An Estimated Model of the Market for Venture Capital*

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Abstract

We model the market for venture capital. VCs have the expetise to assess the profitability of projects, and have liquidity to finance them. The scarcity of VCs enables them to internalize their social value, so that the competitive equilibrium is socially optimal. This optimality obtains on an open set of parameter values.

The scarcity of VCs also leads to an equilibrium return on venture capital higher than the market rate, but our preliminary estimates show this excess return to be negligible. The ability to earn higher returns makes VCs less patient when waiting for a project to succeed; this explains why companies backed by venture capitalists reach IPOs earlier than other start-ups and why they are worth more at IPO.

1 Introduction

Venture capitalists (VCs) draw the highest returns on their investments from ventures that succeed early. When a venture-backed company has its IPO or is acquired at a very young age, the rate of return that the VC earns is very high (Guler 2002, Cochrane 2005). The longer the wait until success, the lower the return, and ventures that are terminated before success entail losses. As a result, VCs look to exit early and devote their capital and time to new, young ventures since it is they that offer the highest returns.

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Information about costs and revenues of a venture fund's portfolio companies is more plentiful for those that succeed or that are terminated early on in their lives. Older companies that have not yet IPO'd or been sold but that are not yet terminated are, however, a drain on the venture fund and typically entail negative returns. Since the winners are not known ahead of time, the overall return on a venture portfolio is an average over all these companies, determined largely by the distribution of the waiting time until success and the waiting time until termination. We use a model to estimate these two distributions and come up with an overall VC return. We find that VC returns are essentially at the market rate, the latter typically taken to be the rate on the S&P 500 portfolio. Returns to investing in private equity (of which venture funds are a subset) are similar to those on the S&P 500, though there is wide variation among funds (Kaplan and Schoar 2005). Any difference between the two represents VC compensation and the payment of direct costs of managing the funds.

The above-normal rate of return that venture capital earns derives, in our model, from its scarcity. In fact, VCs reject most of the proposals that they receive, which suggests that the demand for venture capital far exceeds its supply. In light of this, two questions naturally arise concerning efficiency. First, given that venture capital is scarce and given that a VC wishes to exit early, does the VC allocate capital efficiently, or does he underfund older ventures? I.e., if a venture has not met with success by a certain point, does the VC terminate it too early?

Second, if an entrepreneur knows that a VC will terminate unsuccessful projects early, who applies for VC backing? How poor must an entrepreneur be before she will agree to submitting her project to a venture that may terminate earlier than she would herself terminate it if only she had the funds? If her project is hard for a non-expert – such as a bank – to evaluate, an entrepreneur may not be able to get a bank loan or other debt financing. A VC may be her only recourse for obtaining such funds. But if the VC demands a large share of the project's proceeds and if he also threatens to cut off financing early if the project has not yet succeeded, the entrepreneur may avoid such a deal. Perhaps only the poorest entrepreneurs will seek VC backing, and one wonders if that is efficient.

Thus the two efficiency questions that we address are (i) the efficiency of contracts between VCs and entrepreneurs and (ii) the efficiency of project selection into venture-backed and non-venture-backed (or "solo") projects. We build a model in which the outcome is socially efficient in both dimensions.

We estimate the model and the private return to venture capital. It is likely that only some part of the VC's costs are observed in the data. In particular, the time cost involved in the initial screening and negotiation is missing in the cost data. One then needs a model to calculate the implied returns. Moreover, some of the return data are available only for projects below a certain age, and some assumption is needed about the distribution of waiting times until success for higher ages. Our model enables us to estimate the return to venture capital which turns out to be roughly the market rate which we take to be seven percent. VCs may command a higher rate of return because they are scarce. They are scarce because, presumably, it is costly to generate the human capital, the expertise that it takes to be a good VC.

The model fits fairly well some data on project returns, successes and terminations in a group of 1400 venture-backed companies. It also explains why VCs terminate maturing companies earlier than they would otherwise last. The reason is the scarcity of VCs and the resulting high rate of return that they can, in equilibrium, command on the projects that they back. This makes them impatient and leads them to more quickly terminate a not-yet-successful venture. Establishing the link precisely requires an analysis of equilibrium in the market for VCs.

