Solving for Country Portfolios in Open Economy Macro Models*

Michael B Devereux[†]and Alan Sutherland[‡]

First draft: May 2006
Revised: September 2006

Abstract

Open economy macroeconomics typically abstracts from portfolio structure. But the recent experience of financial globalization makes it important to understand the determinants and composition of gross country portfolios. This paper presents a simple approximation method for computing equilibrium financial portfolios in stochastic open economy macro models. The method is widely applicable, easy to implement, and delivers analytical solutions for optimal gross portfolio positions in any combination of types of asset. It can be used in models with any number of assets, whether markets are complete or incomplete, and can be applied to stochastic dynamic general equilibrium models of any dimension, so long as the model is amenable to a solution using standard approximation methods.

Keywords: Country portfolios, solution methods.

JEL: E52, E58, F41

*We are grateful to Philip Lane, Klaus Adam, Pierpaolo Benigno, Gianluca Benigno, Berthold Herrendorf, Fabrizio Perri, Robert Kollmann, Giancarlo Corsetti, Morten Ravn, Martin Evans and Viktoria Hnatkovska for comments on an earlier draft of this paper. This research is supported by the ESRC World Economy and Finance Programme, award number 156-25-0027. Devereux also thanks SSHRC, the Bank of Canada, and the Royal Bank of Canada for financial support.

†CEPR and Department of Economics, University of British Columbia, 997-1873 East Mall, Vancouver, B.C. Canada V6T 1Z1. Email: devm@interchange.ubc.ca Web: www.econ.ubc.ca/devereux/mdevereux.htm

‡CEPR and School of Economics and Finance, University of St Andrews, St Andrews, Fife, KY16 9AL, UK. Email: ajs10@st-and.ac.uk Web: www.st-and.ac.uk/~ajs10/home.html

1 Introduction

Open economy macroeconomic models typically represent international financial linkages in terms of net foreign assets and the current account. Recent data show, however, that there are large cross-country *gross* asset and liability positions. Lane and Milesi-Ferretti (2001, 2006) show that these gross portfolio holdings have grown rapidly, particularly in the last decade. The existence of large gross positions offers a number of interesting challenges for open economy macro theory. For instance, can international macroeconomic models offer any explanation for the observed structure of portfolio holdings? What are the important macroeconomic determinants of the size and composition of gross portfolio positions? The importance of gross asset positions, however, goes beyond questions about the determinants of portfolio choice. This is because gross asset and liability positions can themselves have important effects on macroeconomic dynamics. For instance a change in the nominal exchange rate or a change in equity prices can give rise to capital gains and losses for gross positions which can have very large effects on the value of net foreign assets.[1]

While these issues are obviously of interest to open economy macroeconomists and policymakers, current theoretical models and current solution methods cannot be used to analyse the implications of gross portfolio holdings in any very systematic way. This is because it is difficult to solve portfolio choice problems within standard general equilibrium macroeconomic models with complex asset markets.

One approach that has recently been adopted is to focus on complete markets structures. Engel and Matsumoto (2005) and Kollmann (2006) represent examples of such an approach. In the case of complete markets, it is possible to solve for a macroeconomic equilibrium without having first to solve for behaviour in asset markets. It is then possible to derive and analyse the implied country portfolios which support the macroeconomic equilibrium.[2]

The complete markets approach certainly offers a useful starting point for analysing

[1] Lane and Milesi-Ferretti (2001) emphasize the quantitative importance of valuation effects on external assets and liabilities. See also subsequent work by Ghironi et al. (2005), Gourinchas and Rey (2005), and Tille (2003, 2004).

[2] For a comprehensive analysis of problems of portfolio choice for investors, see Campbell and Viceira (2005). See also Kray et. al (2005) for an analysis in an international context.

gross asset positions. But a large body of empirical evidence on the failure of risk-sharing across countries throws doubt on the hypothesis that international financial markets are complete (see, for instance, Obstfeld and Rogoff, 2000). Furthermore, there can be no certainty that complete markets are a good approximation to the true position, particularly with regard to the implication for optimal portfolios. It is therefore important to make progress in the analysis of portfolio choice in open economy macroeconomic models with incomplete markets. This presents a number of problems however, principally because standard solution methods cannot be used to analyse macro models with multiple assets but incomplete markets. In such models the equilibrium portfolio allocation depends on macroeconomic outcomes, and macroeconomic outcomes depend on the equilibrium portfolio allocation. But, unlike the complete markets case, a solution for macro outcomes can not be derived without first obtaining a solution for the equilibrium portfolio allocation. This means that models with incomplete markets are intractable in all but the most restricted of cases.[3] These problems are made more acute because it proves infeasible to apply standard first-order and second-order approximation methods to incomplete markets models. The optimal portfolio allocation is generally indeterminate in a first-order approximation of a model. It is also indeterminate in the non-stochastic steady state, which is the natural starting point for standard approximation methods.

In this paper we develop and present a solution method which overcomes these problems. Our method can be applied to any standard open economy model with any number of assets, any number of state variables and complete or incomplete markets. We find a general formula for asset holdings which fits naturally into the standard solution approach for DSGE models. In fact, our solution formula can be applied directly using a standard first-order accurate solution that is generally derived in the analysis of DSGE models. It is not necessary to repeat the derivation of our formula for every model. The technique is simple to implement and can be used to derive either analytical results (for sufficiently small models) or numerical results for larger models. In the case of numerical solutions, the execution time of the solution code is no longer than required to obtain a standard log-linear solution.

A key innovation in our approach is to recognize that, at the level of approximation

[3]Heathcote and Perri (2004) provide one example of an incomplete markets model in which it is possible to derive explicit expressions for equilibrium portfolios. Their model is, however, only tractable for a specific menu of assets and for specific functional forms for preferences and technology.

that open economy macro-economists normally analyze multi-country models, one only requires a solution for steady-state asset holdings. Higher-order aspects of portfolio behaviour are not relevant for the first-order accurate macro dynamics. Another way to say this is that time variation in portfolio shares is irrelevant for all questions regarding first-order responses of macroeconomic variables like consumption, output, etc. in a DSGE model. Therefore, the approximation we derive exhausts all the macroeconomic implications of portfolio choice at this level of approximation.

Using this fact, we characterise the optimal portfolio by a combination of a second-order approximation of the portfolio selection condition with a first-order approximation to the remaining parts of the model.[4] Of course, these two approximations will be interdependent; the endogenous portfolio weights will depend on the variance-covariance matrix of excess returns produced by the general equilibrium model, but that in turn will depend on the portfolio positions themselves. We show that this simultaneous system can be solved to give a simple *closed form* analytical solution for the portfolio weights.

In the existing literature a number of alternative approaches have been developed for analysing incomplete-markets models. Judd *et al* (2002) develop a numerical algorithm based on 'spline collocation' and Evans and Hnatkovska (2005) present a numerical approach that relies on a combination of perturbation, projection and continuous-time approximation techniques. The methods proposed by Judd *et al* (2002) and Evans and Hnatkovska (2005) are designed to handle dynamic general equilibrium models and they are capable of analysing time variation in portfolios. On the other hand these methods are very complex compared to our approach and they represent a significant departure from standard DSGE solution methods. Devereux and Saito (2005) use a continuous time framework which allows some analytical solutions to be derived and allows for time varying portfolios. But their approach can not handle general international macroeconomic

[4] Higher-order aspects of portfolio behaviour can be derived by considering higher-order approximations of the model. This is a relatively straightforward extension of our method. The current paper focuses on the derivation of steady-state portfolios because this represents a distinct and valuable first-step in the analysis of portfolio choice in open-economy DSGE models. In an interesting recent paper, Tille and Van Wincoop (2006) show how higher-order solutions to portfolio behaviour in an open economy model can be obtained numerically via an iterative algorithm. Their approach requires the numerical computation of steady-state portfolios in manner analogous to the analytical solutions derived in this paper. For an analytical approach to the derivation of higher-order solutions to portfolios, see the companion to the present paper (Devereux and Sutherland, 2006b).

models with diminishing-returns technology or sticky nominal goods prices.

In the existing literature our method is most closely related to the work of Samuelson (1970), Judd (1998) and Judd and Guu (2001). Samuelson, who analyses a simple static portfolio allocation problem for a single investor, shows how a mean-variance approximation of a portfolio selection problem is sufficient to identify the optimal portfolio in a near-non-stochastic world. Judd and Guu, who consider a static model of asset market equilibrium, show how the problem of portfolio indeterminacy in the non-stochastic steady state can be overcome by using a Bifurcation theorem in conjunction with the Implicit Function Theorem. This allows them to identify an appropriate approximation point and to construct higher-order Taylor series approximations for equilibrium portfolios which are valid in a neighbourhood of this approximation point. The approximation point they identify is a bifurcation point in the set of non-stochastic equilibria. Our solution approach relies on first-order and second-order approximations of the model, rather than the Implicit Function and Bifurcation Theorems, but the underlying theory described by Judd and Guu (2001) is applicable to our equilibrium solution. In particular, the steady-state gross portfolio holdings derived using our technique correspond to the approximation point derived by the Judd and Guu method. Our equilibrium portfolio can therefore be rationalised in the same way, i.e. it is a bifurcation point in the set of non-stochastic equilibria.[5]

This paper proceeds as follows. The next section sets out a general portfolio choice problem within a generic open economy model. Section 3 develops and describes our solution method. Section 4 presents two examples of how our technique can be use to solve for bond and equity holdings in simple two-country models. Section 5 concludes the paper.

