Non-Diversification Traps in Markets for Catastrophic Risk*

Rustam Ibragimov† Dwight Jaffee‡ Johan Walden§

Preliminary version

January 20, 2006

Abstract

We develop a simple model for markets for catastrophic risk. The model explains why insurance providers may choose not to offer insurance for catastrophic risks and not to participate in reinsurance markets, even though there is large enough market capacity to reach full risk sharing through diversification in a reinsurance market. This is a "nondiversification trap." We show that nondiversification traps may arise when risk distributions have heavy left tails and liability is limited. When they are present, there may be a coordination role for a centralized agency to ensure that risk sharing takes place.

*We thank Steve Evans for help with one of the proofs and Christine Parlour for comments on how to make the manuscript easier to follow.

†Department of Economics, Harvard University, Littauer Center, 1875 Cambridge St., Cambridge, MA 02138. *Email*: ribragim@fas.harvard.edu. *Phone:* +1-203-887-1631. *Fax:* +1-617-495-7730.

‡Haas School of Business, University of California at Berkeley, 545 Student Services Building #1900, CA 94720-1900. *Email*: jaffee@haas.berkeley.edu, *Phone:* +1-510-.642-1273 *Fax:* +1-510-643-7441.

§Haas School of Business, University of California at Berkeley, 545 Student Services Building #1900, CA 94720-1900. *Email*: walden@haas.berkeley.edu, *Phone:* +1-510-643-0547. *Fax:* +1-510-643-1420.

1 Introduction

The nature of markets for catastrophe insurance and the role of governments in supporting these markets is now being actively studied; see Cummins (2005) and Jaffee (2006). Some have argued that catastrophic risks may be "uninsurable" by private markets, opening up an active role for governments. Uninsurable or not, markets for catastrophe insurance seem to deviate from other markets for risky assets and from what is predicted by theory. Specifically,

- The degree of insurance offered is limited, even though there is market capacity to diversify such risks through a reinsurance market (Cummins, Doherty, and Lo 2002).
- "Larger" risks are reinsured to a lower degree than "smaller" risks, contrary to what is predicted by theory (Froot 2001, Froot, Scharsfstein, and Stein 1993).
- Insurance providers support governmental legislation for catastrophe insurance, even though the fiscal commitment from the government is low. The following quote from Edward Liddy, President of Allstate, in the Wall Street Journal, September 6, 2005 illustrates their position: "The insurance industry is designed for those things that happen with great frequency and don't cost that much money when they do. It's the infrequent thing that costs a large amount of money to the country when it occurs – I think that's the role of the federal government."

We suggest that these deviations may be driven by what we call *nondiversification traps* in reinsurance markets for catastrophic risks. The term is loosely related to poverty traps and development traps in economic growth theory (Barro and Sala-i-Martin 2004, Azariadis and Stachurski 2006). It denotes a situation where there are two possible equilibria: a *diversification* equilibrium in which insurance is offered and there is full risk sharing in the reinsurance market, and a *nondiversification* equilibrium, in which the reinsurance market is not used, and no insurance is offered. A move from the nondiversification equilibrium to the diversification equilibrium has to be coordinated by a large number of insurance providers and may therefore be difficult to achieve through a market mechanism. Therefore, there may be a role for a centralized agency to ensure that the diversification equilibrium is reached, for example by mandating that insurance must be offered (as in the case of the Terrorist Risk Insurance Act of 2002 in the United States). Or, the diversification equilibrium may be achieved through structures that are equivalent to a functioning reinsurance market, such as the case when insurance firms are owned by large numbers of relatively small investors, each holding a diversified portfolio of equity positions.

The existence of nondiversification traps depends crucially on there being regions in which diversification is suboptimal for an individual agent. This situation is contrary to the traditional set-up in which diversification is always to be preferred (see, e.g. Samuelson (1967)). Froot and Posner (2002), also within the traditional framework, show that parameter uncertainty is unlikely to be the source of the common failure of catastrophe insurance markets. The traditional set-up is based on concave optimization (e.g. via expected utility), with thin-tailed risks (e.g. normal distributions), and without distortions (unlimited liability, no frictions and no fixed costs). If any of these assumptions fails, the result may not hold.

We will focus on the impact of heavy left-tailed distributions (implying a nonnegligible probability for large negative outcomes) as the defining property of catastrophic risks. As was shown in Ibragimov (2004) and Ibragimov (2005) in a general context, with heavy-tailed risks diversification may be inferior, regardless of the number of (i.i.d.) risks available. In Ibragimov and Walden (2005) it was further shown that with heavy tails and limited liability, diversification may be suboptimal up to a certain number of risks, and then become optimal. We shall show that this may lead to nondiversification traps.

The objective of this paper is to show how this local suboptimality of diversification can lead to nondiversification traps in a simple framework, and to provide an understanding to what is driving the results. The paper is organized as follows. In Section 2, we show how traps can arise. Sections 2.1-2.2 provide an intuitive intuition for the results. Sections 2.3-2.4 are more technical. They provide a game-theoretic set-up for a simple reinsurance market, in which traps can be analyzed. We also introduce the concept of *genuine* nondiversification traps. These traps are severe in that they will not disappear, regardless of the capacity of the insurance market. In Section 3, we show the existence of nondiversification traps and characterize under which conditions they can arise. Finally, we make some concluding remarks in Section 4. All technical details are left to the appendix.

2 Nondiversification traps

2.1 Diversification of heavy-tailed risks

The "value" of diversification under different distributional assumptions is shown conceptually in Figure 1. The Figure provides an intuition for when diversification may be inferior. Consider a situation in which there is a maximum number of risks that an insurance provider can take on, e.g. $n \leq N = 10$, with a diversification curve according to line C in Figure 1. Such a constraint can for example be motivated by capacity constraints, capital requirements, or segmented markets. For any individual insurance provider, diversification will therefore clearly be suboptimal. However, if there are M insurance providers in the market, they could potentially meet in a reinsurance market, pool the risks and reach full diversification with NM risks. For this to be preferred to nondiversification, at least $M \approx 7$ insurance providers must pool the risks. This is a very different situation compared with the traditional situation in line A, in which each individual insurance provider will choose maximal diversification into N risks, and in which two insurance providers can always improve their situation by pooling their risks in a reinsurance market. For line C, there may be a coordination problem.

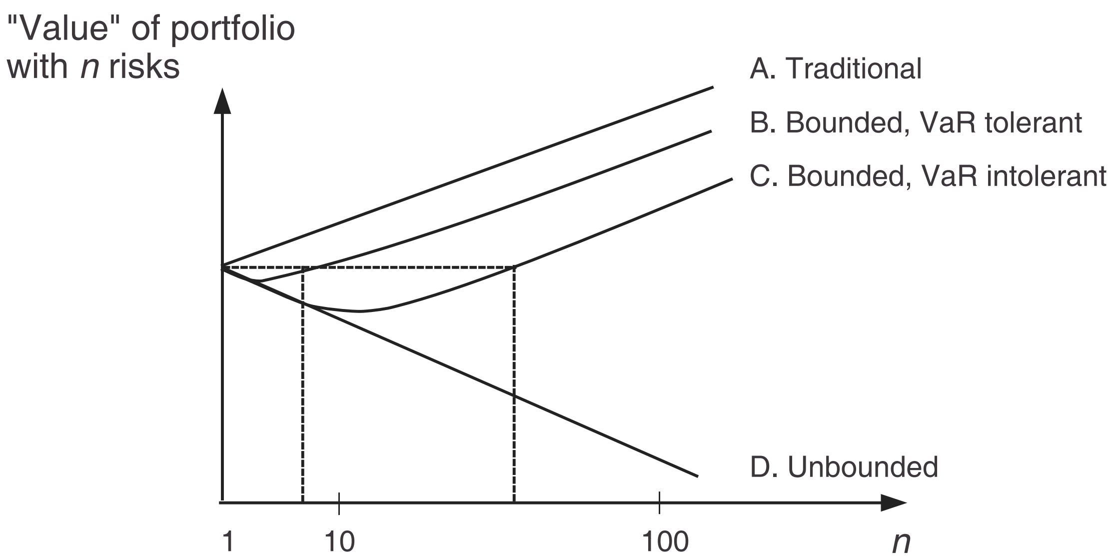

Figure 1: *Value of diversification. A: Traditional situation. The value increases monotonically and it is always preferable to add another risk to portfolio. B-C: Situation in Ibragimov and Walden (2005). Bounded heavy-tailed distributions. Up to a certain number of assets, value decreases with diversification. D: Situation in Ibragimov (2004, 2005). Unbounded heavy-tailed distributions. Value always decreases with diversification.*

We provide an intuition for why diversification may actually increase risk, using the Lévy distribution.

