Income Risk, Risk-Sharing and Household Debt*

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March 27, 2006, Preliminary and Incomplete

Abstract

We investigate possible determinants of the increase of household debt since the 1980s in the US. We use a heterogeneous-agent model, in which labor income is risky and markets are incomplete. Consumers use durables not only as collateral for their debt but also derive utility from their durable stock. We first assume that all debt is secured. That is, debt is collateralized by durable holdings and the lowest attainable labor income flow. In this model financial-market development in terms of lower interest spreads (and lower borrowing rates) or laxer collateral constraints can explain the increase in household debt whereas the buffer-stock saving motive makes higher income risk a less plausible explanation. We then extend the model to unsecured debt, default and risk-sharing with competitive financial intermediaries.

Keywords: household debt, durables, collateral constraint, income risk, incomplete markets, heterogeneous agents, bankruptcy, risk-sharing.

JEL: E21, D91.

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We thank the research group at the *Finance and Consumption in the EU Chair*, sponsored by Findomestic Banca and CETELEM at the European University Institute.

1 Introduction

Household debt has increased substantially in developed countries during the last decades. This has been most dramatic in the US where household debt as a proportion of disposable income has been 46 percentage points higher in 2003 than in 1981; and consumer debt amounted to 113% of households' disposable income in 2003 (see, for example, Iacoviello, 2005). Household debt has increased also in many European countries although starting from lower levels (see ECRI, 2000). Thus, it is important to understand the determinants of households' debt accumulation.

In this paper we want to investigate whether the remarkable increase in household debt since the 1980s in the US can be explained in an incomplete-markets model with consumption-smoothing motives. We assume that labor income is risky and that this risk cannot be insured.[1] In our model consumers derive utility from non-durable and durable consumption. Durables like housing or cars generate utility but also provide collateral against which consumers can borrow.[2] Since most household debt in the US (about 90%, Campbell and Hercowitz, 2005) is mortgage debt or other credit that is secured by collateral and cannot be defaulted upon, we first analyze the case in which all credit needs to be collateralized. We calibrate our model to the US and show how the solution depends on the model's parameters in an intuitive way. We find that financial market development in terms of lower interest spreads (and lower borrowing rates) or an exogenous relaxation of the collateral constraint can explain the upward trend in household debt. Instead an increase in income risk reduces average household debt because of the buffer-stock savings motive. Thus, we get different predictions compared with general equilibrium models which analyze approximations around the non-stochastic steady state (see, for example, Iacoviello, 2005).

An important difference of our model compared with the previous literature is that we allow for an interest spread between the lending and the borrowing rate in financial markets. This generates the empirically realistic finding that a mass of consumers holds no financial assets at all. Such a spread has been analyzed by Carroll (2001, section 3) in a model without durables. As Carroll (2001) we find that there is only a small effect of the spread on non-durable consumption. We find, however, that the effect on the propensity to purchase *durables* is sizeable. This is because durables are an alternative vehicle to transfer resources intertemporally, especially if depreciation rates are low. The effect of the interest spread on the propensity makes durable expenditure more dispersed as income changes.

Since unsecured debt as a fraction of disposable income also increased substantially, from 5 to 9% in the period 1983 to 1998 (see Livshits et al., 2005, Figure 3), we extend our model to allow for costly

[1] See the seminal papers of Deaton (1991), Carroll (1997) and the general equilibrium analysis of Aiyagari (1994) for models of incomplete markets and non-durable consumption; and Diaz and Luengo-Prado (2005) or Gruber and Martin (2003) for models with durables.

[2] It is well known that durables mitigate the precautionary savings motive because they lower the dependence of consumption on income fluctuations. Of course, if downpayments need to be made for durable purchases and durables are illiquid, the precautionary savings motive could be stronger than in our model (see, e.g., Diaz and Luengo-Prado, 2005).

default with risk-sharing financial intermediaries which are perfectly competitive. This allows us to extend our analysis to the evolution of household debt portfolios in terms of unsecured and secured debt. Allowing for default in equilibrium is also an attractive additional feature of the model because consumer bankruptcy has become more important in recent decades in the US. Roughly 1.5% of US households have filed for personal bankruptcy in each recent year; in 2003 households defaulted on approximately $120 billion or $1,100 per household each year (see White, forthcoming).

Our analysis of unsecured debt relates to the general equilibrium model of Chaterjee et al. (2005) and the partial equilibrium model of Athreya (2005).[3] Whereas Chaterjee et al. focus on unsecured debt and do not consider durables and collateralized credit, Athreya does allow for secured debt. However, in his model consumers do not derive utility from the durable and the durable stock is exogenous. This is an important difference to our model which results in a different modelling of bankruptcy.[4] Moreover, both papers are not interested in explaining the trend of debt which we investigate in this paper. More related in this respect is the work by Livshits et al. (2005) who try to explain the rise in consumer bankruptcies using a life-cycle model without durables.

The rest of this paper is structured as follows. In Section 2 we present, solve and calibrate the model with secured debt and discuss possible explanations for the upward trend of household debt. In Section 3 we extend the model to unsecured debt and default. We conclude in Section 4.

2 The model without default

Agents are risk-averse and have an infinite horizon. They derive utility from a durable good d and a non-durable good c. The instantaneous utility is given by $U(c,d) = u(c) + \phi w(d)$ where $u(.)$ and $w(.)$ are both strictly concave, and ϕ is the weight assigned to utility derived from the durable. We assume that the marginal utility $w'(d)$ is well defined at $d = 0$ so that our model is able to generate agents with no durable stock in at least some states of the world, as is realistic. A possible functional form is $w(d) = (d+\underline{d})^{\tau}$, with $\tau \leq 1$ and $\underline{d} > 0$. The asymmetry in the utility function with respect to non-durable and durable consumption is justified in the sense that durables are less essential than non-durable consumption such as food. Note that we implicitly assume that durables can be transformed into non-durable consumption with a linear technology so that the relative price is unity.

In specifying utility as above we have made a number of simplifying assumptions. We assume d to be a homogenous, divisible good. Moreover, utility is separable over time and at each point in time it is separable between durables and non-durables. Both assumptions are made for tractability given

[3] See also the dynamic models of Athreya (2002) and Kubler and Schmedders (2003), and the two-period general equilibrium models of Dubey et al. (2005) and Zame (2003). For a life-cycle model of consumer bankruptcy see Livshits et al. (2004).

[4] Pavan (2005) studies and estimates a structural model of consumer bankruptcy with an endogenous durable stock. As in our model, the durable stock is compared with the legally exempt level in case of bankruptcy. One important difference is that Pavan does not allow for secured and unsecured debt.

that it is more realistic to assume that durables are a bundle of characteristics and that utility derived from durables depends on non-durable consumption in non-trivial ways. Instead, as in much of the literature, we assume that the service flow derived from durables is proportional to the stock where we have normalized the factor of proportionality to 1 (see Waldman, 2003, for a critical review of these common assumptions).

We assume that markets are incomplete so that agents cannot fully diversify their risk. It is well known that in such an environment, it is necessary to assume that agents are impatient, $\beta < 1/(1+r^a)$, where r^a is the lending rate which is taken as given in our small-open economy model. It follows from the results by Deaton and Laroque (1992) that agents hold a finite amount of financial assets a. Because of positive depreciation δ and $\lim_{d\to\infty} w'(d) = 0$, also the durable stock d is bounded from above. The collateral constraint and $d \geq 0$ then imply a compact state space so that standard dynamic programming techniques can be applied (see Araujo et al., 2002, for existence proofs in a general equilibrium context).

We assume that there are transaction costs in the financial market so that the lending rate r^a is smaller than the borrowing rate r^b: $r^a < r^b$. This assumption implies that some agents will hold no financial assets, $a = 0$. As we will see below this has interesting implications for the consumption propensities and the shape of the policy functions.

Timing. We specify our model in discrete time so that we have to make assumptions about the timing. Figure 1 illustrates the time line. First uncertain income y_t is drawn. Then agents derive utility from the durable good d_t before the durable depreciates at rate δ. The agent then makes his choices based on the available cash-on-hand

$$x_t \equiv (1+r^j)a_t + y_t + (1-\delta)d_t,\ j = a, b,$$

where r^b is interest rate on debt and r^a is the interest rate on financial assets a_t, with $r^b > r^a$. Note that the durable stock d_t is predetermined in period t.

The program. Rearranging the budget constraint,

$$c_t = (1+r^j)a_t - a_{t+1} + y_t - (d_{t+1} - (1-\delta)d_t)\ ,$$

we can write the value function as

$$V(x_t, d_t, y_t) = \max_{a_{t+1}, d_{t+1}} \left[u(\underbrace{x_t - a_{t+1} - d_{t+1}}_{c_t}) + \phi w(d_t) + \beta E_t V(x_{t+1}, d_{t+1}, y_{t+1}) \right]$$

We can further simplify the problem by noting that d_t is predetermined in period t and that the additive separable term $\phi w(d_t)$ does not affect the optimal choices of the consumer. Defining

$$\widetilde{V}\left(x_t, y_t\right) \equiv V\left(x_t, d_t, y_t\right) - \phi w\left(d_t\right)$$

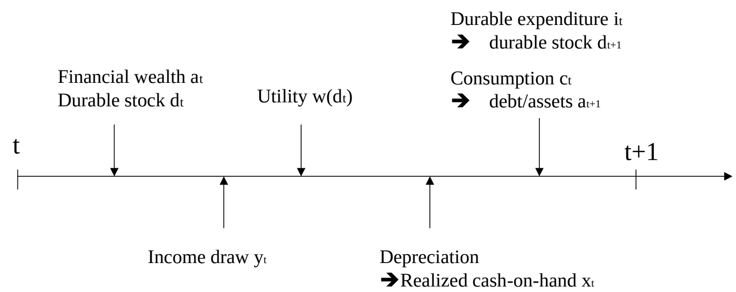

Figure 1: Timing in the model with collateral constraint and no default

the transformed maximization problem is

$$\widetilde{V}(x_t, y_t) = \max_{a_{t+1}, d_{t+1}} \left[u(\underbrace{x_t - a_{t+1} - d_{t+1}}_{c_t}) + \beta\phi w(d_{t+1}) + \beta E_t \widetilde{V}(x_{t+1}, y_{t+1}) \right] \tag{1}$$

under the constraints

$$a_{t+1} = \begin{Bmatrix} (1+r_t^a)a_t + y_t - c_t - i_t \text{ if } a_t \geq 0 \\ (1+r_t^b)a_t + y_t - c_t - i_t \text{ if } a_t < 0 \end{Bmatrix}$$

$$d_{t+1} = (1-\delta)d_t + i_t$$

$$\underbrace{\left(1+r^b\right) a_{t+1} + (1-\delta)d_{t+1} + \underline{y}}_{\underline{x}_{t+1} \equiv x_{t+1}(\underline{y})} \geq 0$$

$$d_{t+1} \geq 0 \ .$$

The first two constraints are the accumulation equations for the financial wealth a and the durable stock d. The third constraint is the collateral constraint. This constraint ensures that the lowest attainable cash-on-hand $\underline{x}_{t+1}$ guarantees full repayment (if income takes its smallest possible value $\underline{y}$). The assumption here is that the lender, who lends at the risk-free rate, knows the financial position (a_t,d_t) and the minimum of the support of the income distribution $\underline{y}$. The lender does not know individual income draws. Note that whether and how much the collateral constraint binds in $t+1$ is entirely determined by the choices in period t. In Section 3 below, we relax this constraint and allow for risky debt with default.