[5] Judd and Guu use their technique to investigate the effects of stochastic noise on the equilibrium portfolio. They therefore solve for the first and higher-order derivatives of portfolio holdings with respect to the standard deviation of the underlying shock. In this sense they are able to derive higher-order approximations of portfolio behaviour around the steady-state portfolio. As in Judd and Guu, our steady-state asset holdings can be used as the starting point for deriving higher-order approximations. As Samuelson (1970) shows, this requires taking higher-order approximations of the portfolio optimality conditions and the model. This allows analysis of the effects of the level of noise on portfolios (is a similar way to Judd and Guu). It also allows an analysis of time variation in equilibrium portfolios.

2 A Generic Open Economy Model with Country Portfolios

The solution process is explained in the context of a two-country open economy model. The model is chosen to be general enough to encompass the range of structures that are widely used in the recent open economy macro literature. However, only those parts of the model necessary for understanding the portfolio selection problem need to be explicitly described here. Other components of the model, such as the labour supply decisions of households and the production and pricing decisions of firms, are not directly relevant to the portfolio allocation problem, so, for the moment, these parts of the model are suppressed. It is important to emphasise from the start, however, that the solution process, and the model used to describe it, are consistent with a wide range of specifications for labour supply, pricing and production. Thus, the non-portfolio parts of the model may be characterised by endogenous or exogenous employment, sticky or flexible prices and wages, local currency pricing or producer currency pricing, perfect competition or imperfect competition, etc.

It is assumed that the world consists of two countries, which will be referred to as the home country and the foreign country. The home country is assumed to produce a good (or a bundle of goods) with aggregate quantity denoted Y_H (which can be endogenous) and aggregate price P_H. Similarly the foreign country produces quantity Y_F of a (potentially differentiated) foreign good (or bundle of goods) at price P_F^*. In what follows foreign currency prices are denoted with an asterisk.

Agents in the home country have a utility function of the form

$$U_t = E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \left[u(C_\tau) + v(.) \right] \tag{1}$$

where C is a bundle of the home and foreign goods and $u(.)$ is a twice continuously differentiable period utility function. The function $v(.)$ captures those parts of the preference function which are not relevant for the portfolio problem.[6] The aggregate consumer price index for home agents is denoted P.

[6] For these other aspects of the preference function to be irrelevant for portfolio selection it is necessary to assume utility is additively separable in $u(C)$ and $v(.)$. Extensions to cases of non-additive separability (e.g. habit persistence in consumption) are straightforward, as will become more clear below. Using (1) allows us to illustrate the method with minimal notation.

It is assumed that there are n assets and a vector of n returns (for holdings of assets from period $t-1$ to t) given by

$$r_t' = \begin{bmatrix} r_{1,t} & r_{2,t} & ... & r_{n,t} \end{bmatrix}$$

Asset payoffs and asset prices are measured in terms of the aggregate consumption good of the home economy (i.e. in units of C). Returns are defined to be the sum of the payoff of the asset and capital gains expressed as percentage of the asset price. It is assumed that the vector of available assets is exogenous and predefined.

The budget constraint for home agents is given by

$$W_t = \alpha_{1,t-1}r_{1,t} + \alpha_{2,t-1}r_{2,t} + ... + \alpha_{n,t-1}r_{n,t} + Y_t - C_t \tag{2}$$

where $[\alpha_{1,t-1}, \alpha_{2,t-1}...\alpha_{n,t-1}]$ are the holdings of the n assets purchased at the end of period $t-1$ for holding into period t. It follows that

$$\sum \alpha_{i,t-1} = W_{t-1} \tag{3}$$

where W_{t-1} is net wealth at the end of period $t-1$. In (2), Y is the total disposable income of home agents expressed in terms of the home consumption good. Thus, Y may be given by $Y_H P_H/P + T$ where T is a fiscal transfer (or tax if negative)[7].

It is simple to show that the budget constraint can be re-written in the following form

$$W_t = \alpha_{t-1}' r_{x,t} + r_{n,t}W_{t-1} + Y_t - C_t \tag{4}$$

where

$$\alpha_{t-1}' = \begin{bmatrix} \alpha_{1,t-1} & \alpha_{2,t-1} & ... & \alpha_{n-1,t-1} \end{bmatrix}$$

and

$$r_{x,t}' = \begin{bmatrix} (r_{1,t} - r_{n,t}) & (r_{2,t} - r_{n,t}) & ... & (r_{n-1,t} - r_{n,t}) \end{bmatrix} = \begin{bmatrix} r_{x,1,t} & r_{x,2,t} & ... & r_{x,n-1,t} \end{bmatrix}$$

Here the nth asset is used as a numeraire and $r_{x,t}$ measures the "excess returns" on the other $n-1$ assets.

[7] The budget constraint is defined so that by default, home residents receive all home income. This means that in a symmetric equilibrium with zero net foreign assets ($W_t = 0$), gross portfolio holdings exactly offset each other in value terms. This convention simplifies the algebra, but it is not an important part of the analysis. It would be easy to assume that direct claims to (some component of) home income was tradable on a stock market, and wage earnings represented the home residents' non-capital income. In this case, even in a symmetric equilibrium with zero net foreign assets, agents in each economy would have non-zero *net* portfolio positions. The method for approximating optimal portfolios applies equally to this environment.

2.1 First-order conditions for portfolio allocation and asset market equilibrium

At the end of each period agents select a portfolio of assets to carry into the following period. Thus, for instance, at the end of period t home country agents select a vector α_t to hold into period $t+1$. There are $n-1$ first-order conditions for the choice of the elements of α_t which can be written in the following form

$$\begin{aligned}
E_t\left[u'(C_{t+1})r_{1,t+1}\right] &= E_t\left[u'(C_{t+1})r_{n,t+1}\right] \\
E_t\left[u'(C_{t+1})r_{2,t+1}\right] &= E_t\left[u'(C_{t+1})r_{n,t+1}\right] \\
&\vdots \\
E_t\left[u'(C_{t+1})r_{n-1,t+1}\right] &= E_t\left[u'(C_{t+1})r_{n,t+1}\right]
\end{aligned} \tag{5}$$

Foreign-country agents face a similar portfolio allocation problem with a budget constraint given by

$$\frac{1}{Q_t}W_t^* = \frac{1}{Q_t}\left[\alpha_{t-1}^{*\prime}r_{x,t} + r_{n,t}W_{t-1}^*\right] + Y_t^* - C_t^* \tag{6}$$

where $Q_t = P_t^* S_t / P_t$ is the real exchange rate. The real exchange rate enters this budget constraint because Y^* and C^* are measured in terms of the foreign aggregate consumption good while (as previously explained) asset holdings and rates of return are defined in terms of the home consumption good.

Foreign agents are assumed to have preferences similar to (1) so the first-order conditions for foreign-country agents' choice of α_t^* are

$$\begin{aligned}
E_t\left[Q_{t+1}^{-1}u'(C_{t+1}^*)r_{1,t+1}\right] &= E_t\left[Q_{t+1}^{-1}u'(C_{t+1}^*)r_{n,t+1}\right] \\
E_t\left[Q_{t+1}^{-1}u'(C_{t+1}^*)r_{2,t+1}\right] &= E_t\left[Q_{t+1}^{-1}u'(C_{t+1}^*)r_{n,t+1}\right] \\
&\vdots \\
E_t\left[Q_{t+1}^{-1}u'(C_{t+1}^*)r_{n-1,t+1}\right] &= E_t\left[Q_{t+1}^{-1}u'(C_{t+1}^*)r_{n,t+1}\right]
\end{aligned} \tag{7}$$

The two sets of first-order conditions, (5) and (7), and the market clearing condition $\alpha_t = -\alpha_t^*$, provide $3(n-1)$ equations which, in principle, can be used to solve for the elements of α_t, α_t^* and $E_t[r_{x,t+1}]$. However, given the non-linear nature of these equations, combined with expectational terms, it is only possible to obtain exact solutions in very special cases. The solution method described below achieves tractability (for the general class of portfolio problems) by replacing the first-order conditions of the home and foreign agents with second-order approximations.

2.2 Other first-order and equilibrium conditions

Clearly, in any given general equilibrium model, there will be a set of first-order conditions relating to intertemporal choice of consumption and labour supply for the home and foreign consumers and a set of first-order conditions for price setting and factor demands for home and foreign producers. Taken as a whole, and combined with an appropriate set of equilibrium conditions for goods and factor markets, this full set of equations will define the general equilibrium of the model. As already explained, the details of these non-portfolio parts of the model are not necessary for the exposition of the solution method, so they are not shown explicitly at this stage. In what follows these omitted equations are simply referred to as the "non-portfolio equations" or the "non-portfolio equilibrium conditions" of the model.