This is mainly for simplicity: the Lévy distribution is one of the few stable distributions[1] for which closed form expressions exist. The p.d.f. of the Lévy distribution with location parameter μ and scale parameter σ is

$$\phi^{\mu,\sigma}(x) = \begin{cases} \sqrt{\frac{\sigma}{2\pi}} e^{-\sigma/2(\mu-x)} (\mu - x)^{-3/2}, & x < \mu, \\ 0, & x \geq \mu, \end{cases}$$

and the c.d.f. is

$$F^{\mu,\sigma}(x) = \begin{cases} \text{Erf}\left(\frac{\sigma}{\sqrt{2(\mu-x)}} \right), & x < \mu, \\ 1, & x \geq \mu. \end{cases} \tag{1}$$

Here, Erf is the error function, see Abramowitz and Stegun (1970). We call the class of such random variables (r.v.'s) $S_{\mu,\sigma}$.

We first show that diversification can be inferior for such distributions. It is clear that if $X \in S_{\mu,\sigma}$, then

$$P\Big(X < -x\Big) \sim \frac{1}{\sqrt{x}}.$$

Here and throughout the paper, $f(x) \sim g(x)$ denotes that there are constants, c and C such that $0 < c \leq f(x)/g(x) \leq C < \infty$ for large x. A stricter condition is $f(x) \simeq g(x)$, denoting that the constants can be chosen arbitrarily close to unity. If we define $f(x) = o(g(x))$ to mean that $f(x)/g(x) \to 0$, as x approaches infinity, then $f(x) \simeq g(x)$ is equivalent to $f(x)/g(x) = 1 + o(1)$. Now consider the portfolio of equal holdings of two i.i.d. random variables, X_1, X_2, both belonging to $S_{\mu,\sigma}$. Clearly,

$$\begin{aligned} P\Big(\frac{X_1 + X_2}{2} &< -x\Big) = P\Big(X_1 + X_2 < -2x\Big) \simeq P\Big(X_1 < -2x\Big) + P\Big(X_2 < -2x\Big) = \\ 2P\Big(X_1 &< -2x\Big) \simeq \frac{2}{\sqrt{2}} P\Big(X_1 < -x\Big) = \sqrt{2} P\Big(X_1 < -x\Big). \end{aligned}$$

Thus, contrary to the traditional case,[2] diversification increases the risk for very negative outcomes. In fact, the larger the number of risks in the portfolio, the higher the probability for extreme negative outcomes. This general result follows from the following diversification rule for portfolios of independent Lévy distributed

[1] That is, distributions that are closed under portfolio formation.

[2] For example represented by normal distributions, $X \sim N(\mu, \sigma)$, and also true for any distribution with $E(|X|) < \infty$. The latter condition is assumed in Samuelson (1967).

risks:

$$X_i \in S_{\mu_i,\sigma_i},\ i=1,\dots,K \qquad \Longrightarrow \qquad \sum_{i=1}^{K} C_i X_i \in S_{\mu,\sigma},\ \mu = \sum_{i=1}^{K} C_i\mu_i,\ \sigma = \Big(\sum_{i=1}^{K} (C_i\sigma_i)^{1/2}\Big)^2.$$

A special case is uniform diversification,

$$X_i \in S_{\mu,\sigma},\ i=1,\dots,K \qquad \Longrightarrow \qquad \frac{\sum_{i=1}^{K} X_i}{K} \in S_{\mu,K\sigma},$$

showing that diversification increases the spread parameter from σ to $K\sigma$, and thereby riskiness, as by (1) increasing the spread parameter leads to first order stochastically dominated risks.

2.2 Risk pooling

We study the potential value of risk sharing between multiple risk-takers under different distributional assumptions on the risks. We make some intuitive arguments about when we may expect there to be hurdles to diversification. In Section 2.3 we prove the results rigorously for a simple model of a reinsurance market.

We study the behavior of risk-takers. These risk-takers may be thought of as insurance companies. We assume that the number of risk-takers is bounded by M and that all risk-takers are expected utility optimizers with identical strictly concave utility functions, u. Firms are usually considered to be riskneutral, but an expected utility set-up can be motivated by agency problems, where the manager of the firm is risk averse. Moreover, if a risk neutral firm faces finance imperfections, then the firm value may be concave transformation of outcomes, as assumed in Froot, Scharsfstein, and Stein (1993). This assumption is effectively identical to our expected utility set-up.

We also assume that there is limited liability. This is modeled by risk-takers only being liable to cover losses up to a certain level, k. If losses exceed k a risk-taker pays k, but defaults on any additional loss; to avoid the complications of any impact on policyholder demand, we assume a third party, perhaps the government, covers the excess losses. Thus, for a random variable, X, the effective outcome under limited liability is

$$V(X) = \begin{cases} (X+k)_+ - k, & k < \infty, \\ X, & k = \infty, \end{cases} \tag{2}$$

where $(X+k)_+ = \max\{X+k, 0\}$. If $k < \infty$, u need only to be defined on $[-k, \infty)$ and we can without loss of generality assume that $u(-k) = 0$.

Assuming i.i.d. risks $X_1, X_2, \ldots$, we wish to study the expected utility of s agents, who share j risks equally. We therefore define the random variable $z_{j,s} = (\sum_{i=1}^{j} X_i)/s$, with p.d.f. $\phi_{j,s}$. The expected utility of such risk sharing is:

$$U_{j,s} \stackrel{\text{def}}{=} Eu(V(z_{j,s})) = \int_{-k}^{\infty} u(x)\phi_{j,s}(x)dx. \tag{3}$$

The expected utility assumption is not crucial. Similar results would arise in a value-at-risk (VaR) framework, for example with agents who trade off VaR versus expected returns for some risk level, α. The specification would be $U_{j,s} = F(\mu, W)$, $\mu = E(V(z_{j,s}))$, $W = VaR_\alpha(V(z_{j,s})))$, with $\partial F/\partial \mu > 0$, and $\partial F/\partial W < 0$.[3] We assume that each risk-taker can maximally bring N risks "to the table." Thus, we have $1 \le s \le M$, $1 \le j \le Ns$.

First set-up: We first study a standard set-up with normal distributions ($X_i \sim Normal(\mu, \sigma^2)$), unlimited liability ($k = \infty$), and CARA utility ($u(x) = -exp(-\theta x)$). In Figure 2, we plot $U_{j,s}$ for parameters $\sigma = 1/10$, $\mu = 4/100$, $N = 20$, $M = 5$ and $\theta = 3$. This may be interpreted as an insurance provider being paid μ to take on a risk with distribution in $Normal(0, \sigma^2)$. As seen in the Figure, the optimal solution is reached when each risk-taker pools N risks and each risk-taker thereby takes on the amount NM/M of a portfolio of NM risks.