Sketch of the model.—In the model, a project entails start-up costs and continuation costs. Whether it is backed by a VC or not, the project has a start-up cost that must be paid before any information about its quality can come in. After that, continuation costs must be paid until the project succeeds or is terminated. Start-up costs entail only capital, but continuation costs entail capital and effort: funds must be supplied and the entrepreneur must exert effort without interruption until the project yields fruit. Project quality in the model has two dimensions: The size of the return, and the waiting time until the return is realized. Neither dimension is known before a contract between a VC and an entrepreneur is signed. After the contract is signed, however, some uncertainty is resolved. After that, either party can, at any time, terminate the project. The entrepreneur can do so by withholding effort, and the VC can do so by withholding capital. The optimal contract is set up so that when a project is terminated, both are better off: The entrepreneur no longer wishes to exert effort, and the VC no longer wishes to lend.

Notes on the literature.—In the theoretical work, some analyses of question (i), notably Bergemann and Hege (1998, forthcoming), deal with a single VC and a single entrepreneur, with their outside options taken as given, and their efficiency implications hinge on the social optimality of the alternative payoffs. In contrast, our model places venture capitalism in a market equilibrium context where outside options are endogenous. We further discuss this paper in the light of our analysis at the end of Section 3.

A market equilibrium is what Holmes and Schmitz (1990) analyze. Their model also determines the reward to firm founders as a function of their scarcity relative to firm developers, or, rather, the scarcity of founding talent relative to managing talent. Their outcome is necessarily optimal because the businesses are traded competitively, and there is no venture capitalist involved.

Our model explains why VC's earn high returns on firms that succeed quickly, and why venture-backed ventures have higher IPO values. We match other features of the data on ventures – we shall describe those in section 4 on the properties of equilibrium. There, we shall also discuss some other models that differ in their implications for efficiency, namely, Muller and Inderst (2004), and Michelacci and Suarez (2004). None of these papers deals with the decision that an entrepreneur faces, namely, whether to seek venture backing or to finance the project in some other way. Ueda (2004) studies this choice in a model where the cost implicit in VC financing is the possibility of having the VC use the information to set up a competing business.

On question (*ii*) we focus on how an entrepreneur's wealth influences her decision about whether to seek VC backing. Not surprisingly, the wealthy entrepreneur does not need VC backing and will go it alone, whereas the poor entrepreneur has no choice but to seek VC backing. On this point, closest to what we do is Basaluzzo (2004), who analyzes partnerships when there are liquidity constraints – his results are similar in that poor entrepreneurs choose to search for partners, with whom they share the ownership and control of their firms. Basaluzzo also studies entrepreneurs' incentives to save, as does Buera (2004). Examples of empirical work on the subject are Lerner (1994), Gompers (1995) and Guler (2003).

It seems fair to say that the work on venture capital consists of theoretical work on the one hand, and empirical work on the other, and not much by way of fitting equilibrium models to the data to see how well they do quantitatively. Cochrane (2004) deals mainly with pricing the income streams to VCs, but takes those income streams to be exogenous.

Plan of the paper.—The next section describes the model, and Section 3 derives the equilibrium contract and shows that the competitive outcome is efficient. Section 4 derives several empirical implications of the model and discusses evidence, mostly from the Corporate Finance literature. Section 5 solves an example by hand and fits it to longitudinal data on VC investments, spanning 1989-2000, and their performance outcomes. Section 6 concludes the paper and the Appendix describes the data and the estimation procedure.

2 Model

There is a measure x of infinitely lived VCs, each able to borrow unlimited amounts of money at the rate r. There if also an inflow at the rate λ of potential projects, each in the possession of a different entrepreneur. The entrepreneurs cannot borrow, and have initial wealth w which is distributed according to the CDF $\Psi(w)$. An entrepreneur can have at most one idea, ever.

A Project

A project can be undertaken by an entrepreneur alone, in which case she must rely on her own wealth only, or together with a VC. For the project to succeed, it requires an immediate payment of a cost C, and after that it also requires k units of investment and a units of effort by the entrepreneur at every instant up until the project yields a return. The project yields a return π at time τ, where both π and τ are random variables, independent of one another.[1] Let F denote the distribution

[1] The independence assumption simplifies the algebra but is inessential for the results until we reach the prediction summarized in Figure 6. We shall coment further on it then.

of τ, and f the corresponding density. Let h denote the hazard rate corresponding to F, that is $h = f/(1-F)$. We assume that the hazard rate h has a bell-shape. It first increases, then decreases. In other words, as time passes without the realization of π, the agents first become more optimistic about a quick realization of π, but then they become more and more pessimistic.[2] These assumptions seem to fit the facts at least roughly; Lerner (1998, p. 738) writes:

> Immediately after a new venture is financed, the probability that there will be significant information inflows is actually likely to be quite low: the entrepreneur is in all probability focusing on the early development of his businesses. At some point thereafter, however, the probability that information will arrive increases dramatically: e.g., the results of the clinical trial will emerge, the prototype will be either be successfully developed or not, or the manufacturing yields from the new production line will become known.

