Catastrophe Bonds, Reinsurance, and the Optimal Collateralization of Risk Transfer*

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Abstract

Catastrophe bonds feature full collateralization of a specific risk, and thus appear to be inefficient risk transfer instruments—ones that completely abandon the modern insurance principle of economizing on collateral through diversification. We confirm this paradox in an idealized world of complete insurance contracts, where catastrophe bonds indeed have no role to play. However, the real world admits a potentially important role. Insurers may find it difficult to contract completely over the division of assets in the event of insolvency, and, more generally, difficult to write contracts with a full menu of state contingent payments. Instead, the basic contract promises to indemnify with reference to a particular customer's loss experience only. In this environment, customers of the insurer will have different levels exposure to default. When contracting constraints limit the insurer's ability to smooth out such differences, there is a potential niche for catastrophe bonds in serving those who would be heavily exposed to default. We show that catastrophe bonds may be useful in mitigating differences in default exposure, which arise with: (1) contractual incompleteness, and (2) heterogeneity among insureds. Heterogeneity is required, because it undermines the efficiency of a mechanical pro rata division of assets that takes place in the event of insurer insolvency.

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*The views expressed in this paper are those of the authors, and do not necessarily reflect the views of the Federal Reserve Bank of New York, the Federal Reserve System, the RAND Corporation, or the RAND Center for Terrorism Risk Management and Policy.

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1 Introduction

Recent disaster experience has produced a flurry of economic inquiry into catastrophe insurance markets. Especially puzzling is the apparent incompleteness of catastrophe risk transfer: The price of risk transfer seems high, risk is not spread evenly among insurers in the manner suggested by Borch's [1] groundbreaking theoretical result, and, in stark contrast to Arrow's well-known characterization of optimal insurance contracts, reinsurance consumers do not purchase coverage for high layers of risk. Froot [5] documents these puzzles and fingers various market imperfections as possible explanations.

Many observers view the catastrophe bond as a promising vehicle for overcoming imperfections in the reinsurance market. In this view, the catastrophe bond opens a direct channel for catastrophe risk to flow to the capital markets, sidestepping the frictions present in the reinsurance market and connecting those who need protection with well-funded investors eager to provide it. However, others are skeptical that catastrophe securitization will be a panacea. Bouriaux and Scott [2] argue that the terms of securitization are unlikely to be attractive to buyers of terrorism coverage and note that, in general, the record of risk-linked capital market instruments has not been encouraging. Indeed, catastrophe bond issuance to date has been underwhelming, even in the aftermath of events that were expected to "push" issuance. While it is far too early to write an epitaph for the catastrophe bond, the experience to date does raise questions about its theoretical foundations and its likely future role.

On closer inspection, the catastrophe bond concept seems paradoxical and almost atavistic. Its current form features full collateralization and links principal forfeiture only to specific risks, thereby retreating from the fundamental, time-tested concept of diversification that allows insurers to protect insured value far in excess of the actual assets held as collateral. In a world where frictional costs (e.g., due to taxes, regulations, moral hazard, etc.,) make capital costly to hold, diversification allows reinsurers to economize on costly collateral. Viewed in this light, a fully collateralized capital instrument seems an unlikely competitor for traditional reinsurance products.

This paper examines this issue by developing a theory of risk collateralization. Specifically, we study the efficient division of risk-bearing assets between reinsurance company assets and catastrophe bond principal (both of which can be used to "collateralize" promises to indemnify consumers). In a narrow sense, the analysis confirms the intuition suggested above. When reinsurance companies can write any type of contract with their insureds and frictional costs associated with catastrophe bond principal are identical to those associated with assets held in reinsurance companies, catastrophe bonds are at best redundant instruments, and at worst welfare-reducing. Intuitively, if the insurer has complete freedom to vary indemnity payments to consumers in every state of the world, it can engineer any possible catastrophe bond pay-out through its contracts.

However, this result is a narrow one, because reinsurers and insurers face contracting constraints in practice. In particular, the contracts typically promise an indemnity payment contingent on the policyholder's experience and do not ordinarily specify rules ex ante for who gets what in the event of insolvency. In bankruptcy, it is often assumed that the receiver will use mechanical rules that assign payouts to insureds on a pro-rata basis that depends on the size of claims relative to assets, or on a first-come-first-served basis. Whatever the rule, companies either do not have the ability or, for practical reasons, do not attempt to specify the rule contractually. As a result, assets are effectively distributed according to inflexible mechanical rules under bankruptcy.

These constraints on asset distribution under default open up a role for catastrophe bonds, even if insurance company assets and catastrophe bond principal have similar frictional costs. When insureds and risk are homogeneous, insurance contracts are similar, and pro rata rules perform well. Heterogeneity, however, exposes the shortcomings of pro rata rules by misallocating assets in the bankruptcy state. Pro rata allocations can be suboptimal when some insureds are more concerned about the bankruptcy state than others, and this will generally be the case: Reinsurance buyers hold policies of differing quality even when purchasing these policies from the same reinsurer. Even if the reinsurer were rated "A"—on the basis, say, of being below some threshold expected policyholder deficit—this is just an *average* across policyholders. If policyholders have different risk profiles, some will be more exposed to default than others, meaning that some are effectively holding "A+" policies (or better), and some are holding "A-" policies (or worse). Those that have greater exposure to bankruptcy risk may desire greater collateralization of their potential claims than can be provided under mechanical rules. This need opens up a role for catastrophe bonds in the risk transfer market.

We show that catastrophe bonds can improve the welfare of insureds when reinsurers face constraints on the distribution of assets in bankruptcy, *and* when they must insure a heterogeneous group of risks. Catastrophe bonds can smooth out capital allocations made ragged by the risk of bankruptcy. Put differently, reinsurance capital may weakly dominate the catastrophe bond in terms of raising *average* policy quality, but such capital can be rendered a blunt instrument by bankruptcy laws. Catastrophe bonds can improve welfare for those insureds most exposed to bankruptcy risk.

The paper is laid out as follows. We begin by providing some historical background on the catastrophe bond as an insurance vehicle. We then develop a simple two consumer model of optimal risk transfer to analyze the trade-off between reinsurance equity and catastrophe bonds. Finally, we generalize our results to the case of N consumers.

2 Background

At the end of 2004, catastrophe bond issuance was running around \$1 billion per year, with outstanding principal in the neighborhood of \$5 billion.[1] These numbers are dwarfed by the comparable figures for reinsurance equity, but, in its short life, the catastrophe bond

[1] Source: *The Growing Appetite for Catastrophe Risk: The Catastrophe Bond Market at Year-End 2004*, MMC Securities.

market is still evolving and, in particular, moving toward higher layers of risk. While the first catastrophe bonds linked forfeiture of principal tied to the issuer's actual losses (an *indemnity trigger*), the typical issue today is done by a reinsurer with forfeiture of principal tied either to industry losses, model output, or to specific parameters of the disaster (e.g., the strength of an earthquake centered in a certain geographic region). Moreover, it is not uncommon for today's deals to feature multiple event triggers—requiring two or more major disasters within a short time period to trigger principal forfeiture (see Woo [8]).

The theory of the insurance firm has made a great deal of progress in understanding the joint determination of multiple line pricing, capital allocation, and the firm's overall default risk. Myers and Read [7], Zanjani [9], and Cummins et al. [3] study various aspects of this problem. The first two papers derive formulae for allocating capital costs across policyholders based on each policyholder's contribution to the firm's default risk (or default value) at the margin. However, these models considered the default risk of the firm *as a whole*. Less progress has been made in studying differences across policyholders in their exposure to default.

These differences across policyholders though are central to the value of catastrophe bonds. If the object of interest is a single default-related financial target for the company as a whole—such as the expected policyholder deficit per dollar of liabilities—a dollar held in the form of a catastrophe bond cannot possibly be preferable to one held as company equity. Since the dollar held as equity will be available in all states of the world, it will be available to pay for all of the losses that will be covered by a catastrophe bond and some losses that are not covered by the catastrophe bond.