We argue that in this situation we can expect a reinsurance market to work well and insurance to be offered for a maximal number of risks, NM. The argument is based on the fact that each risk-taker will choose to diversify fully, regardless of what the other $M-1$ risk-takers do. First, consider a situation where a risk-taker believes that each of the other $M-1$ risk-takers will pool N risks. Clearly, it will be optimal for this risk-taker to also pool N risks, as the globally optimal solution is then reached. Now, consider a situation in which a risk-taker believes that no other risk-taker will pool risks. How many risks should he take on? If he pools, he should choose N risks, as $U_{j,5}$ is strictly increasing in j. If he has the option, he should avoid pooling but still choose N risks as $U_{N,1} > U_{N,5}$ and $U_{j,1}$ is strictly increasing in j. Full diversification is therefore dominant for each risk-taker, regardless of his beliefs about the other risk-takers actions.

The argument is robust to varying the parameters. It is straight-forward to show that $U_{j,s}$ is strictly monotone, or constant in j for each s. Furthermore, if $U_{j,M}$ is constant or decreasing, then $U_{j,s}$ is strictly

[3] A VaR set-up is analyzed in Ibragimov and Walden (2005).

decreasing in j for $s < M$. Therefore, as long as there is any potential for full diversification (i.e., $U_{NM,M} > U_{0,1}$) we expect insurance against the maximal NM risks to be offered.

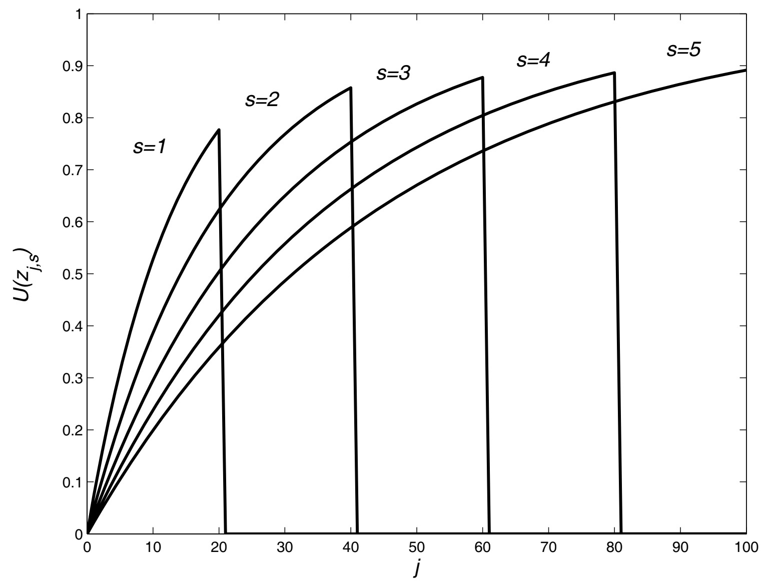

Figure 2: *Expected utility under different risk sharing alternatives. Parameters:* $k = \infty$, $\sigma = 1/10$, $\mu = 4/100$, $N = 20$, $M = 5$ *and* $\theta = 3$.

Second set-up: The situation is very different when we have limited liability and heavy tailed distributions. We consider i.i.d. Bernoulli-Lévy distributed risks, $\tilde{X}_i$, i.e.,

$$\tilde{X}_i = \begin{cases} \mu, & \text{with probability } 1 - q, \\ X,\ X \in S_{\mu,\sigma}, & \text{with probability } q. \end{cases}$$

For such distributions, we say that $\tilde{X}_i \in \tilde{S}^q_{\mu,\sigma}$. For $q << 1$, the distribution is qualitatively similar to distributions for catastrophic risks: There is a small probability for a catastrophe to occur. However, if it does occur, the loss may be very large due to the heavy left tail of the Lévy distribution. We assume limited liability ($k < \infty$) and the utility function $u(x) = (x + k)^{3/4}$.

In Figure 3, we show expected utility for different total number of projects, j, and number of agents involved in risk sharing, s, with parameters $k = 80$, $\sigma = 5$, $\mu = 1$, $N = 30$, $M = 5$ and $q = 0.002$. Clearly, the situation is very different from the first set-up. Specifically, for a moderate number of risks, there is no way to increase expected utility compared with staying away from risks altogether. No risk-taker will therefore choose to invest in risks that can not be pooled. Moreover, if a risk-taker believes that no other risk-taker will pool risks, he will not take on risks, whether he can pool it or not. Thus, even though the situation with full diversification and risk sharing ($U_{NM,M}$) is preferred over the no risk situation ($U_{0,1}$), all five risk-takers must agree to pool risk for risk sharing to be worthwhile.

In this situation there may be a coordination problem: Even though all agents would like to reach $U_{NM,M}$, they may be stuck in $U_{0,1}$. Clearly, the limited liability is important: If liability were unlimited, no agent would take on risk (even though there is only a 0.2% risk for a catastrophe to occur). The situation would be as in Ibragimov (2004) and Ibragimov (2005), where diversification is always inferior. The probability for default in the situation with full pooling and diversification is by no means overwhelming: It is approximately 4%.

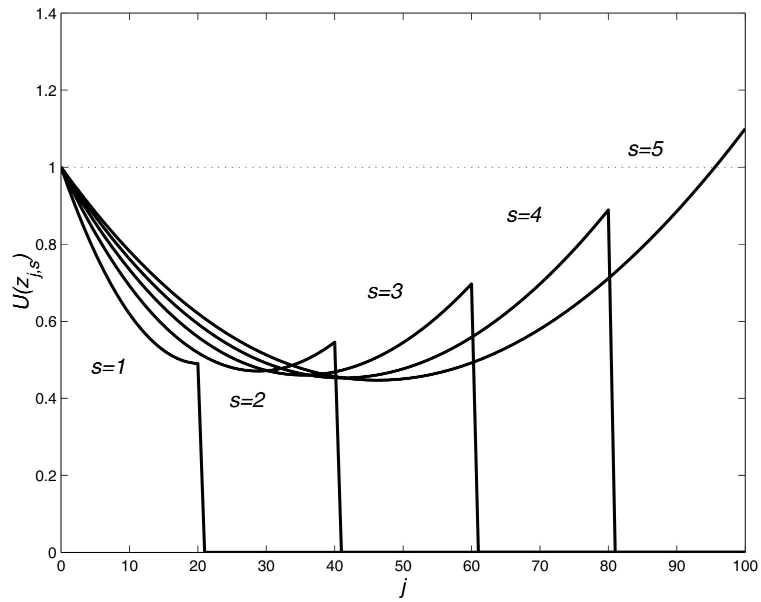

Figure 3: *Expected utility under different risk sharing alternatives. Parameters:* $k = 80$, $\sigma = 5$, $\mu = 1$, $N = 30$, $M = 5$ *and* $q = 0.002$

Our argument so far has been informal. We next make the diversification results rigorous by introducing a simple model of a reinsurance market where coordination plays a role – the *diversification game.* We will show that in the first set-up, indeed the only equilibrium is a diversification equilibrium, where NM risks are insured, whereas in the second set-up there is both a diversification equilibrium, and a nondiversification equilibrium in which no insurance is offered.

2.3 A simple reinsurance market

We analyze a simple market in which insurance providers sell insurance against risks. For simplicity, we model the market in a symmetric setting: participants in reinsurance markets share risks equally. The set-up is a two-stage game that capturers the intuitive idea that insurance has to be offered before reinsurance can be pooled. The decision whether to offer primary insurance will be based on beliefs about how well-functioning (the future) reinsurance markets will be. If a critical number of participants is needed for reinsurance markets to take off, then nondiversification traps can occur.

