpha} \int_\alpha^1 p_{st}(i),$$

where $\kappa = \alpha\gamma/(1-\alpha)$. Now the model is summarized by (1), (2), (5), and (6).

6

We follow the literature and express the sticky price producers' decisions in terms of inflation rates rather than price levels. To do so, let $x_t(i) = p_{st}(i) - p_{t-1}$, and rewrite (5) as

(7) $x_t(i) = E_{t-1}[\pi_t + \gamma y_t]$.

For convenience, we define

(8) $x_t = \dfrac{1}{1 - \alpha} \displaystyle\int_\alpha^1 x_t(i) \; di$

to be the average price set by the sticky price producers relative to the aggregate price level in period $t - 1$. We can then rewrite (6) as

(9) $\pi_t = \kappa y_t + x_t$.

For later use, note that the economy is now summarized by (1), (2), and (7)–(9), so that when checking whether a constructed outcome is an equilibrium, we need check only whether these equations are satisfied.

In the following lemma, we show that this economy produces the key features of a New Classical Phillips curve along the equilibrium path in which

(10) $x_t(i) = x_t$.

(See the discussion below for what happens after deviations from the equilibrium path.)

Lemma 1. Any allocations that satisfy (3)−(5) and (10) satisfy three conditions: (i)

(11) $x_t = E_{t-1}\pi_t$,

(ii) $E_{t-1}y_t = 0$, and (iii) the New Classical Phillips curve:

(12) $\pi_t = \kappa y_t + E_{t-1}\pi_t$,

where again $\kappa = \alpha\gamma/(1 - \alpha)$.

Proof. To prove (i), note that (4) and (5) imply that $p_{st}(i) = E_{t-1}p_{ft}(j)$ for all i and j. Thus we get the result simply by substituting this equation into (3), taking conditional expectations of the resulting equation, and subtracting p_{t-1} from both sides. To prove (ii),

substitute (4) and (5) into (3) and take expectations in period $t - 1$ on both sides of the resulting equation. To prove (iii), substitute $E_{t-1} y_t = 0$ into (9). *Q.E.D.*

Note that when it sets monetary policy, the central bank has to choose to operate under either a money regime or an interest rate regime. In the money regime, the central bank sets money growth μ_t, and the interest rate i_t is residually determined from (1) after the realization of the shock η_t. In the interest rate regime, the central bank sets the interest rate i_t, and money growth μ_t is residually determined from (2) after the realization of the shock η_t. Of course, in both regimes, equations (1) and (2) hold.

B. Competitive Equilibrium

Now we define a notion of *competitive equilibrium* in the spirit of Barro (1979) and Lucas and Stokey (1983). In this equilibrium, allocations, prices, and policies are all defined as functions of the history of exogenous events $s^t = (s_0, \ldots, s_t)$, where $s_t = (\eta_t, \nu_t)$.

The actions of the sticky price producers, inflation, and output can be summarized by $\{x_t(s^{t-1}), \pi_t(s^t), y_t(s^t)\}$. In terms of the policies, we find it convenient to let the regime choice as well as the policy choice within the regime be $\delta_t(s^{t-1}) = (\delta_{1t}(s^{t-1}), \delta_{2t}(s^{t-1}))$, where the first argument $\delta_{1t}(s^{t-1}) \in \{M, I\}$ denotes the regime choice, money (M) or interest rates (I), and the second, $\delta_{2t}(s^{t-1})$, denotes the policy choice within the regime, either money growth $\mu_t(s^{t-1})$ or interest rates $i_t(s^{t-1})$. If the money regime is chosen in t, then the interest rate is determined residually at the end of that period, while if the interest rate regime is chosen in t, then the money growth rate is determined residually at the end of the period. Let $a_t(s^t)$ denote the allocations, prices, and policies in this competitive equilibrium.

A collection of allocations, prices, and policies $a_t(s^t) = \{x_t(s^{t-1}), \pi_t(s^t), y_t(s^t), \delta_t(s^{t-1})\}$ is a *competitive equilibrium* if it satisfies (1), (2), (9), and (11).

C. Sophisticated Equilibrium

We now turn to our notion of *sophisticated equilibrium*. Its definition is similar to that of *competitive equilibrium* except that here we allow allocations, prices, and policies to be functions of the history of aggregate private actions and policies as well as the history of exogenous events.

Observations and Definitions

We make two observations before we turn to our formal definition. One is that our definition of sophisticated equilibrium simply specifies policy rules to be followed by the central bank and does not require that it follow any form of optimality. We specify sophisticated policies in this way in order to show that our result regarding unique implementation does not depend on the objectives of the central bank. One way of thinking of our sophisticated policies is that they are specified at the beginning of period 0, and then the central bank is is committed to follow them.

Our other observation is that the only interesting private agents in this model are the sticky price producers. Their behavior at the beginning of period t depends on what they expect the central bank to do and what other sticky price producers do. The flexible price producers, recall, are described by a simple static rule (4). Their behavior and that of the consumers is summarized by an intertemporal Euler equation (1) and the cash-in-advance constraint (2).

We turn now to defining the histories that private agents and the central bank confront when they make their decisions. The public events that occur in a period are, in chronological order, $q_t = (x_t; \delta_t; s_t; y_t, \pi_t)$. Letting h_t denote the history of these events from period 0 up to and including those in period t, we have that $h_t = (h_{t-1}, q_t)$ for $t \geq 1$ and $h_0 = q_0$. For notational convenience, we focus on perfect public equilibria in which the central bank's strategy is a function of only the public history.

The public history faced by the sticky price producers at the beginning of period t when they set their prices is h_{t-1}. A strategy for the sticky price producers is a sequence of rules $\sigma_s(i) = \{x_t(i, h_{t-1})\}$ for choosing prices for every possible public history, while average prices by these producers are given by $\sigma_x = \{x_t(h_{t-1})\}$.

The public history faced by the central bank when it chooses its regime and sets either its money growth or interest rate policy is $h_{gt} = (h_{t-1}, x_t)$. A strategy for the central bank $\{\delta_t(h_{gt})\}$ is a sequence of rules for choosing the regime as well as the policy within the regime, either $\mu_t(h_{gt})$ or $i_t(h_{gt})$.

If the money regime is chosen in period t (where $\delta_{1t}(h_{gt})$ specifies M), then interest rates $i_t(h_{yt}(M))$, output $y_t(h_{yt}(M))$, and inflation rates $\pi_t(h_{yt}(M))$ are determined residually from (1), (2), (9), and (11) after the relevant shocks are realized, where $h_{yt}(M) = (h_{t-1}, x_t; M, \mu_t; s_t)$

is the history that determines output, inflation, and interest rates in the current period.

If, instead, the interest rate regime is chosen in period t (where $\delta_{1t}(h_{gt})$ specifies I), then the money growth rate $\mu_t(h_{yt}(I))$ as well as output $y_t(h_{yt}(I))$ and inflation $\pi_t(h_{yt}(I))$ are determined residually from (1), (2), (9), and (11) after the relevant shocks are realized, where $h_{yt}(I) = (h_{t-1}, x_t; I, i_t; s_t)$ is the history that determines output, inflation, and money growth in the current period.

We let σ_g denote the strategy of the central bank consisting of the regime choice and the policies under that regime. At the end of period t, output and inflation are determined as functions of the relevant history $h_{yt}(\delta_t)$ according to the rules $y_t(h_{yt}(\delta_t))$ and $\pi_t(h_{yt}(\delta_t))$. We let $\sigma_y = \{y_t(h_{yt}(\delta_t))\}$ and $\sigma_\pi = \{\pi_t(h_{yt}(\delta_t))\}$ denote the sequence of output and inflation rules.

A *sophisticated equilibrium* given the policies here is a collection of strategies $(\sigma_s(i), \sigma_x, \sigma_g)$ and output and inflation rules (σ_y, σ_π) such that, given the other strategies and rules, $\sigma_s(i)$ is optimal for all histories,

$$(13) \quad x_t(h_{t-1}) = \frac{1}{1-\alpha} \int_\alpha^1 x_t(i, h_{t-1}) \, di,$$

and (1), (2), (9), and (11) are satisfied in the manner described above.

In light of condition (13) and the observation that given (σ_g, σ_x), output, inflation, and the residually determined policy are mechanically given by (1), (2), and (9), we summarize a sophisticated equilibrium by (σ_g, σ_x). Note for later, from Lemma 1, that

$$(14) \quad x_t(h_{t-1}) = E[\pi_t | h_{t-1}].$$

Associated with each sophisticated equilibrium $\sigma = (\sigma_g, \sigma_x)$ are the particular stochastic processes for outcomes that occur along the equilibrium path, called *sophisticated outcomes*. These outcomes can be generated from the strategies in the standard recursive fashion and then be written as a function of the history of exogenous events $s^t = (s_0, \ldots, s_t)$, where $s_t = (\eta_t, \nu_t)$. These (on the equilibrium path) outcomes include allocations $a(\sigma) = \{x_t(s^{t-1}; \sigma), \pi_t(s^t; \sigma), y_t(s^t; \sigma), \delta_t(s^{t-1})\}$. We call an allocation $a(\sigma)$ associated with a sophisticated equilibrium σ a *sophisticated outcome*. The following lemma is an immediate consequence of the definitions of competitive equilibrium and sophisticated outcomes.

Lemma 2. (*Equivalence Between Competitive Equilibria and Sophisticated Outcomes*) A sophisticated outcome is a competitive equilibrium, and for any given competitive equilibrium, a sophisticated policy exists that supports the competitive equilibrium as a sophisticated outcome.

Equilibrium with Sophisticated Policies

We now show that any competitive equilibrium can be uniquely implemented with sophisticated policies. The basic idea behind our construction is that the central bank starts by picking any competitive equilibrium allocations and sets its policy on the equilibrium path consistent with this equilibrium. The central bank then constructs its policy off the equilibrium path so that any deviations from these allocations would never be optimal for the deviating agent. In so doing, the constructed sophisticated policies support the chosen allocations as the unique equilibrium allocations.

For convenience, we consider sophisticated policies with *one-period reversion to money.* Under these policies, the central bank discourages deviations by switching to a money regime for one period, and for the rest of the off-the-equilibrium-path policies, it uses the continuation of what it would have done on the equilibrium path. In particular, after a deviation, the central bank switches to a level of the money supply which generates the same expected inflation as in the original equilibrium. (Of course, we could have chosen many other values that also would discourage deviations, but we found this value to be the most intuitive one.[2]) Having the central bank switch to a money regime after a deviation instead of to another interest rate is convenient because in this model outcomes are uniquely determined under a money regime.

We start our analysis by establishing that if the central bank chooses money as its instrument in period t, then output and inflation are uniquely determined in period t. We then show how to construct sophisticated policies that uniquely support any competitive equilibrium allocations.

Lemma 3. (*Controllability of Best Responses*) For any history h_{gt}, if the central bank chooses the money regime with money growth μ_t, then output y_t and inflation π_t are uniquely determined and given by

(15) $\quad y_t = \dfrac{\mu_t + \nu_t + y_{t-1} - x_t}{1 + \kappa}$

(16) $\quad \pi_t = \kappa y_t + x_t.$

11

Proof. The proof is immediate from substituting (2) into (9) and recalling that y_{t-1} and x_t are in the history h_{gt}. *Q.E.D.*

Note that this lemma applies to histories h_{gt} which have been generated off the equilibrium path as well as on it. In particular, it applies to histories in which the sticky price producers' choice of inflation x_t represents a deviation from their strategies (and does not equal their expectations of inflation).