To understand how catastrophe bonds can be used, we must move beyond thinking of a single default-related financial target for the insurance company. Instead, we must think about the company's policies as having varying levels of quality, corresponding to varying levels of exposure to default, and how catastrophe bonds and equity have distributional consequences for recoveries by different policyholder groups in states of default.

3 Optimal Collateralization with Two Consumers

For expositional purposes, we begin by studying the collateralization problem in a world with just two insureds. We later show how the results from this model generalize to a model with multiple insureds.

3.1 The Environment

Consider a world with two consumers. Each risks a loss of fixed size, but the actual loss size and probability may differ across the consumers. Specifically, suppose that consumer 1 loses L_1 with probability p_1, while consumer 2 loses L_2 with probability p_2. We develop our results under the assumption that the risks are independent, but this assumption is not necessary for most of the results (excepting those in Section 4.1).

There are two risk transfer technologies available to insure against losses. First, the consumers can set up an insurance company and issue themselves insurance policies collateralized by the assets of the company. Second, they can issue risk-linked securities (i.e., catastrophe bonds) that pay off (i.e., provide protection to the issuer) in the event of loss.

The insurance company is formed with assets of E, which represents capital contributed by investors and premiums paid by policyholders. In either case, each dollar of assets results in per unit frictional costs of δ_A. Consumers pay for these frictional costs, as well as the expected value of claims associated with the insurance policies, with consumers 1 and 2 paying c_1 and c_2, respectively. In the event of a loss (or losses), the consumers can draw on the assets to pay claims. When there are no losses, all assets revert to the investors.

Throughout our discussion, we think of "assets" as all the resources the insurer can use to pay claims. Therefore, it includes both capital paid in by investors and premiums paid in by consumers. For our purposes, the key characteristics of assets are their frictional cost, and their availability for claims payment.

The consumers can separately issue catastrophe bonds to investors. The principal of the bond is forfeited to the consumer in the event of a loss, but not otherwise. Let B_1 and B_2 be the bond issuance of consumers 1 and 2, respectively. Each dollar of bond principal raised has the frictional cost δ_B, and investors also receive payment for expected losses, just as with insurance company capital. We simplify matters by focusing on *indemnity* triggers and thus avoiding the complexities of optimal trigger design (see Doherty and Mahul [4]).

This focus therefore abstracts from direct modeling of the costs associated with asymmetric information, but such costs can be thought of as being embedded in the frictional cost δ_B.

Any difference in frictional cost (e.g., $\delta_A \neq \delta_B$) will obviously create a potential advantage for one of the technologies, but we will start by considering the case where

$$\delta_A \equiv \delta_B \equiv \delta.$$

Thus, we start by studying how the nature of preferences and risk affect the optimal mix of the two risk transfer technologies.

3.2 Unconstrained Contracting

First, consider an unconstrained contracting world in which insurance policy indemnity payments are fully state-contingent. If consumer 1 suffers the only loss, she receives I_1; if consumer 2 suffers the only loss, she receives I_2; if both suffer losses, consumers 1 and 2 receive I_B^1 and I_B^2, respectively.

A Pareto efficient solution features bond issuance, state-contingent indemnity payments, cost allocations, and capital distributed across the two risk transfer technologies in order to maximize the weighted sum of expected utilities for both consumers. Without loss of generality, we consider the symmetric Pareto optima, where each consumer receives equal Pareto weight. Starting from this point, movements along the Pareto frontier can always be effected using uncontingent transfers from one consumer to another. Formally, the problem is:

$$
\begin{aligned}
&\max_{I_1,I_2,I_B^1,I_B^2 B_1,B_2,c_1,c_2,E} p_1p_2\{U_1(W-L_1+I_B^1-c_1+(1-\delta-p_1)B_1)+ \\
&U_2(W-L_2+I_B^2-c_2+(1-\delta-p_2)B_2)\} \\
&+p_1(1-p_2)\{U_1(W-L_1+I_1-c_1+(1-\delta-p_1)B_1)+U_2(W-c_2-(\delta+p_2)B_2)\} \\
&+p_2(1-p_1)\{U_1(W-c_1-(\delta+p_1)B_1)+U_2(W-L_2+I_2-c_2-(1-\delta-p_2)B_2)\} \\
&+(1-p_1)(1-p_2)\{U_1(W-c_1-(\delta+p_1)B_1)+U_2(W-c_2-(\delta+p_2)B_2)\}
\end{aligned} \tag{1}
$$

subject to non-negativity constraints on the choice variables and the additional constraints (with associated multipliers μ_i),

$$c_1+c_2 \geq \delta E+p_1p_2(I_B^1+I_B^2)+p_1(1-p_2)I_1+p_2(1-p_1)I_2 : [\mu_E] \tag{2}$$

$$E \geq I_B^1+I_B^2 : [\mu_B] \tag{3}$$

$$E \geq I_1 : [\mu_1] \tag{4}$$

$$E \geq I_2 : [\mu_2] \tag{5}$$

When $\delta_A \equiv \delta_B \equiv \delta$ and there are no constraints on the indemnity structure of insurance policies, it is easy to show that catastrophe bonds are redundant risk transfer instruments. In other words, catastrophe bonds cannot strictly improve total welfare relative to what can be achieved with a reinsurance company. This follows from the ensuing theorem, which shows that the welfare level associated with any putative issuance of catastrophe bonds can be replicated by some reinsurance-only solution that eschews catastrophe bonds.

Theorem 1. *Let* $B_1^*, B_2^*, I_1^*, I_2^*, I_B^{1*}, I_B^{2*}, c_1^*, c_2^*, E^*$ *be a set of optimal choices maximizing social welfare as defined in the Pareto problem in (1). If* $B_1^* \neq 0$ *or* $B_2^* \neq 0$*, there exists another set of choices* $B_1^{**}, B_2^{**}, I_1^{**}, I_2^{**}, I_B^{1**}, I_B^{2**}, c_1^{**}, c_2^{**}, E^{**}$ *that also maximize social welfare with* $B_1^{**} = 0$ *and* $B_2^{**} = 0$*.*

Proof (Sketch). Let $E^{**} = E^* + B_1^* + B_2^*$, $I_1^{**} = I_1^* + B_1^*$, $I_2^{**} = I_2^* + B_2^*$, $I_B^{1**} = I_B^{1*} + B_1^*$, $I_B^{2**} = I_B^{2*} + B_2^*$, $c_1^{**} = c_1^* + (\delta + p_1)B_1^*$, $c_2^{**} = c_2^* + (\delta + p_2)B_2^*$, $B_1^{**} = 0$, and $B_2^{**} = 0$. It is easy to verify that these alternative choices yield equivalent welfare and satisfy all constraints. □

This result obtains, because any catastrophe bonds can be replicated at equal or lesser cost by putting assets in the insurance company and manipulating the indemnity payments (if the insurance company faces no contracting constraints).

3.3 Contracting Constraints and the Role of Catastrophe Bonds

There is no point to catastrophe bond issuance when insurance companies have complete freedom in designing state-contingent indemnity schedules, but such unconstrained contracting is not realistic. Insurance contracts rarely specify loss payments that are contingent on the losses of *other* insureds. Of course, there are contract features such as policyholder dividends and assessment provisions that distribute aggregate loss experience across consumers, and annual rate changes could be interpreted as an implicit means of accomplishing the same. However, policyholder dividends and assessment provisions are relatively unimportant in property/casualty insurance (and especially reinsurance) as a whole. Moreover, even where such features are used, they seem to be relatively crude retrospective premium adjustments rather than the detailed configurations of indemnity payments that are theoretically possible.

In particular, typical insurance contracts do not specify distinct indemnification schedules in states of default. Contracts specify the payment in the event of a loss. The actual payment will obviously depend on the loss experiences of other insureds in the event of insolvency, but the nature of that dependence hinges on mechanical rules (e.g., a pro rata payment scheme, or a "first come, first served" scheme). The allocation of company resources in the event of bankruptcy is typically not addressed in the individual contracts.