Problem (1) satisfies Blackwell's sufficient conditions (monotonicity and discounting) for a contraction mapping so that we can apply standard dynamic programming techniques to solve for the stationary equilibrium. Because of stationarity, we drop time indexes and use primes " $'$ " to denote a one-period lead (but for $u'(.)$ or $w'(.)$ which denote first derivatives of the instantaneous utility functions).

Equilibrium definition. A stationary equilibrium is given by the policy functions for non-durable consumption $c(x, y)$, durable investment $i(x, y)$, the accumulation equations $a'(x, y)$ and $d'(x, y)$, and the evolution of the state variable $x'(x, y)$ so that for given prices $\{r^a, r^b\}$

(i) the value function $\widetilde{V}(x, y)$ attains its maximal value.

(ii) the collateral constraint is not violated, i.e., $\underline{x}' \geq 0$.

(iii) the durable stock is weakly positive, $d' \geq 0$.

(iv) the distribution measure $\mu(X, Y)$ over the state space $X \times Y$ of agents is stationary, so that for a transition matrix $\Gamma(y'|y)$

$$\mu(X, Y) = \int_{X \times Y} I_{\{x'=x'(x,y)\}} \Gamma(y'|y) d\mu \ ,$$

where $I_{\{x'=x'(x,y)\}}$ is an indicator function which takes the value 1 if the statement in braces is true.[5]

2.1 Euler equations and analytic results

For later reference, note that in the optimum

$$u'(c) = \beta(1 + r^a) E_y u'(c') \ ,$$

if the agent holds positive financial assets a, and

$$u'(c) = \beta(1 + r^b) \left(E_y u'(c') + \kappa\right)$$

if the agent holds debt and choices are such that the collateral constraint binds in the following period, $\kappa > 0$.[6] Because of the interest spread $r^b > r^a$, both Euler equations can be slack. In this case the intertemporal rate of substitution of non-durable consumption is in-between the lending and the borrowing rate:

$$1 + r^a < \frac{u'(c)}{E_y u'(c')} < 1 + r^b \ .$$

Then, agents hold zero financial assets, $a = 0$.

In the optimum, durable investment is chosen so that it satisfies the condition

[5] See Rios-Rull (1999) for further discussion on the restrictions of admissible income processes which satisfy monotone mixing or the American-dream / American-nightmare condition.

[6] Note again that whether the collateral constraint binds in period $t + 1$ is determined by choices in period t.

$$u'(c) = \beta(1-\delta)E_y u'(c')) + \phi w'(d') + (1-\delta)\kappa + \gamma \ ,$$

where $\gamma \geq 0$ is the multiplier associated with the constraint $d' \geq 0$. As is intuitive, the agent aligns the marginal utility of foregone non-durable consumption today (resulting from durable investment) with the discounted marginal utility derived from the durable tomorrow and the additional marginal utility of non-durable consumption that is afforded by re-selling the durable good (taking into account its depreciation at rate δ).

Note that if the collateral constraint binds, $\kappa > 0$, present consumption is valued less and more resources are transferred to the future period. Defining "permissible income processes" as those processes which ensure that non-durable consumption and the durable stock remain in the domain over which $u(.)$ and $w(.)$ are defined (as in Carroll and Kimball, 1996), we can show the following

Remark 1: *If utility is separable in the durable d and non-durable consumption c, the instantaneous utility functions $u(.)$ and $w(.)$ are strictly concave, of the HARA family, and satisfy prudence so that $u'''(.) \geq 0$ and $w'''(.) \geq 0$, we can show:*

(i) If the constraints are not binding, $c(x,y)$, $d(x,y)$ are concave, $a(x,y)$ is convex and $\partial c(x,y)/\partial x > 0$, $\partial d(x,y)/\partial x > 0$. Moreover, $\partial a(x,y)/\partial x \geq 0$ if $\delta = 1$, and under additional restrictions on concavity also for $0 \leq \delta < 1$.

(ii) If the collateral constraint binds, $\partial a(x,y)/\partial x$ falls and can become negative.

(iii) If the Euler equations for financial assets are slack, $c(x,y)$, $d(x,y)$ can be strictly convex and $a(x,y)$ can be strictly concave over a certain range of x.

Proof: see the Appendix.

Remark 1(i) is an application of Theorem 1 in Carroll and Kimball (1996) to our model with durable and non-durable consumption. The concavity of the non-durable and durable consumption functions in models of incomplete markets is very intuitive. Precautionary motives imply that the consumption propensity falls as agents have more cash-on-hand.

The intuition for Remark 1(ii) is that the possibility of a binding collateral constraint increases the amount of financial wealth a for small values of x so that the slope is flatter. The optimality condition of borrowing agents

$$u'(c) = \beta(1+r^b)\left(E_y u'(c') + \kappa\right)$$

illustrates that as κ falls with more cash-on-hand x (the collateral constraint is less binding), $u'(c)$ decreases, ceteris paribus. The same holds for durable investment. The slope $\partial a(x,y)/\partial x$ can be negative if the propensity of non-durable and durable consumption is larger than 1 and the collateral constraint is relaxed as the durable stock increases.

The intuition for Remark 1(iii) is that the propensity to consume out of cash-on-hand has to increase if the Euler equations for non-durable consumption are slack since $a' = 0$ and $\partial a'/\partial x$ falls so that $\partial a'/\partial x = 0$. Hence, the consumption propensities increase since $\partial c/\partial x + \partial d/\partial x = 1$ if $a' = 0$. The consumption functions are no longer globally concave.

Moreover, the durable stock increases relative to non-durable consumption since the optimality conditions above (without multipliers for the constraints) imply

$$1 + r^a < \beta(1-\delta) + \beta\phi\frac{w'(d')}{E_y u'(c')} < 1 + r^b \; .$$

The expected intra-temporal rate of substitution between durable and non-durable consumption tomorrow equals $\left[1 + r^a - \beta(1-\delta)\right]/(\beta\phi)$ if the agent lends and $\left[1 + r^b - \beta(1-\delta)\right]/(\beta\phi)$ if the agent borrows. Thus, as agents accumulate cash-on-hand in the region where $a' = 0$, $w'(d')/E_y u'(c')$ falls until the intra-temporal rate of substitution equals $1 + r^a$.

The larger propensity for durable investment, for values of cash-on-hand x where $a' = 0$, is intuitive. As long as the depreciation rate is not too high, durables are an imperfect way to transfer resources intertemporally since the rate of transformation is optimally in-between the exogenous interest factors $1 + r^b$ and $1 + r^a$.

That limited access to funds increases the propensity of durable and non-durable consumption is supported by empirical evidence (see, for example, Alessie et al., 1997, for estimates using the period of financial deregulation in the UK in the 1980s).[7] This feature of the model will be of particular interest for the effect of the interest spread on consumption volatility. A smaller fraction of agents with financial assets $a' = 0$ implies smaller changes of consumption in response to income changes.

2.2 Calibration and numerical results

Numerical algorithm. It is well known that problems like ours do not have a closed-form solution for the optimal policies. Therefore, we pursue a numerical approach which relies on value function iteration. While this allows us to conveniently rely on the contraction properties of the Bellman operator, one of the main challenges for this technique is to find a way to get around the curse of dimensionality. This is where the formulation of the problem that reduces the number of state variables to the minimum pays off - by subsuming the portfolio positions and the income realization in the single variable cash-on-hand. Hence, the state variables are cash-on-hand and the state of uncertainty, which is modeled as a 2-state Markov chain. The range of cash-on-hand, x, is restricted to an interval $[0, x_{\max}]$. We perform value function iteration on a grid of 350 points over that interval. The grid is finer at the origin where the value function has more curvature. Our choice of $x_{\max}$ guarantees that, for every x and for every

[7] Bertola et al. (2005) provide alternative microfoundations to explain the higher propensity for durable purchases if there is an interest spread $r^b > r^a$ and agents can be liquidity constrained ($a = 0$). In their model, a monopolist dealer has an incentive to lower the credit price of a durable good to attract liquidity constrained customers.

realization of uncertainty, the equilibrium policy will imply a value for x tomorrow that remains within that interval.[8] We use linear interpolation of the value function between these grid points.

A feature of our algorithm that greatly enhances the accuracy of our solutions is the fact that the maximizing choices for the policy (at each state and each iteration) are not selected from a discretized set of choices, but rather by solving these maximization problems continuously over portfolio choices. We rely on a numerical optimization routine[9], which can also handle the collateral constraint and sign restrictions, to perform this task and to obtain the implicit multipliers on the constraints. The policy functions over the range $[0, x_{\max}]$ are obtained from the optimal policy choices on the grid by interpolation, using cubic splines.

As has become standard in the literature (see, e.g., Judd, 1992, and Aruoba et al., 2006), we evaluate the accuracy of our solutions by the normalized Euler equation errors implied by the policy functions. These are smaller than $4 * 10^{-3}$ over the entire range where the Euler equations apply with equality, and in fact much smaller for most values that the state variables of our problem can assume.

Calibration. We normalize average labor income y to 1, and parametrize the instantaneous utility functions as

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma} \text{ and } w(d) = \frac{(d + \underline{d})^{1-\sigma} - 1}{1-\sigma},$$

where, as mentioned above, $\underline{d} > 0$ allows the consumer to hold no durable stock. We set risk aversion for the non-durable and durable good $\sigma = 2$, which is well within the range of commonly used values, and assume $\underline{d}$= 0.01. It turns out that the parameter $\underline{d}$ is rather unimportant and can be set to negligibly small values without changing the quantitative results much. This is because the region of d close to zero is not important in our simulations. We calibrate the size of the shocks and transition probabilities of our 2-state Markov chain as 0.4. This implies a coefficient of variation of 0.4 and a first-order autocorrelation of 0.86 which is within the range of reasonable values considered by Aiyagari (1994).

We calibrate our model to the US, following previous calibrations by Diaz and Luengo-Prado (2005) and Athreya (2004). Table 1 summarizes the parameters. We calibrate the relative taste for the durable ϕ and the depreciation rate δ so that we match a ratio of the durable stock to disposable income of 1.6 and a ratio of non-durable consumption over durable investment slightly above 6 (see Diaz and Luengo-Prado, 2005, for the discussion of empirical estimates). This results in $\phi = 0.4$ and $\delta = 0.08$. The other parameters are rather standard and their sources are listed in Table 1.

The choice of the depreciation rate merits further discussion. We need a rather high depreciation

[8] In our algorithm, we choose the grid for cash-on-hand so that for an upper bound of cash-on-hand $\overline{x}$, the optimal policies imply that the maximal attainable cash-on-hand, $x'_{\max}$ (for the highest realization of income $y_{\max}$) is smaller than this upper bound: $x'_{\max} = (1+r)\,a' + y_{\max} + (1-\delta)\,d' < \overline{x}$. Using $\underline{x} = 0$ as a lower bound gives us a compact state space (this bound is implied by the collateral constraint).

[9] We are using the Matlab routine fmincon().

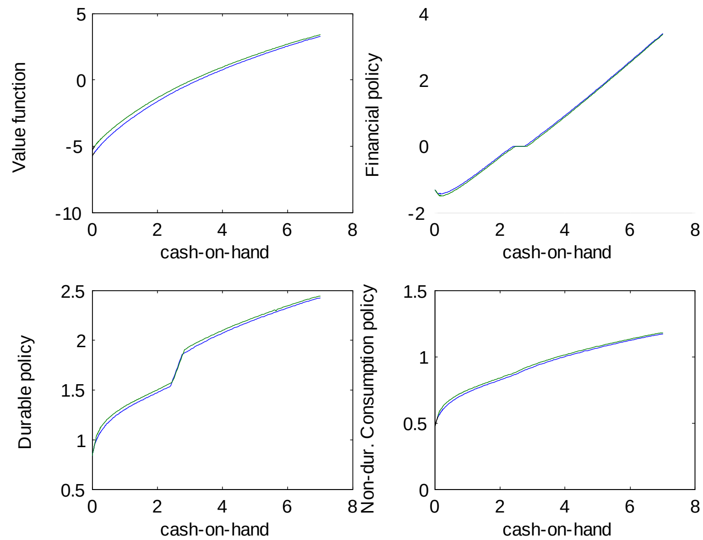

Figure 2: Value and policy functions in the good and bad state

rate so that a durable stock of 1.6, which is realistic empirically, is consistent with a ratio of non-durable consumption over durable investment of 6. Although a depreciation rate $\delta = 0.08$ is less realistic for housing, the rate is below commonly assumed values for other important durables like cars or computers. Thus, we view it as a reasonable approximation for the depreciation of a durable composite. We will also present results for a lower depreciation rate $\delta = 0.04$ which is closer to commonly used depreciation rates as in Campbell and Hercowitz (2005).