The two-stage *diversification game* describes the market. In the first stage, agents choose whether to offer insurance against a set of i.i.d. risks. If they offer insurance, they also choose whether to participate in the reinsurance market or not. In the second stage of the game, named the *participation subgame*, agents who chose not to offer insurance are allowed to participate in the reinsurance market. Finally, all risks of participating agents are pooled, outcomes are realized and shared equally among participating agents. The formal set-up is as follows

Agents and risks: There are $M \geq 2$ agents (also denoted insurance providers, insurance companies and risk-takers). We use, m, $1 \leq m \leq M$ to index these agents. There is a set of i.i.d. risks, $\mathcal{X}$, where each risk has p.d.f. $\phi(x)$. Each agent chooses to take on a specific number of risks, $n \in \{0, 1, 2, \ldots, N\}$, where N denotes the maximum insurance capacity,[4] forming a portfolio of risks $p_m \in P_m$, where $p_m = \sum_{i=1}^{n} X_i$ and $X_i \in \mathcal{X}$. The risks are atomic (indivisible) and each risk can be chosen by at most one agent. We assume that there are enough risks available to exhaust capacity, i.e., $|\mathcal{X}| = NM$.[5] As risks are i.i.d., only the distributional assumptions of the risks matter and we will not care about which insurance provider chooses which risk. The portfolio p_m is therefore completely characterized by the number of risks, n_m. The total number of risk insured is $\overline{N} = \sum_m n_m$. The ordered set of agents who sell insurance is $J = (j_1, j_2, \ldots)$, $1 \leq j_1 < j_2 < \cdots < j_{|J|} \leq M$, with $p_{j_m} \neq \emptyset$ for all $m = 1, \ldots, |J|$, and the ordered set of agents who do not sell insurance is $K = (1, \ldots, M) \backslash J$. Agents have liability to cover losses up to k, where $k \in (0, \infty]$. If losses exceed k for an agent, he defaults, pays k and a third party, possibly the government, steps in and covers excess losses. The effective outcome under limited liability for agent m, taking on risk z_m, is therefore $V(z_m)$, where V is defined in (2). All agents have identical expected utility over risks, $U_m(z_m) = Eu(V(z_m))$, where u is defined and continuous on $[-k, \infty)$, is strictly concave, twice continuously differentiable on $(-k, \infty)$ and,

[4]This constraint could for example be driven by capital requirements.
[5]Here, $|\mathcal{X}|$ denotes the cardinality of $\mathcal{X}$.

if $k < \infty$, satisfies $u(-k) = 0$.

Reinsurance market: There is also a market for pooled risks, i.e., a reinsurance market. Each agent, m, chooses whether to participate in the market or not. The participation decision is represented by a binary variable $q_m \in \{0, 1\} = \mathcal{Q}_m$, where $q_m = 1$ indicates that agent m participates in the reinsurance market. An agent who takes on own risk, (i.e., belongs to J) simultaneously chooses whether to participate in the reinsurance market or not, in which case the whole portfolio is pooled into the market, i.e., $q_m p_m$ is supplied to the reinsurance market by agent m. We use the convention that $q_m = 0$ for $m \in K$. The total pooled risk is $P = \sum_m q_m p_m$ and the number of risks are $R = \sum_m n_m q_m \in \{0, \ldots, NM\}$. This is the first stage of the market, and the outcome of this stage is summarized by (p, q), where $q = (q_1, \ldots, q_M) \in \mathcal{Q}$ and $p = (p_1, \ldots, p_M) \in \mathcal{P}$.

In the second stage of the market, all agents know (p, q). Agents who do not sell insurance (i.e., agents in K) choose whether to participate or not. Formally, let $K = (k_1, \ldots, k_{|K|})$. First, agent k_1 decides whether to participate. This is represented by the binary variable $q'_{k_1} \in \{0, 1\}$, where $q'_{k_1} = 1$ denotes that agent k_1 participates in the reinsurance market and $q'_{k_1} = 0$ otherwise. Then, agent k_2 decides whether to participate, etc. This is repeated until all $|K|$ agents have decided. Previous agents' decisions are observable. If an agent is indifferent between participating or not participating, he will not participate. The variable $\tilde{q}_m \in \{0, 1\}$ summarizes whether an agent participates in the reinsurance market, either in the first or second stage,

$$\tilde{q}_m = \begin{cases} 0, & \text{if } q_m = 0 \text{ and } q'_m = 0, \\ 1, & \text{otherwise.} \end{cases}$$

The two stages are needed to separate the choice of offering insurance, from the creation of a reinsurance market, which can only occur when the risks are already insured. The total number of participating agents in the reinsurance market is $s = \sum_m \tilde{q}_m$. Finally, the pooled risks are split equally among agents in the reinsurance market, i.e., each participating agent receives a fraction $1/s$ of the pooled portfolio, P, with R risks. The second stage of the market, between agents $m \in K$, is called the *participation subgame*. It is a $|K|$-step sequential game with perfect information. It is therefore straight-forward to calculate the unique subgame perfect equilibrium by backward induction (uniqueness being guaranteed by imposing that indifferent agents do not participate). A detailed set-up for the participation subgame is given in the appendix. The equilibrium mapping of the participation game, for a specific first-stage realization, (p, q), is a vector $q' = \mathcal{E}(p,q) \in \{0,1\}^M$. Here, the convention $q'_m = 0$ for $m \notin K$ is used.

The sequence of events is shown in Figure 4 and the structure of the market is shown in Figure 5. The quintuple (u, ϕ, k, N, M) completely characterizes the diversification game.

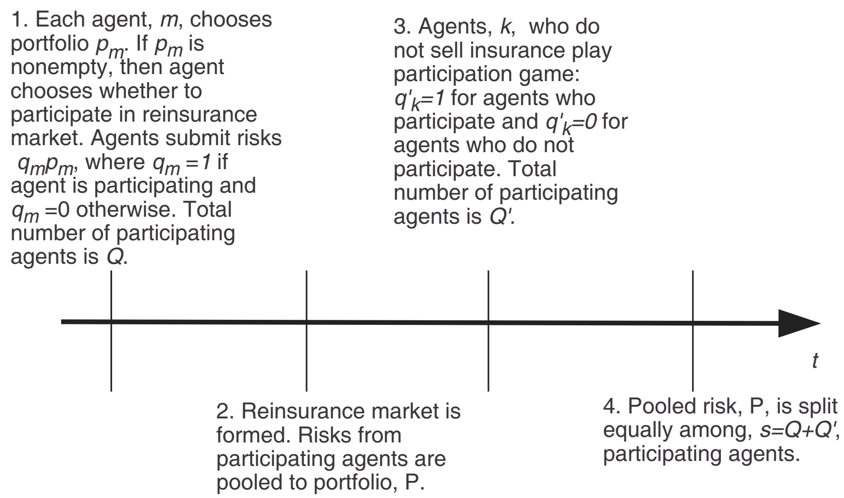

Figure 4: *Sequence of events: 1. Agents choose risk portfolio, p_m. Agents in J choose whether to pool risks in reinsurance, q_m. 2. Reinsurance pool $P = \sum_m q_m p_m$ is formed. 3. Participation game is played between agents in K. Pooled risk is split between s participating agents, each taking on P/s. Nonparticipating agents receive $(1 - \tilde{q}_m)p_m$.*

Action: For elements $p \in \mathcal{P}$ and $q \in \mathcal{Q}$, we define the actions of all agents except agent m:

$$(p)_{-m} = (p_1, \ldots, p_{m-1}, p_{m+1}, \ldots, p_M) \in \prod_{m' \neq m} \mathcal{P}_{m'} \stackrel{\text{def}}{=} \mathcal{P}_{-m},$$
$$(q)_{-m} = (q_1, \ldots, q_{m-1}, q_{m+1}, \ldots, q_M) \in \prod_{m' \neq m} \mathcal{Q}_{m'} \stackrel{\text{def}}{=} \mathcal{Q}_{-m}.$$

An action for agent m consists of a triple: $A = (p_m, q_m, q'_m) \in \mathcal{P}_m \times \mathcal{Q}_m \times \{0,1\}^{\mathcal{P}_{-m} \times \mathcal{Q}_{-m}} = \mathcal{A}$, where p_m is the chosen portfolio of insurance, q_m is the first stage participation choice, and $q'_m : \mathcal{P}_{-m} \times \mathcal{Q}_{-m} \to \{0,1\}$ is the second stage participation choice (which is only of relevance if $q_m = 0$) to participate depending on the realization of the reinsurance market, defined in the first stage.