The non-portfolio equations of the model will normally include some exogenous forcing variables. In the typical macroeconomic model these take the form of AR1 processes which are driven by zero-mean innovations. In what follows the covariance matrix of the innovations is denoted Σ. As is the usual practice in the macroeconomic literature, the innovations are assumed to be i.i.d. This means that Σ is assumed to be non-time-varying.

It is convenient, for the purposes of taking approximations, to assume that the innovations are symmetrically distributed in the interval $[-\epsilon, \epsilon]$. This ensures that any residual in an equation approximated up to order n can be captured by a term denoted $O\left(\epsilon^{n+1}\right)$.[8]

3 The Solution Procedure

The solution procedure proposed here is based on a Taylor-series approximation of the model. The approximation is based around a point where the vector of non-portfolio variables is $\bar{X}$ and the vector of portfolio holdings is $\bar{\alpha}$. In what follows a bar over a variable indicates its value at the approximation point and a hat indicates the log-deviation from the approximation point (except in the case of $\hat{W}$ and $\hat{r}_x$, which are defined below).

The standard log-linear approximation procedures used in macroeconomics can not be directly applied to portfolio problems. This is for two reasons. Firstly, the equilibrium

[8] Clearly there must be a link between Σ and ϵ. The value of ϵ places an upper bound on the diagonal elements of Σ. So an experiment which involves considering the effects of reducing ϵ implicitly involves reducing the magnitude of the elements of Σ.

portfolio is indeterminate in a first-order approximation of the model. And secondly, the equilibrium portfolio is indeterminate in the non-stochastic steady state.

The first of these problems can be overcome by considering higher order approximations of the model. This is the approach we adopt. More specifically, we focus on a second-order approximation of the portfolio problem. A second-order approximation captures the effects of second moments and is therefore sufficient to capture the different risk characteristics of assets.

The second problem (i.e. the indeterminacy of the equilibrium portfolio in the non-stochastic steady state) presents a somewhat greater difficulty. This is because it (apparently) rules out the most obvious approximation point. We overcome this problem by treating the value of $\bar{\alpha}$ as endogenous. Our procedure solves for $\bar{\alpha}$ by looking at the first-order optimality conditions of the portfolio problem in the (stochastic) neighbourhood of the non-stochastic steady state. The solution for $\bar{\alpha}$ is defined to be the one which ensures that the *second-order* approximations of the first-order portfolio optimality conditions are satisfied, within a neighbourhood of $\bar{X}$ and $\bar{\alpha}$. The value of $\bar{X}$, meanwhile, is fixed and pre-specified. For this set of variables we follow the normal practice in the international macro literature and choose $\bar{X}$ based on a non-stochastic steady state of the model.[9]

From this description it might appear that we are approximating two sets of variables around two different approximation points. It will be shown below, however, that the solution derived for $\bar{\alpha}$ can be interpreted as the equilibrium for portfolio holdings in a world with an arbitrarily small amount of stochastic noise, i.e. the equilibrium in a 'near-non-stochastic' world. The use of the non-stochastic equilibrium for the approximation point for non-portfolio variables is therefore mathematically consistent with the use of our solution for $\bar{\alpha}$ as the approximation point for portfolio holdings.[10]

[9] In a sense our solution procedure reverses the normal perturbation methodology. The normal perturbation procedure is to specify an approximation point and to solve for the approximate behaviour of variables around that point. Here the values of some of the variables ($\bar{\alpha}$) are unspecified at the approximation point and are determined endogenously by optimality conditions which hold in the neighbourhood of the approximation point.

[10] As mentioned before, in a non-stochastic world all portfolio allocations are equivalent and can be regarded as valid equilibria. A stochastic world on the other hand (assuming independent asset returns and suitable regularity conditions on preferences) has a unique equilibrium portfolio allocation. If one considers the limit of a sequence of stochastic worlds, with diminishing noise, the equilibrium portfolio tends towards a limit which correspond to one of the many equilibria in the non-stochastic world. As

Before describing the details of the solution method, it is useful to state two important general properties of the approximated form of the model.

Property 1 In a first-order approximation of the non-portfolio parts of a model, the only aspect of the portfolio allocation problem that appears is $\bar{\alpha}$, i.e. portfolio holdings at the approximation point. The deviation of portfolio holdings from their value at the approximation point, $\hat{\alpha}$, does not play any part in first-order accurate macroeconomic dynamics. (This property will be demonstrated in the next subsection.)

Property 2 The solution of a second-order approximation of the portfolio problem only requires the non-portfolio parts of the model to be solved up to first-order accuracy. This is because the only terms that appear in a second-order approximation of the equilibrium conditions of the portfolio problem are second moments, and second-order accurate solutions for second moments can be obtained from first-order accurate solutions for realised values (see Lombardo and Sutherland (2005)).

These two properties will be features of any model of the general form described above. Property 1 is important because it implies that, when studying first-order macroeconomic dynamics, it is sufficient to obtain a solution for $\bar{\alpha}$. Property 2 is important because the first-order behaviour of the non-portfolio parts of a model is easily analysed in terms of the standard theory of linearised macroeconomic models. Solutions for the non-portfolio parts of a model can therefore be easily obtained using standard linear algorithms.

It proves convenient (but is not essential) to use the *symmetric* non-stochastic steady state of the model as the approximation point for non-portfolio variables. Thus $\bar{W} = 0$, $\bar{Y} = \bar{C}$ and $\bar{r}_1 = \bar{r}_2 ... = \bar{r}_n = 1/\beta$. Note that this implies $\bar{r}_x = 0$. As explained above, the objective is to derive a solution for $\bar{\alpha}$.

Taking a second-order approximation of the home-country portfolio first-order condi-

Judd and Guu (2001) point out, this limiting portfolio is a bifurcation point, i.e. it is the point in the set of non-stochastic equilibria which intersects with the sequence of stochastic equilibria. We will show below that our solution corresponds to the portfolio allocation at this bifurcation point.

tions yields

$$
\begin{aligned}
E_t\left[(\hat{r}_{1,t+1}-\hat{r}_{n,t+1})+\tfrac{1}{2}(\hat{r}^2_{1,t+1}-\hat{r}^2_{n,t+1})-\rho\hat{C}_{t+1}(\hat{r}_{1,t+1}-\hat{r}_{n,t+1})\right] &= O\left(\epsilon^3\right)\\
E_t\left[(\hat{r}_{2,t+1}-\hat{r}_{n,t+1})+\tfrac{1}{2}(\hat{r}^2_{2,t+1}-\hat{r}^2_{n,t+1})-\rho\hat{C}_{t+1}(\hat{r}_{2,t+1}-\hat{r}_{n,t+1})\right] &= O\left(\epsilon^3\right)\\
&\vdots\\
E_t\left[(\hat{r}_{n-1,t+1}-\hat{r}_{n,t+1})+\tfrac{1}{2}(\hat{r}^2_{n-1,t+1}-\hat{r}^2_{n,t+1})-\rho\hat{C}_{t+1}(\hat{r}_{n-1,t+1}-\hat{r}_{n,t+1})\right] &= O\left(\epsilon^3\right)
\end{aligned}
\tag{8}
$$

where $\rho \equiv -u''(\bar{C})\bar{C}/u'(\bar{C})$ (i.e. the coefficient of relative risk aversion). Re-writing (8) in vector form yields

$$E_t\left[\hat{r}_{x,t+1}+\frac{1}{2}\hat{r}^2_{x,t+1}-\rho\hat{C}_{t+1}\hat{r}_{x,t+1}\right]=O\left(\epsilon^3\right) \tag{9}$$

where

$$\hat{r}'_{x,t+1} \equiv \left[\begin{array}{cccc} \hat{r}_{1,t+1}-\hat{r}_{n,t+1} & \hat{r}_{2,t+1}-\hat{r}_{n,t+1} & \dots & \hat{r}_{n-1,t+1}-\hat{r}_{n,t+1}\end{array}\right]$$

and

$$\hat{r}^{2\prime}_{x,t+1} \equiv \left[\begin{array}{cccc} \hat{r}^2_{1,t+1}-\hat{r}^2_{n,t+1} & \hat{r}^2_{2,t+1}-\hat{r}^2_{n,t+1} & \dots & \hat{r}^2_{n-1,t+1}-\hat{r}^2_{n,t+1}\end{array}\right]$$

The term $O\left(\epsilon^3\right)$ in (9) is a residual which contains all terms of order higher than two. Applying a similar procedure to the foreign first-order conditions yields

$$E_t\left[\hat{r}_{x,t+1}+\frac{1}{2}\hat{r}^2_{x,t+1}-\rho\hat{C}^*_{t+1}\hat{r}_{x,t+1}+\hat{Q}_{t+1}/\rho\right]=O\left(\epsilon^3\right) \tag{10}$$

The home and foreign optimality conditions, (9) and (10), can be combined to show that, in equilibrium, the following conditions must hold

$$E_t\left[(\hat{C}_{t+1}-\hat{C}^*_{t+1}-\hat{Q}_{t+1}/\rho)\hat{r}_{x,t+1}\right]=0+O\left(\epsilon^3\right) \tag{11}$$

and

$$E\left[\hat{r}_x\right]=-\frac{1}{2}E\left[\hat{r}^2_x\right]+\rho\frac{1}{2}E_t\left[(\hat{C}_{t+1}+\hat{C}^*_{t+1}+\hat{Q}_{t+1}/\rho)\hat{r}_{x,t+1}\right]+O\left(\epsilon^3\right) \tag{12}$$

These two equations express the portfolio optimality conditions in a form which is particularly convenient for the derivation of equilibrium portfolio holdings and excess returns. Equation (11) provides a set of equations which must be satisfied by equilibrium portfolio holdings. And equation (12) shows the corresponding set of equilibrium expected excess returns.