Value function and policy functions. Figure 2 displays the solution for the value function and the policy functions in the bad and good income state. The value function is smooth and concave. Not surprisingly, the function shifts down in the bad state of the world. The policy functions have a slightly non-standard shape consistent with the results of Remark 1. Because of the interest spread $r^b > r^a$, financial assets $a = 0$ for an interval of cash-on-hand values. This local concavity of the financial policy implies local convexities in the policy functions for non-durable consumption and the durable stock. The local convexity is much more pronounced for the durable policy. This depends on whether the depreciation rate is low enough so that durables are a reasonably attractive vehicle to transfer resources intertemporally.

Note that the constraint $d \geq 0$ is never binding whereas the collateral constraint is expected to bind for values of cash-on-hand close to zero. We now simulate our economy to find out more about the mean and distribution of the policy variables in the steady state.

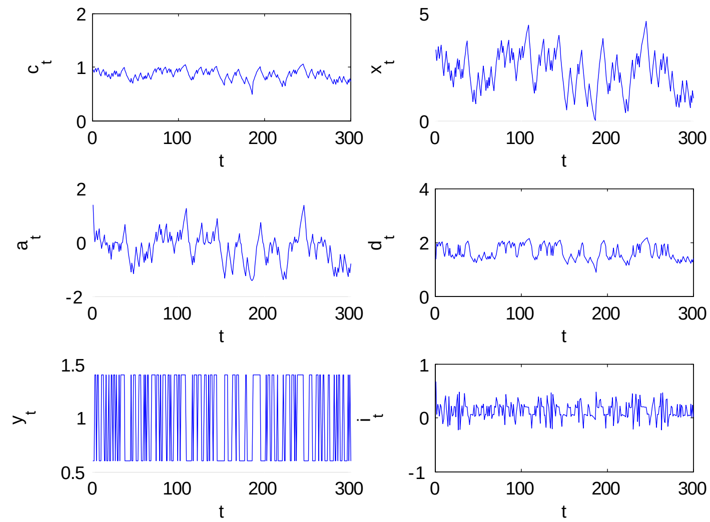

Figure 3: Time-series simulation of the economy without default

Simulations. We simulate our economy for 10,000 periods. Figure 3 displays the results for an arbitrarily chosen subsample of 300 periods. If the exogenous income process y_t implies a long enough sequence of bad-state incomes, the agent accumulates financial debt as he borrows against the durable stock. If the bad shocks persist, the agent might not have the resources to keep the durable stock at his current level so that it decumulates. This tightens the collateral constraint and can sometimes imply that cash-on-hand $x_t = 0$. The collateral constraint $\underline{x}_t \geq 0$, however, is also important for behavior if the realized $x_t > 0$ because of an income larger than $\underline{y}$. The value function incorporates the expectation that the collateral constraint can be become binding in the future with some probability, especially for low values of x.

Note that without income uncertainty, the impatient consumer would always be at his borrowing limit. Income uncertainty implies that the agent does not borrow as much and, if income is persistently good, he even accumulates some buffer-stock of assets, $a_t > 0$. Finally, we observe that durable investment is more volatile than consumption also because of the high propensity to invest if financial assets are zero. We return to this point below.

Table 2 displays the averages in the steady-state equilibrium for the main variables of interest. In column (1) we display the results for our benchmark economy. All values are expressed in average-income equivalents. On average, the consumer holds 2.3 of average income as cash-on-hand and borrows a sixth of average income with financial assets. The size of the durable stock is 1.64 and the ratio of non-durable consumption over durable investment is 6.5 which is in line with empirical evidence for the

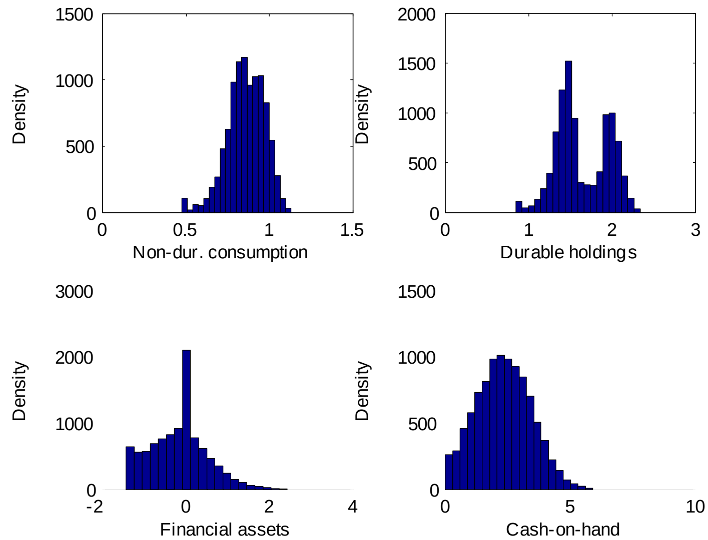

Figure 4: The steady-state distributions

US (see Diaz and Luengo-Prado, 2005).[10]

Given that the income shocks are purely idiosyncratic, the law of large numbers implies that all idiosyncratic risk disappears upon aggregation (see Uhlig, 1996) and the time-series distribution can be used as an approximation of the cross-sectional distribution in the steady state. Figure 4 displays such distributions for non-durable consumption c, durable holdings d, financial assets a, and cash-on-hand x. The density of cash-on-hand is bell-shaped and is truncated at $x = 0$, where the collateral constraint binds. Thus, also the densities of c, d, and a have more mass at their lower bound of the support than would be the case without the constraint. Moreover, financial assets have a mass point at $a = 0$ when the (non-durable) consumption Euler equation is slack for both r^a and r^b. The frequency of agents with zero financial assets in Figure 4 is 11.7%. This is about the same order of magnitude as the 10% of US consumers between age 25 and 50 which hold net non-housing wealth in the range from zero to two weeks' of their permanent income[11] (see the discussion of these statistics based on the 1995 Survey of Consumer Finances in Carroll, 2001). The higher propensity to consume in the range where $a = 0$ implies that both the distribution for non-durable consumption and durable holdings are bimodal. Consistent with the much stronger change in the propensity to purchase durables observed in Figure 2, the bimodality is more pronounced for the distribution of durable holdings.

[10]Note that average disposable income $y + r^j a$ is nearly equal to average income since $r^j a \simeq 0$.

[11]Buffer-stock saving behavior should matter for consumers in this age range.

Changes in parameters. We now investigate how changes of the model's parameters alter the steady-state equilibrium. In Table 2, columns (2) and (3), we compute the average equilibrium for risk-aversion of $\sigma = 1$ and $\sigma = 3$, respectively. Not surprisingly, more risk-aversion increases the buffer-stock saving motive so that consumers hold more financial assets. For $\sigma = 3$, the consumer saves a positive amount on average. Instead, the expenditure for durables decreases. This is interesting since durables relax the collateral constraint and one could have expected that more risk-averse consumers hold a larger durable stock. However, financial assets are a much more direct and return-dominating vehicle to self-insure the consumer. Finally, note that a larger σ shifts consumption towards non-durables. This would not occur in the certainty case when constraints are not binding, since then $c/(d+\underline{d}) = ((1-\beta\phi(1-\delta))/(\beta\phi))^{1/\sigma}$ and the term in brackets is larger than 1 for the chosen parameter values. Under uncertainty more risk aversion reduces the attractiveness of durables as storage device compared with financial assets.

In column (4) we investigate whether the parameter $\underline{d}$ is important in our benchmark equilibrium. We set $\underline{d} = 0$ and find no significant changes. As expected durable holdings increase slightly compared to non-durable consumption because the marginal utility derived from the durable is higher (for a given d). Thus, the ratio c/i falls. The larger durable stock also relaxes the collateral constraint. This allows agents to borrow more so that the average financial-asset position is lower. The increase in debt is not enough, however, to completely offset the increase in d so that cash-on-hand increases. The effect of increasing ϕ from 0.4 to 0.5 is qualitatively the same (see column (5)).

If agents are more impatient ($\beta = 0.9$), the consumers borrow more (see column (6)). At the same time the ratio c/i increases since non-durable consumption generates utility today whereas durable investment only generates utility tomorrow. Thus, the durable stock falls which also tightens the collateral constraint. Since consumers borrow more, the collateral constraint binds much more often.

When calibrating the model, we have mentioned that a depreciation rate $\delta = 0.08$ is rather high. In column (7) we lower the depreciation rate to $\delta = 0.04$. This increases the durable stock and non-durable consumption and lowers durable investment which is only a tenth of non-durable consumption. The larger cash-on-hand relaxes the collateral constraint and allows agents to borrow more in bad times so that the average financial asset position is lower.

2.2.1 On the evolution of household debt

We now apply our model to investigate whether the model can explain the rise in household debt in the US in the last decades. We consider the following explanations as candidates: (i) a fall of the interest rate on debt and/or a fall of the interest spread in financial markets, (ii) laxer collateral constraints or (iii) an increase in income risk.

A fall of the interest rate on debt. If we lower the borrowing rate r^b to 0.02, not surprisingly agents borrow more (see Table 2, column (8)). Cheaper borrowing also allows consumers to afford a larger durable stock. Total cash-on-hand decreases, however, because of more consumer debt. The fall in the borrowing rate also reduces the spread in the financial market so that agents hold zero financial assets less frequently and the kinks in the policy functions of durables and financial assets become less pronounced. This implies that the frequency of consumers with financial assets $a = 0$ is 2.3% which is similar to the empirically observed frequency of 2.5% for consumers holding precisely zero net non-housing worth in the 1995 Survey of Consumer Finances in the US (see Carroll, 2001). The lower frequency implies in our model that the distribution of durable holdings becomes less bimodal (the figures are not reported but are available upon request).

A fall of the interest spread. In order to distinguish between changes in the average interest rate (lending and borrowing rate) and changes in the spread, we try to disentangle both effects in columns (9) and (10). We first lower the interest spread but keep the average interest rate constant (see column (9)). Decreasing the spread by 50% has a very small effect on the steady-sate averages. Alternatively, we keep the spread constant but increase both the lending and borrowing rate so that the average interest rate increases (see column (10)). The results in column (10) differ from the benchmark in the opposite way as in column (8) where we decreased the borrowing rate. There is only one subtle difference. The durable stock increases in column (8) because of cheaper borrowing rates. Instead in column (10), higher average interest rates imply that the distribution of financial assets shifts upwards so that the wealth effect allows consumers to afford more expenditure.

An increase in income risk. We find that an increase in income risk cannot explain the increase in debt. The reason is that higher risk (in terms of shock size or persistence) increases the buffer-stock saving motive and thus *decreases* the debt holdings of agents. The results are in Table 2, columns (11)-(13). In column (11) we increase the size of shocks from 0.4 to 0.5, which implies an increase of the standard deviation of log-income by 12 percentage points. This is about the increase of the cross-sectional standard deviation of log-earnings in the US (15 basis points) in the period between 1981 and 2003. As can be seen in column (11), consumers hold more financial assets as buffer stock and also, conditional on holding debt, average debt decreases from -0.35 to -0.22. The average durable stock increases slightly. The results are qualitatively the same if the shocks are more persistent (see column (12) where the transition probability falls from $p = 0.4$ to $p = 0.2$).