If the project is either not invested into or effort is not exerted, the project cannot yield a positive return, ever. Neither party knows π and τ, but their distributions are common knowledge. In a venture-backed firm, after the contract is signed and after a cost C is incurred, π becomes known to both parties. This is where the VC has the advantage over a bank which lack the needed expertise and cannot learn π before date τ. However, no information about τ is received. In a solo venture, the entrepreneur alone incurs C at the outset, and thereby she learns π. Since the solo entrepreneur also has to pay C, it is not a project-screening cost but should instead be thought of as a lumpy initial investment.

Let G denote the distribution of π, and g the corresponding density. The expected social value to implementing projects is assumed to be positive.

Preferences

The entrepreneur and the VC are risk neutral and both discount the future at the rate r. The VC maximizes the expected discounted present value of his net income. The entrepreneur maximizes the expected discounted present value of her income minus her disutility, a_t, from exerting effort.

We choose units of a and k so that the amounts required to keep the project alive sums up to one: $a+k=1$. This normalization has no bearing on the analysis because a doubling of all costs and benefits leaves unchanged all the variables that we shall consider, namely the duration of projects and their rates of return.

Market Structure

[2] Our theoretical results also hold if the hazard declines monotonically throughout.

When an entrepreneur gets an idea, she has to decide whether to invest with a bank (at a risk-free interest r), or to seek VC-backing, or to go solo, i.e., to implement her project alone. This decision is irreversible.

Suppose at time t there is a measure n of VCs who is not in a contractual relationship with entrepreneurs and a measure of m of entrepreneurs who wishes to be financed by a VC. Then the number $\min\{n, m\}$ of VCs and entrepreneurs are randomly matched and can enter into a contractual relationship.

Timing

1. Entrepreneur chooses whether to (i) invest her wealth with a bank, (ii) develop her project on her own, or (iii) sign with a VC, in which case they sign a contract that we describe in detail presently

2. Under option (ii) or (iii) a cost C is paid immediately

3. π is fully revealed but not τ

4. No further signals come in about τ until it is realized

Contracting

Feasible Contracts —. The contract the VC can offer to the entrepreneur consists of a pair of positive numbers: (p, s). The number p is an up-front payment the entrepreneur pays the VC right after signing a contract. The number s, specifies how to share the return if the project succeeds. If the project yields return π, the entrepreneur gets $s\pi$ and the VC gets $(1-s)\pi$. Neither the effort of the entrepreneur nor the investment of the VC can be contracted on. On the other hand the payments p and $s\pi$ are enforceable.

After the transfer p, this is a pure equity contract. We could allow for more complicated contracts, where s depends on τ and π. We shall show, however, that these simple contracts already induce socially efficient decisions. Moreover, the equilibrium outcome of a game with more complicated contracts would be identical to ours.

*Timing of the Contractual Relationship —*First, the VC offers a contract, (p, s), to the entrepreneur. If the entrepreneur refuses the contract the game between these two parties ends; the entrepreneur has to leave the market and invest with a bank, the VC seeks to be matched with an other entrepreneur. If the entrepreneur signs the contract she pays p to the VC up front. We interpret p as the amount that the entrepreneur pays towards financing C. The VC will finance the remaining part of C and both parties then immediately learn the value of π.

After a length of time t, if the return has not yet been realized, both parties must decide whether to continue supporting the project or not. That is, the entrepreneur has to decide whether to exert effort and the VC has to decide whether to invest. One can assume that the parties can observe the history of investments and effort up to time t, when making these decisions.[3] If either party decides not to support the project the game between the two parties ends, otherwise it continues. When it does end, the VC is free to devote his time to another project. The entrepreneur, on the other hand, must leave the market and invest with a bank.

If the project yields a return π at time t, the entrepreneur gets $s\pi$ the VC gets $(1-s)\,\pi$ and the game ends between the two parties. Again, the VC seeks to be matched with new entrepreneurs, and the entrepreneur leaves the market.