To capture this, we impose the constraint that payments to each individual in the joint-loss state must be a fixed fraction f of payments in the single-loss states, where f is fixed for both consumers. If the firm chooses $f = 1$, it holds enough assets to eliminate bankruptcy risk. If, however, $f < 1$, this indicates that claimants will be paid at equivalent rates on the dollar during bankruptcy, according to the firm's available assets. Constraints such as this can create an opportunity for catastrophe bonds. Formally, the problem now becomes:

$$\max_{I_1,I_2,B_1,B_2,c_1,c_2,E,f} p_1p_2\{U_1(W-L_1+fI_1-c_1+(1-\delta-p_1)B_1)+$$
$$U_2(W-L_2+fI_2-c_2+(1-\delta-p_2)B_2)\}$$
$$+p_1(1-p_2)\{U_1(W-L_1+I_1-c_1+(1-\delta-p_1)B_1)+U_2(W-c_2-(\delta+p_2)B_2)\} \tag{6}$$
$$+p_2(1-p_1)\{U_1(W-c_1-(\delta+p_1)B_1)+U_2(W-L_2+I_2-c_2-(1-\delta-p_2)B_2)\}$$
$$+(1-p_1)(1-p_2)\{U_1(W-c_1-(\delta+p_1)B_1)+U_2(W-c_2-(\delta+p_2)B_2)\}$$

$$s.t.\ c_1+c_2 \geq \delta E+p_1p_2f(I_1+I_2)+p_1(1-p_2)I_1+p_2(1-p_1)I_2 : [\mu_E] \tag{7}$$
$$E \geq fI_1+fI_2 : [\mu_B] \tag{8}$$
$$E \geq I_1 : [\mu_1] \tag{9}$$
$$E \geq I_2 : [\mu_2] \tag{10}$$

Since indemnity payments in the joint-loss state must be less than assets, the firm will make joint-loss payments according to the mechanical rule:

$$\text{Payout to Consumer i} = E\left(\frac{I_i}{I_1+I_2}\right) \tag{11}$$

At this contract-constrained optimum, the following first order conditions obtain:[2]

[2]We use shorthand for the utility of consumption. $U_i^{j,k}$ is the utility of consumer i in state $j,k : j = 0$

$$[B_1] : p_1 p_2 \frac{\partial U_1^{1,1}}{\partial W}(1-\delta-p_1) + p_1(1-p_2)\frac{\partial U_1^{1,0}}{\partial W}(1-\delta-p_1) - \\ p_2(1-p_1)\frac{\partial U_1^{0,1}}{\partial W}(\delta+p_1) - (1-p_1)(1-p_2)\frac{U_1^{0,0}}{\partial W}(\delta+p_1) \leq 0 \tag{12}$$

$$[B_2] : p_1 p_2 \frac{\partial U_2^{1,1}}{\partial W}(1-\delta-p_2) + p_2(1-p_1)\frac{\partial U_2^{0,1}}{\partial W}(1-\delta-p_2) - \\ p_1(1-p_2)\frac{\partial U_2^{1,0}}{\partial W}(\delta+p_2) - (1-p_1)(1-p_2)\frac{\partial U_2^{0,0}}{\partial W}(\delta+p_2) \leq 0 \tag{13}$$

$$[c_1] : -p_1 p_2 \frac{\partial U_1^{1,1}}{\partial W} - p_1(1-p_2)\frac{\partial U_1^{1,0}}{\partial W} - p_2(1-p_1)\frac{\partial U_1^{0,1}}{\partial W} - (1-p_1)(1-p_2)\frac{\partial U_1^{0,0}}{\partial W} = -\mu_E \tag{14}$$

$$[c_2] : -p_1 p_2 \frac{\partial U_2^{1,1}}{\partial W} - p_1(1-p_2)\frac{\partial U_2^{1,0}}{\partial W} - p_2(1-p_1)\frac{\partial U_2^{0,1}}{\partial W} - (1-p_1)(1-p_2)\frac{\partial U_2^{0,0}}{\partial W} = -\mu_E \tag{15}$$

$$[E] : -\delta\mu_E + \mu_B + \mu_1 + \mu_2 = 0 \tag{16}$$

$$[I_1] : p_1 p_2 f \frac{\partial U_1^{1,1}}{\partial W} + p_1(1-p_2)\frac{\partial U_1^{1,0}}{\partial W} - \mu_E p_1 - \mu_B - \mu_1 + (1-f)(\mu_B + \mu_E p_1 p_2) = 0 \tag{17}$$

$$[I_2] : p_1 p_2 f \frac{\partial U_2^{1,1}}{\partial W} + p_2(1-p_1)\frac{\partial U_2^{0,1}}{\partial W} - \mu_E p_2 - \mu_B - \mu_2 + (1-f)(\mu_B + \mu_E p_1 p_2) = 0 \tag{18}$$

$$[f] : (I_1 + I_2)(\mu_B + p_1 p_2 \mu_E) = p_1 p_2 \left(\frac{\partial U_1^{1,1}}{\partial W} I_1 + \frac{\partial U_2^{1,1}}{\partial W} I_2 \right) \tag{19}$$

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indicates that consumer 1 experienced no loss, while $j = 1$ means that consumer 1 experienced a loss; k performs a similar function for consumer 2. For example, $U_2^{1,0}$ represents the utility of consumer 2 in the state where consumer 1 has a loss and consumer 2 has no loss.

The first-order condition for I_1 can be rewritten as:

$$p_1p_2\frac{\partial U_1^{1,1}}{\partial W} + p_1(1-p_2)\frac{\partial U_1^{1,0}}{\partial W} - \mu_E p_1 - \mu_B - \mu_1 = -(1-f)[\mu_B + p_1p_2\mu_E - p_1p_2\frac{\partial U_1^{1,1}}{\partial W})] \quad (20)$$

Substituting the first-order condition for E transforms this into:

$$p_1p_2\frac{\partial U_1^{1,1}}{\partial W} + p_1(1-p_2)\frac{\partial U_1^{1,0}}{\partial W} - \mu_E(p_1+\delta) = -(1-f)[\mu_B + p_1p_2\mu_E - p_1p_2\frac{\partial U_1^{1,1}}{\partial W})] - \mu_2 \quad (21)$$

Notice that the left-hand side of this equation is exactly equal to the marginal utility of B_1.

Finally, substituting the first-order condition for f into the right-hand side yields:

$$p_1p_2\frac{\partial U_1^{1,1}}{\partial W} + p_1(1-p_2)\frac{\partial U_1^{1,0}}{\partial W} - \mu_E(p_1+\delta) = (1-f)p_1p_2\left[\frac{\partial U_1^{1,1}}{\partial W} - \frac{\frac{\partial U_1^{1,1}}{\partial W}I_1 + \frac{\partial U_2^{1,1}}{\partial W}I_2}{I_1+I_2}\right] - \mu_2 \quad (22)$$

The expression on the right-hand side characterizes the marginal utility of catastrophe bonds for agent 1. More generally, the marginal utility of bond issuance by agent i is:

$$\boxed{R_i = (1-f)p_1p_2\left[\frac{\partial U_i^{1,1}}{\partial W} - \frac{\frac{\partial U_1^{1,1}}{\partial W}I_1 + \frac{\partial U_2^{1,1}}{\partial W}I_2}{I_1+I_2}\right] - \mu_j} \quad (23)$$

To understand the circumstances under which catastrophe bonds are welfare-improving, it is useful to imagine (23) as representing the marginal utility of catastrophe bonds in an equilibrium that features only insurance policies. Put differently, this marginal utility answers the question: Can the "bond-free" solution to (6) be improved upon by an issuance of bonds?[3]

Equation (23) shows that a necessary condition for bond issuance to be strictly welfare-improving is for $\frac{\partial U_1^{1,1}}{\partial W} \neq \frac{\partial U_2^{1,1}}{\partial W}$.[4] In other words, in the joint-loss state (where the insurance company defaults), one consumer must value coverage more than the other. Intuitively, if both consumers valued coverage in the joint loss state equally, catastrophe bond issuance would have no advantage over increasing the capitalization of the insurance company. This benefit of bond issuance (the first term on the right hand side of (23) for the high-valuation consumer) is rising in the probability of joint-loss occurrence ($p_1 p_2$) and the size of the default "haircut" $(1-f)$ applied to the indemnity payment. The consumer with the higher valuation will then enjoy a potential benefit associated with bond issuance, while the lower-valuation consumer cannot possibly benefit from bond issuance.