Suppose now that interest rates are the chosen policy instrument. Fix a desired competitive equilibrium outcome path $\left(x_t^*(s^{t-1}), \pi_t^*(s^t), y_t^*(s^t)\right)$ together with central bank policies $i_t^*(s^{t-1})$. Consider the following trigger-type policy that supports these outcomes as unique equilibria: If sticky price producers choose x_t in period t to coincide with the desired outcomes $x_t^*(s^{t-1})$, then let central bank policy in t be $i_t^*(s^{t-1})$. If not and these producers deviate to some $\tilde{x}_t(s^{t-1}) \neq x_t^*(s^{t-1})$, then for that period t, let the central bank switch to a money regime with money growth set so that the expected inflation for that period equals the expected level of inflation in the original equilibrium, namely, $x_t^*(s^{t-1})$. To determine the required level of money growth, use (15) and (16) to calculate

$$(17) \quad \tilde{\mu}_t = \tilde{x}_t(s^{t-1}) - y_{t-1} + \frac{1+\kappa}{\kappa}\left[x_t^*(s^{t-1}) - \tilde{x}_t(s^{t-1})\right].$$

From period $t+1$ on along this deviation path, let the central bank use what it would have done if producers had not deviated. Then, from period $t+1$ on along the equilibrium path, let the central bank continue on with the analog of the policies just described.

We use these policies to establish the following proposition:

PROPOSITION 1. UNIQUE IMPLEMENTATION WITH SOPHISTICATED POLICIES. Any competitive equilibrium outcome can be implemented as a unique equilibrium with sophisticated policies with one-period reversion to money.

Proof. Consider the sophisticated policies described above, and suppose that in period t the sticky price producers deviate to $\tilde{x}_t(s^{t-1}) \neq x_t^*(s^{t-1})$. Then the central bank sets money growth according to (17), and the resulting inflation, by construction, is $\pi_t^*(s^t)$, and the resulting output is

$$(18) \quad \tilde{y}_t = \frac{\tilde{\mu}_t + \nu_t + y_{t-1} - \tilde{x}_t(s^{t-1})}{1+\kappa},$$

12

where we have used (15). Substituting for $\tilde{\mu}_t$ from (17) gives that

$$(19) \quad E_{t-1}\tilde{y}_t = \frac{1}{\kappa}\left[x_t^*(s^{t-1}) - \tilde{x}_t(s^{t-1})\right].$$

We need to show that given these levels of inflation and output, a sticky price producer will not find the deviation described above optimal. That is, the sticky price producer will set $x_t(i)$ to some value other than $\tilde{x}_t$. From (7), we can see that the best response of a sticky price producer is

$$(20) \quad x_t(i) = E_{t-1}\left[x_t^*(s^{t-1}) + \gamma\tilde{y}_t\right],$$

where we have used the fact that $\tilde{\mu}_t$ is constructed to generate a level of inflation equal to $x_t^*(s^{t-1})$. Combining (19) and (20), we have that the best response of the sticky price producer is

$$x_t(i) = x_t^*(s^{t-1}) + \frac{1}{\kappa}\left[x_t^*(s^{t-1}) - \tilde{x}_t(s^{t-1})\right].$$

Since $\kappa > 0$, clearly $x_t(i) \neq \tilde{x}_t(s^{t-1})$ whenever $\tilde{x}_t(s^{t-1}) \neq x_t^*(s^{t-1})$. That is, an optimizing individual sticky price producer will follow the deviation $\tilde{x}_t(s^{t-1})$ whenever $\tilde{x}_t(s^{t-1})$ is indeed a deviation from $x_t^*(s^{t-1})$. Q.E.D.

The logic of the proof of the proposition makes clear that in order for reversions to money to uniquely implement equilibrium outcomes, sophisticated policies must have a key *controllability* property: after a deviation, the central bank can choose policies so as to make an optimizing an individual price-setter not go along with the deviation. A sufficient condition for this property is that an individual price-setter's best response is monotone in the money growth rate. The construction of money growth given in (17) shows that in the simple model, after a deviation, monetary policy can be chosen in such a way that the best response of any individual price-setter can be controlled.

Implications for the Taylor Principle

Now that we've established in our simple model that any competitive equilibrium can be uniquely implemented by sophisticated policies, we turn now to the welfare implications of the Taylor principle.

We begin by characterizing equilibrium with policies that are restricted to be the same on and off the equilibrium path. We label such policies *restricted policies.* We follow the literature in focusing on Taylor-type rules in which interest rates are linear functions of inflation and output. We show that if such rules satisfy the Taylor principal, then they are inefficient.

Equilibrium with Restricted Policies We show here that Taylor rule type standard policies produce the standard result. Policies that satisfy the Taylor principle produce a continuum of equilibria rather than a unique one. In all but one of these equilibria, the economy has explosive inflation rates. In the nonexplosive equilibrium, the economy has a constant expected inflation rate. Policies that violate the Taylor principle produce a continuum of bounded equilibria.

One way of specifying Taylor rules is to write them as

$$(21) \quad i_t = \phi E_{t-1}\pi_t + b E_{t-1} y_t,$$

where ϕ and b are parameters representing the responsiveness of interest rates to expected inflation and expected output. When the parameter $\phi > 1$, such policies are said to satisfy the *Taylor principle*, namely, that the central bank should raise its interest rate more than one-for-one with increases in inflation. When $\phi < 1$, such policies are said to violate that principle.

Of course, the Taylor rule is not a well-defined function of histories until we fill in how expectations are formed. To do so, we begin with a simple lemma. The lemma shows that under any interest rate rule, the expected inflation rate is uniquely determined by the policy, but the realized inflation rate may not be.

Lemma 4. In any history h_{t-1},

$$(22) \quad E\left[y_t|h_{t-1}\right] = 0.$$

If that history leads to an interest rate regime, then

$$(23) \quad E\left[\pi_{t+1}|h_{t-1}\right] = i_t(h_{gt}),$$

where $h_{gt} = \left(h_{t-1}, x_t(h_{t-1})\right).$

Proof. Note that (22) is simply a restatement of part (ii) of Lemma 1. Taking expectations of the Euler equation (1) with respect to h_{t-1} gives that

$$(24) \quad E\left[y_t|h_{t-1}\right] = E\left[y_{t+1}|h_{t-1}\right] - \psi\big(i_t(h_{t-1}) - E\left[\pi_{t+1}|h_{t-1}\right]\big).$$

Using the law of iterated expectations gives that $E\left[y_{t+1}|h_{t-1}\right] = 0$. From (24), we then have (23), that $E\left[\pi_{t+1}|h_{t-1}\right] = i_t(h_{gt})$. *Q.E.D.*

From this lemma we know that $E\left[y_t|h_{t-1}\right] = 0$. Since $E\left[\pi_t|h_{t-1}\right] = x_t$, policies of the Taylor rule form can be written as

$$(25) \quad i_t = \phi x_t.$$

We follow the literature in focusing on equilibria in which all outcomes are linear functions of the history. We restrict attention to equilibrium in which inflation is bounded below (with probability 1). The rationale for this restriction is that the nominal interest rate must be nonnegative.

PROPOSITION 2. INDETERMINACY OF EQUILIBRIUM UNDER RESTRICTED POLICIES. The linear equilibria with interest rate rules of the Taylor rule form (25) have outcomes of the form

$$(26) \quad x_{t+1} = i_t + c\eta_t, \quad \pi_t = x_t + \kappa(1 + \psi c)\eta_t, \quad \text{and} \quad y_t = (1 + \psi c)\eta_t.$$

For every ϕ, the economy has a continuum of equilibria indexed by the parameter c and by x_0. For every $\phi \geq 1$, the economy has a continuum of unbounded equilibria indexed by c and $x_0 \geq 0$ as well as a unique bounded equilibrium with $c = 0$ and $x_0 = 0$. For $\phi < 1$, all the equilibria are bounded.

Proof. In order to verify that the outcomes which satisfy (26) are part of an equilibrium, we need to check that they satisfy (1), (9), and (14). That they satisfy (14) follows by taking expectations of both sides of the proposition's middle equation $\pi_t = x_t + \kappa(1+\psi c)\eta_t$. Substituting for x_{t+1} from (26) and i_t from (25) into (1), we obtain that $y_t = (1 + \psi c)\eta_t$, as required by (26). Inspecting the expressions for π_t and y_t in (26) shows that they satisfy (9).

To discuss boundedness, we first substitute from (25) into the first equation in (26) to obtain a difference equation in expected inflation:

(27) $x_{t+1} = \phi x_t + c\eta_t.$

If $\phi \geq 1$, then clearly $x_t = 0$ when $c = 0$ and $x_0 = 0$, and x_t is unbounded otherwise. If $\phi < 1$, then clearly x_t is bounded. *Q.E.D.*

In the literature, researchers often restrict attention to bounded equilibria. The proposition shows that under such a restriction, policies that obey the Taylor principle lead to a unique equilibrium, and policies that violate it lead to indeterminacy. This observation leads researchers in this literature to prefer policies that satisfy the Taylor principle on the grounds that they are less risky.

We argue, however, that policies that satisfy the Taylor principle are at least as risky as those that violate it. In the equilibria that satisfy the Taylor principle, these risks arise because inflation can potentially explode. The explosive behavior of inflation, in turn, is due to the explosive behavior of the money supply.

To see this, consider the growth rate of the money supply given in (2). For simplicity, suppose that $\eta_t = \nu_t = 0$ for all t. Using (22), we know that $y_t = 0$ for all t, and hence, the growth of the money supply is given by

(28) $\mu_t = x_t = \phi^t x_0.$

Thus, in these equilibria, inflation explodes because money growth explodes. Each equilibrium is indexed by a different initial value of the endogenous variable x_0. This endogenous variable depends solely on expectations of future policy and is not pinned down by any initial condition or transversality condition.

The idea that the central bank's printing of money at an ever-increasing rate leads to a hyperinflation is at the core of most monetary models. In these equilibria, inflation does not arise from the speculative reasons analyzed by Obstfeld and Rogoff (1983) but from the conventional money printing reasons analyzed by Cagan (1956). In this sense, the theory predicts for perfectly standard and sensible reasons that if the central bank follows a Taylor rule that satisfies the Taylor principle, then the economy can suffer from any one of a continuum of very undesirable paths for inflation.

Now consider an economy with stochastic shocks in which η_t and ν_t are not restricted to be zero. When $\phi \geq 1$, the economy has two kinds of indeterminate equilibrium. In one kind, $c = 0$ and expected inflation grows in a deterministic fashion. In the other kind, $c \neq 0$ and expected inflation grows in a stochastic fashion with mean growth rate ϕ. When $\phi < 1$, the economy has a continuum of bounded equilibria. In one kind, $c = 0$ and expected inflation gradually reverts to 0. In the other kind, $c \neq 0$ and expected inflation fluctuates, and its mean value reverts to 0. The intuitive idea behind the multiplicity of stochastic equilibria in Proposition 2 associated with $c \neq 0$ is that interest rates pin down only expected inflation and not the state-by-state realizations indexed by the parameter c.

In Proposition 2, we focused on linear equilibria which can be described as time-invariant linear functions of the shocks. Clearly, there are other equilibria in which the coefficients of the allocation rules depend on period t as well as the history of the shocks. There are also equilibria in which the allocations depend on exogenous sunspots. Our theorems about supporting outcomes with sophisticated policy rules apply to all of these equilibria as well.

Welfare Here we examine the efficiency properties of the various outcomes that can arise under the Taylor rule.