3.1 Time variation

Before proceeding with a detailed description of how equations (11) and (12) can be used to derive equilibrium, it is important to discuss the effects of time on portfolio equilibrium in the approximated model. In the form equations (11) and (12) are currently presented it may appear that there is a separate set of equilibrium conditions for each time-period and thus a separate solution for asset holdings in each time-period. It is, however, simple to show that no element of the approximated portfolio problem is time varying at the level of approximation employed here. It follows, therefore, that the time subscripts in the above expressions can be omitted. This leaves a single non-time-varying set of equations and a single set of non-time-varying unknowns.

The absence of time variation arises from a combination of Property 2 (stated above) and two further properties of the approximated model:

Property 3 Expected excess returns, $E_t\left[\hat{r}_{x,t+1}\right]$, are zero in all time periods in a first-order approximation of the model. This is obvious from a first-order approximation of the portfolio first-order conditions (i.e. the first-order parts of equations (9) and (10)). This implies that the expected cross product of excess returns with any variable is equal to the covariance of excess returns with that variable, e.g.. for any variable z it must be true that $E_t\left[\hat{z}_{t+1}\hat{r}_{x,t+1}\right] = Cov_t\left[\hat{z}_{t+1}, \hat{r}_{x,t+1}\right]$.

Property 4 The conditional one-period-ahead second moments generated by the first-order approximation of the non-portfolio parts of the model are non-time-varying if the covariance matrix of the innovations, Σ, is non-time-varying. (This is a standard property of any linearised stochastic rational expectations model with homoskedastic forcing processes.)

Property 3 implies the following:

$$\begin{aligned} E_t\left[(\hat{C}_{t+1} - \hat{C}^*_{t+1} - \hat{Q}_{t+1}/\rho)\hat{r}_{x,t+1}\right] &= Cov_t\left[(\hat{C}_{t+1} - \hat{C}^*_{t+1} - \hat{Q}_{t+1}/\rho), \hat{r}_{x,t+1}\right] \\ E_t\left[(\hat{C}_{t+1} + \hat{C}^*_{t+1} + \hat{Q}_{t+1}/\rho)\hat{r}_{x,t+1}\right] &= Cov_t\left[(\hat{C}_{t+1} + \hat{C}^*_{t+1} + \hat{Q}_{t+1}/\rho), \hat{r}_{x,t+1}\right] \end{aligned} \tag{13}$$

and $E_t\left[\hat{r}^2_{x,t+1}\right]$ is given by

$$E_t\left[\hat{r}^2_{x,t+1}\right] = \begin{bmatrix} Var_t[\hat{r}_{1,t+1}] - Var_t[\hat{r}_{n,t+1}] \\ Var_t[\hat{r}_{2,t+1}] - Var_t[\hat{r}_{n,t+1}] \\ \vdots \\ Var_t[\hat{r}_{n-1,t+1}] - Var_t[\hat{r}_{n,t+1}] \end{bmatrix} \tag{14}$$

Property 2 implies that second-order accurate solutions for the second-moments in these expressions can be obtained from first-order accurate solutions for realised values of $\hat{r}_{x,t+1}$, $\hat{C}_{t+1}$, $\hat{C}^*_{t+1}$ and $\hat{Q}_{t+1}$. Property 4 implies that the solutions for these second-moments will be non-time varying provided Σ is non-time-varying. Thus all the terms in equations (11) and (12) are non-time-varying.

3.2 Partial equilibrium in asset markets

The next subsection will describe the derivation of portfolio equilibrium within a full general equilibrium of the model. Before considering general equilibrium, however, it is insightful briefly to consider a partial equilibrium solution to the portfolio problem. A partial equilibrium solution can be derived by substituting for $\hat{C}$ and $\hat{C}^*$ in (11) using the home and foreign budget constraints. Note that, as stated in Property 2, the budget constraint need only be approximated up to first-order accuracy. This is because $\hat{C}$ and $\hat{C}^*$ only appear in (11) in second-order terms. First-order approximations of the home and foreign budget constraints (in period $t+1$) imply

$$\hat{C}_{t+1} = \tilde{\alpha}'\hat{r}_{x,t+1} + \frac{1}{\beta}\hat{W}_t - \hat{W}_{t+1} + \hat{Y}_{t+1} + O\left(\epsilon^2\right) \tag{15}$$

and

$$\hat{C}^*_{t+1} = -\tilde{\alpha}'\hat{r}_{x,t+1} - \frac{1}{\beta}\hat{W}_t + \hat{W}_{t+1} + \hat{Y}^*_{t+1} + O\left(\epsilon^2\right) \tag{16}$$

where the market clearing conditions, $\tilde{\alpha} = -\tilde{\alpha}^*$ and $\hat{W} = -\hat{W}^*$ have been imposed and $\tilde{\alpha} = \bar{\alpha}/(\beta\bar{Y})$ and $\hat{W}_t = (W_t - \bar{W})/\bar{C}$. The term $O\left(\epsilon^2\right)$ in these equations is a residual which contains all terms of order higher than one. Note that, as stated in Property 1, in (15) and (16) there are no terms in $\hat{\alpha}$ (the deviations of gross asset holdings from their values at the approximation point).[11]

[11] This can the thought of as a type of envelope theorem result. Given that, at an optimal choice of $\overline{\alpha}$, expected returns are equal (in equilibrium) up to the first order, time variation in the portfolio (i.e. $\widehat{\alpha}$) can only affect net wealth at the second-order level.

Using (15) and (16) to substitute for $\hat{C}_{t+1}$ and $\hat{C}^*_{t+1}$ in (11) and solving for $\tilde{\alpha}$ yields

$$\tilde{\alpha} = -\frac{1}{2}V_{xx}^{-1}V_{xD} + O(\epsilon) \tag{17}$$

where

$$V_{xx,t} = E_t\left[\hat{r}_{x,t+1}\hat{r}'_{x,t+1}\right], \quad V_{xD} = Cov\left[\hat{r}_x, (\hat{Y} - \hat{Y}^* - 2\Delta\hat{W} - \hat{Q}/\rho)\right]$$

where $\Delta\hat{W}_{t+1} = (\hat{W}_{t+1} - \hat{W}_t/\beta)$.[12] The corresponding (partial equilibrium) solution for excess returns is

$$E[\hat{r}_x] = -\frac{1}{2}E\left[\hat{r}_x^2\right] + \rho\frac{1}{2}V_{xA} + O\left(\epsilon^3\right) \tag{18}$$

where

$$V_{xA} = Cov\left[\hat{r}_x, (\hat{Y} + \hat{Y}^* + \hat{Q}/\rho)\right]$$

Notice that (17) and (18) are very similar to the solutions for asset holdings and expected returns that would emerge from a mean-variance model of portfolio allocation. Thus some of the intuition that applies to models of that type is also applicable to the approximate solution for portfolio holdings proposed here. Of course (17) and (18) are not full general equilibrium solutions for $\tilde{\alpha}$ and $E[\hat{r}_x]$ because V_{xx}, V_{xD} and V_{xA} all depend on $\tilde{\alpha}$ via the impact of gross portfolio holdings on net wealth. Gross asset holdings affect net wealth via the budget constraints (15) and (16). In turn this equation interacts with first-order conditions for intertemporal allocation of consumption and work effort and potentially many other components of a general equilibrium model. Thus the full solution of the portfolio allocation problem requires a solution for the general equilibrium of the entire model.

3.3 A general equilibrium solution for portfolio holdings

The derivation of a full general equilibrium solution for $\tilde{\alpha}$ is now described.[13] The objective is to find values for the vector of portfolio allocations, $\tilde{\alpha}$, and solutions for the behaviour of $\hat{C}$, $\hat{C}^*$, $\hat{Q}$ and $\hat{r}_x$, which satisfy equation (11) and all the non-portfolio equations of the model.

[12] Note that V_{xx} is a second-order term, so $V_{xx}^{-1} \times O\left(\epsilon^3\right)$ is a first-order term and thus the residual in (17) is of order one. This is consistent with the definition of $\tilde{\alpha}$ as the point of approximation.

[13] The solution procedure will be described in terms of deriving a solution for $\tilde{\alpha}$. The corresponding solution for $\bar{\alpha}$ is obviously given by $\bar{\alpha} = \tilde{\alpha}\beta\bar{Y}$.