Laxer collateral constraint. Institutional financial market reforms that allow consumers to collateralize more of their debt are a more plausible explanation for the higher debt levels in our model. We tighten the collateral constraint exogenously in column (13) where we no longer allow consumers to collateralize their durable stock. This implies qualitatively similar changes as an increase of income risk. Thus, *relaxing* collateral constraints, does *increase* consumer debt. Lower collateral requirements are thus a possible explanation for higher consumer debt. See Campbell and Hercowitz (2005) for a discussion on how market innovations that followed the Monetary Control Act of 1980 and the Garn-St.Germain Act of 1982 relaxed collateral constraints on household debt in the US.[12]

2.2.2 Steady state dispersion

Since agents are heterogenous we are not only interested in the mean but also in the dispersion in the steady state. Thus, we display the coefficient of variation for the main variables of interest in Table 3. In the benchmark calibration (see column (1)), durable investment is most dispersed, followed by financial assets and cash-on-hand. The durable stock and non-durable consumption have the smallest variation since the agents try to smooth out the income fluctuations. The durable stock is slightly more dispersed than non-durable consumption because the interest spread makes the propensity to accumulate durables much higher than the propensity for non-durable consumption if financial assets $a = 0$ (see Figure 2 above). If we reduce the interest spread as in column (2) or (3), the volatilities of d and c are much more alike. The variation of c increases slightly in column (2) since more debt and less cash-on-hand make the collateral constraint binding more often. This effect is outweighed by the smaller non-linearity of the policy function for durable expenditure. Thus, financial market development that results in a reduction of the spread reduces variation in consumption mostly for durables (see column (3)). But if also the borrowing rate falls, higher consumer debt of impatient consumers *increases* the importance of the collateral constraint with an opposite effect on dispersion. Thus, it is important for the results on dispersion to allow the collateral constraint to become more or less binding in equilibrium. This is not always done in general equilibrium models where a fixed share of agents is assumed to be at the constraint for tractability (see Campbell and Hercowitz, 2005, or Iacoviello, 2005).

Higher average interest rates reduce the variation of consumption and the durable stock because a higher financial asset position implies that the collateral constraint binds less often (see column (4)). Interestingly also the dispersion of durable expenditure falls. Consumers make smaller durable adjustments as they do not need to postpone investment due to binding constraints. Indeed, if we exogenously tighten the collateral constraint in column (7), where consumers can no longer collateralize durables, the dispersion of the durable stock and expenditure increases. Interestingly, the variation of non-durable consumption remains nearly unchanged as agents accumulate more cash-on-hand to self-insure.

Not surprisingly, a larger size of the income shock increases the variation of all variables (see column (5)). The variation of non-durable consumption increases by 20% compared with the 25% increase of income variation. If shocks are more persistent, this is qualitatively similar but for durable investment (see column (6)). The dispersion of the durable stock occurs at a higher level, however, so that it is less

[12] Financial market development, in terms of a lower spread, and more income risk are both consistent with the empirical upward trend in the US of durable expenditure compared with non-durable consumption. The ratio c/i falls in columns (8)-(10) compared with the benchmark. Instead for a laxer collateral constraint the opposite is the case for our parameter values (since c/i decreases slightly for a *tighter* constraint in column (11)).

costly in marginal utility terms.

The main message of our results has been so far that income risk alone cannot explain the increase in household debt in our model. Financial market development in terms of lower interest spreads (and lower borrowing rates) or exogenous relaxation of the collateral constraint can explain the larger household debt. However, we cannot fully dismiss the hypothesis that more idiosyncratic income risk increased consumer debt for at least two reasons:

(i) In our small-open economy model interest rates are exogenous. A general equilibrium effect in a closed economy as in Aiyagari (1994) would imply that interest rates have to fall until the asset market clears. This would reduce the strength of the buffer-stock saving motive. We are currently investigating how sensitive our results are to the small-open economy assumption.

(ii) The access to borrowing and idiosyncratic risk maybe endogenously related. For example in Krueger and Perri (2005), limited enforcement of credit contracts implies that financial market development interacts with income volatility. If more volatile income makes the exclusion from credit markets in case of default more costly, this might foster financial market development. In this case, more volatile income will induce a higher buffer-stock but with respect to a laxer borrowing limit. Whether this implies more or less debt depends on which effect dominates quantitatively and is *a priori* unclear.

We now extend our model in this direction and allow for limited commitment in credit contracts and costly default. This is particularly interesting because unsecured debt has increased substantially over the 1990s in the US. Thus, a joint analysis of secured and unsecured debt and their determinants is warranted. We now introduce unsecured debt, default and risk-sharing intermediaries which are perfectly competitive.

3 The model with default

As before agents have access to risk-free secured debt $a^s \leq 0$, which is backed by collateral and bears interest rate r^b, and risk-free positive assets $a^u \geq 0$ which bear interest r^a ($r^b > r^a$). Creditors of secured debt have priority for the payment of their debt principal and interest. The new feature of the model with default is that agents can also borrow unsecured debt $a^u < 0$.[13] This debt does not need to be backed by collateral so that agents possibly default on that debt depending on their income draw. Unsecured debt is priced actuarially fairly by a risk-neutral intermediary which perfectly diversifies the idiosyncratic risk applying the law of large numbers. We will derive the price of unsecured debt below.

[13] Modeling risk-free savings as $a^u \geq 0$ and unsecured debt as $a^u < 0$ has the advantage that this structure abstracts from strategic default. If we modelled risk-free savings as $a^s \geq 0$, agents would have an incentive to accumulate risk-free assets before they default and we would need a more complicated specification of exemption levels and bankruptcy procedures to prevent this from happening (as is done in reality).

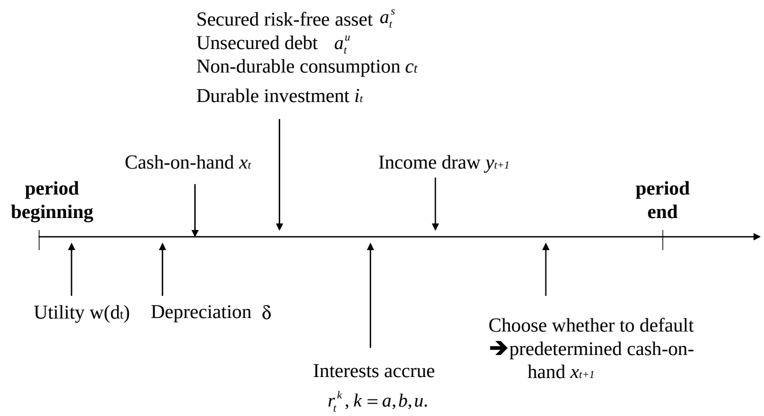

Figure 5: Timing for the model with default

Timing. Figure 5 illustrates the time line. We assume that agents first derive utility from the durable good before the durable depreciates. Agents then make choices about their financial asset portfolio $(\frac{a_t^s}{1+r_t^b}, \frac{a_t^u}{1+r_t^j})$,[14] where $r_t^j = r_t^a$ if $a_t^u > 0$ and $r_t^j = r_t^u$ otherwise. The interest rate of unsecured debt, r_t^u, is determined below. The agents also choose the durable stock target d_{t+1}^* and non-durable consumption c_t. After the consumption and portfolio decisions, the interest for the financial assets accrues. Then the uncertain income is drawn. After this draw, agents decide whether to declare bankruptcy. The timing implies that the collateral constraint for secured debt is

$$a_t^s \geq -(1-\delta)d_{t+1}^* - \underline{y} \ . \tag{2}$$

Bankruptcy. We model consumer bankruptcy assuming that consumers can keep no resources but durables up to an *exemption level* $d^\dagger$. This level is motivated by real-world bankruptcy legislation for US households (chapter 7 and 11 of the federal bankruptcy act). The US law defines the value of the property that is protected from creditors that claim *unsecured* debt. The property collateralizing *secured* debt cannot be used to service unsecured debt. Only the remaining property (net of secured-debt claims) which is above the exemption level can then be used to service unsecured debt (see, for example, Athreya, 2005, or Grant and Koeniger, 2005, for more information on the US legislation).

[14] Writing the asset choices in this discounted way (see also Athreya, 2005), allows us to express cash-on-hand in the next period without any interest factor. This is very helpful for technical reasons since the contraction mapping needs to be applied only to the value function and not also to the unknown interest rate factor of unsecured debt (which depends on the agent's asset position).

The most important value items that are defined as exempt in the US legislation are durables such as housing, cars and other household durable goods.

At the time of bankruptcy the agent needs to reveal information on d^*_{t+1}, a^*_{t+1} and y_{t+1} to the bankruptcy judge, where

$$d^*_{t+1} = (1-\delta)d_t + i_t \ ,$$

$$a^*_{t+1} = a^s_t + a^u_t \ ,$$

and

$$a^u_t < 0$$

if agents default. In particular, the judge will know the composition of financial assets a^s_t and a^u_t, the minimum of the support of the income distribution $\underline{y}$ and the exemption level $d^\dagger$. The judge then secures the durable stock for all secured debt $a^s_t/\,(1-\delta)$. He computes the remaining durable stock available for honoring the unsecured debt as

$$d^s_{t+1} = \max\left\{d^*_{t+1} - a^s_t/\,(1-\delta)\,, 0\right\} \ ,$$

where $a^s_t \leq 0$.[15] Thus, all secured assets a^s_t are carried forward into the next period.

The bankruptcy judge then continues to satisfy the outstanding unsecured debt where the remaining resources are d^s_{t+1} and

$$\widetilde{y}_{t+1} \equiv y_{t+1} - \underline{y} + \max\{\underline{y} + \min\left\{a^s_t + (1-\delta)d^*_{t+1}, 0\right\}, 0\} \ .$$

The last expression consists of the labor income which remains after the collateralizable income has been (partly) used to repay secured debt. The bankruptcy judge then uses d^s_{t+1} and $\widetilde{y}_{t+1}$ to honor unsecured debt. The remaining durable stock after doing that is

$$d^+_{t+1} = \max\left\{\min\{d^\dagger, d^s_{t+1}\}, d^s_{t+1} + a^u_t\right\} \ ,$$

where $a^u_t < 0$. If $d^s_{t+1} \leq d^\dagger$, nothing of the durable stock can be used to honor unsecured debt. If $d^s_{t+1} > d^\dagger$, the durable stock above the exemption level is used to repay unsecured debt.

The cash-on-hand in the next period is then given by

$$x^+_{t+1} = (1-\delta)d^+_{t+1} + \max\{\widetilde{y}_{t+1} + \min\{a^u_t + d^s_{t+1} - d^\dagger, 0\}, 0\} + a^s_t \ ,$$

where the claims of unsecured debt that cannot be honored with durables have to be repaid with the remaining resources $\widetilde{y}_{t+1}$, if possible. After the judge has satisfied creditors as much as possible according to these rules, he sets $a^u_t = 0$.

[15] Note that secured debt collateralizes the durable target net of the depreciation rate since payments are due after the depreciation realizes (see the discussion of the timing above and the collateral constraint (2)). Moreover, the minimum of the support of the income distribution $\underline{y}$ is only used to service secured debt if the durable stock does not suffice. The maximum operator is necessary because also $\underline{y}$ can be collateralized.