We evaluate these outcomes using a quadratic approximation to the utility of the representative consumer, given by

$$(29) \quad E_0 \sum \beta^t [(\pi_t - E_{t-1}\pi_t)^2 + \gamma y_t^2],$$

where β is the discount factor. (For details, see Woodford 2003, Chap. 6, Prop. 6.2.)

Interestingly, equilibria which satisfy the Taylor principle turn out to be inefficient: they are dominated by an equilibrium which violates that principle. Given the form of the objective function in (29), the best equilibrium minimizes the appropriately weighted sum of the variances of unexpected inflation and output. From the form of the equilibrium in Proposition 2, we know that the variance of unexpected inflation is $\kappa^2(1 + \psi c)^2 var(\eta)$, and that of output is $(1 + \psi c)^2 var(\eta)$. Clearly, the value of c which maximizes welfare is $c = -1/\psi$.

We summarize this discussion with a proposition:

PROPOSITION 3. INEFFICIENCY OF RULES SATISFYING THE TAYLOR PRINCIPLE. In

the simple model with one-period price stickiness, the outcomes under a Taylor rule of the form (25) with $\phi > 1$ are dominated by outcomes of an equilibrium with $\phi = 0$.

To get some intuition for this proposition, think of η_t as a type of demand shock, a shock that distorts the relationship between the intertemporal marginal rate of substitution and the return on government debt. (The nonlinear model in Appendix A formalizes this interpretation.) When η_t is positive, the Euler equation (1) implies that for a given value of expected consumption and real interest rates, desired consumption in t rises. In a determinate equilibrium, $c = 0$. Hence, when desired consumption in t rises, actual consumption rises one-for-one with it. For the economy to produce this increased output, the Phillips curve (9) implies, unexpected inflation must rise one-for-one as well. The opposite happens when the shock to marginal utility is negative. Hence, in this economy, output and inflation simply inherit the variability of the demand shock.

Now consider an equilibrium of the sort considered in Proposition 2 with c negative. In this equilibrium, when the demand shock is positive, real interest rates in (1) rise because expected inflation rises. This rise in real interest rates dampens the rise in desired consumption. Actual consumption then rises less than one-for-one with the shock, and from the Phillips curve, we know that inflation does too. Thus, in this equilibrium, the variability of output and that of inflation are lower than in the best equilibrium with $\phi > 1$.

As we have shown, any restricted policy is associated with a wide variety of outcomes because equilibrium is indeterminate. As we have stressed, this indeterminacy occurs regardless of whether or not the Taylor principle is satisfied.

Finally, combining Propositions 1 and 3 immediately gives the following corollary:

Corollary. (*Optimality with Sophisticated Policies*) The optimal allocations in the simple sticky price model can be uniquely supported by sophisticated polices which specify that along the equilibrium path the central bank follow Taylor rules which violate the Taylor principle and after deviations specify the sophisticated policies described above.

Thus far we have focused on welfare under interest rate regimes. Note that in the efficient equilibrium under such regimes (with $c = -1/\psi$) the variances of output and unexpected inflation are both zero. (Of course, in this equilibrium, inflation π_t itself is variable: it is given

by $-\eta_{t-1}/\psi$, so that it has variance $var(\eta)/\psi^2$.) Note that the best equilibrium under an interest rate regime is better than any equilibrium under a money regime. To see this result, recall that under a money regime, there is a unique equilibrium. In that equilibrium, $y_t = \nu_t/(1 + \kappa)$ and $\pi_t - E_{t-1}\pi_t = \kappa\nu_t/(1 + \kappa)$. This equilibrium clearly has lower welfare than the efficient equilibrium under an interest rate regime.

2. A New Keynesian Model

We turn now to a version of our simple model with staggered price-setting. Our main point here is to show that the simple model's primary result, that sophisticated policies can uniquely implement any equilibrium allocation, carries through to this widely used setting. To make this point in the simplest way, we first abstract from aggregate uncertainty. Later we add uncertainty, in order to demonstrate that our secondary result also holds in this model; under sufficient conditions policies that obey the Taylor principle do not support the best outcomes. Our stochastic model with aggregate uncertainty is closely related to the New Keynesian model in Woodford (2003, Chap. 3, Sec. 2), but allows for a different class of stochastic shocks.

A. A Deterministic Version

We begin by setting up the deterministic model. We show in this model that sophisticated policies can uniquely implement any competitive equilibrium, whether the policies after deviations revert to money or alternative interest rate regimes.

Setup: Changes from the Simple Model

Consider, then, a model with no aggregate shocks in which prices are set in a staggered fashion as in the work of Calvo (1983). At the beginning of each period, a fraction $1 - \alpha$ of producers are randomly chosen and allowed to reset their prices. After that, the central bank makes its decisions, and then, finally, consumers make theirs. This economy has no flexible price producers. The nonlinear economy is described in Appendix A.

The linearized equations for this model are similar to those in the simple model. The Euler equation (1) and the money growth equation (2) are unchanged except that they have no shocks, η_t, ν_t. The price set by a sticky price producer which is permitted to reset its price

19

is given by the analog of (5), which is

$$(30) \quad p_{st}(i) = (1 - \alpha\beta) \left[\sum_{r=0}^{\infty} (\alpha\beta)^{r-t} (\gamma y_r + p_r) \right].$$

Here, again, Taylor's γ is the elasticity of the equilibrium real wage with respect to output. Letting p_{st} denote the average price set by firms that are permitted to reset their prices in period t, we can recursively rewrite this equation as

$$(31) \quad p_{st}(i) = (1 - \alpha\beta) [\gamma y_t + p_t] + \alpha\beta p_{st+1},$$

together with a type of transversality condition $\lim_{T\to\infty} (\alpha\beta)^T p_{sT}(i) = 0$. The aggregate price level can be written as

$$(32) \quad p_t = \alpha p_{t-1} + (1 - \alpha) p_{st}.$$

To make our analysis parallel to the literature, we again express the decisions of the sticky price producers in terms of the inflation rate rather than prices. Letting $x_t(i) = p_{st}(i) - p_{t-1}$, with some manipulation, we can rewrite (31) as

$$(33) \quad x_t(i) = (1 - \alpha\beta)\gamma y_t + \pi_t + \alpha\beta x_{t+1}.$$

We can also rewrite (32) as

$$(34) \quad \pi_t = (1 - \alpha) x_t,$$

where x_t is the average across i of $x_t(i)$.

The transversality-type condition can be rewritten in terms of inflation rates as

$$(35) \quad \lim_{T\to\infty} (\alpha\beta)^T x_t(i) = 0.$$

In equilibrium, since $x_t(i) = x_t$ and (34) holds, this restriction is equivalent to

$$(36) \quad \lim_{T\to\infty} (\beta\alpha)^T \pi_t = 0.$$

In the following lemma, we show that this economy produces the key features of a New Keynesian Phillips curve along the equilibrium path in which

$$(37) \quad x_t(i) = x_t.$$

Lemma 5. Any allocations that satisfy (33)–(37) also satisfy the New Keynesian Phillips curve:

$$(38) \quad \pi_t = \kappa y_t + \beta \pi_{t+1},$$

where $\kappa = (1 - \alpha)(1 - \alpha\beta)\gamma/\alpha$.

Proof. To prove (38), substitute for x_t using (34) and (37) into (33). Collecting terms yields (38). *Q.E.D.*

We then have that a competitive equilibrium must satisfy (1), (2), (36), and (38). In addition to these conditions, we now argue that a competitive equilibrium must satisfy two boundedness conditions. Such conditions are controversial in the literature. Standard analyses of New Keynesian models impose strict boundedness conditions, that both output and inflation must be bounded both above and below in any reasonable equilibrium. Cochrane (2007) has forcefully criticized this practice, arguing that any boundedness conditions must have a solid economic rationale. Here we provide rationales for two such conditions. We think there are solid arguments for requiring that output y_t be bounded above, so that

$$(39) \quad y_t \leq \bar{y} \text{ for some } \bar{y},$$

and inflation be bounded below, so that

$$(40) \quad \pi \geq \underline{\pi} \text{ for some } \underline{\pi}.$$

The rationale for output being bounded above is that in this economy, there is a finite amount of labor to produce the output. The rationale for requiring that inflation be bounded below comes from the restriction that the nominal interest rate must be nonnegative.[3]

We think of the boundedness conditions (39) and (40) as being minimal. These bounds allow for outcomes in which y_t, the log of output, falls without bound (so that the level of output converges to zero). The bounds also allow for outcomes in which inflation rates rise without bound. For completeness, below we provide conditions under which our unique implementation result holds with stricter and weaker boundedness conditions as well.

With our current restrictions, a *competitive equilibrium* is a sequence of inflation rates and output which satisfy the deterministic versions of (1) and (2) as well as (36), (38), (39),

and (40). Clearly, this definition is analogous to that for a deterministic version of the competitive equilibrium in the simple model; the definition of a sophisticated equilibrium here is also analogous to that in the simple model. It should thus be clear that the equivalence of competitive equilibria and sophisticated outcomes, as in Lemma 2, holds here.

Unique Implementation of Sophisticated Policies

Now we turn to the construction of sophisticated policies which uniquely implement any competitive equilibrium. We analyze this construction under two options for the central bank: reversion to a money regime or to an interest rate regime.

Reversion to a Money Regime   In our construction of sophisticated policies with reversion to a money regime, it is convenient to consider sophisticated policies with *infinite reversion to money*. Under these policies, along the equilibrium path, the central bank chooses the prescribed interest rate i_t^*. If, instead, sticky price producers deviate by setting $\tilde{x}_t \neq x_t^*$, then the central bank switches to a money regime with money growth set so that the profit-maximizing value of $x_t(i)$ is such that $x_t(i) \neq \tilde{x}_t$.

To illustrate the details of our construction of monetary policy after a deviation, we suppose that in the nonlinear economy, preferences are given by $U(c, l) = \log c + b(1 - l)$, where c is consumption and l is labor supply, so that in the linearized economy, Taylor's γ equals one. We also suppose that after a deviation, the central bank reverts to a constant money supply $m = \log M$. With a constant money supply, it is convenient to use the original formulation of the economy with price levels rather than inflation rates. Now the cash-in-advance constraint implies that $y_r + p_r = m$ for all r, so that with $\gamma = 1$, (30) reduces to

$$(41) \quad p_{st}(i) = (1 - \alpha\beta) \left[\sum_{r=0}^{\infty} (\beta\alpha)^{r-t} m \right] = m.$$

That is, if after a deviation the central bank chooses a constant level of the money supply m, then sticky price producers optimally choose their prices to be m.

We can use (41) to show how a sophisticated policy with infinite reversion to money deters deviations. To see that, consider a history in which price-setters in period t deviate from p_{st}^* to $\tilde{p}_{st}$. Clearly, (41) implies that for any history, the central bank can effectively control the best response of any price-setter by the appropriate choice of monetary policy. Specifically, the central bank can make the optimal choice for an individual price-setter be $p_{st}(i) \neq \tilde{p}_{st}$.

The following proposition then follows immediately:

PROPOSITION 4. UNIQUE IMPLEMENTATION WITH REVERSION TO MONEY. Suppose that $\gamma = 1$. Then any competitive equilibrium, that is, any sequence of inflation and output that satisfies the deterministic versions of (1), (2), (36), (38), (39), and (40), can be implemented as a unique equilibrium with sophisticated policies with an infinite reversion to money.

The logic of the proof of the proposition again makes clear that in order for reversions to money to uniquely implement equilibrium outcomes, sophisticated policies must have a controllability property. Equation (41) makes clear that the best response of each individual price-setter is controllable.