Properties 1 and 2, stated above, play a crucial role in allowing a general equilibrium solution to be derived. Property 2 implies that, in order to analyse equation (11) at the level of second-order accuracy, it is only necessary to derive first-order accurate solutions for the behaviour of $\hat{C}$, $\hat{C}^*$, $\hat{Q}$ and $\hat{r}_x$. Property 1 implies that $\tilde{\alpha}$ is the only aspect of portfolio behaviour that affects the first-order accurate behaviour of $\hat{C}$, $\hat{C}^*$, $\hat{Q}$ and $\hat{r}_x$. Two further important properties of the approximated model make it possible to obtain a solution.

> **Property 5** Portfolio holdings, $\tilde{\alpha}$, only directly enter the first-order approximation of the non-portfolio side of the model via budget constraints. In fact, because of Walras' Law, only one budget constraint is relevant. Here we focus on the home country budget constraint, which, in its linearised form, is given by $\hat{W}_t = \hat{W}_{t-1}/\beta + \hat{Y}_t - \hat{C}_t + \tilde{\alpha}'\hat{r}_{xt}$.

> **Property 6** Up to a first order of accuracy, realised excess asset returns, $\hat{r}_{xt}$, are zero-mean i.i.d. random variables. This follows from Property 3.

Property 6 implies that the total realised excess return on the portfolio (i.e. $\tilde{\alpha}'\hat{r}_{xt}$) is also a zero-mean random variable (up to a first order of accuracy). This, in turn, implies that the value of $\tilde{\alpha}$ does not have any effect on the eigenvalues or eigenvectors of the non-portfolio equations of the model.

These properties can now be used to derive a full general equilibrium solution for $\tilde{\alpha}$. From Property 6, we may initially treat the realised excess return on the portfolio as an exogenous independent mean-zero i.i.d. random variable denoted ξ_t. The home country budget constraint can therefore be written in the form

$$\hat{W}_t = \frac{1}{\beta}\hat{W}_{t-1} + \hat{Y}_t - \hat{C}_t + \xi_t + O\left(\epsilon^2\right) \tag{19}$$

and the entire first-order approximation of the non-portfolio equations of the model can be summarised in a matrix equation of the form

$$A_1 \begin{bmatrix} s_{t+1} \\ E_t\left[c_{t+1}\right] \end{bmatrix} = A_2 \begin{bmatrix} s_t \\ c_t \end{bmatrix} + A_3 x_t + B\xi_t + O\left(\epsilon^2\right) \tag{20}$$

$$x_t = N x_{t-1} + \varepsilon_t$$

where s is a vector of predetermined variables, c is a vector of jump variables, x is a vector of exogenous forcing processes and ε is a vector of i.i.d. shocks and B is a column vector with unity in the row corresponding to the equation for the evolution of net wealth (19) and zero in all other rows.[14]

The state-space solution to (20) can be derived using any standard solution method for linear rational expectations models. It can be written as follows

$$\begin{aligned} s_{t+1} &= F_1 x_t + F_2 s_t + F_3 \xi_t + O\left(\epsilon^2\right) \\ c_t &= P_1 x_t + P_2 s_t + P_3 \xi_t + O\left(\epsilon^2\right) \end{aligned} \tag{21}$$

This form of the solution shows explicitly, via the F_3 and P_3 matrices, how the first-order accurate behaviour of all the model's variables depend on exogenous i.i.d. innovations to net wealth.

By extracting the appropriate rows from (21) it is possible to write the following expression for the first-order accurate relationship between excess returns, $\hat{r}_{xt+1}$, and ε_{t+1} and ξ_{t+1}

$$\hat{r}_{xt+1} = R_1 \xi_{t+1} + R_2 \varepsilon_{t+1} + O\left(\epsilon^2\right) \tag{22}$$

where the matrices R_1 and R_2 are formed from the appropriate rows of (21). Equation (22) shows how first-order accurate realised excess returns depend on the exogenous i.i.d. shocks, ε_{t+1}, and the excess return on the portfolio, ξ_{t+1}.[15]

Now recognize that rather than being exogenous, ξ_{t+1} is determined by the endogenous excess portfolio returns via the relationship

$$\xi_{t+1} = \tilde{\alpha}' \hat{r}_{xt+1} \tag{23}$$

where the vector of portfolio allocations, $\tilde{\alpha}$, is to be determined. This equation, together with (22), can be solved to yield expressions for ξ_{t+1} and $\hat{r}_{xt+1}$ in terms of the exogenous

[14] As in many open economy macro models, there will be a unit root in the dynamics of net foreign assets, $W_{t.}$. This means that we would not be able to compute unconditional second moments from the model. But, as shown above, the optimal portfolio requires only conditional moments, which always exist. It would be easy to amend the model using methods suggested by Schmitt-Grohe and Uribe (2003) so as to render the distribution of W_t stationary. This has no bearing on the use of our method for computing optimal portfolios.

[15] Notice that, as follows from Property 6, $\hat{r}_{xt+1}$ depends only on exogenous i.i.d. innovations and does not depend on the values of the state variables contained in x_t or s_t.

innovations as follows

$$\xi_{t+1} = \tilde{H}\varepsilon_{t+1} \tag{24}$$

$$\hat{r}_{xt+1} = \tilde{R}\varepsilon_{t+1} + O\left(\epsilon^2\right) \tag{25}$$

where

$$\tilde{H} = \frac{\tilde{\alpha}' R_2}{1 - \tilde{\alpha}' R_1}, \qquad \tilde{R} = R_1 \tilde{H} + R_2 \tag{26}$$

Equation (25), which shows how realised excess returns depend on the exogenous i.i.d. innovations of the model, provides one of the relationships necessary to evaluate the left-hand side of (11). The other relationship required is the link between $(\hat{C}_{t+1} - \hat{C}^*_{t+1} - \hat{Q}_{t+1}/\rho)$ and the vector of exogenous innovations, ε_{t+1}. This relationship can derived in a similar way to (25). First extract the appropriate rows from (21) to yield the following

$$\left(\hat{C}_{t+1} - \hat{C}^*_{t+1} - \hat{Q}_{t+1}/\rho\right) = D_1\xi_{t+1} + D_2\varepsilon_{t+1} + D_3 \begin{bmatrix} x_t \\ s_{t+1} \end{bmatrix} + O\left(\epsilon^2\right) \tag{27}$$

where the matrices D_1, D_2 and D_3 are formed from the appropriate rows of (21). After substituting for ξ_{t+1}, this implies

$$\left(\hat{C}_{t+1} - \hat{C}^*_{t+1} - \hat{Q}_{t+1}/\rho\right) = \tilde{D}\varepsilon_{t+1} + D_3 \begin{bmatrix} x_t \\ s_{t+1} \end{bmatrix} + O\left(\epsilon^2\right) \tag{28}$$

where

$$\tilde{D} = D_1 \tilde{H} + D_2 \tag{29}$$

Using (25) and (28) it is now simple to derive the following expression

$$\begin{aligned} &E_t\left[(\hat{C}_{t+1} - \hat{C}^*_{t+1} - \hat{Q}_{t+1}/\rho)\hat{r}_{x,t+1}\right] = \\ &\qquad Cov_t[\hat{C}_{t+1} - \hat{C}^*_{t+1} - \hat{Q}_{t+1}/\rho, \hat{r}_{x,t+1}] = \tilde{R}\Sigma\tilde{D}' + O\left(\epsilon^3\right) \end{aligned} \tag{30}$$

where Σ is the covariance matrix of ε.[16] The equilibrium value of $\tilde{\alpha}$ is that which satisfies the following equation

$$\tilde{R}\Sigma\tilde{D}' = 0 \tag{31}$$

This matrix equation defines $(n-1)$ equations in the $(n-1)$ elements of $\tilde{\alpha}$.

To solve for $\tilde{\alpha}$ first substitute for $\tilde{R}$ and $\tilde{D}$ in (31) and expand to yield

$$R_1\tilde{H}\Sigma\tilde{H}'D_1' + R_2\Sigma\tilde{H}'D_1' + R_1\tilde{H}\Sigma D_2' + R_2\Sigma D_2' = 0 + O\left(\epsilon^3\right) \tag{32}$$

[16] Notice D_3 does not appear in this expression because, by assumption, $E_t(\varepsilon_{t+1}x_t) = E_t(\varepsilon_{t+1}s_{t+1}) = 0$.

Substituting for $\tilde{H}$ and $\tilde{H}'$ and multiplying by $(1-\tilde{\alpha}'R_1)^2$ yields

$$\begin{aligned} & R_1\tilde{\alpha}'R_2\Sigma R_2'\tilde{\alpha}D_1' + R_2\Sigma R_2'\tilde{\alpha}D_1'(1-\tilde{\alpha}'R_1) \\ & \quad + R_1\tilde{\alpha}'R_2\Sigma D_2'(1-\tilde{\alpha}'R_1) + R_2\Sigma D_2'(1-\tilde{\alpha}'R_1)^2 = 0 + O\left(\epsilon^3\right) \end{aligned} \tag{33}$$

Note that $\tilde{\alpha}'R_1$, $(1-\tilde{\alpha}'R_1)$ and D_1 are all scalars. It therefore follows that $\tilde{\alpha}'R_1 = R_1'\tilde{\alpha}$ and $D_1' = D_1$. Using these facts (33) simplifies to

$$D_1R_2\Sigma R_2'\tilde{\alpha} - R_2\Sigma D_2'R_1'\tilde{\alpha} + R_2\Sigma D_2' = 0 + O\left(\epsilon^3\right) \tag{34}$$

which can be solved to yield the following expression for the equilibrium $\tilde{\alpha}$

$$\tilde{\alpha} = \left[R_2\Sigma D_2'R_1' - D_1R_2\Sigma R_2'\right]^{-1} R_2\Sigma D_2' + O\left(\epsilon\right) \tag{35}$$

Notice that the residual in this expression is a first-order term. As previously noted, the solution for $\bar{\alpha}$ is simply given by $\bar{\alpha} = \tilde{\alpha}\beta\bar{Y}$.