The benefit may still be outweighed by an important potential drawback, which is captured in the second term on the right-hand side of (23). This term is positive if the indemnity payment to the *other* consumer in the single-loss state is constrained by the asset holdings of

[3]If the second-order conditions hold globally, this is equivalent to determining whether the equilibrium with bonds dominates the equilibrium without them.

[4]Put differently, the bracketed term must be non-zero. Note that the bracketed term represents $\frac{\partial U_i^{1,1}}{\partial W}$ minus a weighted average of $\frac{\partial U_1^{1,1}}{\partial W}$ and $\frac{\partial U_2^{1,1}}{\partial W}$. Thus, it can either be 1) zero for both consumers, or 2) positive for one consumer and negative for the other.

the insurance company. If this is the case, there are opportunities for diversification benefits associated with risk transfers to the insurance company that were not optimal to pursue. If this term is zero, however, there are no remaining opportunities for diversification across the two consumers, and this key disadvantage of the catastrophe bond will not be in play at the margin.

3.3.1 Homogenous Risk and Preferences

The foregoing discussion suggests that asymmetries in preferences or risk across consumers will be needed to create opportunities for issuance of catastrophe bonds, and this is in fact turns out to be the case under binary risk. When risk and preferences are identical across consumers, catastrophe bonds are redundant securities even when insurance contracting is constrained as described above.

Homogenous risk, preferences, and Pareto weights imply a symmetric solution, and this means that marginal utilities for each consumer will be equal in the joint loss state. Then, (23) reduces to:

$$R_i = -\mu_j \leq 0.$$

Thus, it is evident that catastrophe bonds will be strictly suboptimal if there are remaining opportunities for diversification within the insurance company. That is, it cannot be optimal for a catastrophe bond to be issued if the assets in the insurance company are insufficient to indemnify either consumer in the single loss state. If this were the case, the rewards to increasing the insurance company assets and simultaneously increasing indemnity payments to both consumers would exceed the rewards to catastrophe bond issuance.

If opportunities for diversification have been exhausted (i.e., $\mu_1 = \mu_2 = 0$), then $R_1 = R_2 = 0$. While this does imply that catastrophe bond issuance is not necessarily suboptimal, it does not imply that catastrophe bond issuance is necessary. In the region where diversification possibilities have been exhausted, both consumers are being fully indemnified in the single-loss state—and additions to insurance company capital thus serve only to collateralize payments in the joint-loss state. In such a circumstance, catastrophe bonds could be used in conjunction with insurance policies and insurance equity to yield an optimum, but this optimum is not unique and can be replicated without catastrophe bonds. Catastrophe bonds are viable when dealing with risk that is undiversifiable in this case, but not essential.

This following theorem proves the result more formally:

Theorem 2. *Suppose $p_1 = p_2 = p$, $L_1 = L_2 = L$, and identical utility functions. Let $B_1^*, B_2^*, I_1^*, I_2^*, I_B^{1*}, I_B^{2*}, c_1^*, c_2^*, E^*$ be a set of optimal choices maximizing social welfare as defined in the Pareto problem in (6). If $B_1^* \neq 0$ or $B_2^* \neq 0$, there exists another set of choices $B_1^{**}, B_2^{**}, I_1^{**}, I_2^{**}, I_B^{1**}, I_B^{2**}, c_1^{**}, c_2^{**}, E^{**}$ that also maximize social welfare with $B_1^{**} = 0$ and $B_2^{**} = 0$.*

Proof (Sketch). We start by proving that, with homogenous risk and preferences, a symmetric solution (with $I_1^* = I_2^*$, $B_1^* = B_2^*$, $c_1^* = c_2^*$) dominates an asymmetric one.

Suppose the opposite and denote the (asymmetric) optimal choices that maximize 6 by $I_1^*, I_2^*, B_1^*, B_2^*, c_1^*, c_2^*, E^*$. Define:

$$V_1(I,B,c) = p^2 U_1(W - L + fI - c + (1-\delta-p)B) + (1-p)^2 U_1(W - c - (\delta+p)B) \\ + p(1-p)U_1(W - L + I - c + (1-\delta-p)B) + p(1-p)U_1(W - c - (\delta+p)B) \tag{24}$$

$$V_2(I,B,c) = p^2 U_2(W - L + fI - c + (1-\delta-p)B) + (1-p)^2 U_2(W - c - (\delta+p)B) \\ + p(1-p)U_2(W - L + I - c + (1-\delta-p)B) + p(1-p)U_2(W - c - (\delta+p)B) \tag{25}$$

Since the consumers are ex ante identical, the solution is reversible in the sense that quantities allocated to each consumer could be swapped. In other words, the maximized objective function

$$V_1((I_1^*, B_1^*, c_1^*) + V_2(I_2^*, B_2^*, c_2^*)$$

is equivalent to:

$$V_2((I_1^*, B_1^*, c_1^*) + V_1(I_2^*, B_2^*, c_2^*)$$

Now consider alternative choices formed by equally weighting the optimal choices for the two consumers, as in:

$$I_\theta = 0.5 * I_1^* + 0.5 * I_2^*,$$

$$B_\theta = 0.5 * B_1^* + 0.5 * B_2^*,$$

$$c_\theta = 0.5 * c_1^* + 0.5 * c_2^*.$$

$$E_\theta = E^*$$

It is easily verified that these alternative choices satisfy the constraints satisfied by $I_1^*, I_2^*, B_1^*, B_2^*, c_1^*$, and c_2^*. BUT concavity of the utility functions allows us to apply Jensen's Inequality, implying that:

$$\begin{aligned} & V_1(I_\theta, B_\theta, c_\theta) + V_2(I_\theta, B_\theta, c_\theta) > \\ & 0.5 * V_1(I_1^*, B_1^*, c_1^*) + 0.5 * V_1(I_2^*, B_2^*, c_2^*) + 0.5 * V_2(I_1^*, B_1^*, c_1^*) + 0.5 * V_2(I_2^*, B_2^*, c_2^*) = \\ & V_1((I_1^*, B_1^*, c_1^*) + V_2(I_2^*, B_2^*, c_2^*) \end{aligned} \tag{26}$$

which implies the contradiction: The alternative symmetric choices yield a higher objective function value than the asymmetric choices while still satisfying the constraints. Thus, solutions under homogenous preferences and risk cannot be asymmetric.

With a symmetric solution, let $E^{**} = E^* + B_1^* + B_2^*$, $I_1^{**} = I_1^* + B_1^*$, $I_2^{**} = I_2^* + B_2^*$, $c_1^{**} = c_1^* + (\delta + p_1)B_1^*$, $c_2^{**} = c_2^* + (\delta + p_2)B_2^*$, $B_1^{**} = 0$, and $B_2^{**} = 0$. It is easy to verify that these alternative choices yield equivalent welfare and satisfy all constraints. Going further, it is possible for a symmetric solution without catastrophe bond issuance to strictly dominate one with issuance if the optimal equity level is less than $2L$. □

3.3.2 Heterogeneous Losses

Equation 23 implies that catastrophe bonds are optimal for at most one consumer, and will be written for the consumer with the higher marginal utility of consumption in the joint-loss state. We now prove a theorem demonstrating that, all else equal, this will be the consumer with the larger loss. We begin with the following useful lemma.