Evolution of state variables. For completeness we summarize the evolution of the state variables which depend on the bankruptcy decision. The timing implies that the targeted durable stock $d^*_{t+1} = (1-\delta)d_t + i_t$ is predetermined after the investment decision i_t. Of course, the realized durable stock in the next period d_{t+1} depends on the default decision and thus is uncertain:

$$d_{t+1} = \begin{cases} d^*_{t+1} = (1-\delta)d_t + i_t \text{ if no default} \\ d^+_{t+1} = \max\left\{\min\{d^\dagger, d^s_{t+1}\}, d^s_{t+1} + a^u_t\right\} \text{ if default } (a^u_t < 0) \end{cases} . \quad (3)$$

Financial assets evolve according to

$$a_{t+1} = \begin{cases} a^*_{t+1} \equiv a^s_t + a^u_t \text{ if no default} \\ a^s_t \text{ if default} \end{cases} . \quad (4)$$

In words, agents default on unsecured debt (principal and accrued interest). The accumulation of cash-on-hand is then defined as

$$x_{t+1} = \begin{cases} x^*_{t+1} \equiv a^*_{t+1} + y_{t+1} + (1-\delta)d^*_{t+1} \text{ if no default} \\ x^+_{t+1} \equiv (1-\delta)d^+_{t+1} + \max\{\widetilde{y}_{t+1} + \min\{a^u_t + d^s_{t+1} - d^\dagger, 0\}, 0\} + a^s_t \text{ if default} \end{cases} . \quad (5)$$

The pricing of unsecured debt. We assume that the lender knows $\underline{y}$ and observes the portfolio $(a^u_t, a^s_t, d^*_{t+1})$ before the income draw y_{t+1} (Note that d^*_{t+1} is predetermined in period t). The lender is not able to observe the income draw. Denoting ν_t as the vector of the portfolio $(a^u_t, a^s_t, d^*_{t+1})$, we define the interest factor for unsecured debt as $R^u(\nu_t) \equiv 1 + r^u$. Thus, the financial intermediary computes a price-schedule conditional on the portfolio position. This is because the portfolio choice changes the probability of default. Since we assume perfectly competitive financial intermediaries (as in Chaterjee et al., 2005), there is no cross-subsidization and consumers with different portfolios receive a different interest quote for unsecured debt.

Since we introduce a random utility cost of default, as discussed below, the agent will possibly repay in each income state. The lender takes this into account when pricing the loan. We define the probability of default for a given portfolio ν_t, conditional on the income state s with $y_{t+1}(s)$, as $\pi(\nu_t|s)$. We denote the unconditional probability for each income state as $\theta(y_{t+1}(s))$. Then the zero-profit condition implies that for an additional unit of unsecured debt, $a^u_t < 0$,

$$\begin{aligned} &(1-\pi(\nu_t))\, R^u(\nu_t) \\ &+ \frac{\Sigma_s\ \theta(y_{t+1}(s))\pi(\nu_t|s) \min\left\{-a^u_t, \max\{d^s_{t+1} - d^\dagger, 0\} + \widetilde{y}_{t+1}(s)\right\}}{\left|\frac{a^u_t}{1+r^u}\right|} \\ &= 1 + r^s \ . \end{aligned}$$

Hence,

$$R^u(\nu_t) = \frac{1 + r^s}{1 - \pi(\nu_t) + \frac{\Sigma_s\ \theta(y_{t+1}(s))\pi(\nu_t|s) \min\left\{-a^u_t, \max\{d^s_{t+1} - d^\dagger, 0\} + \widetilde{y}_{t+1}(s)\right\}}{|a^u_t|}} . \quad (6)$$

The minimum operator does compare the amount of outstanding unsecured debt (transformed into a positive number) with the resources that unsecured creditors receive from the durable sales $d^s_{t+1} - d^\dagger$ and the other available resources $\widetilde{y}_{t+1}(s)$ which depend on the realized value of labor income. Of course, the creditors receive at most all their outstanding debt and accrued interest a^u_t.

The program. If the consumer has not declared bankruptcy in the last period, the budget constraint[16] is

$$\frac{a^s_t}{1+r^j_t} + \frac{a^u_t}{1+r^u_t} + c_t + d^*_{t+1} \leq x^*_t = a^*_t + y_t + (1-\delta)d^*_t \ .$$

We can rewrite the value function, if the agent has not defaulted in the last period, as

$$V(x^*_t, d^*_t, y_t) = \max_{a^s_t, a^u_t, d^*_{t+1}} \left[\begin{array}{c} u(\underbrace{x^*_t - \frac{a^s_t}{1+r^j_t} - \frac{a^u_t}{1+r^u_t} - d^*_{t+1}}_{c_t})) + \phi w(d^*_t) \\ +\beta E_{\nu,y} \max[V(x^*_{t+1}, d^*_{t+1}, y_{t+1}), V(x^+_{t+1}, d^+_{t+1}, y_{t+1})] \end{array} \right] .$$

If the agent has defaulted in the last period, the budget constraint is

$$\frac{a^s_t}{1+r^j_t} + \frac{a^u_t}{1+r^u_t} + c_t + d^*_{t+1} \leq x^+_t$$

and

$$V(x^+_t, d^+_t, y_t) = \max_{a^s_t, a^u_t, d^*_{t+1}} \left[\begin{array}{c} u(\underbrace{x^+_t - \frac{a^s_t}{1+r^j_t} - \frac{a^u_t}{1+r^u_t} - d^*_{t+1}}_{c_t})) + \phi w(d^+_t) \\ +\beta E_{\nu,y} \max[V(x^*_{t+1}, d^*_{t+1}, y_{t+1}), V(x^+_{t+1}, d^+_{t+1}, y_{t+1})] \end{array} \right] .$$

We can further simplify the problem by noting that d^n_t, $n \in \{*, +\}$, is predetermined in period t and that the additive separable term $\phi w(d^n_t)$ does not affect the optimal choices of the consumer. We define

$$\widetilde{V}(x^*_t) \equiv V(x^*_t, d^*_t) - \phi w(d^*_t)$$

and

$$\widetilde{V}\left(x^+_t\right) \equiv V\left(x^+_t, d^+_t\right) - \phi w\left(d^+\right) \ .$$

Note that the functions are the same and we only need to keep track of default for the evolution of x and d, which reduces the computational burden. Thus, the value function of the transformed maximization problem is

$$\widetilde{V}(x^n_t, y_t) = \max_{a^s_t, a^u_t, d^*_{t+1}} \left[\begin{array}{c} u(\underbrace{x^n_t - \frac{a^s_t}{1+r^j_t} - \frac{a^u_t}{1+r^u_t} - d^*_{t+1}}_{c_t}) \\ +\beta E_{\nu,y} \max[\widetilde{V}(x^*_{t+1}, y_{t+1}) + \phi w\left(d^*_{t+1}\right), \widetilde{V}(x^+_{t+1}, y_{t+1}) + \phi w\left(d^+_{t+1}\right)] \end{array} \right] ,$$

[16] Writing the budget constraint this way has the advantage that x is not a direct function of R^u. Since R^u is unknown until the optimal portfolio is determined, we would otherwise need convergence over both V and R^u in our algorithm. In the current formulation instead standard contraction mapping over V can be employed.

where $n \in \{*, +\}$. Note that the bankruptcy decision has a utility cost since $w\left(d_{t+1}^{+}\right) \leq w\left(d_{t+1}^{*}\right)$.[17] Since the durable stock cannot be adjusted after the default decision and choices in the next period are made after utility is derived from the durable, consumers who have a taste for durable consumption will find default costly. The cost of default varies according to how long it takes until the durable can be readjusted. In reality, this depends on the length of court procedures and legally set time spans after which consumers can deviate from their exemption levels. In our model, the cost depends on how we calibrate the length of one discrete period.

In most of the literature the cost of bankruptcy is either modelled as an exogenous utility cost (see, for example, Athreya, 2005) or by assuming that defaulting consumers are excluded from the credit market for a certain number of periods (see, for example, Chaterjee et al., 2005). Since the empirical evidence for such exclusion is not overwhelming (see Musto, 1999, or Staten, 1993), and consumers that just have defaulted should be attractive for lenders because they cannot default again for some years, we find our alternative modeling of the cost of bankruptcy on the "durable-side" worth studying. As we will discuss now, we also need a random bankruptcy cost in terms of utility for computational reasons.

It remains to address one technical issue which complicates the numerical solution. As we mentioned above when discussing our numerical algorithm in Section 3, the accuracy and speed rely on solving maximization problems *continuously* over portfolio choices. Given that income only varies between 2 states, the probability of default is no longer continuous. This translates into discontinuities of the interest factor $R^u(\nu_t)$ and thus the right-hand side of the Bellman equation. Hence, we introduce some additional "randomness" to make the probability of default $\pi(\nu_t)$ continuous. We assume a random cost ψ which can be interpreted as uncertain lawyer cost, psychological pain or stigma (see Athreya, 2005, for a deterministic utility cost). Thus, we rewrite the program as

$$\widehat{V}\left(x_t^n, y_t\right) = \max_{a_t^s, a_t^u, d_{t+1}^*} \left[\begin{array}{c} \underbrace{u(x_t^n - \frac{a_t^s}{1+r_t^j} - \frac{a_t^u}{1+r_t^u} - d_{t+1}^*)}_{c_t} \\ +\beta E_{\nu,y} \max[\widehat{V}(x_{t+1}^*, y_{t+1}) + \phi w\left(d_{t+1}^*\right), \widehat{V}(x_{t+1}^+, y_{t+1}) - \psi + \phi w\left(d_{t+1}^+\right)] \end{array} \right].$$

Together with the equations for the evolution of the state variables (3), (4), (5) and the constraints $d_t \geq 0$ and $a_t^u \leq 0$, this completes the set-up of the program.

Note that for a given portfolio ν_t and for each realization of y_{t+1}, we compute the critical $\psi^*(\nu_t, y_{t+1})$ at which the consumer is indifferent between declaring bankruptcy or not. Assuming that the additive separable random cost is exponentially distributed allows us to compute the expected value of the cost (conditional on defaulting) in closed form. Importantly, the default probability is higher for low values

[17] In the literature often two value functions need to be computed: one for case of default and another for the case of no default. This is because dynamic costs of default, such as the exclusion from the credit market for n periods, imply different present discounted values. In our model default "only" matters for the level of x and d once and for all so that we do not need to distinguish two value functions.

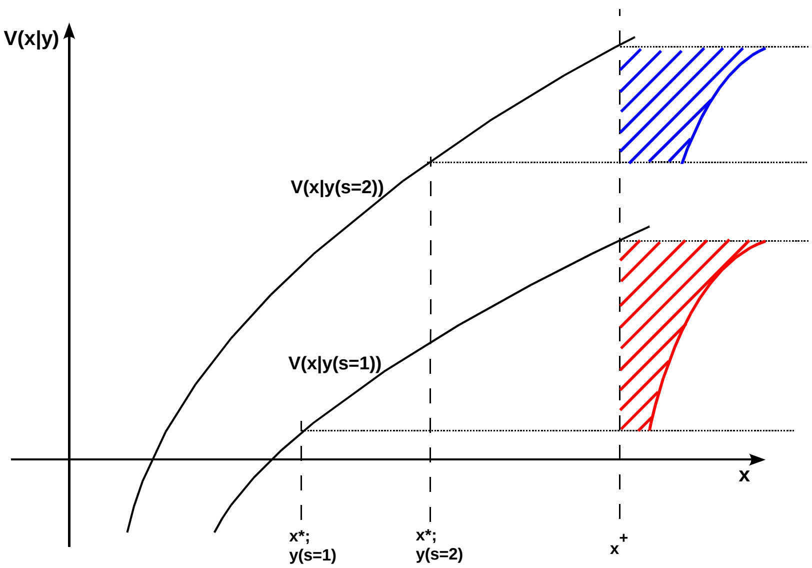

Figure 6: The default probability for two different income states (without durables, $\phi = 0$).

of income as illustrated in Figure 6, where we set $\phi = 0$. The probability mass below the function $\widehat{V}(x^+, y(s = 1))$ is larger than the mass below the function $\widehat{V}(x^+, y(s = 2))$ (both are marked with dashed lines in the figure). The critical value $\psi^*(\nu_t, y_{t+1}(s))$, for income state s, at which the consumer would refrain from bankruptcy is $\psi^* = \widehat{V}(x^+, y(s)) - \widehat{V}(x^*, y(s))$.