3.4 $\bar{\alpha}$ in a 'near-non-stochastic' world

It is now possible to demonstrate that our solution for $\bar{\alpha}$ is consistent with the use of the non-stochastic steady state as the approximation point for non-portfolio variables.

Suppose that the covariance matrix of the innovations is given by $\Sigma = \zeta\Sigma_0$ where $\zeta > 0$ is a scalar and Σ_0 is a valid covariance matrix. Notice that the solution for $\tilde{\alpha}$ given in (35) is independent of ζ. So the value of $\tilde{\alpha}$ given by (35) (and therefore the value of $\bar{\alpha}$) is equivalent to the value that would arise in the case of an arbitrarily small, but non-zero, value of ζ - i.e. the value of $\tilde{\alpha}$ that would arise in a world which is arbitrarily close to a non-stochastic world.

Furthermore, notice that as ϵ tends to zero (which is equivalent to ζ tending to zero) the size of the residual in (35) tends to zero. So, as the amount of noise tends to zero, the value of $\tilde{\alpha}$ becomes arbitrarily close to the true value of portfolio holdings in the non-approximated model.

Our solution for $\bar{\alpha}$ can therefore be thought of as the true portfolio equilibrium in a world which is arbitrarily close to the non-stochastic equilibrium. So using our solution for $\bar{\alpha}$ as the approximation point for portfolio holdings is mathematically consistent with using the non-stochastic steady state as an approximation point for the non-portfolio variables.[17]

[17] In the terminology used by Judd and Guu (2001), it is clear that our solution corresponds to a

3.5 Summary of the procedure

It should be emphasized that implementing this procedure requires *only* that the user apply (35), which needs only information from the first-order approximation of the model in order to construct the D and R matrices. So long as the model satisfies the general properties described above, the other details of the model, such as production, labour supply, and price setting can be varied without affecting the implementation. The derivations used to obtain (35) do not need to be repeated. In summary, the solution for equilibrium $\tilde{\alpha}$ has three steps:

1. Solve the non-portfolio equations of the model in the form of (20) to yield a solution in the form of (21).

2. Extract the appropriate rows from this solution to form R_1, D_1, R_2 and D_2.

3. Calculate $\tilde{\alpha}$ using (35).

4 Applications of the Method

This section presents two simple examples of how the above solution technique can be applied.

4.1 Example 1: A two-country endowment model with trade in nominal bonds

Consider a one-good, two-country endowment economy where the utility of home households is given by

$$U_t = E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \frac{C_t^{1-\rho}}{1-\rho} \tag{36}$$

where C is consumption of the single good. There is a similar utility function for foreign households.

bifurcation point in the set of non-stochastic equilibria. So the portfolio allocation defined by (35) corresponds to one of the many possible non-stochastic equilibria and is thus consistent with the non-stochastic steady-state values of the non-portfolio variables.

The home and foreign endowments of the single good are auto-regressive processes of the form

$$\log Y_t = \zeta_Y \log Y_{t-1} + \varepsilon_{Y,t}, \quad \log Y_t^* = \zeta_Y \log Y_{t-1}^* + \varepsilon_{Y^*,t} \tag{37}$$

where $0 \leq \zeta_Y \leq 1$ and ε_Y and ε_{Y^*} are i.i.d. shocks symmetrically distributed over the interval $[-\epsilon, \epsilon]$ with $Var[\varepsilon_Y] = Var[\varepsilon_{Y^*}] = \sigma_Y^2$.

Asset trade is restricted to home and foreign nominal bonds. The budget constraint of home agents is given by

$$W_t = \alpha_{B,t-1} r_{B,t} + \alpha_{B^*,t-1} r_{B^*,t} + Y_t - C_t \tag{38}$$

where W is net wealth, α_B and α_{B^*} are holdings of home and foreign bonds and $r_{B,t}$ and $r_{B^*,t}$ are the real returns on bonds. By definition, net wealth is the sum of bond holdings, i.e.

$$W_t = \alpha_{B,t} + \alpha_{B^*,t} \tag{39}$$

Real returns on bonds are given by

$$r_{B,t} = R_{B,t} \frac{P_{t-1}}{P_t} \quad r_{B^*,t} = R_{B^*,t} \frac{P_{t-1}^*}{P_t^*} \tag{40}$$

where P and P^* are home and foreign currency prices for the single tradeable good and R_B and R_{B^*} are the nominal returns on bonds. The law of one price holds so $P = SP^*$ where S is the nominal exchange rate (defined as the home currency price of foreign currency).

Consumer prices are assumed to be determined by simple quantity theory relations of the following form

$$M_t = P_t Y_t, \quad M_t^* = P_t^* Y_t^* \tag{41}$$

where home and foreign money supplies, M and M^*, are assumed to be exogenous auto-regressive processes of the following form

$$\log M_t = \zeta_M \log M_{t-1} + \varepsilon_{M,t}, \quad \log M_t^* = \zeta_M \log M_{t-1}^* + \varepsilon_{M^*,t} \tag{42}$$

where $0 \leq \zeta_M \leq 1$ and ε_M and ε_{M^*} are i.i.d. shocks symmetrically distributed over the interval $[-\epsilon, \epsilon]$ with $Var[\varepsilon_M] = Var[\varepsilon_{M^*}] = \sigma_M^2$.

To make the example easy, the four shock processes are assumed to be independent from each other. So the covariance matrix of the vector of innovations, $\varepsilon_t =$

$\left[\begin{array}{cccc} \varepsilon_{Y,t} & \varepsilon_{Y^*,t} & \varepsilon_{M,t} & \varepsilon_{M^*,t} \end{array}\right]$, is given by

$$\Sigma = \left[\begin{array}{cccc} \sigma_Y^2 & 0 & 0 & 0 \\ 0 & \sigma_Y^2 & 0 & 0 \\ 0 & 0 & \sigma_M^2 & 0 \\ 0 & 0 & 0 & \sigma_M^2 \end{array}\right]$$

The first-order conditions for home and foreign consumption and bond holdings are

$$C_t^{-\rho} = \beta E_t \left[C_{t+1}^{-\rho} r_{B^*,t+1}\right], \quad C_t^{*-\rho} = \beta E_t \left[C_{t+1}^{*-\rho} r_{B^*,t+1}\right] \tag{43}$$

$$E_t \left[C_{t+1}^{-\rho} r_{B,t+1}\right] = E_t \left[C_{t+1}^{-\rho} r_{B^*,t+1}\right], \quad E_t \left[C_{t+1}^{*-\rho} r_{B,t+1}\right] = E_t \left[C_{t+1}^{*-\rho} r_{B^*,t+1}\right] \tag{44}$$

Finally, equilibrium consumption plans must satisfy the resource constraint

$$C_t + C_t^* = Y_t + Y_t^* \tag{45}$$

There are four sources of shocks in this model and only two independent assets. Assets markets are incomplete.

4.1.1 First-order approximation

Application of the solution procedure requires a solution of the log-linear form of this model. First-order approximation of (43) implies the following

$$-\rho \hat{C}_t = E_t \left[-\rho \hat{C}_{t+1} + \hat{r}_{B^*,t+1}\right] + O\left(\epsilon^2\right), \quad -\rho \hat{C}_t^* = E_t \left[-\rho \hat{C}_{t+1}^* + \hat{r}_{B^*,t+1}^*\right] + O\left(\epsilon^2\right) \tag{46}$$

while approximation of (44) implies

$$E_t \left[\hat{r}_{B,t+1}\right] = E_t \left[\hat{r}_{B^*,t+1}\right] + O\left(\epsilon^2\right) \tag{47}$$

First order approximation of the resource constraint, the budget constraint and the quantity theory relations yields

$$\hat{C}_t + \hat{C}_t^* = \hat{Y}_t + \hat{Y}_t^* + O\left(\epsilon^2\right) \tag{48}$$

$$\hat{W}_t = \frac{1}{\beta} \hat{W}_{t-1} + \hat{Y}_t - \hat{C}_t + \tilde{\alpha}_B \hat{r}_{x,t} + O\left(\epsilon^2\right) \tag{49}$$

$$\hat{M}_t - \hat{P}_t = \widehat{Y}_t, \quad \hat{M}_t^* - \hat{P}_t^* = \widehat{Y}_t^* \tag{50}$$

where foreign bonds are treated as the reference asset and $\hat{r}_{x,t}$ is the excess return on home bonds, defined as

$$\hat{r}_{x,t} = \hat{r}_{B,t} - \hat{r}_{B^*,t} + O\left(\epsilon^2\right) \tag{51}$$