Lemma 3. *Suppose $p_1 = p_2 = p$, $L_1 > L_2$, and consumers have identical utility functions. In an reinsurance-only equilibrium, $I_1 > I_2$ or $I_1 = I_2 = E$.*

Proof. We first prove that $I_1 \geq I_2$. Assume instead that $I_1 < I_2$, so that $\mu_2 \geq 0$ and $\mu_1 = 0$. In this case, $L_1 - fI_1 > L_2 - fI_2$. Since $\mu_1 = 0$, $\mu_2 \geq 0$, and $p_1 = p_2$, the first order conditions for I_1 and I_2 imply that:

$$p_1 p_2 f \frac{\partial U_1^{1,1}}{\partial W} + p_1(1-p_2)\frac{\partial U_1^{1,0}}{\partial W} \leq p_1 p_2 f \frac{\partial U_2^{1,1}}{\partial W} + p_2(1-p_1)\frac{\partial U_2^{0,1}}{\partial W} \tag{27}$$

The above expression can only be true if $c_1 < c_2$. This then implies that $\frac{\partial U_1^{0,1}}{\partial W} < \frac{\partial U_2^{1,0}}{\partial W}$ and $\frac{\partial U_1^{0,0}}{\partial W} < \frac{\partial U_2^{0,0}}{\partial W}$. These conditions, coupled with the first order conditions for c_1 and c_2 imply that

$$p_2 p_1 \frac{\partial U_1^{1,1}}{\partial W} + p_1(1-p_2)\frac{\partial U_1^{1,0}}{\partial W} > p_2 p_1 \frac{\partial U_2^{1,1}}{\partial W} + p_2(1-p_1)\frac{\partial U_2^{0,1}}{\partial W} \tag{28}$$

Inequalities 27 and 28 can coexist only if $(1-f)p_1p_2\frac{\partial U_1^{1,1}}{\partial W} > (1-f)p_1p_2\frac{\partial U_2^{1,1}}{\partial W}$, which implies in turn that $\frac{\partial U_1^{1,1}}{\partial W} > \frac{\partial U_2^{1,1}}{\partial W}$. Applied to inequality 27, this implies that $\frac{\partial U_1^{1,0}}{\partial W} < \frac{\partial U_2^{0,1}}{\partial W}$. The latter implies that $I_1 + c_1 - L_1 > I_2 + c_2 - L_2$. Since $L_1 > L_2$, and since we proved above that $c_1 < c_2$, this implies that $I_1 > I_2$. This is a contradiction. Therefore, $I_1 \geq I_2$.

Suppose instead that $I_1 = I_2 = I$. Clearly, if $I < E$, the expression in 27 holds at equality, and a contradiction is derived exactly as above. Therefore, if $I_1 = I_2 = I$, it follows that $I = E$. □

With this lemma in hand, we can now prove the following theorem.

Theorem 4. *Suppose $p_1 = p_2 = p$, $L_1 > L_2$, and consumers have identical utility functions. Let $B_1^*, B_2^*, I_1^*, I_2^*, I_B^{1*}, I_B^{2*}, c_1^*, c_2^*, E^*$ be a set of optimal choices maximizing social welfare as defined in the Pareto problem in (6). This must imply that $B_1^* \geq 0$ and $B_2^* = 0$.*

Proof. Lemma 3 implies that, under the conditions of the theorem, $I_1 \geq I_2$ in an equity-only optimum. We now decompose the proof into the analysis of two cases for the equity-only optimum: $I_1 > I_2$, and $I_1 = I_2$.

Suppose $I_1 > I_2$. Since $\mu_2 = 0$ and $\mu_1 \geq 0$, $B_1^* > 0$ and $B_2^* = 0$ if $\frac{\partial U_1^{1,1}}{\partial W} > \frac{\partial U_2^{1,1}}{\partial W}$ in an equity-only optimum. Therefore, suppose that $\frac{\partial U_1^{1,1}}{\partial W} \leq \frac{\partial U_2^{1,1}}{\partial W}$ in an equity-only optimum. Since $\mu_1 \geq 0$, $\mu_2 = 0$, and $p_1 = p_2$, the first order conditions for I_1 and I_2 imply that:

$$p_1p_2f\frac{\partial U_1^{1,1}}{\partial W} + p_1(1-p_2)\frac{\partial U_1^{1,0}}{\partial W} \geq p_1p_2f\frac{\partial U_2^{1,1}}{\partial W} + p_2(1-p_1)\frac{\partial U_2^{0,1}}{\partial W} \tag{29}$$

Since $\frac{\partial U_1^{1,1}}{\partial W} \leq \frac{\partial U_2^{1,1}}{\partial W}$, inequality 29 implies that $\frac{\partial U_1^{1,0}}{\partial W} \geq \frac{\partial U_2^{0,1}}{\partial W}$. Therefore, it must be true that $I_1 - L_1 - c_1 \leq I_2 - L_2 - c_2$. However, since $\frac{\partial U_1^{1,1}}{\partial W} < \frac{\partial U_2^{1,1}}{\partial W}$, it must also be true that $fI_1 - L_1 - c_1 > fI_2 - L_2 - c_2$. These two conditions can only be met if $(1-f)I_1 < (1-f)I_2$, but this contradicts the case we are considering.

The second case is that in which $I_1 = I_2$. $B_1^* \geq 0$ and $B_2^* = 0$ if $\frac{\partial U_1^{1,1}}{\partial W} > \frac{\partial U_2^{1,1}}{\partial W}$ in an equity-only optimum. To prove this, assume that $\frac{\partial U_1^{1,1}}{\partial W} \leq \frac{\partial U_2^{1,1}}{\partial W}$ in an equity-only optimum. In this case, $L_1 - fI_1 > L_2 - fI_2$, and $L_1 - I_1 > L_2 - I_2$. The only way the first order conditions for c_1 and c_2 could hold at equality in the presence of these conditions would be if $c_1 < c_2$. This then implies that $\frac{\partial U_1^{0,1}}{\partial W} < \frac{\partial U_2^{1,0}}{\partial W}$ and $\frac{\partial U_1^{0,0}}{\partial W} < \frac{\partial U_2^{0,0}}{\partial W}$. These conditions, coupled with the first order conditions for c_1 and c_2 imply that

$$p_2 p_1 \frac{\partial U_1^{1,1}}{\partial W} + p_1(1-p_2)\frac{\partial U_1^{1,0}}{\partial W} > p_2 p_1 \frac{\partial U_2^{1,1}}{\partial W} + p_2(1-p_1)\frac{\partial U_2^{0,1}}{\partial W} \tag{30}$$

Inequality 30, coupled with our assumption that $\frac{\partial U_1^{1,1}}{\partial W} \leq \frac{\partial U_2^{1,1}}{\partial W}$, implies that $\frac{\partial U_1^{1,0}}{\partial W} > \frac{\partial U_2^{0,1}}{\partial W}$. Therefore, $I_1 - L_1 - c_1 < I_2 - L_2 - c_2$, but since $\frac{\partial U_1^{1,1}}{\partial W} \leq \frac{\partial U_2^{1,1}}{\partial W}$, it must be true that $fI_1 - L_1 - c_1 \geq fI_2 - L_2 - c_2$. These two conditions can only be true if $(1-f)I_1 > (1-f)I_2$, which is a contradiction in this case. □

Corollary 5. *Under the conditions of Theorem 4,* $B_1^* = 0$ *only if* $I_1 = I_2 = E$.