Reversion to an Interest Rate Regime We now discuss uniquely implementing equilibrium outcomes using sophisticated policies with reversion to an interest rate regime, in particular, *reversion to Taylor rules*. Under these policies, along the equilibrium path, the central bank chooses the prescribed interest rate i_t^*. If, instead, sticky price producers deviate by setting $\tilde{x}_t \neq x_t^*$, then the central bank switches to a Taylor rule set so that the optimal relative price set by any sticky price producer i, is different from the deviation, namely, $x_t(i) \neq \tilde{x}_t$.

With such policies, uniquely implementing any equilibrium outcome requires that under interest rate regimes, best responses be controllable. Here a sufficient condition for controllability is that the continuation equilibrium is unique. To verify that the continuation equilibrium is unique, we need to describe continuation outcomes after a deviation in an arbitrary period s. We begin with a description of what happens from $s+1$ onward and then turn to what happens in period s.

Consider, then, sophisticated policies in which, after a deviation in period s, the central bank uses reversion policies of the Taylor rule form

$$(42) \quad i_t = \phi \pi_t$$

for $t \geq s + 1$. Note that this rule is a *strategy* in our sense: that is, it depends on things that have happened before the central bank makes its decision.

We will show that for $t \geq s + 1$, the Taylor rule coefficient ϕ can be chosen in such a way that the continuation equilibrium is unique and that in this equilibrium $y_t = \pi_t = 0$ for all $t \geq s + 1$. We will then construct reversion policies in period s that discourage deviations.

To verify uniqueness of the continuation equilibrium for an appropriate choice of ϕ, we begin by solving (1), (38), and (42) without imposing the transversality-like condition (36) (or any boundedness conditions). To do so, we substitute out i_t in (1), using (42), to get

$$(43) \quad y_{t+1} + \psi \pi_{t+1} = y_t + \psi \phi \pi_t,$$

which together with (38) defines a dynamical system. Letting $z_t = (y_t, \pi_t)'$, with some manipulation we can stack these equations to give $z_{t+1} = A z_t$, where

$$A = \begin{bmatrix} a & b \\ \frac{-\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}$$

and where $a = 1 + \kappa\psi/\beta$ and $b = \psi(\phi - 1/\beta)$. The solutions to this system are that

$$(44) \quad \begin{aligned} y_t &= \lambda_1^{t-s-1} \omega_{1s+1} + \lambda_2^{t-s-1} \omega_{2s+1} \quad \text{and} \\ \pi_t &= \lambda_1^{t-s-1} \left(\frac{\lambda_1 - a}{b} \right) \omega_{1s+1} + \lambda_2^{t-s-1} \left(\frac{\lambda_2 - a}{b} \right) \omega_{2s+1}, \end{aligned}$$

where $\lambda_1 < \lambda_2$, the eigenvalues of A, are given by

$$(45) \quad \lambda_1, \lambda_2 = \frac{1}{2} \left(\frac{1 + \kappa\psi}{\beta} + 1 \right) \pm \frac{1}{2} \sqrt{ \left(\frac{1 + \kappa\psi}{\beta} - 1 \right)^2 - 4(\phi - 1)\frac{\kappa\psi}{\beta} },$$

and $\omega_{1s+1} = \left[(\frac{\lambda_2 - a}{b}) y_{s+1} - \pi_{s+1} \right] / \Delta$, $\omega_{2s+1} = \left[(\frac{a - \lambda_1}{b}) y_{s+1} + \pi_{s+1} \right] / \Delta$, where Δ is the determinant of A. Here and throughout we restrict attention to values of $\phi \in [0, \phi_{\max}]$, where $\phi_{\max}$ is the largest value of ϕ that yields real eigenvalues. (That is, at $\phi_{\max}$ the discriminant in (45) is zero.)

For a continuation outcome to be part of an equilibrium outcome, it must satisfy the transversality-like condition (36) and the boundedness conditions (39) and (40) as well as (44). The restrictions imposed by condition (36) on the solutions described in (44) can be derived by substituting for π_t in (36), using (44), to get

$$(46) \quad \lim_{T \to \infty} (\beta\alpha)^T \left[\lambda_1^T \left(\frac{\lambda_1 - a}{b} \right) \omega_{1s+1} + \lambda_2^T \left(\frac{\lambda_2 - a}{b} \right) \omega_{2s+1} \right] = 0.$$

24

The boundedness conditions can be rewritten using (44) as

$$
\begin{aligned}
(47) \quad y_s &= \lambda_1^{t-s-1}\omega_{1s+1} + \lambda_2^{t-s-1}\omega_{2s+1} \leq \bar{y} \text{ and} \\
\pi_s &= \lambda_1^{t-s-1}\left(\frac{\lambda_1 - a}{b}\right)\omega_{1s+1} + \lambda_2^{t-s-1}\left(\frac{\lambda_2 - a}{b}\right)\omega_{2s+1} \geq \underline{\pi}.
\end{aligned}
$$

The "initial" conditions $\omega_{1s+1}, \omega_{2s+1}$ satisfying (46) and (47) determine the continuation outcomes from (44).

Next we discuss a lemma which shows that under the condition

$$
(48) \quad \alpha(1 + \kappa\psi) > 1,
$$

some value of $\bar{\phi} > 1$ exists such that after any history, if the central bank switches to a Taylor rule with $\phi \in (1, \bar{\phi})$, the resulting continuation is unique. In particular, under (48), the initial conditions $\omega_{1s+1}, \omega_{2s+1}$ satisfying (46) and (47) are unique and equal to 0 for a range of values of the Taylor coefficient $\phi > 1$.

The idea of the proof is that we use the transversality condition to eliminate the large root indeterminacy associated with the initial condition ω_{2s+1}. We also eliminate the small root indeterminacy associated with the initial condition ω_{1s+1}, using the boundedness condition.

Before we develop these conditions, note that if $\phi < 1$, then we know from (45) that $\lambda_1 < 1$, and the small root indeterminacy cannot be eliminated because there exists a continuum of values of ω_{1s+1} which satisfy (46) and (47). Consider, then, the following lemma which is proved in Appendix B:

Lemma 6. Suppose that (48) is satisfied. Then some value of $\bar{\phi} > 1$ exists such that if the central bank chooses a reversion policy of the Taylor rule form with $\phi \in (1, \bar{\phi})$, then the resulting continuation is unique, and the associated output and inflation rates are zero in all periods $t \geq s + 1$, where the deviation occurs in period s.

We now use Lemma 6 to prove controllability after a deviation by price-setters in period s. We define sophisticated policies with a reversion to the Taylor rule that support a given equilibrium outcome $\{x_t^*, \pi_t^*, y_t^*, i_t^*\}$ as follows. Along the equilibrium path, the central bank chooses the prescribed interest rates $\{i_t^*\}$. After a deviation $\tilde{x}_s$ in period s, let $\tilde{\pi}_s = (1 - \alpha)\tilde{x}_s$

denote the corresponding inflation. If $\tilde{\pi}_s \neq 0$, then let the sophisticated policy specify reversion to a Taylor rule of the form

$$(49) \quad i_r = \phi \pi_r \text{ for all } r \geq s,$$

with some $\phi \in (1, \bar{\phi})$ that satisfies $\phi \neq 1/(1-\alpha)\kappa\psi$. If $\tilde{\pi}_s = 0$, then let the sophisticated policy specify that at s, i_s is some nonzero number, and for $r \geq s+1$, let policy be given by (49).

We now show that the policies after a deviation, parameterized by ϕ and i_s, can be chosen so that the best response $x_s(i)$ of an individual price-setter is unique and controllable. From (33) and (34), the best response $x_s(i)$ given the inflation $\tilde{\pi}_s$ induced by the deviation is that

$$(50) \quad x_s(i) = (1-\alpha\beta)\gamma y_s + \frac{\tilde{x}_s}{1-\alpha} + \alpha\beta x_{s+1}.$$

Note that $x_{s+1} = 0$ because Lemma 6 implies that for all periods after the one with the deviation, output and inflation are zero; that is, $y_r = \pi_r = x_r = 0$ for all $r \geq s+1$. Next note that substituting $y_{s+1} = \pi_{s+1} = 0$ into the Euler equation (1) gives that $y_s = -\psi i_s$. Using both of these results, we can rewrite (50) as

$$(51) \quad x_s(i) = -(1-\alpha\beta)\gamma\psi i_s + \frac{\tilde{x}_s}{1-\alpha}.$$

Using (34) and the form of the sophisticated policy which implies that $i_s = \phi\tilde{\pi}_s$, we can rewrite (51) as

$$(52) \quad x_s(i) = \frac{1-\alpha(1-\alpha)\kappa\psi\phi}{1-\alpha}\tilde{x}_s.$$

The condition that $\phi \neq 1/(1-\alpha)\kappa\psi$ implies that $x_s(i) \neq \tilde{x}_s$ unless $\tilde{x}_s = 0$. If $\tilde{x}_s = 0$, then recall that the policy rule specifies that i_s is some nonzero number; thus, from (51), we know that $x_s(i)$ is not equal to zero.

We then have proved the following proposition:

PROPOSITION 5. UNIQUE IMPLEMENTATION WITH REVERSION TO TAYLOR RULES. Suppose (48) is satisfied. Then a sophisticated policy with reversion to Taylor rules after a deviation uniquely implements any competitive equilibrium outcome.

The basic idea of our construction is that by reverting to a Taylor rule with ϕ in the determinate region, the central bank uniquely pins down the continuation values of output and

inflation from $s+1$ on. By varying the policy in period s, the central bank can uniquely control any best response and thereby discourage any deviation. Thus, here, as before, sophisticated policies can be used to control best responses.

If (48) is violated, then it can be shown that interest rate rules produce indeterminacy for all $\phi \in [0, \phi_{\max}]$. For such economies, sophisticated policies which specify reversion to Taylor rules do not uniquely implement outcomes. Doing that may still be possible by specifying reversion to other interest rate rules or to money rules.

Other Views on Bounds   So far we have considered one view on bounds. Since the issue of what bounds to impose is controversial, we now briefly discuss other views. Adding bounds reduces the region of indeterminacy and expands the region of determinacy. The bounds increase the applicability of these policies, but reduce their need. As the region of determinacy expands, so does the range of parameter values for which sophisticated policies can be used for unique implementation. As the region of indeterminacy shrinks, however, the range of parameter values for which sophisticated policies are needed does too.

Consider the standard view in the literature, the *strict bound* view. In this view, only outcomes that are bounded both above and below are considered reasonable. Then the range of Taylor rule coefficients which yield uniqueness expands to include all values of $\phi \in (1, \phi_{\max})$. To see the expansion in the range, note from (45) that $\lambda_1 > 1$ when $\phi > 1$. Since $\lambda_2 \geq \lambda_1 > 1$ for $\phi > 1$, (44) and the boundedness conditions imply that $\omega_{1s+1} = \omega_{2s+1} = 0$. Hence, the continuation equilibrium is unique for all $\phi > 1$. Here, we can choose the Taylor rule parameter in a reversion to any value of $\phi > 1$; the analog of Proposition 5 thus holds even for parameter values that violate (48). Clearly, the strict bound view expands the applicability of sophisticated policies by expanding the range of ϕ such that the equilibrium is determinate relative to our view. It also reduces the range for which these policies are needed.

Another possibility is what we call the *no bounds* view, that imposing any bounds other than transversality is not appropriate. We now discuss sufficient conditions for the analog of Lemma 6 under this view.