Notice that (47) implies that the expected excess return is zero (up to a first-order approximation) i.e. $E_t\left[\hat{r}_{x,t+1}\right] = O(\epsilon^2)$. This is a demonstration of Property 3 in the context of this model. Notice that this implies that nominal returns on bonds must satisfy

$$\hat{R}_{B,t} - \hat{R}_{B^*,t} = \left(E_{t-1}\left[\hat{P}_t\right] - \hat{P}_{t-1}\right) - \left(E_{t-1}\left[\hat{P}^*_t\right] - \hat{P}^*_{t-1}\right) + O\left(\epsilon^2\right) \tag{52}$$

i.e. the nominal interest differential must equal the expected inflation differential. Combined with (51) this implies that the realised excess return in period t is

$$\hat{r}_{x,t} = \left(E_{t-1}\left[\hat{P}_t\right] - \hat{P}_t\right) - \left(E_{t-1}\left[\hat{P}^*_t\right] - \hat{P}^*_t\right) + O\left(\epsilon^2\right) \tag{53}$$

i.e. the realised excess return is given by the difference between home and foreign price surprises. Price surprises, by definition, can only depend on exogenous i.i.d. innovations. This is a demonstration of Property 6 in the context of this model. Note that, since the law of one price holds, (53) is also equal to the negative of the unexpected change in the exchange rate.

In order to write the model in the form of a first-order system it is useful to define the following relationships

$$\hat{P}^E_t = E_{t-1}[\hat{P}_t], \qquad \hat{P}^{*E}_t = E_{t-1}[\hat{P}^*_t] \tag{54}$$

$$\hat{r}^E_t = E_t\left[\hat{r}_{B,t+1}\right] = E_t\left[\hat{r}_{B^*,t+1}\right] \tag{55}$$

where $\hat{P}^E_t$, $\hat{P}^{*E}_t$ and $\hat{r}^E_t$ represent expected home and foreign prices and the expected real return on bonds.

The equations of the model can now be collected together in the form of matrix equation system (20) where the vectors s, c and x are defined as follows

$$s'_t = \left[\begin{array}{ccc} \hat{P}^E_{t-1} & \hat{P}^{E*}_{t-1} & \hat{W}_{t-1} \end{array}\right]$$

$$c'_t = \left[\begin{array}{cccccc} \hat{C}_t & \hat{C}^*_t & \hat{r}^E_t & \hat{P}_t & \hat{P}^*_t & \hat{r}_{x,t} \end{array}\right]$$

$$x'_t = \left[\begin{array}{cccc} \hat{Y}_t & \hat{Y}^*_t & \hat{M}_t & \hat{M}^*_t \end{array}\right]$$

and the coefficient matrices are drawn from equations (46) to (55).

4.1.2 Solution for bond holdings

The model is now in a form where it is straightforward to apply the solution procedure outlined in the previous section. Any standard linear solution algorithm can be applied to the first-order system to yield a state-space solution in the form of (21). In the case of the above model, the resulting expressions for the matrices R_1, R_2, D_1 and D_2 are given by

$$R_1 = [0]\,, \quad R_2 = \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}$$

$$D_1 = [2(1-\beta)]\,, \quad D_2 = \begin{bmatrix} \frac{1-\beta}{1-\beta\zeta_Y} & -\frac{1-\beta}{1-\beta\zeta_Y} & 0 & 0 \end{bmatrix}$$

Finally, applying (35) yields the following expression for bond holdings

$$\tilde{\alpha}_B = -\tilde{\alpha}_{B^*} = -\frac{\sigma_Y^2}{2(\sigma_M^2 + \sigma_Y^2)(1-\beta\zeta_Y)}$$

Home consumers take a negative position in home currency bonds, balanced by a positive position in foreign currency bonds. The home price level is countercyclical, so that home currency bonds have a relatively high payoff when home output is high. This makes home currency bonds a relatively bad hedge against home output risk, while foreign currency bonds are a relatively good hedge. An equivalent statement is that the home country exchange rate appreciates in response to a positive home output shock, increasing the return on home bonds relative to foreign bonds in this state. It is also noteworthy that monetary policy volatility reduces the gross holdings of bonds. Although prices are fully flexible, monetary volatility is costly because it reduces the usefulness of nominal bonds as a risk-hedging instrument.

4.2 Example 2: A two-country production model with trade in equities

Now we extend the model of the previous example to allow for endogenous production, with productivity and fiscal policy shocks. Assume now that households supply labour, and the utility of home households is

$$U_t = E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \left(\frac{C_t^{1-\rho}}{1-\rho} - KL_t \right) \tag{56}$$

where now C_t is a composite consumption aggregate over home and foreign good categories, L_t is labour supply, and K is a constant. C is defined as

$$C = \left[\left(\frac{1}{2}\right)^{\frac{1}{\theta}} C_H^{\frac{\theta-1}{\theta}} + \left(\frac{1}{2}\right)^{\frac{1}{\theta}} C_F^{\frac{\theta-1}{\theta}} \right]^{\frac{\theta}{\theta-1}} \tag{57}$$

where C_H and C_F are indices of individual home and foreign-produced goods with an elasticity of substitution between individual goods denoted ϕ, where $\phi > 1$. The parameter θ in (57) is the elasticity of substitution between home and foreign composite goods. The aggregate consumer price index for home agents is therefore

$$P = \left[\frac{1}{2} P_H^{1-\theta} + \frac{1}{2} P_F^{1-\theta} \right]^{\frac{1}{1-\theta}} \tag{58}$$

where P_H (P_F) is the price index of home (foreign) goods. There is a similar utility function for foreign households.

In this example, the focus is on trade in equities. The home budget constraint is given by

$$W_t = \alpha_{E,t-1} r_{E,t} + \alpha_{E^*,t-1} r_{E^*,t} + w_t L_t + \pi_t - T_t - C_t \tag{59}$$

where again W is real net wealth, w is the real wage, π is real profit, α_E and α_{E^*} are holdings of home and foreign equities and r_E and r_{E^*} are the real returns on equities. In addition, T is a lump-sum tax which is used to finance government consumption. As before, net wealth satisfies.

$$W_t = \alpha_{E,t} + \alpha_{E^*,t} \tag{60}$$

Firms produce differentiated products. The production function for each differentiated home good is linear in labour with productivity A_t, which is a stochastic productivity shock. The foreign country has an analogous production function with productivity shock A_t^*. The home and foreign productivity shocks are given by

$$\log A_t = \zeta_A \log A_{t-1} + \varepsilon_{A,t}, \qquad \log A_t^* = \zeta_A \log A_{t-1}^* + \varepsilon_{A^*,t} \tag{61}$$

where $0 \leq \zeta_A \leq 1$, and ε_A and ε_{A^*} are i.i.d. shocks symmetrically distributed over the interval $[-\epsilon, \epsilon]$ with $Var[\varepsilon_A] = Var[\varepsilon_{A^*}] = \sigma_A^2$. Firms maximise profits, and all prices are fully flexible ex-post.

The home government is assumed to purchase a bundle of goods, denoted G, with the same composition as C, with budget constraint $T_t = G_t$. Similarly the foreign government purchases an amount G^* of a bundle of foreign goods with the same composition as C^*. We assume that government spending satisfies

$$\log G_t = \overline{G}(1-\zeta_G) + \zeta_G \log G_{t-1} + \varepsilon_{G,t}, \ \log G_t^* = \overline{G}(1-\zeta_G) + \zeta_G \log G_{t-1}^* + \varepsilon_{G^*,t}, \quad (62)$$

where $0 \leq \zeta_G \leq 1$ and ε_G and ε_{G^*} are i.i.d. shocks symmetrically distributed over the interval $[-\epsilon, \epsilon]$ with $Var[\varepsilon_G] = Var[\varepsilon_{G^*}] = \sigma_G^2$. The non-stochastic equilibrium level $\overline{G}$ is set to match a given value for g_y, the ratio of government spending to GDP in a symmetric, non-stochastic equilibrium.

Home equities represent a claim on home aggregate profits. The real payoff to a unit of the home equity is defined to be $\pi_t = \Pi_t/P_t$, where Π_t are nominal profits. In a symmetric equilibrium, nominal profits of each home firm will be $(P_{Ht} - w_t P_t)Y_{Ht}$, which are positive so long as $\phi > 1$. The real price of a unit of home equity is denoted $Z_{E,t-1}$. Thus the gross real rate of return on the home equity is $r_{E,t} = (\pi_t + Z_{E,t})/Z_{E,t-1}$.

The first-order conditions for home and foreign consumption and equity holdings are as in the previous example, simply replacing $r_{B,t+1}$with $r_{E,t+1}$, etc. The first order condition governing labour supply is

$$C_t^{-\rho} w_t = K$$

Profit maximisation by firms implies

$$\frac{P_{H,t}}{P_t} = \frac{\phi}{\phi - 1} w_t$$

Finally, the market clearing conditions are

$$C_{Ht} + C_{Ht}^* + G_{Ht} + G_{Ht}^* = Y_{Ht}$$

$$C_{Ft} + C_{Ft}^* + G_{Ft} + G_{Ft}^* = Y_{Ft}^*$$

As before, the four shocks are assumed to be independent from each other. As in the previous example, asset markets are incomplete.