Banks

In our model, the only role of the banks is to guarantee a risk-free interest rate, but they do not finance projects. This is because VCs are assumed to have two advantages over banks. First, banks lack the expertise of the VCs which is necessary to learn π after paying the cost C. Hence banks can only learn π at the date of success, τ, but not before. Second, banks also lack the monitoring ability of the VCs which ensures that the entrepreneurs do not divert investment to private consumption. As a result, banks do not offer contracts to entrepreneurs, for otherwise anybody could pretend to be an entrepreneur and the banks would make negative profit.

2.0.1 Comment on the informational structure

Before analyzing the model, let us comment on the informational structure .

First, why insist on the fixed cost C being paid before the VC or the entrepreneur sees π? And, second, why is there no advance information on τ?

On the one hand, it is much easier to treat the case where information is symmetric about π. But more to the point, if the VC or the entrepreneur knew τ or π before paying C it would be hard reconcile the following three facts.

1. The VC accepts about one percent of the proposals that he gets
2. The overall return on venture funds is not significantly higher than the S&P 500 return
3. A large fraction of the companies never reach IPO or acquisition and impose losses on the VC

The payment of C before τ and π are known means that returns on some ventures can be negative, while on others they can be astronomically high. The payment of k and a while waiting for success leads to terminations of ventures that have not yet succeeded.

[3] So as to avoid coordination problems, we assume that at time t the VC observes the history of efforts on the interval $[0,t]$ and that the entrepreneur observes the history of investments on the interval $[0,t)$.

3 Analysis

First, we characterize the socially optimal outcome of our model. Then we show that this outcome is the unique outcome in the competitive market conditional on some distributional assumption on the wealth of the entrepreneurs.

3.1 Socially Optimal Decisions

Our strategy of characterizing the socially optimal outcome is the following. First, we analyze the optimal decision regarding the time an individual project should be supported. This decision depends on whether the project is venture-backed , or supported by a solo entrepreneur. Second, we characterize the socially optimal decision whether an entrepreneur should go solo, seek VC-backing, or invest with a bank.

3.1.1 The termination problem of a venture-backed project

The VC has "unlimited wealth" the transfer of which over periods he values at the market rate of interest r. His time, however, can be devoted to only one project at a time, and this is where the bottleneck will arise. How long should the VC and the entrepreneur support a project? Since the value π is learnt at time zero, we derive the optimal time until the project should be supported, denoted by $T^*(\pi)$.

Let W denote the social value of a free VC. Once π is known and C has been sunk, the planner solves

$$V(\pi) \equiv \max_T \int_0^T \left(\pi + W - \frac{1}{h(t)}\right) e^{-rt} f(t)\, dt + e^{-rT}(1 - F[T])\, W \qquad (1)$$

An interior solution for this problem, $T^*(\pi)$, must solve for t the first-order condition

$$\begin{aligned} 0 &= \left(\pi + W - \frac{1}{h(t)}\right) f(t) - f(t)\, W - r(1 - F[t])\, W \\ &= \pi - \frac{1 + rW}{h(t)}, \end{aligned}$$

i.e.,

$$h(t) = \frac{1 + rW}{\pi}.$$

The local second-order condition is $h'(T^*(\pi)) < 0$. The bell-shaped hazard rate assumption guarantees that the local second-order condition is also sufficient, as shown in Figure 1.

Notice however, that T may be at a corner: π may be so low that the project yields a negative return. Let $\pi_{\min}$ be the smallest value of π for which it is worth supporting the project. A project should be funded if and only if $V(\pi) \geq W$. That is, $\pi_{\min}$ solves