Let $\tilde{\phi}$ be defined so that

(53) $\beta \alpha \lambda_1(\tilde{\phi}) = 1$ for some $\phi \in [0, \phi_{\max}]$,

and if no such value exists, then let $\tilde{\phi} = \phi_{\max}$. Suppose that $\tilde{\phi} < \phi_{\max}$. For that to be true, inspection of (45) makes clear, a necessary and sufficient condition is that

$$(54) \quad \frac{\beta\alpha}{2}\left(\frac{1+\kappa\psi}{\beta}+1\right) > 1,$$

which holds if $\kappa\psi$ is sufficiently large. We then have the following lemma:

Lemma 7. Consider an alternative definition of competitive equilibrium which does not impose (39) or (40). Suppose (54) is satisfied and suppose a Taylor rule with coefficient $\phi \in (1,\tilde{\phi})$ is followed from $s+1$ onward after any history. Then the continuation equilibrium is indeterminate for $\phi \le \tilde{\phi}$ and unique for $\phi > \tilde{\phi}$.

Proof. The restrictions imposed by the transversality condition (36) on the solutions described in (44) are given by (46). Suppose first that $\phi > \tilde{\phi}$. Since (45) implies that the smaller root $\lambda_1(\phi)$ is increasing in ϕ, we know that $\beta\alpha\lambda_1(\phi) > 1$ for all $\phi > \phi^*$. Since $\lambda_2(\phi) > \lambda_1(\phi)$, $\beta\alpha\lambda_2(\phi) > 1$ for all $\phi > \phi^*$. Then (46) implies that $\omega_{1s+1} = \omega_{2s+1} = 0$, and (44) then implies that the unique equilibrium is $y_t = \pi_t = 0$ for all t.

If $\phi < \phi^*$, then $\beta\alpha\lambda_1(\phi) < 1$, and there is clearly a continuum of solutions indexed by ω_{1s+1} that satisfy (44) and (36). *Q.E.D.*

Under the view that no bounds other than transversality are appropriate, the analog of Proposition 5 with (54) in place of (48) holds. The proof is identical except that we use Lemma 7 rather than Lemma 6 to obtain controllability of best responses.

B. A Stochastic Version

Now we add aggregate shocks to the deterministic New Keynesian model. Clearly, our implementation result applies here. Our goal is to show that efficient outcomes violate the Taylor principle along the equilibrium path; hence, these outcomes lie in the indeterminate region with restricted policies. Restricted policies which attempt to generate efficient outcomes are therefore risky, and in this sense, sophisticated policies are needed in order to support good outcomes.

Consider, then, a version of the model with aggregate uncertainty in the form of additive shocks to the Euler equation. We assume that the Euler equation shock is the sum of two components, one realized near the beginning of the period and the other near its end. The

timing of the model is as follows. At the beginning of the period, the first component η_{1t} is realized; then new prices are set, and then the central bank chooses its policy. Then the second component η_{2t} is realized, and then, finally, consumers make their decisions. The rest of the model is identical to the deterministic model, and the definition of sophisticated equilibrium is the obvious analog of that in the simple model.

In order to help clarify the nature of optimal policy in the stochastic model, we allow for shock both before and after the central bank sets its policies. The central bank can and will offset the effects of the first shock η_{1t} by allowing interest rates to feed back on it. If this were the only shock, then in the efficient outcome, inflation would be identically zero and the coefficient ϕ in the Taylor rule, irrelevant. But the central bank cannot fully offset the other shock η_{2t}. In an economy with this type of shock as well, while inflation can be kept identically equal to zero, doing so is not optimal, and the coefficient ϕ on the Taylor rule is relevant.

Clearly, the analog of Proposition 5 holds for this economy. Here, as in the simple model, associated with each sophisticated equilibrium are the particular stochastic processes for outcomes that occur along the equilibrium path. Suppressing explicit dependence on the strategies, we can write these outcomes as a function of the history of exogenous events $s^t = (s_0, \ldots, s_t)$, denotes the shocks that have occurred before that allocation is chosen. Here $s_t = (\eta_{1t}, \eta_{2t})$. (Note that we suppress explicit dependence on the velocity shock ν_t since under an interest rate regime, that shock affects only the residually determined money growth rate.)

Recalling that the period t prices and interest rates are chosen after η_{1t} but before η_{2t}, we know that the inflation rate and interest rate in period t are of the form $\pi_t(s^{t-1}, \eta_{1t})$ and $i_t(s^{t-1}, \eta_{1t})$. Since output is chosen at the end of period t, after η_{2t}, it is of the form $y_t(s^t)$. More precisely, the equilibrium outcomes satisfy these conditions: the Euler equation

$$(55) \quad y_t(s^t) = E\left[y_{t+1}(s^{t+1})|s^t\right] - \psi\left(i_t(s^{t-1}, \eta_{1t}) - E\left[\pi_{t+1}(s^t, \nu_t)|s^t\right]\right) + \eta_{1t} + \eta_{2t},$$

the New Keynesian Phillips curve

$$(56) \quad \pi_t(s^{t-1}, \eta_{1t}) = \kappa E\left[y_t(s^t)|s^{t-1}, \eta_{1t}\right] + \beta E\left[\pi_{t+1}(s^t, \eta_{1t+1})|s^{t-1}, \eta_{1t}\right],$$

the Taylor rule

$$(57) \quad i_t(s^{t-1}, \eta_{1t}) = \phi_\pi \pi_t(s^{t-1}, \nu_t) + \phi_1 \eta_{1t},$$

and the analogs of the transversality and boundedness conditions on output and inflation (36), (39), and (40), where $s_t = (\eta_{1t}, \eta_{2t})$.

Inspection of (55) and (57) makes clear that if the central bank sets $\phi_1 = -1/\psi$, then it will completely offset the effects of the η_{1t} shock in the Euler equation. Clearly, doing so is optimal because this shock simply adds inefficient variation to the economy. Because of this observation, we will simplify the exposition by setting η_{1t} identically equal to zero.

Consider now the welfare criterion. Obviously, the only candidates for efficient outcomes have a steady-state inflation rate of zero. Hence, we consider approximations only around a steady state with zero inflation. Using a quadratic approximation to the utility of the representative consumer, we can express welfare as

$$(58) \quad E_0 \sum \beta^t [\pi_t^2 + \gamma y_t^2].$$

(For details, see Woodford 2003, Chapter 6, Prop. 6.4.)

The following proposition, proved in Appendix B, shows that the outcomes that satisfy the Taylor principle along the equilibrium path are inefficient.

PROPOSITION 6. INEFFICIENCY OF RULES SATISFYING THE TAYLOR PRINCIPLE. In a stochastic version of a model with staggered price-setting, the outcomes under a Taylor rule of the form (25) with $\phi > 1$ are dominated by outcomes of an equilibrium with $\phi = 0$.

Recall that in the deterministic version of this model, if $\phi > 1$, then the equilibrium has either $y_t = 0$ and $\pi_t = 0$ or unbounded paths for these variables. In the stochastic model, the analog of this result is that when $\phi > 1$, the equilibrium has either $y_t(s^t) = \eta_{2t}$ and $\pi_t(s^{t-1}) = 0$ or unbounded paths. The unbounded paths are clearly inefficient. In the bounded paths, the variance of output is $var(\eta)$, and that of inflation is zero. In Proposition 6, we have shown that a small increase in the variance of inflation increases welfare.

3. Trembles and Imperfect Monitoring

Thus far we have shown that sophisticated policies can uniquely implement any equilibrium outcome. With sophisticated policies, deviations in private actions lead to changes in the policy regime. That leads to the questions, how should sophisticated policies be constructed if we allow for private or central bank decisions which are not implemented as intended (trembles)

or if we acknowledge that the central bank can monitor private decisions only imperfectly? The answer to both questions is, not significantly differently. These concerns do not affect our result.

A. Trembles

Consider first allowing for trembles in private decisions by supposing that the actual price chosen by a price-setter, $x_t(i)$, differs from the intended price, $\hat{x}_t(i)$, by an additive error $\varepsilon_t(i)$, so that

$$x_t(i) = \hat{x}_t(i) + \varepsilon_t(i).$$

If $\varepsilon_t(i)$ is independently distributed across agents, then it simply washes out in the aggregate and is irrelevant. Even if $\varepsilon_t(i)$ is correlated across agents, say, because it has both aggregate and idiosyncratic components, our argument goes through unchanged if the central bank can observe the aggregate component, say, with a random sample of prices.

Trembles in central bank decisions are also irrelevant. To see this, suppose that the central bank trembles in setting its interest rates, so that the actual interest rate is different from the intended rate by a mean zero additive error. This tremble acts effectively like the error in the Euler equation above, so does not affect our implementation result.

B. Imperfect Monitoring

The most interesting potential concern for sophisticated policies is when the central bank monitors prices only imperfectly.

Specifically, consider the deterministic New Keynesian model. Suppose that in each period the central bank observes the aggregate action of price-setters x_t with probability q and observes nothing with probability $1 - q$. Of course, if the central bank could see some other variable, such as output or interest rates on private debt, then it could infer what the private agents did. In this sense, we can think of this setup as giving the central bank minimal amounts of information relative to what actual central banks have. We will show that even in this extreme case, sophisticated policies can be used to support desirable outcomes. We do so by showing that sophisticated policies can deter deviations. We restrict attention to deviations which generate bounded paths for inflation, with the rationale that the central bank can easily figure out if the economy is on an unbounded path.

We prove the following proposition in Appendix B:

PROPOSITION 7. UNIQUE IMPLEMENTATION WITH IMPERFECT MONITORING. If the detection probability q is sufficiently high, so that

$$(59) \quad \frac{1}{1-q} > 1 + \beta q + (1-q)\kappa\psi,$$

then sophisticated policies with infinite reversion to money can uniquely implement any competitive equilibrium outcome. Also, under conditions (48) and (59), sophisticated policies with reversion to interest rates can uniquely implement any equilibrium outcome.

The sophisticated policies we use to prove this result are as follows. If the central bank detects a deviation, then it switches to a suitably chosen policy that yields uniqueness. Such a policy could be either a reversion to a money regime or a reversion to an interest rate regime in the determinate region. With such policies in place, the dynamical system after undetected deviations can easily be worked out. If the detection probability satisfies (59), then the dynamical system has a unique solution, so that the best response is controllable.

Notice that for any values of the other parameters, there is always a detection probability strictly less than one that satisfies (59).

Now suppose that the central bank perfectly monitors prices every K periods. An argument similar to that in Proposition 7 can then be used to obtain unique implementation. The essential idea behind both this result and that in Proposition 7 is that indeterminacy arises in the New Keynesian model because the associated dynamical system lacks a terminal condition. Periodic monitoring provides the needed terminal condition, and probabilistic monitoring acts as a form of discounting that effectively provides that condition.

So far we have focused on environments in which the central bank, either with probability q in each period or every K periods, gets to see the aggregate actions perfectly. We have shown, under a set of conditions, that we can uniquely implement any desired competitive equilibrium. In particular, we have shown that the outcome produced by following a sophisticated policy coincides exactly with the desired outcome.

An alternative environment is one in which the central bank never sees aggregate actions perfectly but instead receives a noisy signal of these actions. In this environment, we conjecture that exact implementation is not possible but that approximate implementation is possible. If the noise in the signal is small, then the welfare effects of indeterminacy can be limited by

ensuring that none of the equilibrium outcomes yield welfare substantially different from that of the desired outcome.

4. Concluding Remarks

We have defined and illustrated what we have called *sophisticated policies* for monetary economies and have shown how they can uniquely implement any competitive equilibrium. The logic of our argument should extend to applications other than monetary policy as well, for example, to analyses of how to construct optimal fiscal policies and how to respond to special circumstances like the recent financial crises.