4.2.1 First-order approximation

First-order approximation of the model follows very closely that of the last example. The excess return on foreign equity is given by

$$\hat{r}_{x,t} = [(1-\beta)\hat{\pi}_t^* + \beta \widehat{Z}_{E,t}^* - \widehat{Z}_{E,t-1}^*] - [(1-\beta)\hat{\pi}_t + \beta \widehat{Z}_{E,t} - \widehat{Z}_{E,t-1}] + O\left(\epsilon^2\right) \quad (63)$$

As in the previous example, the equations of the model can be collected together in the form of matrix equation system (20). Then any standard linear solution algorithm can be applied to the first-order system to yield a state-space solution in the form of (21).

4.2.2 Solution for equity holdings

In order to illustrate the solutions for equity holdings, we make the further assumption that $\zeta_A = \zeta_G$. In this case we compute the following expressions for the matrices R_1, R_2, D_1 and D_2

$$R_1 = \left[\frac{2\rho(\theta-1)(1-\beta)}{\Theta} \right]$$

$$R_2 = \left[\begin{array}{cccc} -\frac{(1-g_y)(1-\beta)(\theta-1)}{\Theta(1-\beta\zeta_A)} & \frac{(1-g_y)(1-\beta)(\theta-1)}{\Theta(1-\beta\zeta_A)} & -\frac{g_y\rho(1-\beta)(\theta-1)}{\Theta(1-\beta\zeta_A)} & \frac{g_y\rho(1-\beta)(\theta-1)}{\Theta(1-\beta\zeta_A)} \end{array} \right]$$

$$D_1 = \left[\frac{2(1-\beta)}{\Theta} \right]$$

$$D_2 = \left[\begin{array}{cccc} \frac{(1-\beta)(\theta-1)}{\Theta(1-\beta\zeta_A)} & -\frac{(1-\beta)(\theta-1)}{\Theta(1-\beta\zeta_A)} & -\frac{g_y(1-\beta)}{\Theta(1-\beta\zeta_A)} & \frac{g_y(1-\beta)}{\Theta(1-\beta\zeta_A)} \end{array} \right]$$

where $\Theta = (1 - g_y + \rho(\theta - 1))$. Applying (35) yields the following expression for equity holdings

$$\tilde{\alpha}_E = -\tilde{\alpha}_{E^*} = -\frac{1}{2(1-\beta)} \left(1 - \frac{\rho g_y^2}{(\theta-1)(1-g_y)} \frac{\sigma_G^2}{\sigma_A^2} \right) \tag{64}$$

In the absence of government spending shocks, home households will hold a perfectly pooled portfolio of home and foreign equity. Since the status quo embodied in budget constraint (59) implies that home households receive all home profits, this requires that $\tilde{\alpha}_E$ is negative.

When $g_y > 0$ however, home equity represents a good hedge against the consumption risk of government spending shocks, since home profits are relatively high when government spending is high. In this case, households will hold less than a fully pooled equity portfolio. In fact, the presence of government spending shocks may explain either partial or full home bias in equity portfolios.

The method can also be very easily applied to sticky price open economy models of the type developed by, for instance Obstfeld and Rogoff (1995), Benigno and Benigno (2003), Devereux and Engel (2003) and Corsetti and Pesenti (2005). Devereux and Sutherland (2006a) show how the solution method can be used to analyze the impact of alternative monetary policy rules on asset holdings in sticky-price models of this type.

5 Conclusion

Portfolio structure has become a central issue in open economy macroeconomics and international finance. Despite this, existing models and solution methods are not well-suited to analyzing portfolio choice in policy-relevant general equilibrium environments. This paper has developed a simple approximation method for portfolio choice problems in open economy models. Our approach is extremely easy to implement and can be used in any of the existing models that rely on first-order approximation methods. If the researcher is primarily interested in the implications of portfolio choice for the first order properties of macro variables (such as GDP, consumption, or the real exchange rate), either through impulse response analysis or by computing second moments so as to describe volatility and comovement, then the solution method outlined here allows a full answer to these questions. Since the overwhelming majority of the research in international finance and macroeconomics is carried out at the level of first order approximation, the method is widely applicable. It can be used to study many empirical questions in the interface between international finance and macroeconomics. Moreover, the method allows us to study the macroeconomic determinants of optimal steady state portfolio holdings for any asset or combination of assets, whether markets are complete or incomplete.

The current paper focuses on the derivation of steady-state portfolios. If one is interested in the time-variation in portfolio holdings (which, following from the analysis of this paper, have only a second order effect on macroeconomic variables), it is necessary to approximate the model to a higher order. A companion paper (Devereux and Sutherland 2006b) shows how a straightforward extension of the methods in this paper to higher-order approximations allows analysis of higher-order aspects of portfolio behaviour, including the impact of time-variation on portfolio holdings.

References

[1] Benigno, G and P Benigno (2003a) "Price Stability in Open Economies" *Review of Economics Studies*, 70, 743-764.

[2] Campbell, J and L Viceira (2002) *Strategic Asset Allocation* Oxford, Oxford University Press.

[3] Corsetti, G and P Pesenti (2005) "International Dimensions of Optimal Monetary Policy" *Journal of Monetary Economics*, 52, 281-305.

[4] Devereux, M and C Engel (2003) "Monetary Policy in an Open Economy Revisited: Price Setting and Exchange Rate Flexibility" *Review of Economic Studies*, 70,765-783.

[5] Devereux, M and M Saito (2005) "A Portfolio Theory of International Capital Flows" unpublished manuscript, UBC.

[6] Devereux, M and A Sutherland (2006a) "Monetary Policy Rules and International Portfolio Choice" unpublished manuscript, UBC.

[7] Devereux, M and A Sutherland (2006b) "Country Portfolio Dynamics" unpublished manuscript, UBC.

[8] Engel, C and A Matsumoto (2005) "Portfolio Choice in a Monetary Open-Economy DSGE Model" unpublished manuscript, University of Wisconsin and IMF.

[9] Evans, M and V Hnatkovska (2005) "International Capital Flows, Returns and World Financial Integration" NBER Working Paper 11701.

[10] Ghironi, F, J Lee and A Rebucci (2005) "The Valuation Channel of External Adjustment" unpublished manuscript, Boston College.

[11] Gourinchas, P and H Rey (2005) "International Financial Adjustment" NBER Working Paper no. 11155.

[12] Heathcote, J and F Perri (2004) "The International Diversification Puzzle is Not as Bad as You Think" unpublished manuscript, Georgetown University and New York University.

[13] Judd, K L (1998) *Numerical Methods in Economics* Cambridge: MIT Press.

[14] Judd, K L and S-M Guu (2001) "Asymptotic Methods for Asset Market Equilibrium Analysis" *Economic Theory,* 18, 127-157.

[15] Judd, K L, F Kubler and K Schmedders (2002) "A Solution Method for Incomplete Asset Markets with Heterogeneous Agents" unpublished manuscript, Stanford University.

[16] Kraay, A, N Loayza, L Serven and J Ventura (2005) "Country Portfolios" *Journal of the European Economic Association*, 3, 914-945.

[17] Kollmann, R (2006) "International Portfolio Equilibrium and the Current Account" unpublished manuscript, University of Paris XII.

[18] Lombardo, G. and A. Sutherland (2005) "Computing Second-Order-Accurate Solutions for Rational Expectation Models using Linear Solution Methods" European Central Bank Working Paper No. 487.

[19] Lane, P, and G M Milesi-Ferretti (2001) "The External Wealth of Nations: Measures of Foreign Assets and Liabilities for Industrial and Developing Countries" *Journal of International Economics* 55, 263-94.

[20] Lane, P, and G M Milesi-Ferretti (2006) "The External Wealth of Nations Mark II" IMF Working Paper no 06-69.

[21] Obstfeld, M and K Rogoff (1995) "Exchange Dynamics Redux" *Journal of Political Economy,* 103, 624-660.

[22] Obstfeld, M and K Rogoff (2000) "The Six Major Puzzles in International Macroeconomics: Is there a Common Cause" *NBER Macroeconomics Annual*, 15, 339-390.

[23] Samuelson, P A (1970) "The Fundamental Approximation Theorem of Portfolio Analysis in terms of Means, Variances and Higher Moments" *Review of Economic Studies*, 37, 537-542.

[24] Schmitt-Grohe, S. and M. Uribe (2003) "Closing Small Open Economy Models", *Journal of International Economics,* 59, 137-59.

[25] Tille, C (2003) "The Impact of Exchange Rate Movements on U.S. Foreign Debt" *Current Issues in Economics and Finance* 9, 1-7.

[26] Tille, C (2004) "Financial Integration and the Wealth Effect of Exchange Rate Fluctuations" unpublished manuscript, Federal Reserve Bank of New York.

[27] Tille, C and E van Wincoop (2006) "International Capital Flows", unpublished manuscript, University of Virginia.