Belief set: Agent m has a belief set about the other agents' first stage actions, $B_m = (p_{-m}, q_{-m}) \in$

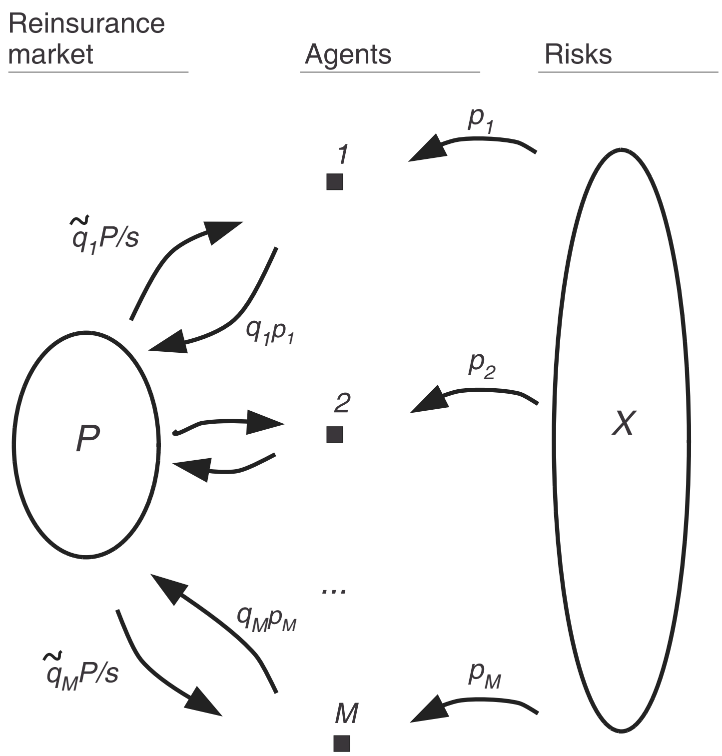

Figure 5: *Market: Each of agent, $m = 1, \ldots, M$, chooses a portfolio, p_m, from the set of risks $\mathcal{X}$ and submits $q_m p_m$ to the reinsurance markets. Reinsurance risk, R is shared equally by s agents, who have $\tilde{q}_m = 1$.*

$\mathcal{P}_{-m} \times \mathcal{Q}_{-m} = \mathcal{B}$. The inferred outcome of an action $A_m = (p_m, q_m, q'_m)$, conditioned on a belief set, B_m, is

$$z_m = \begin{cases} p_m, & \text{if } p_m \neq \emptyset \text{ and } q_m = 0, \\ p_m, & \text{if } p_m = \emptyset \text{ and } q'_m(p_{-m}, q_{-m}) = 0, \\ P/s, & \text{if } p_m \neq \emptyset \text{ and } q_m = 1, \\ P/s, & \text{if } p_m = \emptyset \text{ and } q'_m(p_{-m}, q_{-m}) = 1, \end{cases}$$

where $P = \sum_{m'} p_{m'} q_{m'}$, and $s = \sum_{m'} \tilde{q}_{m'} = \sum_{m'} q_{m'} + \sum_{m'} \left(\mathcal{E}(\prod_{m''} p_{m''}, \prod_{m''} q_{m''})\right)_{m'}$.

Equilibrium: An M-tuple of actions, $(A_1, \dots, A_M) \in \mathcal{A}^M$, and belief sets $(B_1, \dots, B_M) \in \mathcal{B}^M$ (where $A_m = (p_m, q_m, q'_m)$ and $B_m = (p_{-m}, q_{-m})$), defines an equilibrium if

1. Maximized actions: For each agent, m, $A_m \in \arg\max_{A \in \mathcal{A}} U_m(z_m(A|B_m))$.
2. Consistent beliefs: For each agent, m, for all $m' \neq m$
   (a) $(p_{-m})_{m'} = p_{m'}$. (Consistent beliefs about insurance offered by other agents)
   (b) $(q_{-m})_{m'} = q_{m'}$. (Consistent beliefs about participation in reinsurance market by other agents)
3. Equilibrium of participation game: For all $p \in \mathcal{P}$ for all $q \in \mathcal{Q}$: $\prod_{m=1}^{M} q'_m((p)_{-m}, (q)_{-m}) = \mathcal{E}(p, q)$.

This concludes the formal definition of the diversification game.

2.4 Classification of equilibria

We are interested in diversification and nondiversification equilibria to a diversification game $\mathcal{G} = (u, \phi, k, N, M)$. We define

Definition 1 *A diversification equilibrium of a diversification game $\mathcal{G}$, is an equilibrium in which insurance against all risks in $\mathcal{X}$ is offered, i.e., $\overline{N} = NM$.*

Definition 2 *A diversification equilibrium of a diversification game $\mathcal{G}$, is risk sharing if all risk insured is pooled in the reinsurance market, i.e., $R = NM$.*

Definition 3 *A nondiversification equilibrium of a diversification game $\mathcal{G}$ is an equilibrium, in which no insurance against risk is offered, i.e., $\overline{N} = 0$.*

Definition 4 *A nondiversification trap exists in a diversification game $\mathcal{G}$, if there is both a nondiversification equilibrium and a risk sharing diversification equilibrium.*

We are especially concerned about cases when nondiversification traps may arise, even though there is a large risk bearing capacity of the market as a whole. This might arise if the market is fragmented so coordination problems may be present, i.e., M is large. We therefore define

Definition 5 *A genuine nondiversification trap to the quadruple (u, ϕ, k, N) exists if there exists a M_0, such that for all $M \geq M_0$, the diversification game $\mathcal{G} = (u, \phi, k, N, M)$ has a nondiversification trap.*

The type of equilibria that can arise is closely related to $U_{j,s}$, defined in equation (3). Clearly, under the following assumption, an agent would never offer insurance if the reinsurance market were not available:

Assumption 1 $U_{j,1} < U_{0,1}$ *for all* $j \in \{1, \ldots, N\}$.

A stronger assumption is that even if there is a reinsurance market, there is no way to increase expected utility by risk sharing if only one agent contributes with risk, i.e.,

Assumption 2 $U_{j,s} < U_{0,1}$ *for all* $j \in \{1, \ldots, N\}$ *and all* $s \in \{1, \ldots, M\}$.

We shall see that a sufficient condition for there to be an equilibrium in which full diversification and risk sharing is achieved is

Assumption 3

- $U_{NM,M} > U_{j,1}$ *for all* $j \in \{0, \ldots, N\}$ *and*
- $U_{NM,M} > U_{j,M}$ *for all* $j \in \{N(M-1), \ldots, NM-1\}$.

With these definitions and assumptions, we can completely classify under which conditions nondiversification traps and genuine nondiversification traps can occur.