Our main message here is that in designing any policy, researchers and policymakers should use the Ramsey approach in order to determine the best competitive equilibrium and then check whether best responses to deviations are controllable. If they are, then sophisticated policies of the kind we have constructed can uniquely implement the Ramsey outcome. If they are not, then researchers and policymakers have no choice but to accept indeterminacy.

Our new approach to policy design also has stark implications for the Taylor principle, which has lately become prominent in the literature on monetary policy and indeterminacy. One implication, of course, is that the Taylor principle is irrelevant as a device to implement a unique equilibrium. We also show that obeying the Taylor principle may actually be inefficient.

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References

Adão, Bernardino, Isabel Correia, and Pedro Teles, "Monetary Policy with Single Instrument Feedback Rules," Manuscript, Bank of Portugal, 2006.

Barro, Robert J., "On the Determination of the Public Debt," *Journal of Political Economy,* 87 (October 1979), 940–71.

Bassetto, Marco, "A Game-Theoretic View of the Fiscal Theory of the Price Level," *Econometrica*, 70 (November 2002), 2167–95.

Cagan, Phillip, "The Monetary Dynamics of Hyperinflation," in *Studies in the Quantity Theory of Money*, ed. Milton Friedman, 75–117, Chicago: University of Chicago Press, 1956.

Calvo, Guillermo A. "Staggered Prices in a Utility-Maximizing Framework," *Journal of Monetary Economics*, 12 (September, 1983), 383–98.

Chari, V. V.; Christiano, Lawrence J.; Kehoe, Patrick J., "Optimality of the Friedman Rule in Economies with Distorting Taxes" *Journal of Monetary Economics*, April 1996, v. 37, iss. 2, pp. 203–23.

Clarida, Richard, Jordí Gali, and Mark Gertler, "Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory," *Quarterly Journal of Economics*, 115 (February 2000), 147–80.

Cochrane, John H., "Inflation Determination with Taylor Rules: A Critical Review," NBER Working Paper 13409, National Bureau of Economic Research, 2007.

Correia, Isabel Horta; Nicolini, Juan Pablo; Teles, Pedro, "Optimal Fiscal and Monetary Policy: Equivalence Results," *Journal of Political Economy*, 2008.

Lubik, Thomas A., and Frank Schorfheide, "Computing Sunspot Equilibria in Linear Rational Expectations Models," *Journal of Economic Dynamics and Control,* 28 (November 2003) 273–85.

Lucas, Robert E., Jr., "Equilibrium in a Pure Currency Economy," *Economic Inquiry*, 18 (April 1980), 203–20.

Lucas, Robert E., Jr., and Nancy L. Stokey, "Optimal Fiscal and Monetary Policy in an Economy Without Capital," *Journal of Monetary Economics*, 12 (July, 1983), 55–93.

McCallum, Bennett T., "Price Level Determinacy with an Interest Rate Policy Rule and Rational Expectations," *Journal of Monetary Economics*, 8 (November 1981), 319–29.

Obstfeld, Maurice, and Kenneth Rogoff, "Speculative Hyperinflations in Maximizing Models: Can We Rule Them Out?" *Journal of Political Economy*, 91 (August 1983), 675–87.

Sargent, Thomas J., and Neil Wallace, " 'Rational' Expectations, the Optimal Monetary Instrument, and the Optimal Money Supply Rule," *Journal of Political Economy*, 83 (April 1975), 241–54.

Schmitt-Grohe, Stephanie; Uribe, Martin, "Optimal Fiscal and Monetary Policy under Sticky Prices," *Journal of Economic Theory*, February 2004, v. 114, iss. 2, pp. 198-230

Siu, Henry E., "Optimal Fiscal and Monetary Policy with Sticky Prices, *Journal of Monetary Economics*, April 2004, v. 51, iss. 3, pp. 575-607.

Wallace, Neil, "A Hybrid Fiat-Commodity Monetary System," *Journal of Economic Theory*, 25 (1981), 421–430.

Woodford, Michael, *Interest and Prices: Foundations of a Theory of Monetary Policy,* Princeton, NJ: Princeton University Press, 2003.

Notes

[1]An extensive literature has used the Ramsey approach to discuss optimal monetary policy. See, among others, Chari, Christiano, and Kehoe (1996), Schmitt-Grohe and Uribe (2004),Siu (2004), and Corrreia, Nicolini, and Teles (2008).

[2]We choose this part of the policy as a clear demonstration that after a deviation the central bank is not doing anything exotic, like producing a hyperinflation. The central bank is simply getting the economy back on the track it had been on before the deviation threatened to shift it in another direction.

[3]Note that even though the real value of consumer's holdings of bonds must satisfy a transversality condition, this condition does not impose any restrictions on the paths of y_t and π_t. The reason is that in our nonlinear model, the government has access to lump-sum taxes, so that government debt can be arbitrarily chosen to satisfy any transversality condition.

5. Appendix A: The Nonlinear Economies

Here we describe the nonlinear economies discussed above that when linearized give the equilibrium conditions described in there.

A. *The Simple Sticky Price Model*

This model is a monetary economy populated by a large number of identical, infinitely lived consumers, flexible price and sticky price intermediate good producers, final good producers, and a central bank. In each period t, the economy experiences one of finitely many events s_t. We denote by $s^t = (s_0, \ldots, s_t)$ the history of events up through and including period t. The probability, as of period zero, of any particular history s^t is $g(s^t)$. The initial realization s_0 is given.

The timing within a period is as follows. At the beginning of the period, sticky price producers set their prices, and the government chooses its monetary policy, either by setting interest rates or by choosing the quantity of money. The event s_t is then realized. At the end of the period, flexible price producers set their prices, and consumers and final good producers make their decisions. The event s_t is associated with a *flight to quality* shock $[1 - \tau(s^t)]$ that affects the attractiveness of government debt relative to private claims.

In each period t, the commodities in this economy are labor, a consumption good, money, and a continuum of intermediate goods indexed by $i \in [0, 1]$. The technology for producing final goods from intermediate goods at history s^t is

$$(60) \quad y(s^t) = \left[\int y(i, s^t)^\theta \, di \right]^{\frac{1}{\theta}},$$

where $y(s^t)$ is the final good, $y(i, s^t)$ is an intermediate good of type i, and θ is a parameter governing the elasticity of substitution between goods. The technology for producing each intermediate good i is simply

$$(61) \quad y(i, s^t) = l(i, s^t),$$

where $l(i, s^t)$ is the input of labor.

Intermediate good producers behave as imperfect competitors. Fraction α of intermediate good producers have flexible prices; they set their prices in period t after the realization of the shock s_t. Fraction $1 - \alpha$ have sticky prices; they set their prices in period t before the realization

of the shock s_t. Let $P_f(i, s^t)$ denote the price set by a flexible price producer $i \in [0, \alpha]$ and $P_s(i, s^t)$, the price set by a sticky price producer $i \in [\alpha, 1]$.

Final good producers behave competitively. In each period t, they choose inputs $y(i, s^t)$, for all $i \in [0, 1]$, and output $y(s^t)$ in order to maximize profits given by

$$(62) \quad \max \ P(s^t)y(s^t) - \int_0^\alpha P_f(i, s^t)y(i, s^t) \, di - \int_\alpha^1 P_s(i, s^{t-1})y(i, s^t) \, di$$

subject to (60), where $P(s^t)$ is the price of the final good in period t. Solving the problem in (62) gives the input demand functions:

$$(63) \quad y^d(i, s^t) = \left[\frac{P(s^t)}{P(i)} \right]^{\frac{1}{1-\theta}} y(s^t),$$

where $P(i)$ is the price charged by the intermediate good producer i. The zero profit condition implies that

$$(64) \quad P(s^t) = \left[\int_0^\alpha P_f(i, s^t)^{\frac{\theta}{\theta-1}} \, di + \int_\alpha^1 P_s(i, s^{t-1})^{\frac{\theta}{\theta-1}} \, di \right]^{\frac{\theta-1}{\theta}}.$$

Using (61), we can see that the problem faced by the flexible price producers is to choose $P_1(i, s^t)$ in order to maximize

$$(65) \quad \left[P_f(i, s^{t-1}) - W(s^t) \right] y^d(i, s^t)$$

subject to (63), where $W(s^t)$ is the nominal wage rate. The resulting optimal price is given as a markup over the nominal wage rate:

$$(66) \quad P_s(i, s^t) = \frac{1}{\theta} W(s^t).$$

The problem faced by the sticky price producers is to choose $P_s(i, s^{t-1})$ in order to maximize

$$(67) \quad \sum_{s^t} Q(s^t | s^{t-1}) \left[P_2(i, s^{t-1}) - W(s^t) \right] y^d(i, s^t)$$

subject to (63), where $Q(s^t | s^{t-1})$ is the price of a dollar at s^t in units of a dollar at s^{t-1}. The resulting optimal price for these producers is given as a markup over weighted expected marginal costs:

$$(68) \quad P_s(i, s^{t-1}) = \frac{1}{\theta} \frac{\sum_{s^t} Q(s^t | s^{t-1}) P(s^t)^{\frac{1}{1-\theta}} W(s^t) y(s^t)}{\sum_{s^t} Q(s^t | s^{t-1}) P(s^t)^{\frac{1}{1-\theta}} y(s^t)}.$$

The consumer side of the economy is a variant of the standard cash-in-advance formulation, as in Lucas (1980), but with two modifications. One is that we assume that the government pays interest on wages at the private market interest rate. This modification ensures that the consumer's first-order condition for labor supply is undistorted, as in the cashless economies of Woodford (2003). Our other modification is that we allow for flight to quality shocks which affect the value of government debt relative to private debt.

Consumer preferences are given by

$$(69) \quad \sum_{t=0}^{\infty} \sum_{s^t} \beta^t g(s^t) \, U\big(c(s^t), l(s^t)\big),$$

where U is utility and $c(s^t)$ and $l(s^t)$ are aggregate consumption and labor. In each period $t = 0, 1, \ldots$, consumers face a cash-in-advance constraint in which purchases of consumption goods are constrained by their holdings of nominal money balances $M(s^t)$ according to

$$(70) \quad P(s^t)c(s^t) = M(s^t)$$

as well as by a sequence of budget constraints

$$(71) \quad M(s^t) + \frac{B(s^t)}{R(s^t)} = R_p(s^{t-1})(1+\tau_l)W(s^{t-1})l(s^{t-1}) + \big[1 - \tau(s^{t-1})\big]B(s^{t-1}) + T(s^t) + \Pi(s^t),$$

where $B(s^t)$ is government debt with price $1/R(s^t)$, $R_p(s^t)$ is the rate of return on private debt, τ_l is a subsidy to labor income, $T(s^t)$ is nominal transfers, and $\Pi(s^t)$ is the nominal profits of the intermediate good producers, and where the right side of (71) is given in period 0. The subsidy τ_l is set, as is standard in the literature, to undo the inefficiency in a steady state due to monopoly power. Specifically, $(1 + \tau_l) = 1/\theta$. Note that we have imposed that the cash-in-advance constraint holds with equality.