Equilibrium definition. A stationary equilibrium is given by the policy functions for non-durable consumption $c(x, y)$, durable investment $i(x, y)$, the accumulation equations $a'(x, y)$ and $d'(x, y)$, and the evolution of the state variable $x'(x, y)$ so that for given prices $\{r^a, r^b\}$ for secured debt and deposits:

(i) the value function $\widehat{V}\left(x^i, y\right)$, $i \in \{*, +\}$, attains its maximal value.

(ii) the price for unsecured debt $R^u(\nu)$ satisfies the arbitrage equation (6).

(iii) the accumulation equations (3), (4) and (5) for d, a and x are satisfied.

(iv) the collateral constraint for secured assets is not violated, and $d' \geq 0$, $a^u \leq 0$.

(v) the distribution measure $\mu(X, Y)$ over the state space $X \times Y$ of agents is stationary, so that for a transition matrix $\Gamma(y'|y)$

$$\mu(X, Y) = \int_{X \times Y} I_{\{x' = x'(x,y)\}} \Gamma(y'|y) d\mu \ .$$

In the Appendix we derive the optimality conditions for the consumer's portfolio choice for the problem without random cost ψ. The main change, compared with Section 3, is that portfolio choices affect the interest factor R^u. Thus, accumulation of durables can be more attractive if this makes default less likely and unsecured debt becomes 'cheaper'. Moreover, the effect of today's portfolio choice on tomorrow's cash-on-hand depends on the bankruptcy decision. Thus, the implicit rate of return for each asset changes as soon as the probability of default is larger than zero.

3.1 Calibration and numerical results

Numerical algorithm. [To be completed]

Calibration. [To be completed]

4 Conclusion

We have studied and solved heterogenous agent models where income risk cannot be insured and consumers derive utility from non-durable and durable consumption. We first study a version of the model in which consumers only have access to secured debt that is collateralized by durables. We apply this model to investigate the determinants of the increase in household debt in the US since the 1980s. We find that income risk alone cannot explain the increase in household debt in our model. Financial market development in terms of lower interest spreads (and lower borrowing rates) or exogenous relaxation of the collateral constraint can explain the larger household debt. Finally, we extend our model to unsecured debt and consumer bankruptcy. [To be completed]

Appendix

I. Proof of Remark 1

The proof is based on results of Carroll and Kimball (1996). In order to simplify notation we drop income y_t as an argument of the functions.

Claim (i): If the constraints are not binding, $c(x)$, $d(x)$ are concave and $a(x)$ is convex and $\partial c(x)/\partial x > 0$, $\partial d(x)/\partial x > 0$, $\partial a(x)/\partial x \geq 0$.

Proof: We want to show that if $u(.)$ and $w(.)$ are HARA utility functions and $u'(.) > 0$, $u''(.) < 0$, $u'''(.) \geq 0$, and $w'(.) > 0$, $w''(.) < 0$, $w'''(.) \geq 0$, then $c(x)$, $d(x)$ are concave and $a(x)$ is convex and $\partial c(x)/\partial x > 0$, $\partial d(x)/\partial x > 0$, $\partial a(x)/\partial x \geq 0$.

Our problem is

$$\widetilde{V}_t(x_t) = \max_{a_{t+1},d_{t+1}} \left[u(\underbrace{x_t - a_{t+1} - d_{t+1}}_{c_t}) + \beta\phi w(d_{t+1}) + \beta E_t \widetilde{V}_{t+1}(x_{t+1}) \right]$$

where $x_t \equiv (1 + r^j)a_t + y_t + (1 - \delta)d_t$ so that the budget constraint

$$c_t = x_t - a_{t+1} - d_{t+1} \ .$$

To start we also assume a finite horizon so that we have the terminal condition

$$c_T = x_T.$$

We then proceed analogously as in Carroll and Kimball and prove Lemmas 1-3. For this we define as $\xi_t((1+r^j)a_{t+1}(x_t) + (1-\delta)d_{t+1}(x_t)) \equiv \beta E_t \widetilde{V}_{t+1}(x_{t+1})$, where

$$x_{t+1} \equiv (1+r^j)a_{t+1} + y_{t+1} + (1-\delta)d_{t+1}.$$

Note that $\xi_t(.)$ is written as a function of choice variables.

The first lemma shows that the property of prudence is conserved when aggregating across states of nature.

Lemma 1: If $\widetilde{V}'''_{t+1}\widetilde{V}'_{t+1} / \left[\widetilde{V}''_{t+1}\right]^2 \geq k$, *then* $\xi'''_t \xi'_t / \left[\xi''_t\right]^2 \geq k$.

Proof: see Carroll and Kimball, p. 985.

The second lemma shows that the property of prudence is conserved when aggregating intertemporally.

Lemma 2: If $\xi'''_t \xi'_t / \left[\xi''_t\right]^2 \geq k$ and $u'''u' / \left[u''\right]^2 \geq k$, $w'''w' / \left[w''\right]^2 = k$, then $\widetilde{V}'''_t \widetilde{V}'_t / \left[\widetilde{V}''_t\right]^2 \geq k$.

Proof: Following Carroll and Kimball, p. 985/986, we denote the marginal utility of non-durable consumption at the optimal consumption level with $z_t = u'(c^*_t(x_t)$. Neglecting the collateral constraint and interest spread, we know that in our problem the following equations hold in the optimum:

$$\begin{aligned}
z_t &= u'(c^*_t(x_t)) \ , \\
u'(c^*_t(x_t)) &= \widetilde{V}'_t(x_t) \ , \\
u'(c^*_t(x_t)) &= \beta(1+r^j)E_t\widetilde{V}'_{t+1}(x_{t+1}) = (1+r^j)\xi'_t \ , \\
u'(c^*_t(x_t)) &= \beta\phi w'(d_{t+1}) + (1-\delta)\xi'_t \ ,
\end{aligned}$$

where $\xi_t((1+r^j)a_{t+1}(x_t) + (1-\delta)d_{t+1}(x_t))$. We then define the functions $f_t(z_t)$, $g_t(z_t)$, $h_t(z_t)$, $l_t(z_t)$ as

$$\begin{aligned}
f_t(z_t) &= u'^{-1}(z_t) = c_t \ , \\
h_t(z_t) &= \widetilde{V}'^{-1}_t(z_t) = x_t \ , \\
l_t(z_t) &= w'^{-1}\left(\frac{z_t - (1-\delta)\xi'_t(.)}{\beta\phi}\right) = d_{t+1} \ , \\
g_t(z_t) &= \xi'^{-1}_t\left(\frac{z_t}{1+r^j}\right) - (1-\delta)l_t(z_t) = (1+r^j)a_{t+1} \ .
\end{aligned}$$

Noting from the last equation that

$$(1+r^j)a_{t+1} + (1-\delta)d_{t+1} = \xi'^{-1}_t\left(\frac{z_t}{1+r^j}\right) \ ,$$

we use this expression in as the argument of $\xi_t'(.)$ in the second equation which then simplifies to

$$l_t(z_t) = w'^{-1}\left(\frac{r^j+\delta}{\beta\phi\left(1+r^j\right)}z_t\right) = d_{t+1}$$

Dropping time indexes for functions f, g, l, h, we have

$$\begin{aligned} f'(z) &= \frac{1}{u''(c(z))}\ , \\ f'' &= -\frac{u'''(c)}{\left[u''(c)\right]^2}\underbrace{f'}_{\partial c/\partial z} = -\frac{u'''}{\left[u''\right]^3}\ , \end{aligned}$$

so that

$$-\frac{zf''}{f'} = \frac{u'''u'}{\left[u''\right]^2} \geq k\ .$$

Similarly,

$$-\frac{zh''}{h'} = \frac{\widetilde{V}_t'''\widetilde{V}_t'}{\left[\widetilde{V}_t''\right]^2}\ .$$

Furthermore,

$$\begin{aligned} l' &= \frac{r^j+\delta}{\beta\phi\left(1+r^j\right)w''}\ , \\ l'' &= -\frac{\left(r^j+\delta\right)w'''}{\beta\phi\left(1+r^j\right)\left[w''\right]^2}l'\ , \end{aligned}$$

so that

$$-\frac{zl''}{l'} = \frac{w'''w'}{\left[w''\right]^2} \geq k\ ,$$

where we use that

$$\frac{r^j+\delta}{\beta\phi\left(1+r^j\right)}z_t = w'(d_{t+1})\ .$$

Finally,

$$\begin{aligned} g' &= \frac{1}{(1+r^j)\xi''\left(\xi_t'^{-1}\left(\frac{z_t}{1+r^j}\right)\right)} - (1-\delta)\frac{r^j+\delta}{\beta\phi\left(1+r^j\right)w''}\ , \\ g'' &= -\frac{\xi'''}{(1+r^j)^2\left[\xi''\right]^3} + (1-\delta)\frac{\left(r^j+\delta\right)w'''}{\beta\phi\left(1+r^j\right)\left[w''\right]^2}l'\ . \end{aligned}$$

Thus,

$$-\frac{zg''}{g'} = \frac{\frac{\xi'''\xi'}{(1+r^j)\left[\xi''\right]^3} - (1-\delta)\frac{w'''w'}{\left[w''\right]^2}l'}{\frac{1}{(1+r^j)\xi''} - (1-\delta)l'}\ .$$

For $\delta = 1$, this simplifies to

$$-\frac{zg''}{g'} = \frac{\xi'''\xi'}{\left[\xi''\right]^2} \geq k\ ,$$

For $0 < \delta < 1$,

$$-\frac{zg''}{g'} = \frac{g'}{g'-(1-\delta)l'}\frac{\xi'''\xi'}{\left[\xi''\right]^2} - \frac{(1-\delta)l'}{g'-(1-\delta)l'}\frac{w'''w'}{\left[w''\right]^2}\ .$$

If we assume HARA utility so that $w'''w'/\left[w''\right]^2 = k$, then $\xi_t'''\xi_t'/\left[\xi_t''\right]^2 \geq k$ implies that

$$-\frac{zg''}{g'} \geq \frac{g'}{g' - (1-\delta)l'}k - \frac{(1-\delta)l'}{g' - (1-\delta)l'}k = k \; .$$

Now note that since

$$c_t = x_t - a_{t+1} - d_{t+1}$$

and

$$a_{t+1} = \frac{g}{(1+r^j)} - (1-\delta)l \; ,$$

we have

$$\begin{aligned} h &= f + \frac{g}{(1+r^j)} - (1-\delta)l + l \\ &= f + \frac{g}{(1+r^j)} + \delta l \; . \end{aligned}$$

That is, h is an additive function of f, g and l, so that

$$h' = f' + \frac{g'}{(1+r^j)} + \delta l'$$

and

$$h'' = f'' + \frac{g''}{(1+r^j)} + \delta l'' \; .$$

This implies that

$$\begin{aligned} -\frac{zh''}{h'} &= -z\frac{f'' + \frac{g''}{(1+r^j)} + \delta l''}{f' + \frac{g'}{(1+r^j)} + \delta l'} \\ &= \underbrace{\frac{f'}{f' + \frac{g'}{(1+r^j)} + \delta l'}}_{>0}\underbrace{\left(-\frac{zf''}{f'}\right)}_{\geq k} + \underbrace{\frac{\frac{g'}{(1+r^j)}}{f' + \frac{g'}{(1+r^j)} + \delta l'}}_{>0}\underbrace{\left(-\frac{zg''}{g'}\right)}_{\geq k} + \underbrace{\frac{\delta l'}{f' + \frac{g'}{(1+r^j)} + \delta l'}}_{>0}\underbrace{\left(-\frac{zl''}{l'}\right)}_{\geq k} \\ &\geq k \; , \end{aligned}$$

since this is a weighted average of expressions that are larger or equal than k.