3 Existence of traps

Our first set of results relate the existence of nondiversification traps to the expected utilities $\{U_{j,s}\}_{0 \leq j \leq NM, 1 \leq s \leq M}$, defined in (3). These results are fully in line with the arguments in Section 2.2. We have:

Proposition 1 *If Assumption 2 is satisfied, then there is a nondiversification equilibrium.*

The implication can almost reversed, as shown in

Proposition 2 *If Assumption 2 fails strictly, i.e., if* $U_{n,s} > U_{0,1}$ *for some* $j \in \{1, \ldots, N\}$ *and* $s \in \{1, \ldots, M\}$*, then there is no nondiversification equilibrium.*

Remark 1 *Clearly, if* $U_{0,1} > U_{j,s}$ *for all* (j, s) *such that* $j \in \{1, \ldots, Ns\}$ *and* $s \in \{1, \ldots, M\}$*, then the nondiversification equilibrium is unique. Under these conditions, the risks are genuinely uninsurable. Such a situation may correspond to the "globally uninsurable" risks mentioned in Cummins (2005). Under such conditions, we can have no hopes for an insurance market to work: The risks are simply too large.*

Proposition 3 *If Assumption 3 is satisfied, then there is a risk sharing diversification equilibrium.*

These results, together with the results in Section 2.2, immediately imply the following Corollaries:

Corollary 1 *In the first set-up of Section 2.2 (CARA utility, normal distributions and unlimited liability), there will never be a nondiversification trap regardless of parameter values.*

Corollary 2 *In the second set-up in Section 2.2 (Bernoulli-Lévy distributions), there is a nondiversification trap.*

Corollary 2 can be strengthened: the nondiversification trap is *genuine*:

Proposition 4 *The nondiversification trap in the second set-up in Section 2.2 (Bernoulli-Lévy distributions) is genuine.*

We next move on to classifying general distributional properties of the primitive risks that permit traps. It turns out that traps will only arise under quite specific conditions: First, nondiversification traps will not arise in a mean-variance framework with unlimited liability. Second, genuine nondiversification traps can only arise with distributions that have heavy tails (i.e., infinite second moments).

Proposition 5 *If utility is on the form* $Eu(X) = E(X) - \gamma Var(X)$*, and* $k = \infty$*, then a nondiversification trap can not occur.*

It turns out that non-genuine nondiversification traps can arise under standard conditions. For example, the diversification game with

$$u(x) = xI_{x\leq 0} + \log(1+x)I_{x>0},$$

$$X = 130Ber(1/2) - 50,$$

$$k = \infty, \qquad N = 20, \qquad M = 5,$$

(where $Ber(p)$ is the two-point Bernoulli distribution with probability p and $1-p$ for 0 and 1 respectively, and I_A is the indicator function on the set A, $x \in A \Rightarrow I_A(x) = 1$, $x \notin A \Rightarrow I_A(x) = 0$), has a nondiversification trap. However, *genuine* nondiversification traps only arise if distributions have heavy tails, as shown by the following three propositions:

Proposition 6 *If $k = \infty$ and the risks $X \in \mathcal{X}$ have finite second moments, i.e., $E(X^2) < \infty$, then a genuine nondiversification trap can not occur.*

Proposition 7 *If $k < \infty$, the risks $X \in \mathcal{X}$ have $E(X) \neq 0$ and $E(X^2) < \infty$ then a genuine nondiversification trap can not occur.*

Proposition 8 *If $k < \infty$, the risks $X \in \mathcal{X}$ have $E(X) = 0$ and $E(X^{2+\epsilon}) < \infty$, for some arbitrary small $\epsilon > 0$, then a genuine nondiversification trap can not occur.*

These general results can also be viewed from an approximation perspective. If M is large, but finite, then nondiversification traps can only arise with distributions that have left tails that are "approximately" heavy, i.e., decay slowly up until a certain point (even though their real support may be bounded). For details on this type of argument, see Ibragimov and Walden (2005).

4 Concluding remarks

Catastrophic risks seem ideal for insurance markets with large aggregate capacity: They are basically independent over types and geography, and there are few, if any, informational asymmetries to hinder well-functioning markets for pooled risks. The nonexistence of markets for catastrophe insurance is therefore quite puzzling.

We offer an explanation to this puzzle based on the one unique property of catastrophic risks: the non-negligible probability for extremely negative outcomes, i.e., the heavy left tails. The value of diversification decreases drastically when distributions are heavy tailed. In some cases, it vanishes completely or can even be negative. The heavier the tails, the less we can therefore rely on standard mean-variance analysis and normal distributions in our analysis.

In a simple model of a reinsurance market, we have shown that under some distributional assumptions, there can be nondiversification traps. These traps occur when the value of diversification is U-shaped in the number of risks – starting out negative, but eventually becoming positive. In such situations, the value of diversification may be negative on the scale of the individual insurance company, but positive on a market scale. Diversification must therefore be coordinated by a large number of companies, which could motivate a role for a central agency in coordinating and ensuring that diversification is reached.

Appendix

Results in Section 2.2

In the first set-up, expected utility will be:

$$U_{j,s} = -e^{-\frac{\theta}{2}(2j\mu/s - \theta j\sigma^2/s^2)}. \tag{4}$$

Clearly, the term $2\mu - \theta\sigma^2/s$ decides whether $U_{j,s}$ is increasing, constant or decreasing, and the results on strict monotonicity in j for fixed s follow accordingly.

In the second setup, expected utility will be:

$$U_{j,s} = \sum_{n=0}^{j} \binom{j}{n} q^n (1-q)^{j-n} W_{j,n,s}, \tag{5}$$

where

$$W_{j,n,s} = \sqrt{\frac{\tilde{\sigma}}{2\pi}} \int_0^v (v-x)^{3/4} e^{-\tilde{\sigma}/(2x)} x^{-3/2} dx, \qquad \tilde{\sigma} = n^2\sigma/s \text{ and } v = k + j\mu/s.$$

The closed form solution for the integral is

$$W_{j,n,s} = v^{3/4} F(-3/4, 1/2, -\tilde{\sigma}/(2v)) - \frac{\sqrt{2\tilde{\sigma}}\Gamma(7/4)}{\Gamma(5/4)} v^{1/4} F(-1/4, 3/2, -\tilde{\sigma}/(2v)).$$

Here, F is the confluent hypergeometric function of the first kind, i.e., the Kummer function, and Γ is the gamma-function (Abramowitz and Stegun 1970).

Figures 2 and 3 are plotted using equations 4 and 5.

Results in Section 2.3

The K-person participation subgame: $K \geq 1$ agents decide sequentially whether to participate or not. Previous decisions are observable. The outcome is represented by $q = (q_1, \ldots, q_K) \in \{0,1\}^K$, where $q_k = 1$ indicates participation. The payoff to not participating, (i.e. choosing $q_k = 0$) is 0. The payoff to participating is $F(Q)$, where Q denotes the number of participating agents, $Q = \sum_k q_k$, and $F : \{1, \ldots, K\} \to \mathbb{R}$ is any function. To ensure uniqueness, we assume that if agents are indifferent between participating and not participating then they do not participate. We call this the "laziness" assumptions. We define $0 \leq w \leq K$

by

$$w = \min\Big(\{K\} \cup \{k : 0 \le k \le K-1, \text{ and } F(k') \le 0 \text{ for all } k' > k\}\Big). \tag{6}$$

We have

Lemma 1 *Any K-person participation subgame has a unique subgame perfect equilibrium, satisfying the laziness assumption. The form of the equilibrium is that agents $1, \ldots, w$ participate and agents $w+1, \ldots, K$ do not participate, i.e., $q_1, \ldots, q_w = 1$ and $q_{w+1}, \ldots, q_K = 0$.*

Proof:

i) It is an equilibrium: Clearly, there is no incentive for any of the first w agents to deviate, as they receive a strictly positive outcome compared with 0 if they deviate. If $w < K$, then $F(w+1) \le 0$ and there is clearly no reason for any of the $K-w$ last agents to deviate, as they can never increase their payoff by deviating.

ii) The equilibrium is subgame perfect: Any subgame starting in any node after $K-j$ steps is a j step participation game. Therefore, if the equilibrium is subgame perfect for the j step participation games, then it is subgame perfect for the $j+1$ step participation games. The equilibrium to the one-person game is subgame perfect, so by induction, the equilibrium to the K-person game is subgame perfect.

iii) It is the unique subgame perfect equilibrium satisfying the laziness assumption: Assume that we have proved uniqueness for all $K-1$-person games. We consider candidates for alternative equilibria for the K-person game. We identify three cases:

- $Q > w$: By (6), such an equilibrium would either have $F(Q) < 0$, which is inferior to not participating, or $F(Q) = 0$, which would violate the laziness assumption.