Proof. The analysis of the $I_1 > I_2$ case in the proof of theorem 4 demonstrated that $\frac{\partial U_1^{1,1}}{\partial W} > \frac{\partial U_2^{1,1}}{\partial W}$ in the equity-only optimum. Moreover, since $\mu_2 = 0$ in this case, the return to a catastrophe bond must be strictly positive. Finally, Lemma 3 demonstrated that $I_1 = I_2 = I$ only if $I = E$, completing the proof. □

3.3.3 Frictional Costs

The results above investigated the pure risk-spreading characteristics of bonds versus insurer assets. Equation 23 can be rewritten for the case of different frictional costs simply as:

$$R_i = (1-f)p_1p_2\left[\frac{\partial U_i^{1,1}}{\partial W} - \frac{\frac{\partial U_1^{1,1}}{\partial W}I_1 + \frac{\partial U_2^{1,1}}{\partial W}I_2}{I_1 + I_2}\right] - \mu_j + \mu_E(\delta_A - \delta_B) \tag{31}$$

If bonds are cheaper than insurer assets ($\delta_B < \delta_A$), one dollar of bond issuance lowers the frictional cost of insurance provision. Note, however, that frictional costs are but one element of the return to catastrophe bonds.

If frictional costs are generated purely by intrinsic costs, this is a sufficient characterization of the problem. However, if taxation and regulatory policy contribute to frictional costs, it is important to study them further. Catastrophe bonds are often advanced as a method for sidestepping the frictions in the reinsurance market. But, of course, a different set of frictional costs exist in the catastrophe bond market. The key policy question concerns whether supply-side initiatives to promote catastrophe risk transfer are best focused on the frictional costs in the reinsurance market or those in the catastrophe bond market.

At first glance, it seems that the opportunities for welfare gains are much greater when reducing frictional costs in the reinsurance market. The value of reducing the frictional costs of insurer assets by one unit is the derivative of the Lagrangian with respect to δ_A, or,

$$V_A \equiv \mu_E E,$$

while an analogous reduction in the frictional costs of catastrophe bond principal yields

$$V_B \equiv \mu_E (B_1 + B_2).$$

Thus, the marginal benefit of reducing frictional costs in each market is directly proportional to the assets deployed in the respective market: Since far more collateral is held in the form of reinsurer assets than the form of catastrophe bond principal, the marginal impact of frictional cost reductions in the reinsurance market should be far greater than similar reductions in the catastrophe bond market.

On the other hand, the cost side of the policy equation—i.e., what resources must be sacrificed to reduce frictional costs in each market—is less clear. Indeed, the frictional cost reduction technologies could differ substantially across the markets. In particular, since the catastrophe bond market is young, there may be "low hanging fruit." For example, investments in investor education or primary and secondary bond market infrastructure could offer much larger frictional cost reductions in the catastrophe bond market than could be possible in the more mature reinsurance market.

However, it is important to stress that frictional costs constitute only one dimension of the competition between catastrophe bonds and reinsurance equity, so frictional cost reductions will not necessarily translate perfectly into corresponding movements in market performance. In particular, the drawback of diversification inefficiencies (the second term on the right-hand side of (31)) could dominate any frictional cost advantage held by the catastrophe bond.

4 Multiple Consumers

The intuition of the two-person model can be recovered in an N person model. We start with the simple example of homogenous risk and preferences before tackling the notational complexity of the general case.

4.1 Homogenous Risk and Preferences

An approach analogous to that used in the two person case (see Theorem 2) can be used to establish that the solution under homogeneity will be symmetric across consumers. With this focus, the choice problem can be simplified in recognition that all consumers have the same contracts and bond issuance. We set $p_1 = p_2 = p$ and $L_1 = L_2 = L$ and seek to solve:

$$\max_{B,I,E,c,\{f_l\}} \sum_{l=0}^{N} \binom{N}{l} p^l (1-p)^{N-l} \left[\begin{array}{c} l * U(W - L + f_l I + (1 - \delta - p)B - c) + \\ (N - l) * U(W - (\delta + p)B - c) \end{array} \right]$$

subject to the following constraints, with their associated multipliers,

$$[\mu] \quad : \quad Nc \geq \delta E + \sum_{l=0}^{N} \binom{N}{l} p^l (1-p)^{N-l} l f_l I \tag{32}$$

$$[\lambda_l] \quad : \quad l f_l I \leq E, \forall l \tag{33}$$

$$[\phi_l] \quad : \quad f_l \leq 1, \forall l \tag{34}$$

The contracting constraints are embodied in the f_l factors (which are equivalent to $1 - r$ in the two person case),which allow the indemnity payments to be scaled back in any state of the world (e.g., in states of default), but restricts any discounting to apply evenly across policyholders. Note that f_l is allowed to vary with l, the number of insureds experiencing a loss.

The first order conditions for this problem are as follows (where we use the notation U_l' to denote the marginal utility of a consumer who experienced a loss along with $l - 1$ other consumers, and $\bar{U}'$ is marginal utility in the no loss state):

$$
\begin{aligned}
[B] &: \sum_{l=0}^{N} \binom{N}{l} p^l (1-p)^{N-l} \left[l U_l' (1-\delta-p) - (N-l)\bar{U}'(\delta+p) \right] \leq 0 && (35)\\
[c] &: \sum_{l=0}^{N} \binom{N}{l} p^l (1-p)^{N-l} \left[-l U_l' - (N-l)\bar{U}' \right] + N\mu = 0 && (36)\\
[I] &: \sum_{l=0}^{N} \binom{N}{l} p^l (1-p)^{N-l} \left(U_l' - \mu \right) l f_l - \sum l f_l \lambda_l \leq 0 && (37)\\
[E] &: -\delta\mu + \sum \lambda_l \leq 0 && (38)\\
[f_l] &: \binom{N}{l} p^l (1-p)^{N-l} \left(U_l' - \mu \right) lI - \lambda_l l I - \phi_l = 0 && (39)
\end{aligned}
$$

Note that the optimality condition for $[c]$ can be used to rewrite $[B]$ as:

$$
-(\delta+p)N\mu + \sum_{l=0}^{N} \binom{N}{l} p^l (1-p)^{N-l} l U_l' = -\delta\mu + \sum_{l=0}^{N} \binom{N}{l} p^l (1-p)^{N-l} \left(\frac{l}{N} \right) [U_l' - \mu] \leq 0. \tag{40}
$$

To rewrite this, we relied on the fact that, with N consumers,

$$
p = p^1(1-p)^{N-1} + (N-1)p^2(1-p)^{N-2} + ... + p^N = \sum_{i=1}^{N} \binom{N-1}{i-1} p^i (1-p)^{N-i},
$$

And, going further, that:

$$Np = N\sum_{i=1}^{N}\binom{N-1}{i-1}p^i(1-p)^{N-i} = N\sum_{i=1}^{N}\frac{N-1!}{N-i!i-1!}p^i(1-p)^{N-i} = \sum_{i=1}^{N}\binom{N}{i}ip^i(1-p)^{N-i}$$

But $[E]$ can be rewritten as:

$$-\delta\mu + \sum_{l=0}^{N}\binom{N}{l}p^l(1-p)^{N-l}\left[U_l' - \mu\right] \leq 0, \tag{41}$$

which is identical to the left hand side of (40) *except* for the weights $\left(\frac{l}{N}\right)$.

Note further that 1) $[f_l]$ implies that $[U_l' - \mu] \geq 0$ for all $l \neq 0$, and 2) $\left(\frac{l}{N}\right) < 1$ for $l < N$; $\left(\frac{l}{N}\right) = 1$ for $l = N$. This implies that left hand side of (40) will be strictly less than the left hand side of (41) unless $U_l' - \mu = 0$ for $l < N$.

In other words, in a result that echoes the two person case, cat bond issuance will be strictly suboptimal except in one particular case where it is just equivalent to insurance policies. The social planner will be indifferent between an insurance-only equilibrium and one featuring cat bonds if consumers are fully indemnified in every state of the world *except* the state where *everyone* experiences a loss. If the social planner finds it desirable to indemnify consumers in this manner, she will be indifferent between cat bonds and insurance policies as a means of providing additional coverage in the N-loss state.

Thus, the N consumer example exposes the extreme disadvantage of cat bonds with respect to diversification. Even if catastrophe bonds were cheaper than equity (i.e., if $\delta_B < \delta_A$), they could still be strictly suboptimal if the welfare-maximizing solution involved tolerance of default beyond the absolute worst case scenario of N losses.