As in Carroll and Kimball we move on to show Lemma 3, where we exploit again that HARA utility implies $w'''w'/\left[w''\right]^2 = k$ and $u'''u'/\left[u''\right]^2 = k$ with equality.

Lemma 3: If $\widetilde{V}_t'''\widetilde{V}_t'/\left[\widetilde{V}_t''\right]^2 \geq k$, $w'''w'/\left[w''\right]^2 = k$ and $u'''u'/\left[u''\right]^2 = k$, then the optimal consumption policy rules $c(x)$ and $d(x)$ are concave and liquid assets $a(x)$ are convex.

Proof: Note that

$$c_t(x) = f_t(h_t^{-1}(x)) \; .$$

Thus,

$$\frac{\partial c}{\partial x} = \frac{f'(h^{-1})}{h'(h^{-1})} = \frac{\widetilde{V}''}{u''} > 0$$

if $u'' < 0$, $\widetilde{V}'' < 0$ and

$$\begin{aligned}\frac{\partial^2 c}{\partial x^2} &= \frac{\left(f''(h^{-1})/h'(h^{-1})\right)\left(h'(h^{-1})\right) - \left(f'(h^{-1})\right)\left(h''(h^{-1})/h'(h^{-1})\right)}{\left[h'(h^{-1})\right]^2} \\ &= \frac{f'(h^{-1})}{\left[h'(h^{-1})\right]^2}\left[\frac{f''(h^{-1})}{f'(h^{-1})} - \frac{h''(h^{-1})}{h'(h^{-1})}\right] .\end{aligned}$$

Applying Lemma 2 we find

$$\frac{\partial^2 c}{\partial x^2} = \frac{f'(h^{-1})}{\left[h'(h^{-1})\right]^2}\frac{1}{z}\left[\underbrace{-\frac{zh''(h^{-1})}{h'(h^{-1})}}_{\geq k} - \underbrace{\frac{-zf''(h^{-1})}{f'(h^{-1})}}_{=k}\right] .$$

The sign of this derivative is smaller or equal than zero if $sgn(f'(h^{-1})) < 0$. Recalling that $f'(h^{-1}) = f'(z) = 1/u'' < 0$, this is the case for a strictly concave utility function. Analogous manipulations for $d_t(x) = l_t(h_t^{-1}(x))$ prove $\partial d(x)/\partial x > 0$ and $\partial^2 d(x)/\left(\partial x\right)^2 \leq 0$.

Since $a_{t+1}(x) = x_t - c_t(x) - d_{t+1}(x)$,

$$\frac{\partial a}{\partial x} = 1 - \frac{\partial c(x)}{\partial x} - \frac{\partial d(x)}{\partial x}$$

and

$$\frac{\partial^2 a}{\partial x^2} = -\frac{\partial^2 c(x)}{\partial x^2} - \frac{\partial^2 d(x)}{\partial x^2} \geq 0.$$

Thus, financial wealth increases or decreases with x, depending on whether the marginal propensity to consume $\partial c(x)/\partial x + \partial d(x)/\partial x \gtreqless 1$. The second derivative is certainly positive so that $a(x)$ is convex.

We now investigate the properties of the consumption propensities further. In particular, do we know whether $\partial c(x)/\partial x + \partial d(x)/\partial x > 1$?

Noting that

$$h' = f' + \frac{g'}{(1+r^j)} + \delta l'$$

we can write

$$\frac{\partial c}{\partial x} = \frac{f'(h^{-1})}{f'(h^{-1}) + \frac{g'(h^{-1})}{(1+r^j)} + \delta l'(h^{-1})}$$

and

$$\frac{\partial d}{\partial x} = \frac{l'(h^{-1})}{f'(h^{-1}) + \frac{g'(h^{-1})}{(1+r^j)} + \delta l'(h^{-1})} .$$

Thus,

$$\frac{\partial c}{\partial x} + \frac{\partial d}{\partial x} = \frac{f'(h^{-1}) + l'(h^{-1})}{f'(h^{-1}) + \frac{g'(h^{-1})}{(1+r^j)} + \delta l'(h^{-1})} < 1 ,$$

if $\delta = 1$ and $g'(h^{-1}) > 0$.

We now compute the derivative of $a(x) = g(h^{-1}(x))/\left(1 + r^j\right)$:

$$\begin{aligned}\frac{\partial a}{\partial x} &= \frac{1}{1+r^j} g'(h^{-1}(x))/h'(h^{-1}(x)) \\ &= \frac{\widetilde{V}_t''}{1+r^j}\left(\frac{1}{(1+r^j)\xi''} - (1-\delta)\frac{r^j+\delta}{\beta\phi\left(1+r^j\right)w''}\right) ,\end{aligned}$$

which is certainly positive if $\delta = 1$ since $\widetilde{V}_t'' < 0, \xi'' < 0$. For $\delta < 1$, we need to impose an additional condition on the curvature

$$\begin{aligned}\frac{1}{(1+r^j)\xi''} - (1-\delta)\frac{r^j+\delta}{\beta\phi\left(1+r^j\right)w''} &< 0 \text{ or} \\ \frac{\xi''}{\beta\phi w''} &< \frac{r^j+\delta}{1-\delta} .\end{aligned}$$

In general the sign of $\partial a/\partial x$ depends on the relative curvature of the value function expected tomorrow, ξ_t'', and instantaneous utility derived from the durable, w''. Intuitively, a larger δ makes durables less useful to transfer utility and thus increase the marginal propensity of financial assets to transfer resources.

The lemmas derived above imply Theorem 1 as in Carroll and Kimball (1996). Note that the second-order derivatives for the policy functions hold with strict equality if $k > 0$ and there is some labor income uncertainty.

Carroll and Kimball show results for a finite horizon. In a finite horizon, we have that in the last period $V_T = u(c) + \phi w(d)$ so that prudence of $u(.)$ and $w(.)$ trivially also apply to V_T. Then one iterates forward using Lemma 1 and 2. To extend these results to the infinite horizon one needs to apply the contraction property of V, for $T \to \infty$. Since cash on hand is finite, agents discount and V satisfies monotonicity, $\lim_{T\to\infty} V_t(x) = V(x)$ for all x (see Lucas and Stokey, 1989, ch. 3). Pointwise convergence implies that the properties of V_t are conserved as V_t converges towards V. ■

Claim (ii): If the collateral constraint binds, $\partial a(x)/\partial x$ falls and can become negative.

Proof: Intuitively, the value function will be more concave if the collateral constraint holds. The expression for the propensities derived above, then imply that $\partial c(x)/\partial x + \partial d(x)/\partial x$ increases if $\widetilde{V}''$ falls (i.e., increases in absolute value). This can imply $\partial a(x)/\partial x < 0$, which we now want to derive more formally. Adding the multiplier κ for the collateral constraint and γ for the constraint $d > 0$, the four equations used in Lemma 2 change to

$$
\begin{aligned}
z_t &= u'(c_t^*(x_t) \ , \\
u'(c_t^*(x_t) &= \widetilde{V}_t'(x_t) \ , \\
u'(c_t^*(x_t) &= (1+r^j)\,(\xi_t' + \kappa) \ , \\
u'(c_t^*(x_t) &= \beta\phi w'(d_{t+1}) + (1-\delta)\,(\xi_t' + \kappa) + \gamma \ ,
\end{aligned}
$$

so that

$$
\begin{aligned}
f_t(z_t) &= u'^{-1}(z_t) = c_t \ , \\
h_t(z_t) &= \widetilde{V}_t'^{-1}(z_t) = x_t \ , \\
l_t(z_t) &= w'^{-1}\left(\frac{z_t - (1-\delta)\,(\xi_t'(.) + \kappa) - \gamma}{\beta\phi}\right) = d_{t+1} \ , \\
g_t(z_t) &= \xi_t'^{-1}\left(\frac{z_t}{1+r^j} - \kappa\right) - (1-\delta) l_t(z_t) = (1+r^j) a_{t+1} \ .
\end{aligned}
$$

Observing that

$$
(1+r^j)a_{t+1} + d_{t+1} = \xi_t'^{-1}\left(\frac{z_t}{1+r^j} - \kappa\right) \ ,
$$

the third equation can be rewritten as

$$
l_t(z_t) = w'^{-1}\left(\frac{\frac{r^j+\delta}{1+r^j} z_t - \gamma}{\beta\phi}\right) = d_{t+1} \ .
$$

Thus, a binding collateral constraint does not directly affect d_{t+1} . Instead if the constraint $d = 0$ is expected to bind this lowers $w'(d_{t+1})$ and thus induces a larger d_{t+1}, ceteris paribus.

More interestingly, let us investigate how the marginal propensity of $a(x)$ changes if the collateral constraint is binding (we neglect the constraint $d \geq 0$ for simplicity). Recall that $a(x) = g(h^{-1}(x))/\left(1+r^j\right)$:

$$
\begin{aligned}
\frac{\partial a}{\partial x} &= \frac{1}{1+r^j} g'(h^{-1}(x))/h'(h^{-1}(x)) \\
&= \frac{\widetilde{V}_t''}{1+r^j}\left(\frac{\frac{1}{1+r^j} - \frac{\partial\kappa}{\partial z}}{\xi''} - (1-\delta)\frac{r^j+\delta}{\beta\phi\,(1+r^j)\,w''}\right) \ .
\end{aligned}
$$

Since a larger $z = u'(c^*(x))$ means a smaller c and x, $\partial\kappa/\partial z > 0$, i.e. the collateral constraint is more binding for smaller x and thus larger z. Then, this derivative shows that the propensity $\partial a/\partial x$ falls if the collateral constraint binds in the next period. In particular, the propensity need no longer be positive. The intuition is that the binding collateral constraint increases the amount of financial wealth for small values of x so that the slope is flatter. ■

Claim (iii): If the Euler equations for non-durable consumption are slack, $c(x)$, $d(x)$ can be locally strictly convex and $a(x)$ can be locally strictly concave.

Proof: We show that $c(x)$, $d(x)$ are locally strictly convex and $a(x)$ is locally strictly concave in the range where $a = 0$. In particular, $\partial c(x)/\partial x|_{a=0} > \partial c(x)/\partial x$ and $\partial d(x)/\partial x|_{a=0} > \partial d(x)/\partial x$ for given x, and $w'(d')/E_y\mu'$ falls.

If $a_{t+1}(x) = 0$,

$$c_t = x_t - d_{t+1}$$

and thus

$$h = f + l \ .$$

Hence,

$$-\frac{zh''}{h'} = \underbrace{\frac{f'}{f' + l'}}_{>0}\underbrace{\left(-\frac{zf''}{f'}\right)}_{\geq k} + \underbrace{\frac{l'}{f' + l'}}_{>0}\underbrace{\left(-\frac{zl''}{l'}\right)}_{\geq k}$$

so that the curvature of $w(.)$ becomes much more important for the curvature of the value function. Also

$$\frac{\partial c}{\partial x} + \frac{\partial d}{\partial x} = \frac{f'(h^{-1}) + l'(h^{-1})}{f'(h^{-1}) + l'(h^{-1})} = 1 \ ,$$

so that the propensities increase since $\partial a(x)/\partial x > 0$ to the left of the range where $a(x) = 0$. The local increase of the propensities implies local convexity of the consumption functions. Moreover, $\partial a(x)/\partial x > 0$ is locally concave.