- $Q < w$: A nonparticipating agent would be strictly better off by participating, so this can not be an equilibrium.

- $Q = w$: It must be that $0 < w < K$ for an alternative candidate to exist. Define $j = \max\{k : q_k = 0\}$ and $m = \max\{k : q_k = 1\}$. By assumption, $j < m \le w+1 \le K$. Clearly, such an equilibrium can not be subgame perfect and satisfy the laziness assumption, as if agent j chooses to participate, agent m will choose not to participate in the unique subgame perfect equilibrium satisfying the laziness assumption of the remaining $K-j-1$ participation game. Thus, j will deviate and this is not an equilibrium. ■

Results in Section 3

Proof of Proposition 1: As Assumption 2 implies Assumption 1 it is clearly not optimal for any agent who does not participate in the reinsurance market to offer insurance. Moreover, if agent m believes that no other agent will offer nontrivial risks into the pooled market, Assumption 2 implies that it is optimal for agent m to not offer nontrivial risk, as any risk sharing with up to N risks is inferior to not taking on risk. Thus, it is an equilibrium for no one to offer insurance. ■

Proof of Proposition 2: If $U_{n,s} > U_{0,1}$ for $n \geq 1$ and $s = 1$, then clearly any agent will strictly improve by taking on n risks. For $U_{n,s} > U_{0,1}$ with $s > 1$, the proof is a direct consequence of the equilibrium structure of the participation game. For example, agent 1 strictly improves by pooling n risks into the reinsurance market, as agent $2, \ldots, w+1$ will then choose to participate in the participation game, where $w \geq s-1$. This leads to a strict improvement for all agents $1, \ldots, w+1$. Thus, nondiversification can not be an equilibrium. ■

Proof of Proposition 3: Under Assumption 3, if agent m believes that all other agents will participate in the reinsurance market, by choosing N risks and participating, $U_{NM,M}$ can be achieved. This clearly dominates any alternative strategy of not participating in the market, which will lead to $U_{n,1}$, or participating and offering fewer risks, which will lead to $U_{n,M}$, for $N(M-1) \leq n \leq NM-1$. Both these strategies are strictly dominated by the strategy leading to $U_{NM,M}$. ■

Proof of Corollary 1: Clearly, for there to be a risk-sharing diversification equilibrium, we must have $U_{NM,M} > U_{0,1}$. However, the strict monotonicity of $U_{j,M}$ in j then implies that $U_{1,M} > U_{0,1}$. By Proposition 2, this contradicts there being a nondiversification equilibrium, which would require that $U_{1,M} \leq U_{0,1}$. ■

Proof of Corollary 2: The Corollary follows by checking that the second set-up satisfies Assumptions 2 and 3 simultaneously. ■

Proof of Proposition 4: To be completed. Depends on the asymptotic convergence of distributions as M grows. ■

Proof of Proposition 5: Identical to the proof in the normal case, as $U_{j,s} = j/s(\mu - \gamma/s\sigma^2)$, which is monotone in j. ■

Proof of Proposition 6: Assume that there exists a nondiversification traps to the game (u, ϕ, ∞, N, M) for arbitrarily large M, for a strictly concave twice continuously differentiable utility function u and distribution ϕ, satisfying:

$$\int_{-\infty}^{\infty} x^2 \phi(x) dx = C < \infty. \tag{7}$$

Without loss of generality, we can assume $u(0) = 0$, $u'(0) = 1$. For a genuine nondiversification trap to exist, it must be the case that for arbitrarily large M,

$$U_{NM,M} > 0, \tag{8}$$

and

$$U_{1,M} < 0. \tag{9}$$

However, a necessary condition for (9) to hold for arbitrarily large M is that $EX \leq 0$, as seen by the following argument: Assume that $\mu = EX > 0$. We define

$$U(\epsilon) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} u(\epsilon x) \phi(x) dx.$$

We decompose

$$u(x) \stackrel{\text{def}}{=} x - t(x) \stackrel{\text{def}}{=} x - x^2 z(x),$$

where $t(0) = t'(0) = 0$, $t'' > 0$, $t(x) < x$, $z(x)$ is continuous and both t and z are nonnegative. We then have

$$U(\epsilon) = \epsilon\mu - \Big(\int_{-\infty}^{-1/\epsilon} t(\epsilon x)\phi(x)dx + \int_{-1/\epsilon}^{1/\epsilon} (\epsilon x)^2 z(\epsilon x)\phi(x)dx + \int_{1/\epsilon}^{\infty} t(\epsilon x)\phi(x)dx \Big). \tag{10}$$

The $\int_{1/\epsilon}^{\infty}$-term is clearly $o(\epsilon)$,[6] as

$$\int_{1/\epsilon}^{\infty} t(\epsilon x)\phi(x)dx \leq \int_{1/\epsilon}^{\infty} \epsilon x\phi(x)dx \leq \epsilon \int_{1/\epsilon}^{\infty} x\phi(x)dx \leq C_2\epsilon^2.$$

Furthermore, as $z(x)$ is continuous, it is bounded on $[-1,1]$, so Hölder's inequality can be used to bound the $\int_{-1/\epsilon}^{1/\epsilon}$ term by

$$\int_{-1/\epsilon}^{1/\epsilon} (\epsilon x)^2 z(\epsilon x)\phi(x)dx \leq \epsilon^2 \max_{-1\leq y\leq 1} |z(x)| \times C = C_3\epsilon^2,$$

so the second term is also of $o(\epsilon)$. Finally, the $\int_{-\infty}^{-1/\epsilon}$-term is also $o(\epsilon)$, as

$$\int_{-\infty}^{-1/\epsilon} t(\epsilon x)\phi(x)dx = \int_{-\infty}^{-1/\epsilon} \frac{t(\epsilon x)}{t(x)} t(x)\phi(x)dx \leq \epsilon t'(-1) \int_{-\infty}^{-1/\epsilon} t(x)\phi(x)dx = o(\epsilon),$$

where we use Hölder's inequality to move the $t(\epsilon x)/t(x)$ outside of the integral, and the inequality

$$\frac{t(\epsilon x)}{t(x)} \leq \epsilon t'(-1),$$

which must hold for $x \geq 1/\epsilon$, as t is convex. Finally,

$$\int_{-\infty}^{-1/\epsilon} t(x)\phi(x)dx = o(1),$$

as the integral $Eu(X)$ could otherwise not exist. This altogether implies that $U(\epsilon) = \epsilon\mu - o(\epsilon)$, which is strictly positive for small enough ϵ. Therefore, if $EX > 0$, then $U_{1,M}$ will be strictly positive for large enough M, and no genuine nondiversification trap can therefore exist.

However, if $EX \leq 0$, then a nondiversification trap can not exist as Jensen's inequality implies that $U_{NM,M}$ is strictly negative for arbitrary $M > 0$ and $N > 0$ and thus $U_{NM,M} < U_{0,1}$. ■

Proof of Proposition 7: Assume that there exists a nondiversification traps to the game (u, ϕ, k, N, M) for arbitrarily large M, for a strictly concave twice continuously differentiable utility function u and distribution ϕ, satisfying:

$$\int_{-\infty}^{\infty} x^2\phi(x)dx = C < \infty. \tag{11}$$

[6]The term $f(\epsilon) = o(\epsilon)$ denoting that $\lim_{\epsilon\searrow 0} f(\epsilon)/\epsilon = 0$.

Without loss of generality, we can assume $u(0) = 0$, $u'(0) = 1$.