The consumer's problem is to maximize utility, subject to the cash-in-advance constraint, the budget constraint, and borrowing constraints $B(s^{t+1}) \geq \bar{B}$ for some large negative number $\bar{B}$. For notational simplicity, we have suppressed decisions on holdings of private state-contingent debt with the price $Q(s^t|s^{t-1})$ and private state-uncontingent debt with the private market interest rate $R_p(s^t)$. Clearly, $1[R_p(s^t)] = \sum_{s_{t+1}} Q(s^{t+1}|s^t)$ and

$$Q\left(s^{t+1}|s^t\right) = \beta g(s^{t+1}|s^t) \frac{U_c(s^{t+1})P(s^t)}{U_c(s^t)P(s^{t+1})}.$$

The first-order conditions for the consumer's problem imply that

$$-\frac{U_l(s^t)}{U_c(s^t)} = \frac{(1+\tau_l)W(s^t)}{P(s^t)}$$

$$(72) \quad \frac{1}{R(s^t)} = \left[1 - \tau(s^t)\right] \sum_{s^{t+1}} \beta g(s^{t+1}|s^t) \frac{U_c(s^{t+1})}{U_c(s^t)} \frac{P(s^t)}{P(s^{t+1})}.$$

If we log-linearize this economy, then we can obtain the equations in the body for the simple model. Setting $(1 + \tau_l) = 1/\theta$, we obtain the quadratic approximation to welfare used in the body.

B. *The New Keynesian Model*

The nonlinearized version of the New Keynesian model is nearly identical to the simple model above. The main differences are that in this new model there are no flexible price producers, and each producer can reset prices in each period with probability $1 - \alpha$.

In this model, the problem of a producer who is allowed to reset prices is to

$$\max_{P_s(s^t)} \sum_{r=t}^{\infty} \sum_{s^r} \alpha^{r-t} Q(s^r|s^t) \left[P_s(s^t)C_s(s^r) - W(s^r)C_s(s^r)\right]$$

subject to

$$C_s(s^t) = \left(\frac{P_s(s^t)}{P(s^t)}\right)^{-\theta} C(s^t),$$

where $C(s^t)$ is aggregate consumption. The first-order conditions imply that

$$P_s(s^t) = \frac{\theta}{\theta - 1} \frac{\sum_{r=t}^{\infty} \sum_{s^r} \alpha^{r-t} Q(s^r|s^t) \frac{W(s^r)}{P(s^r)} \left(\frac{1}{P(s^r)}\right)^{-\theta-1} C(s^r)}{\sum_{r=t}^{\infty} \sum_{s^r} \alpha^{r-t} Q(s^r|s^t) \left(\frac{1}{P(s^r)}\right)^{-\theta} C(s^r)}.$$

The consumer side of this model is identical to that of the simple model. When linearized, this staggered price-setting model gives the equilibrium conditions described in the body.

6. Appendix B: Proofs of Lemma 6 and Propositions 6 and 7

Here we prove Lemma 6 and Propositions 6 and 7 discussed and used in the body of this work.

A. Another Lemma

To help prove all of these, we will use the following Lemma. Let $\lambda_1(\phi)$ and $\lambda_2(\phi)$ be defined from (45). Then consider

Lemma A. The smaller eigenvalue $\lambda_1(\phi)$ is increasing in ϕ, and the larger eigenvalue $\lambda_2(\phi)$ is decreasing in ϕ. Furthermore, for all $\phi \in [1, 1/\beta)$, the smaller eigenvalue satisfies $\lambda_1(\phi) > 1$ and $[\lambda_1(\phi) - a]/b > 0$.

Proof. With some manipulation from (45), we have that the smaller eigenvalue is

$$(73) \quad \lambda_1(\phi) = \frac{1}{2}\left(1 + \frac{1 + \kappa\psi}{\beta}\right) - \frac{1}{2}\sqrt{\left(1 + \frac{1 + \kappa\psi}{\beta}\right)^2 - 4\frac{(1 + \phi\psi\kappa)}{\beta}}.$$

and the larger eigenvalue is

$$(74) \quad \lambda_2(\phi) = \frac{1}{2}\left(1 + \frac{1 + \kappa\psi}{\beta}\right) + \frac{1}{2}\sqrt{\left(1 + \frac{1 + \kappa\psi}{\beta}\right)^2 - 4\frac{(1 + \phi\psi\kappa)}{\beta}}.$$

Clearly, λ_1 is an increasing function of ϕ and λ_2 is a decreasing function of ϕ.

To prove that $\lambda_1(\phi) > 1$ for $\phi \in [1, 1/\beta)$, note that

$$\lambda_1(1) = \frac{1}{2}\left(\frac{1 + \kappa\psi}{\beta} + 1\right) - \frac{1}{2}\sqrt{\left(\frac{1 + \kappa\psi}{\beta} - 1\right)^2} = 1$$

while

$$\lambda_1\left(\frac{1}{\beta}\right) = \frac{1}{2}\left(1 + \frac{\kappa\psi}{\beta} + \frac{1}{\beta}\right) - \frac{1}{2}\sqrt{\left(1 + \frac{\kappa\psi}{\beta} + \frac{1}{\beta}\right)^2 - \frac{4}{\beta}\left(1 + \frac{\kappa\psi}{\beta}\right)}$$

$$= \frac{1}{2}\left(1 + \frac{\kappa\psi}{\beta} + \frac{1}{\beta}\right) - \frac{1}{2}\sqrt{\left(1 + \frac{\kappa\psi}{\beta} - \frac{1}{\beta}\right)^2} = \frac{1}{\beta},$$

so that $\lambda_1(\phi) > 1$ for all $\phi \in (1, 1/\beta)$.

Next, to prove that $[\lambda_1(\phi) - a]/b > 0$ for $\phi \in [1, 1/\beta)$, note that straightforward algebra gives that

$$\lambda_1(\phi) - \left(1 + \frac{\kappa\psi}{\beta}\right) = \left\{ \begin{array}{l} -\frac{\kappa\psi}{\beta} < 0 \text{ for } \phi = 1 \\ \frac{1}{\beta} - (1 + \frac{\kappa\psi}{\beta}) < 0 \text{ for } \phi = \frac{1}{\beta} \end{array} \right\}.$$

Since $a = 1 + \kappa\psi/\beta$ and $b = \phi - 1/\beta$, we have shown that $[\lambda_1(\phi) - a]/b > 0$ for all $\phi \in [1, 1/\beta)$. *Q.E.D.*

41

B. Lemma 6

Recall Lemma 6:

Lemma 6. Suppose that (48) is satisfied. Then some value of $\bar{\phi} > 1$ exists such that if the central bank chooses a reversion policy of the Taylor rule form with $\phi \in (1, \bar{\phi})$, then the resulting continuation is unique, and the associated output and inflation rates are zero in all periods $t \geq s + 1$, where the deviation occurs in period s.

Proof. We now develop sufficient conditions under which the initial conditions $\omega_{1s+1}, \omega_{2s+1}$ satisfying (46) and (47) are unique, and equal to 0, for a range of values of the Taylor rule coefficient $\phi > 1$.

We eliminate large root indeterminacy by finding values for the Taylor rule coefficient ϕ for which the transversality condition rules out paths for inflation that explode at rate λ_2, so that equilibria must have $\omega_{2s+1} = 0$. To see how we find such values, let ϕ^* be defined by

$$(75) \quad \beta\alpha\lambda_2(\phi^*) = 1$$

if $\beta\alpha\lambda_2(\phi_{\max}) \leq 1$ and by $\phi_{\max}$ if there is no value of $\phi \in [0, \phi_{\max}]$ for which $\beta\alpha\lambda_2(\phi) = 1$. We now show that under (48), $\phi^* > 1$. To see this, note from (74) that $\lambda_2(1) = (1 + \kappa\psi)/\beta$, so that $\beta\alpha\lambda_2(1) = \alpha(1 + \kappa\psi)$, which by (48) is greater than one. Since $\lambda_2(\phi)$ is decreasing it follows that if $\beta\alpha\lambda_2(\phi^*) = 1$ is satisfied for some point ϕ^* in $[1, \phi_{\max}]$, then $\phi^* > 1$. If no such point exists, then $\phi^* = \phi_{\max}$, which is also greater than 1. Either way, $\phi^* > 1$. Hence, $\beta\alpha\lambda_2(\phi) > 1$ for all $\phi \in [0, \phi^*)$, and the transversality condition, written as (46), is satisfied only if $\omega_{2s+1} = 0$ for all $\phi \in [0, \phi^*)$.

We eliminate small root indeterminacy by finding values for the Taylor rule coefficient for which the smaller root $\lambda_1(\phi)$ is larger than one and the coefficient on the initial condition on the small root ω_{1s+1}, namely, $[\lambda_1(\phi) - a]/b > 0$. For such values of ϕ, the bound on output in (47) requires that $\omega_{1s+1} \leq 0$, and the bound on inflation in (47) requires that $\omega_{1s+1} \geq 0$, so that $\omega_{1s+1} = 0$. From Lemma A, we have that the required interval is $[1, 1/\beta)$ because for all $\phi \in [1, 1/\beta)$, we have that $\lambda_1(\phi) > 1$ and $[\lambda_1(\phi) - a]/b > 0$.

Combining the two parts of the argument for Lemma 6, we have that if ϕ satisfies both $\phi \in [0, \phi^*)$ and $\phi \in [1, 1/\beta)$, then both large root indeterminacy and small root indeterminacy

are eliminated. The intersection of these intervals is contained in $(1, \bar{\phi})$, where

$$\bar{\phi} = \min\left[\phi^*, \frac{1}{\beta}\right].$$

Since $\phi^* > 1$ and $1/\beta > 1$, clearly $\bar{\phi} > 1$.

In sum, we have shown that a $\bar{\phi} > 1$ exists such that for $\phi \in (1, \bar{\phi})$, the initial conditions for the dynamical system $\omega_{2s+1} = \omega_{1s+1} = 0$ that starts after the deviation. Hence, from (44) we have that $y_t = \pi_t = 0$ for all $t \geq s + 1$. $Q.E.D.$

C. Proposition 6

Now recall Proposition 6:

PROPOSITION 6. INEFFICIENCY OF RULES SATISFYING THE TAYLOR PRINCIPLE. In a stochastic version of a model with staggered price-setting, the outcomes under a Taylor rule of the form (25) with $\phi > 1$ are dominated by outcomes of an equilibrium with $\phi = 0$.

Preliminaries

Before we get to the proof of this proposition, we work out the stochastic processes for y_t and π_t that are implied by the dynamical system. For notational simplicity, we write η_{2t} as simply η_t. We begin with the dynamical system that arises with $\phi = 0$. We can write this system as

$$(76) \quad y_t = E_t y_{t+1} + \psi \pi_{t+1} + \eta_t$$

$$(77) \quad \pi_{t+1} = \beta E_t \pi_{t+2} + \kappa E_t y_{t+1}.$$

We solve this system using the method exposited by Lubik and Schorfheide (2003). For convenience, let $u_t = \pi_{t+1}$ and let the forecast errors be defined by $\varepsilon_{yt} \equiv y_t - E_{t-1} y_t$ and $\varepsilon_{ut} \equiv u_t - E_{t-1} u_t$. After some manipulation, we can rewrite (76) and (77) as

$$(78) \quad E_t z_{t+1} = \Gamma E_{t-1} z_t + \Psi \eta_t + \Pi \varepsilon_t,$$

where $z_t = [E_t y_{t+1}, E_t u_{t+1}]'$, $\varepsilon_t = [\varepsilon_{yt}, \varepsilon_{ut}]'$ and

$$(79) \quad \Gamma = \begin{bmatrix} 1 & -\psi \\ -\frac{\kappa}{\beta} & \frac{\kappa\psi+1}{\beta} \end{bmatrix}, \quad \Psi = \begin{bmatrix} -1 \\ \frac{\kappa}{\beta} \end{bmatrix}, \quad \Pi = \begin{bmatrix} 1 & -\psi \\ -\frac{\kappa}{\beta} & \frac{\kappa\psi+1}{\beta} \end{bmatrix}.$$

Now let $J \wedge J^{-1} = \Gamma$ be the Jordan decomposition of Γ. Letting $w_t = J^{-1}E_t z_{t+1}$, we can write this system as

$$w_t = \wedge\, w_{t-1} + J^{-1}\Psi\eta_t + J^{-1}\Pi\varepsilon_t,$$

with eigenvalues $\lambda_1 \leq \lambda_2$

$$\lambda_1, \lambda_2 = \frac{1}{2}\left(1 + \frac{1+\kappa\psi}{\beta}\right) \pm \frac{1}{2}\sqrt{\left(1 - \frac{1+\kappa\psi}{\beta}\right)^2 + 4\frac{\kappa\psi}{\beta}},$$

and eigenvectors

$$J = \begin{bmatrix} 1 & 1 \\ (1-\lambda_1)/\psi & (1-\lambda_2)/\psi \end{bmatrix}.$$

It is immediate that since $\kappa\psi > 0$, $0 \leq \lambda_1 < 1 < \lambda_2$, so that a continuum of solutions exists. More precisely, since the number of explosive roots, here 1, is less than the number of expectation errors, here 2, the system has one degree of indeterminacy.