4.2 General Case

We start by introducing notation. Define a row vector $\mathbf{x}$ of length N, with the elements all taking a value of zero or one: $\mathbf{x}(i) = 1$ means that Consumer i experienced a loss, while $\mathbf{x}(i) = 0$ means that she did not. Let Ω denote the set of all such vectors of length N with the elements taking values of one or zero. Each element of Ω corresponds to a complete description of one possible state of the world. The entire set Ω contains all possible such states. The following set definitions are useful:

$$\Omega^i = \{\mathbf{x} : \mathbf{x}(i) = 1\},$$

the set of all states in which agent i suffers a loss, and

$$\Gamma(\mathbf{x}) = \{i : x(i) = 1\},$$

the set of all agents that suffer a loss in state $\mathbf{x}$.

Additionally, the probability of state $\mathbf{x}$ can be defined as,

$$\Pr(\mathbf{x}) = \prod_{i \in \Gamma(\mathbf{x})} p_i \prod_{i \notin \Gamma(\mathbf{x})} (1 - p_i).$$

We can now define utility for Consumer i as

$$EU_i = \sum_{\mathbf{x}\in\Omega^i} \Pr(\mathbf{x}) U_i \left(W - L + f_{\mathbf{x}} I_i + (1-\delta-p_i)B_i - c_i\right) + \sum_{\mathbf{x}\notin\Omega^i} \Pr(\mathbf{x}) U_i \left(W - (\delta+p_i)B_i - c_i\right),$$

where $f_{\mathbf{x}}$ represents the proportion of the indemnity payment actually paid in state $\mathbf{x}$.

The Pareto problem can now be written as:

$$\max_{E,\{B_i\},\{c_i\},\{I_i\},\{f_{\mathbf{x}}\}} \sum_i EU_i$$

subject to:

$$[\mu] \quad : \quad \sum c_i \geq \delta E + \sum_{\mathbf{x}\in\Omega} \left(\Pr(\mathbf{x}) f_{\mathbf{x}} \sum_{i\in\Gamma(\mathbf{x})} I_i \right) \tag{42}$$

$$[\lambda_{\mathbf{x}}] \quad : \quad f_{\mathbf{x}} \sum_{i\in\Gamma(\mathbf{x})} I_i \leq E, \forall \mathbf{x} \tag{43}$$

$$[\phi_{\mathbf{x}}] \quad : \quad f_{\mathbf{x}} \leq 1, \forall \mathbf{x} \tag{44}$$

The optimality conditions are as follows (where we use the notation $U_i^{\mathbf{x}}$ to denote the utility of consumer i in state $\mathbf{x}$):

$$[B_i] \quad : \quad \sum_{\mathbf{x}\in\Omega^i} \Pr(\mathbf{x})\frac{\partial U_i^{\mathbf{x}}}{\partial W}(1-\delta-p_i) - \sum_{\mathbf{x}\notin\Omega^i} \Pr(\mathbf{x})\frac{\partial U_i^{\mathbf{x}}}{\partial W}(\delta+p_i) \leq 0 \tag{45}$$

$$[c_i] \quad : \quad -\sum_{\mathbf{x}\in\Omega} \Pr(\mathbf{x})\frac{\partial U_i^{\mathbf{x}}}{\partial W} + \mu = 0 \tag{46}$$

$$[I_i] \quad : \quad \sum_{\mathbf{x}\in\Omega^i} \Pr(\mathbf{x}) f_{\mathbf{x}} \left(\frac{\partial U_i^{\mathbf{x}}}{\partial W} - \mu\right) - \sum_{\mathbf{x}\in\Omega^i} f_{\mathbf{x}}\lambda_{\mathbf{x}} = 0 \tag{47}$$

$$[E] \quad : \quad -\delta\mu + \sum_{\mathbf{x}\in\Omega} \lambda_{\mathbf{x}} = 0 \tag{48}$$

$$[f_{\mathbf{x}}] \quad : \quad \sum_{i\in\Gamma(\mathbf{x})} \Pr(\mathbf{x}) I_i \left(\frac{\partial U_i^{\mathbf{x}}}{\partial W} - \mu\right) - \lambda_{\mathbf{x}} \sum_{i\in\Gamma(\mathbf{x})} I_i - \phi_{\mathbf{x}} = 0 \tag{49}$$

We start by observing that $\phi_{\mathbf{x}} = 0$ for all states $\mathbf{x}$ (i.e., $\forall \mathbf{x}$, the constraint $f_{\mathbf{x}} \leq 1$ fails to bind). To see this, multiply $[I_i]$ by I_i and sum over i to obtain:

$$\sum_{i=1}^{N} \sum_{\mathbf{x}\in\Omega^i} \Pr(\mathbf{x}) f_{\mathbf{x}} I_i \left(\frac{\partial U_i^{\mathbf{x}}}{\partial W} - \mu\right) - \sum_{i=1}^{N} \sum_{\mathbf{x}\in\Omega^i} f_{\mathbf{x}} I_i \lambda_{\mathbf{x}} = 0 \tag{50}$$

Next, multiply $[f_{\mathbf{x}}]$ by $f_{\mathbf{x}}$ and sum over $\mathbf{x}$ to obtain:

$$\sum_{\mathbf{x}\in\Omega} \sum_{i\in\Gamma(\mathbf{x})} \Pr(\mathbf{x}) f_{\mathbf{x}} I_i \left(\frac{\partial U_i^{\mathbf{x}}}{\partial W} - \mu\right) - \sum_{\mathbf{x}\in\Omega} \sum_{i\in\Gamma(\mathbf{x})} f_{\mathbf{x}} I_i \lambda_{\mathbf{x}} - \sum_{\mathbf{x}\in\Omega} f_{\mathbf{x}}\phi_{\mathbf{x}} = 0 \tag{51}$$

After noting that $\Gamma(\mathbf{x})$ is a null set for the state where nobody experiences a loss (i.e., where $\mathbf{x}$ is a vector of zeroes), it is evident that the first two terms of (50) are equal to the first two terms of (51),[5] implying that $\sum_{\mathbf{x}\in\Omega} f_{\mathbf{x}}\phi_{\mathbf{x}} = 0$. Thus, it is clear that $\phi_{\mathbf{x}} = 0$ for all $\mathbf{x}$.

We now derive the N-consumer analog of (23)—marginal utility of catastrophe bond issuance for the case of two consumers. After multiplying by $f_{\mathbf{x}}$, using the above result on $\phi_{\mathbf{x}}$, and rearranging, note that $[f_{\mathbf{x}}]$ can be written as:

$$\sum_{i\in\Gamma(\mathbf{x})} \Pr(\mathbf{x}) w_i^{\mathbf{x}} f_{\mathbf{x}} \left(\frac{\partial U_i^{\mathbf{x}}}{\partial W} - \mu \right) - f_{\mathbf{x}}\lambda_{\mathbf{x}} = 0,$$

where

$$w_i^{\mathbf{x}} = \frac{I_i}{\sum_{i\in\Gamma(\mathbf{x})} I_i}.$$

Summing over $\mathbf{x} \in \Omega^i$:

$$\sum_{\mathbf{x}\in\Omega^i} \Pr(\mathbf{x}) f_{\mathbf{x}} \left(\sum_{j\in\Gamma(\mathbf{x})} w_j^{\mathbf{x}} \frac{\partial U_j^{\mathbf{x}}}{\partial W} - \mu \right) - \sum_{\mathbf{x}\in\Omega^i} f_{\mathbf{x}}\lambda_{\mathbf{x}} - \sum_{\mathbf{x}\in\Omega^i} \frac{f_{\mathbf{x}}\phi_{\mathbf{x}}}{\sum_{j\in\Gamma(\mathbf{x})} I_j} = 0,$$

which is the same as $[I_i]$ except that $\frac{\partial U_i^{\mathbf{x}}}{\partial W}$ in each state is replaced by $\sum_{i\in\Gamma(\mathbf{x})} w_i^{\mathbf{x}} \frac{\partial U_i^{\mathbf{x}}}{\partial W}$ (a weighted average of the marginal utilities of all consumers who lost in that state). In summary, we have

$$\sum_{\mathbf{x}\in\Omega^i} f_{\mathbf{x}}\lambda_{\mathbf{x}} = \sum_{\mathbf{x}\in\Omega^i} \Pr(\mathbf{x}) f_{\mathbf{x}} \left(\sum_{j\in\Gamma(\mathbf{x})} w_j^{\mathbf{x}} \frac{\partial U_j^{\mathbf{x}}}{\partial W} - \mu \right). \tag{52}$$

[5] Intuitively, the summations $\sum_{i=1}^{N} \sum_{\mathbf{x}\in\Omega^i}$ represent the sum of all states in which agent i suffers a loss, summed across all agents i. This is equivalent to the sum of all agents suffering a loss in state $\mathbf{x}$, summed across all states $\mathbf{x}$, which is depicted by the double summation $\sum_{\mathbf{x}\in\Omega} \sum_{i\in\Gamma(\mathbf{x})}$.