More formally, if $\partial a(x)/\partial x = 0$, the collateral constraint is certainly not binding and

$$\begin{aligned} z_t &= u'(c_t^*(x_t) \ , \\ u'(c_t^*(x_t) &= \widetilde{V}_t'(x_t) \ \ , \\ (1 + r^a)\xi_t' &< u'(c_t^*(x_t) < (1 + r^b)\xi_t' \\ u'(c_t^*(x_t) &= \beta\phi w'(d_{t+1}) + (1 - \delta)\xi_t' \ , \end{aligned}$$

so that

$$f_t(z_t) = u'^{-1}(z_t) = c_t \ ,$$

$$h_t(z_t) = \widetilde{V}_t'^{-1}(z_t) = x_t \ ,$$

$$l_t(z_t) = w'^{-1}\left(\frac{z_t - (1 - \delta)\xi_t'(.))}{\beta\phi}\right) = d_{t+1} \ ,$$

$$g_t(z_t) = \xi_t'^{-1}(z_t + \lambda^b) - (1 - \delta)l_t(z_t) = (1 + r^b)a_{t+1}$$

or

$$g_t(z_t) = \xi_t'^{-1}(z_t - \lambda^a) - (1 - \delta)l_t(z_t) = (1 + r^a)a_{t+1}$$

with $\lambda^a > 0$ and $\lambda^b > 0$.

This implies

$$\begin{aligned}\frac{\partial a}{\partial x} &= \frac{1}{1+r^b} g'(h^{-1}(x))/h'(h^{-1}(x)) \\ &= \frac{\widetilde{V}_t''}{1+r^b}\left(\frac{1+\frac{\partial \lambda^b}{\partial z}}{(1+r^b)\xi''} - (1-\delta)\frac{r^b+\delta}{\beta\phi\,(1+r^b)\,w''}\right) .\end{aligned}$$

For the range $a_{t+1}(x) = 0$, $\partial\lambda^b/\partial z < 0$ so that $\partial a/\partial x = 0$ (Note that $\partial\lambda^b/\partial x > 0$.). Similarly, for the lending Euler-equation,

$$\frac{\partial a}{\partial x} = \frac{\widetilde{V}_t''}{1+r^b}\left(\frac{1-\frac{\partial \lambda^a}{\partial z}}{(1+r^b)\xi''} - (1-\delta)\frac{r^b+\delta}{\beta\phi\,(1+r^b)\,w''}\right) ,$$

with $\partial\lambda^a/\partial z > 0$ (Note that $\partial\lambda^a/\partial x < 0$.). ■

II: Derivations of optimality conditions for model with default

Note that we need to take into account that the interest rate r^u changes with the portfolio decision (a^u,a^s,i) in each period. Instead within the expectation operator $E_{\nu,y}$, the change of the probability of default $\pi(\nu)$ does not affect choices because of the envelope theorem. Recall that the portfolio choice (a^s,a^u,i) is done before the income shock realizes and the decision whether to declare bankruptcy. Moreover, the budget constraint implies that $\partial c/\partial a^s = -1/(1+r^j)$, $j = a, b$, $\partial c/\partial a^u = -1/(1+r^u)$ and $\partial c/\partial i = -1$; and the evolution of x implies that $\partial x'/\partial a^k = 1$ and $\partial x'/\partial i = 1-\delta$, $k = s, u$, if there is no default. If agents default, $\partial x'/\partial a^s = 1$ if $a^s \leq 0$, since secured debt cannot be defaulted upon. Moreover, $\partial x'/\partial a^s = 1$, if $a^s > 0$, since agents never default if they hold positive risk-free assets. Instead, $1 \geq \partial x'/\partial a^u \geq 0$ and $1 \geq \partial x'/\partial i \geq 0$. The latter two derivatives are zero if default is complete and all investment is seized. For simplicity, let us assume that there are only two income states where one implies default and the other does not. Then,

$$\begin{aligned}\frac{\partial}{\partial a^s}:\quad & -u'(c)\left(\frac{(R^u)^2 - \frac{\partial R^u}{\partial a^s}a^u\left(1+r^j\right)}{\left(1+r^j\right)(R^u)^2}\right) \\ & + \beta E_y\left((1-\pi(\nu))\frac{\partial \widetilde{V}(x')}{\partial x'}|_{[x'=(x^*)']} + \pi(\nu)\frac{\partial \widetilde{V}(x')}{\partial x'}|_{[x'=(x^+)']}\right) \\ & = 0,\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial a^u}:\quad & -u'(c)\left(\frac{R^u - \frac{\partial R^u}{\partial a^u}}{(R^u)^2}\right) \\ & + \beta E_y\left((1-\pi(\nu))\frac{\partial \widetilde{V}(x')}{\partial x'}|_{[x'=(x^*)']} + \pi(\nu)\frac{\partial x'}{\partial a^u}\frac{\partial \widetilde{V}(x')}{\partial x'}|_{[x'=(x^+)']}\right) \\ & = 0,\end{aligned}$$

$$
\begin{aligned}
\frac{\partial}{\partial i}: \quad & -u'(c)\left(\frac{(R^u)^2 - \frac{\partial R^u}{\partial i}a^u}{(R^u)^2}\right) \\
& +\beta E_y \left(\begin{array}{c} (1-\pi(\nu))\left(\phi w'(d')|_{[d'=(d^*)']} + (1-\delta)\frac{\partial \widetilde{V}(x')}{\partial x'}|_{[x'=(x^*)']}\right) \\ +\pi(\nu)\left(\phi w'(d')|_{[d'=(d^+)']} + \frac{\partial x'}{\partial i}\frac{\partial \widetilde{V}(x')}{\partial x'}|_{[x'=(x^+)']}\right) \end{array} \right) \\
& = 0 \ .
\end{aligned}
$$

Since

$$
\frac{\partial \widetilde{V}(x)}{\partial x}|_{[x=x^+]} = u'(c'((x^+)')) \text{ and } \frac{\partial \widetilde{V}(x)}{\partial x}|_{[x=x^*]} = u'(c'((x^*)')),
$$

we can define

$$
E_{\nu,y}u(c') \equiv (1-\pi(\nu))u'(c'((x^*)')) + \pi(\nu)u'(c'((x^+)'))
$$

and

$$
E_{\nu,y}w'(d') \equiv (1-\pi(\nu))w'(d'((x^*)')) + \pi(\nu)w'(d'((x^+)'))
$$

so that the optimality conditions are

$$
u'(c) = \beta E_{\nu,y}u'(c')\left(\frac{(1+r^j)(R^u)^2}{(R^u)^2 - \frac{\partial R^u}{\partial a^s}a^u(1+r^j)}\right),
$$

$$
u'(c) = \beta\left\{E_{\nu,y}u'(c') + \pi(\nu)E_y\left\{\left(\frac{\partial x'}{\partial a^u} - 1\right)u'(c'((x^+)'))\right\}\right\}\left(\frac{(R^u)^2}{R^u - \frac{\partial R^u}{\partial a^u}}\right)
$$

and

$$
u'(c) = \beta\left\{E_{\nu,y}\left\{u'(c')(1-\delta) + \phi w'(d')\right\} + \pi(\nu)E_y\left\{\left(\frac{\partial x'}{\partial i} - (1-\delta)\right)u'(c'((x^+)'))\right\}\right\}\left(\frac{(R^u)^2}{(R^u)^2 - \frac{\partial R^u}{\partial i}a^u}\right) \ .
$$

Note that if $\partial R^u/\partial k = 0$, $k \in \{a^s, a^u, i\}$ and $\partial x'/\partial a^u = 1$ and $\partial x'/\partial i = 1-\delta$, we would be back to the optimality conditions for the no-default case (note that $R^u = 1 + r^s$ in this case).

If the collateral constraint for secured debt is binding we have to add $\beta\kappa$ on the right-hand-side of the Euler equation for secured debt and replace $u'(.)$ with $(u'(.) + \kappa)$ on the right-hand-side of the Euler equation for investment.

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Parameters Values	*Sources / Targets*
lending rate: $r^a = 0.01$	Mehra and Prescott (1985)
borrowing rate: $r^b = 0.044$	Athreya (2004)
discount factor: $\beta = 0.96$	Aiyagari (1994)
risk aversion: $\sigma = 2$	for example, Aiyagari (1994)
transition probability $p = 0.4$	$\rightarrow$ coefficient of variation of 0.4, e.g. Aiyagari (1994)
size of the shock 0.4	$\rightarrow$ 1st order autocorrelation 0.86, e.g. Aiyagari (1994)
minimum durable: $\underline{d}$= 0.01	-
depreciation rate: $\delta = 0.08$	$\rightarrow$ ratio $c/i \in 6 - 6.5$, Diaz and Luengo-Prado (2005)
weight of durable utility: $\phi = 0.4$	$\rightarrow$ durable stock $d \in 1.4 - 1.6$, DLP (2005)

Table 1: Parameter values for the calibration.

Variables	Benchmark	$\sigma = 3$	$\sigma = 1$	$\underline{d} = 0$	$\phi = 0.5$	$\beta = 0.9$	
	(1)	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**	
cash-on-hand x	2.330	2.524	2.794	2.335	2.417	1.207	
financial assets a	-0.170	0.250	-0.585	-0.172	-0.227	-1.085	
durable stock d	1.641	1.386	2.610	1.649	1.799	1.453	
durabl. inv. i	0.131	0.111	0.208	0.132	0.144	0.116	
non-d. cons. c	0.859	0.888	0.769	0.858	0.845	0.840	
ratio c/i	6.541	8.006	3.682	6.501	5.869	7.228	
	$\delta = 0.04$	$r^b = 0.02$	$r^a = 0.02$ $r^b = 0.034$	$r^a = 0.03$ $r^b = 0.064$	shock size 0.5	$p = 0.2$	no collat. d
	(7)	**(8)**	**(9)**	**(10)**	**(11)**	**(12)**	**(13)**
cash-on-hand x	2.684	1.789	2.299	3.633	2.799	2.886	2.671
financial assets a	-0.358	-0.755	-0.204	1.069	0.239	0.295	0.132
durable stock d	2.142	1.691	1.641	1.664	1.696	1.717	1.673
durabl. inv. i	0.086	0.135	0.131	0.133	0.136	0.137	0.134
non-d. cons. c	0.900	0.853	0.862	0.900	0.864	0.873	0.866
ratio c/i	10.499	6.304	6.568	6.758	6.366	6.359	6.476

Table 2: Steady-state averages for the model without default

	Benchmark	$r^b = 0.02$	$r^a = 0.02$ $r^b = 0.034$	$r^a = 0.03$ $r^b = 0.064$	shock size 0.5	$p = 0.2$	no collat. d
	(1)	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**	**(7)**
cash-on-hand x	0.4686	0.5701	0.5208	0.4578	0.4861	0.5456	0.3721
financial assets a	0.8450	0.5845	0.7287	0.9759	0.8987	0.8256	0.9612
durable stock d	0.1913	0.1549	0.1534	0.1482	0.2149	0.2359	0.2121
durabl. inv. i	1.0839	0.9533	0.8033	0.6763	1.2171	1.0068	1.3013
non-d. cons. c	0.1309	0.1435	0.1303	0.1152	0.1595	0.1817	0.1304

Table 3: Steady-state coefficients of variation for the model without default (NB: variation of absolute value for financial assets)