If $EX > 0$, then the same argument as in the proof of Proposition 6 rules out a genuine nondiversification trap, as the limited liability *increases* $U_{1,M}$ compared with the unlimited liability case. Thus, for M large enough, $U_{1,M}$ must be strictly positive and a genuine nondiversification trap can not exist.

If $EX = \mu < 0$, then we use the law of large numbers to show that as M becomes large, $X_{NM} = (NM)^{-1}\sum_{i=1}^{NM} X_i$ converges in distribution to μ. Thus, $\lim_{M\to\infty} E((X_{NM} + kNM)_+ - kNM) = \mu$, so for some large enough M_0, $E((X_{NM} + kNM)_+ - kNM) < 0$ for all $M \geq M_0$. Jensen's inequality therefore again implies that $U_{NM,M}$ is strictly negative for $M \geq M_0$ and thus that $U_{NM,M} < U_{0,1}$, so there can be no genuine nondiversification trap. ■

Proof of Proposition 8: Without loss of generality, we can assume $u(0) = 0$, $u'(0) = 1$.

We prove that $U_{NM,M} < U_{0,1} = 0$ for large M. We define:

$$\gamma = \min_{x\in[-k/2,k/2]} u''(x).$$

As u is strictly concave and twice continuously differentiable, $\gamma > 0$. We define

$$\tilde{u}(x) = x - \frac{\gamma}{2}x^2 I_{[-k/2,k/2]},$$

implying that $u(x) \leq \tilde{u}(x)$ for all $x \in [-k, \infty)$. Similar to $U_{j,s}$, we define $\tilde{U}_{j,s}$, the "utility" of sharing j risks equally among s agents, for agents with "utility" functions $\tilde{u}$. Clearly, $U_{j,s} \leq \tilde{U}_{j,s}$, so if $\tilde{U}_{NM,M} < 0$ for large M, then $U_{j,s} < 0$ for large M and there cannot be a genuine nondiversification trap.

We next define $Y_1 = \sum_{i=1}^{N} X_i$ and study uniform portfolios of i.i.d. risks $Y_1, \ldots, Y_M$ by defining $\overline{Y}_M \stackrel{\text{def}}{=} (\sum_{m=1}^{M} Y_m)/M$. As $E(\overline{Y}_M) = 0$, the condition $\tilde{U}_{NM,M} < 0$ for large M can be written:

$$\tilde{U}_{NM,M} = E\left(\overline{Y}_M I_{[-k,\infty)}\right) - \frac{\gamma}{2} E\left(\overline{Y}_M^2 I_{[-k/2,k/2]}\right) < 0. \tag{12}$$

We begin by bounding $E\left(\overline{Y}_M^2 I_{[-k/2,k/2]}\right)$ from below. From the central limit theorem, we know that $Z_M \stackrel{\text{def}}{=} \sqrt{M}\,\overline{Y}_M$ converges in distribution to $Z \sim Normal(0, \sigma^2)$, so $E(Z_M^2 I_{[-k/2,k/2]}) \to C > 0$, as M grows. As $ME\left(\overline{Y}_M^2 I_{[-k/2,k/2]}\right) \geq ME\left(\overline{Y}_M^2 I_{[-k/2\sqrt{M},k/2\sqrt{M}]}\right) = E\left(Z_M^2 I_{[-k/2,k/2]}\right)$, we can therefore conclude that for large M,

$$\frac{\gamma}{2}E\left(\overline{Y}_M^2 I_{[-k/2,k/2]}\right) \geq \frac{C'}{M}, \qquad C' > 0. \tag{13}$$

We next bound $E\left(\overline{Y}_M I_{[-k,\infty)}\right)$ from above. As, $E\left(\overline{Y}_M\right) = 0$, we have $E\left(\overline{Y}_M I_{[-k,\infty)}\right) = -E\left(\overline{Y}_M I_{(-\infty,-k)}\right)$. From the Cauchy-Schwarz inequality, we know that

$$-E\left(\overline{Y}_M I_{(-\infty,-k)}\right) \leq E\left(\overline{Y}_M^2\right)^{1/2} E\left(I_{(-\infty,-k)}\right)^{1/2},$$

(as $I_{(-\infty,-k)}^2 = I_{(-\infty,-k)}$). Of course, $E\left(\overline{Y}_M^2\right) = \sigma^2/M$. Moreover, Rosenthal's inequality (Rosenthal 1970) implies that $E\left(\overline{Y}_M^{2+\epsilon}\right) \leq C''/M^{1+\epsilon/2}$, and by Markov's inequality, we therefore know that

$$E\left(I_{(-\infty,-k)}\right) = P(x < -k) \leq \frac{E\left(\overline{Y}_M^{2+\epsilon}\right)}{k^{2+\epsilon}} \leq \frac{C''k^{-(2+\epsilon)}}{M^{1+\epsilon/2}}.$$

Altogether, this implies that

$$E\left(\overline{Y}_M I_{(-k,\infty)}\right) \leq \sqrt{\frac{\sigma^2}{M}} \sqrt{\frac{C''k^{-2+\epsilon}}{M^{1+\epsilon/2}}} = \frac{C'''}{M^{1+\epsilon/4}}.$$

The bounds in (12) are therefore

$$\tilde{U}_{NM,M} \leq \frac{C'''}{M^{1+\epsilon/4}} - \frac{C'}{M}, \qquad C' > 0,$$

which is strictly negative for large M. Thus, as $U_{NM,M} \leq \tilde{U}_{NM,M}$, we know that $U_{NM,M} \leq U_{0,1} = 0$ for large M. Therefore, there can be no genuine nondiversification trap in this case either. ■

References

Abramowitz, M., and I. A. Stegun, 1970, *Handbook of Mathematical Functions* (Dover).

Azariadis, C., and J. J. Stachurski, 2006, Poverty traps, in Aghion P., and Durlauf S., ed.: *Handbook of Economic Growth*. Elsevier.

Barro, R. J., and X. Sala-i-Martin, 2004, *Economic growth* (MIT Press).

Cummins, J.D., N. Doherty, and A. Lo, 2002, Can insurers pay for the 'big one'? measuring the capacity of the insurance market to respond to catastrophic losses, *Journal of Banking and Finance* 26, 557–583.

Cummins, J. D., 2005, Should the government provide insurance for catastrophes, The Wharton School, University of Pennsylvania, Working paper.

Froot, K., and S. Posner, 2002, The pricing of event risks with parameter uncertainty, The Geneva Papers on Risk and Insurance Theory, 27-153-165.

Froot, K.A., D. Scharsfstein, and J. Stein, 1993, Risk management: coordinating corporate investment and financing decisions, *Journal of Finance* 48, 1629–1658.

Froot, K. A., 2001, The market for catastrophe risk: a clinical examination, *Journal of Economic Theory* 60, 529–571.

Ibragimov, R., 2004, Portfolio diversification and value at risk under thick-tailedness, Harvard Institute of Economic Research Discussion Paper #2086. Available at http://post.economics.harvard.edu/hier/2005papers/HIER2086.pdf.

——— , 2005, New majorization theory in economics and martingale convergence results in econometrics, Ph.D. dissertation, Yale University.

——— , and J. Walden, 2005, The limits of diversification when losses may be large, Harvard University, Working paper.

Jaffee, D., 2006, Should the government provide insurance for catastrophes?, comments on a paper by J. David Cummins", presented at the 30th Federal Reserve Bank of St. Louis Policy Conference, Revised January 2006.

Rosenthal, H. P., 1970, On the subspaces of L^p $(p > 2)$ spanned by sequences of independent random variables, *Israel Journal of Mathematics* 8, 273–303.

Samuelson, P. A., 1967, General proof that diversification pays, *The Journal of Financial and Quantitative Analysis* 2, 1–13.