The best outcome clearly has bounded output and inflation, so that we need to choose both the initial condition on ω_{20} and the shocks so as to never put weight on the explosive root. These restrictions can be summarized by a condition on the deterministic component of the system

(80) $\quad [J^{-1}E_t z_{t+1}]_{2.} = 0$

and a condition on the stochastic component

(81) $\quad [J^{-1}\Psi]_{2.}\eta_t + [J^{-1}\Pi]_{2.}\,\varepsilon_t = 0,$

where $[A]_2$ denotes the second row of matrix A. With some algebra, we can write these conditions as

(82) $\quad \dfrac{\lambda_1 - 1}{\psi}E_{t-1}y_t + E_{t-1}u_t = 0$ and

(83) $\quad \left(\dfrac{1-\lambda_1}{\psi} + \dfrac{\kappa}{\beta}\right)(\eta_t - \varepsilon_{yt}) + \left(1 - \lambda_1 + \dfrac{1}{\beta}(\kappa\psi + 1)\right)\varepsilon_{ut} = 0.$

For later use, let D be defined from (83) so that

(84) $\quad \varepsilon_{ut} = D(\nu_t - \varepsilon_{yt}).$

Since we have chosen w_{2t} to be identically zero, we can write the solution to the system as

(85) $\quad w_{1t} = \lambda_1 w_{1t-1} + \left[J^{-1}\Psi\right]_1 \eta_t + \left[J^{-1}\Pi\varepsilon_t\right]_1 .$

Recall that

$$w_{1t} = \left[J^{-1}E_t z_{t+1}\right]_1 = \frac{1}{\Delta}\left(\frac{1-\lambda_2}{\psi}E_t y_{t+1} - E_t u_{t+1}\right).$$

Using (82) in (85), we have, after some manipulation, that

$$
\begin{aligned}
E_t u_{t+1} &= \lambda_1 E_{t-1} u_t + \left(\frac{\lambda_2 - 1}{\lambda_1 - 1} - 1\right)^{-1}\left(\frac{\lambda_2 - 1}{\psi} - \frac{\kappa}{\beta}\right)\eta_t \\
&\quad + \left(\frac{\lambda_2 - 1}{\lambda_1 - 1} - 1\right)^{-1}\left(\frac{1-\lambda_2}{\psi} + \frac{\kappa}{\beta}\right)\varepsilon_{yt} \\
&\quad + \left(\frac{\lambda_2 - 1}{\lambda_1 - 1} - 1\right)^{-1}\left(\lambda_2 - 1 - \frac{\kappa\psi + 1}{\beta}\right)\varepsilon_{ut}
\end{aligned}
$$

and

$$
\begin{aligned}
E_t y_{t+1} &= \lambda_1 E_{t-1} y_t \\
&\quad + \frac{\psi}{1 - \lambda_1}\left(\frac{\lambda_1 - 1}{\lambda_2 - \lambda_1}\right)\left[\left(\frac{\lambda_2 - 1}{\psi} - \frac{\kappa}{\beta}\right)(\eta_t - \varepsilon_{yt}) + \left(\lambda_2 - 1 - \frac{\kappa\psi + 1}{\beta}\right)\varepsilon_{ut}\right].
\end{aligned}
$$

Using (84), we can write this latter equation as

(86) $\quad E_t y_{t+1} = \lambda_1 E_{t-1} y_t + F(\eta_t - \varepsilon_{yt}),$

where

$$F = \frac{b}{\lambda_2 - \lambda_1}\left[\left(-\frac{\lambda_2 - a}{b} - \frac{\kappa}{\beta}\right) + \left(\lambda_2 - a - \frac{1}{\beta}(\kappa\psi + 1)\right)D\right]$$

and ε_{yt} is a free random variable which captures the stochastic indeterminacy of the system. The solution for y_{t+1} is, then,

$$y_{t+1} = E_t y_{t+1} + \varepsilon_{yt+1},$$

where $E_t y_{t+1}$ is given by (86). Using (82), we have that

(87) $\quad u_{t+1} = \frac{1 - \lambda_1}{\psi}E_t y_{t+1} + D(\eta_{t+1} - \varepsilon_{yt+1}).$

45

Proof of Proposition of 6.

Proof. We compute welfare using

$$(88) \quad E_0 \sum \beta^t \left(\gamma y_t^2 + \pi_t^2 \right)$$

and the system described above. To do so, we assume that y_0 is drawn from the invariant distribution, so that from (86) we have that

$$(89) \quad \mathrm{var}(E_t y_{t+1}) = \left(\frac{F^2}{1 - \lambda^2} \right) \mathrm{var}(\varepsilon_t - \eta_{yt}).$$

From the definition of the forecast error η_{yt+1}, we have that

$$(90) \quad \mathrm{var}(y_{t+1}) = \mathrm{var}(E_t y_{t+1}) + \mathrm{var}(\varepsilon_{yt+1}),$$

while from (87) we have that

$$(91) \quad \mathrm{var}(u_{t+1}) = \left(\frac{1 - \lambda_1}{\psi} \right)^2 \mathrm{var}(E_t y_{t+1}) + D^2 \mathrm{var}(\eta_{t+1} - \varepsilon_{yt+1}).$$

Using (89)–(91), we have that (88) is proportional to

$$(92) \quad \mathrm{var}(\eta_{t+1} - \varepsilon_{yt+1}) \left(\frac{F^2}{1 - \lambda^2} \left[\gamma + \left(\frac{1 - \lambda_1}{\psi} \right)^2 \right] + D^2 \right) + \gamma \mathrm{var}(\varepsilon_{yt+1}).$$

Choose $\varepsilon_{yt} = A\eta_t$. Then (92) is proportional to

$$(1 - A)^2 \left(\frac{F^2}{1 - \lambda^2} \left[\gamma + \left(\frac{1 - \lambda_1}{\psi} \right)^2 \right] + D^2 \right) + A^2.$$

The $\phi > 1$ solution corresponds to $A = 1$. Since $\left(\frac{F^2}{1-\lambda^2} \left[1 + \left(\frac{1-\lambda_1}{\psi} \right)^2 \right] + D^2 \right) \neq \infty$, it is clear that $A = 1$ is not optimal. *Q.E.D.*

D. *Proposition 7*

Now recall Proposition 7:

PROPOSITION 7. UNIQUE IMPLEMENTATION WITH IMPERFECT MONITORING. If the detection probability q is sufficiently high, so that

$$(93) \quad \frac{1}{1 - q} > 1 + \beta q + (1 - q)\kappa\psi,$$

then sophisticated policies with infinite reversion to money can uniquely implement any competitive equilibrium outcome. Also, under conditions (48) and (59), sophisticated policies with reversion to interest rates can uniquely implement any equilibrium outcome.

Proof. Consider the sophisticated policies of the form used in the proof of Proposition 5 except that the reversion phase is triggered only if a deviation is detected. By construction, we know a unique equilibrium continuation follows a detected deviation.

We now show that without detection, a unique equilibrium also follows. Consider the dynamical system when a deviation occurs but is not detected. For notational simplicity, imagine that the deviation occurs in period 0. In period t, the deviation at 0 is detected with probability q. Let y_t^m and π_{t+1}^m denote output and inflation under the reversion policy when the period 0 deviation is first detected in period t. Choose the reversion policy so that $y_t^m = \pi_{t+1}^m = 0$ for all $t \geq 1$. The resulting system is, then,

(94) $y_t = (1 - q)(y_{t+1} + \psi \pi_{t+1})$

(95) $\pi_t = (1 - q)(\beta \pi_{t+1} + \kappa y_t).$

A sequence of output and inflation is part of a continuation equilibrium if and only if it satisfies (94), (95), (36), (39), and (40). Letting $z_t = (y_t, \pi_t)'$, with some manipulation we can stack these equations to give $z_{t+1} = A' z_t$, where

$$A' = \begin{bmatrix} a' & b' \\ \frac{-\kappa}{\beta} & \frac{1}{\beta(1-q)} \end{bmatrix}$$

and where $a' = 1/(1-q) + \kappa\psi/\beta$, $b' = -\psi/[\beta(1-q)]$. The solutions to this system are

(96) $y_t = \lambda_1^t \omega_1' + \lambda_2^t \omega_2'$ and $\pi_t = \lambda_1^t \left(\frac{\lambda_1 - a'}{b'}\right) \omega_1' + \lambda_2^t \left(\frac{\lambda_2 - a'}{b'}\right) \omega_2'$

for $t \geq 1$.

The eigenvalues of the transition matrix A' are

$$\lambda_1, \lambda_2 = \frac{1}{2(1-q)} + \frac{1}{2\beta}\left(\kappa\psi + \frac{1}{1-q}\right)$$
$$\pm \left[\left(\frac{1}{2(1-q)} + \frac{\kappa\psi}{2\beta}\right)^2 + \frac{1}{4(1-q)^2\beta^2} + \frac{\kappa\psi}{(1-q)\beta^2}\right]^{1/2}$$

With some algebra, we can show that condition (93) implies that $\lambda_2 \geq \lambda_1 > 1$, so that the paths of output and inflation given by (96) do not have bounded indeterminacy. Thus, the only possible solutions are $y_t = \pi_t = 0$, or paths in which output or inflation are unbounded.

We rule out large root indeterminacy by using the transversality condition. In particular, straightforward algebra shows that if (48) is satisfied, then $\beta\alpha\lambda_2 > 1$, so that the transversality condition implies that the system does not have large root indeterminacy, that is, that $\omega_2' = 0$.

To rule out small root indeterminacy, we show that the only unbounded sequences satisfying (94) and (95) have either output going to plus infinity or inflation to minus infinity, so that these sequences violate the boundedness conditions. In particular, with some algebra, we can show that $(\lambda_1 - a')/b' > 0$. Therefore, (96) implies that, if $\omega_1' > 0$, then y_t converges to plus infinity, and if $\omega_1' < 0$, then π_t converges to minus infinity. Thus, $\omega_1' = \omega_2' = 0$, so that (96) implies that $y_t = \pi_t = 0$ for all $t \geq 1$. An argument identical to that in the proof of Proposition 5 shows that reversion policies can be designed in period 0 to make best responses controllable, so that the sophisticated policies uniquely implement any competitive equilibrium. $Q.E.D.$