Note that we did not need to multiply by $f_{\mathbf{x}}$. Omitting this step leads to:

$$\sum_{\mathbf{x}\in\Omega^i}\lambda_{\mathbf{x}} = \sum_{\mathbf{x}\in\Omega^i}\Pr(\mathbf{x})\left(\sum_{j\in\Gamma(\mathbf{x})} w_j^{\mathbf{x}}\frac{\partial U_j^{\mathbf{x}}}{\partial W} - \mu\right) \tag{53}$$

Working with $[B_i]$ and $[c_i]$ yields the following recharacterization of $[B_i]$:

$$R_i = -(\delta + p_i)\mu + \sum_{\mathbf{x}\in\Omega^i}\Pr(\mathbf{x})\frac{\partial U_i^{\mathbf{x}}}{\partial W} \leq 0.$$

The first term is the marginal cost of issuance—including both the frictional cost per dollar of collateral and the expected loss on the bond—and the second term is the marginal benefit, which amounts to an extra dollar of consumption in all of the loss states. Adding the left-hand side of the first order condition $[I_i]$ to the above expression, and noting that $p_i = \sum_{x\in\Omega^i}\Pr(x))$ yields the following:

$$R_i = \sum_{\mathbf{x}\in\Omega^i}\Pr(\mathbf{x})\left[1 - f_{\mathbf{x}}\right]\left(\frac{\partial U_i^{\mathbf{x}}}{\partial W} - \mu\right) + \sum_{\mathbf{x}\in\Omega^i} f_{\mathbf{x}}\lambda_{\mathbf{x}} - \delta\mu.$$

or

$$R_i = \sum_{\mathbf{x}\in\Omega^i}\Pr(\mathbf{x})\left[1 - f_{\mathbf{x}}\right]\left(\frac{\partial U_i^{\mathbf{x}}}{\partial W} - \mu\right) + \sum_{\mathbf{x}\in\Omega^i}\left[f_{\mathbf{x}} - 1\right]\lambda_{\mathbf{x}} + \sum_{\mathbf{x}\in\Omega^i}\lambda_{\mathbf{x}} - \delta\mu.$$

Substituting in from $[E]$ yields:

$$R_i = \sum_{\mathbf{x}\in\Omega^i} \Pr(\mathbf{x}) \left[1 - f_{\mathbf{x}}\right] \left(\frac{\partial U_i^{\mathbf{x}}}{\partial W} - \mu\right) + \sum_{\mathbf{x}\in\Omega^i} \left[f_{\mathbf{x}} - 1\right] \lambda_{\mathbf{x}} - \sum_{\mathbf{x}\notin\Omega^i} \lambda_{\mathbf{x}}.$$

Subtracting (53) from (52) and substituting in the resulting expression for $\sum_{\mathbf{x}\in\Omega^i} \left[f_{\mathbf{x}} - 1\right] \lambda_{\mathbf{x}}$ yields:

$$\boxed{R_i = \sum_{\mathbf{x}\in\Omega^i} \Pr(\mathbf{x}) \left[1 - f_{\mathbf{x}}\right] \left(\frac{\partial U_i^{\mathbf{x}}}{\partial W} - \sum_{j\in\Gamma(\mathbf{x})} w_j^{\mathbf{x}} \frac{\partial U_j^{\mathbf{x}}}{\partial W}\right) - \sum_{\mathbf{x}\notin\Omega^i} \lambda_{\mathbf{x}}} \tag{54}$$

This is exactly analogous to the two person case. Catastrophe bonds can be useful only for those consumers for whom the expected marginal utility of consumption in multiple loss states exceeds the average of other consumers experiencing a loss in those states (i.e., the first term on the right hand side is positive). Moreover, the viability of catastrophe bonds also depends on having exhausted diversification possibilities, as captured in the second term on the right hand side. If that term is positive, it means those possibilities still exist: There are other consumers who might enjoy benefits from increasing the capitalization of the insurance company, and this makes it more difficult for the catastrophe bond to be the preferable instrument for addressing the the risk transfer needs of the consumer in question.

5 Conclusions

In theory, catastrophe bonds can potentially be useful to ameliorate the effects of the contracting constraints faced by insurers. These constraints include the difficulty of writing contracts that can be enforced at a company's insolvency, or of contracts that are contingent on the loss experiences of other insureds. These constraints can bind when insureds are heterogeneous. Therefore, catastrophe bonds can be welfare-improving when: (1) Reinsurers face constraints on contracting, and (2) Insureds are heterogeneous. We have derived these results from models of efficient collateral allocation with two or more insureds, when the frictional costs associated with catastrophe bond issuance mirror those associated with holding assets in insurance companies.

If catastrophe bond issuance is a cheaper option (with respect to frictional costs), additional opportunities for welfare-improvement arise. However, because of the catastrophe bond's relative inefficiency in the realm of diversification, it is possible for the catastrophe bond to be cheaper and still inefficient. Thus, while frictional costs associated with underdeveloped market infrastructure and the basis risk faced by issuers are often fingered as the main roadblocks to growth in the catastrophe bond market, this analysis suggests that a more fundamental obstacle—costs deriving from the instrument's full collateralization—may ultimately place limits on its potential in the absence of further innovation.

The binary risk model used in this paper, of course, is too crude to allow detailed analysis of the microstructure of risk transfer—including how different layers of risk are allocated across the two instruments. We plan to address this in future research by studying the optimal collateralization of risk transfer in settings where the loss distributions have weight on more than two outcomes.

References

[1] Borch, K. (1962), "Equilibrium in a Reinsurance Market," *Econometrica* 30, 424-44.

[2] Bouriaux, S., and Scott, W.L. (2004), "Capital Market Solutions to Terrorism Risk Coverage: A Feasibility Study," *Journal of Risk Finance* 5(4).

[3] Cummins, J.D., Phillips, R.D., and Allen, F. (1998), "Financial Pricing of Insurance in the Multiple-Line Insurance Company," *Journal of Risk and Insurance* 65, 597-636.

[4] Doherty, N.A., and Mahul, O. (2001), "Mickey Mouse and Moral Hazard: Uninformative but Correlated Triggers," working paper, Wharton.

[5] Froot, K.A. (2001), "The Market for Catastrophe Risk: A Clinical Examination," *Journal of Financial Economics* 60, 529-71.

[6] Froot, K.A. (ed.) (1999), *The Financing of Catastrophe Risk.* Chicago and London: University of Chicago Press.

[7] Myers, S.C. and Read, J.A. (2001), "Capital Allocation for Insurance Companies," *Journal of Risk and Insurance* 68, 545-80.

[8] Woo, G. (2004), "A Catastrophe Bond Niche: Multiple Event Risk," Paper presented at the 2004 meeting of the NBER Insurance Project Group.

[9] Zanjani, G. (2002), "Pricing and Capital Allocation in Catastrophe Insurance," *Journal of Financial Economics* 65, 283-305.