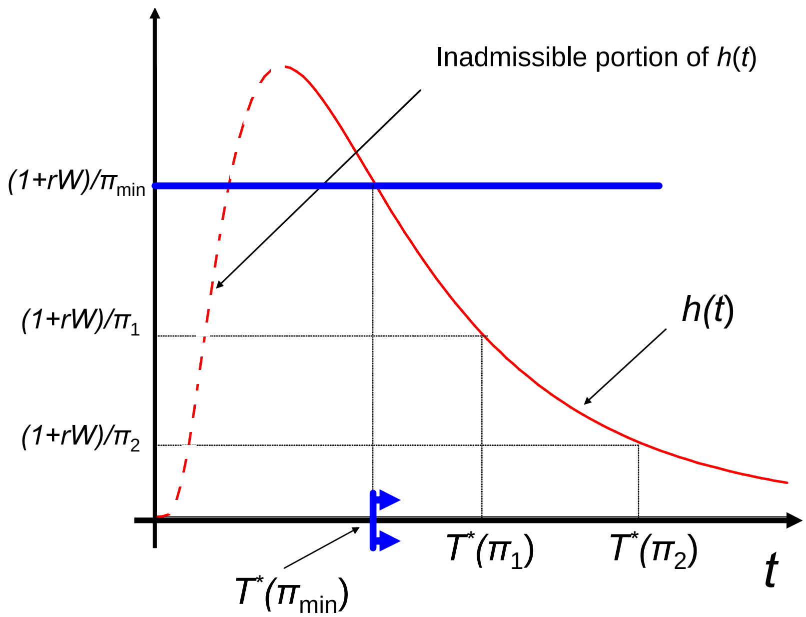

Figure 1: The determination of $T^*(\pi)$

$$V(\pi) = W. \tag{2}$$

Therefore the optimal stopping time of funding a project of quality π, $T^*(\pi)$ is defined as follows

$$T^*(\pi) = \begin{cases} h^{-1}\left(\frac{1+rW}{\pi}\right) & \text{if } \pi > \pi_{\min}, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

3.1.2 The solo entrepreneur's termination problem

For the solo entrepreneur we assume that π is also drawn from the same G as the venture-backed projects' π. More controversially, we shall assume that π is drawn independently of w; in doing so we implicitly shut off any influence that the entrepreneur's wealth may exert on the scale of businesses, an effect that Evans and Jovanovic (1989) highlight. On the other hand, an entrepreneur's wealth will generally raise the probability that the project survives, an effect that Holtz-Eakin *et al.* (1994) document.

First we solve for the optimal stopping time of a solo entrepreneur, $T^S(\pi)$, who has enough money to finance her project forever. Then $T^S(\pi)$ solves the following maximization problem:

$$\max_T \int_0^T \left(\pi - \frac{1}{h(t)}\right) e^{-rt} f(t)\, dt.$$

Hence, $T^S(\pi)$ is either equal to zero (if the value of the previous maximization problem is negative), or satisfies the first-order condition, $h\left(T^S(\pi)\right) = 1/\pi$, along with the second-order condition $h'\left(T^S[\pi]\right) < 0$. Let $\pi^S_{\min}$ denote the smallest realization of π, which should be supported by a solo entrepreneur with no budget constraint. Hence,

$$T^S(\pi) = \begin{cases} h^{-1}\left(\frac{1}{\pi}\right) & \text{if } \pi > \pi^S_{\min}, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Next, we turn to solving the decision regarding a solo entrepreneur's project. The law of motion for the solo entrepreneur's wealth (assuming she does not consume out of it but only covers her business expenses)

$$\frac{dw_t}{dt} = rw_t - k$$

for $t < T$, where w_t denotes the wealth of the entrepreneur's wealth at time t. The solution is

$$w_t = \frac{k}{r} + \left(w_0 - C - \frac{k}{r}\right)e^{rt}. \tag{5}$$

This is because at time zero, she has to incur the cost C in order to learn π, so that initial wealth is effectively $w_0 - C$. Let $\tau(w)$ be the date at which a solo entrepreneur's wealth runs out conditional on no success until then. In other words, the function $\tau(w)$ is the project's maximum financial life if it does not succeed. Then $\tau(w)$ solves for t the equation $k/r + (w - C - k/r)e^{rt} = 0$. Its solution is

$$\tau(w) = \begin{cases} \frac{1}{r}\ln\left(\frac{k}{k - r(w-C)}\right) & \text{if } w < \frac{k}{r} + C \\ +\infty & \text{otherwise} \end{cases} \tag{6}$$

The date-zero value of the solo entrepreneur's decision problem now is

$$\begin{aligned} & q(\pi, w_0) \\ \equiv\ & \max_T \int_0^{\min(\tau,T)} \left(\pi + w_t - \frac{1-k}{h(t)}\right) e^{-rt} f(t)\,dt + \left(1 - F\left[\min(\tau, T)\right]\right) e^{-r\min(\tau,T)} w_{\min(\tau,T)} \end{aligned} \tag{7}$$

using (5).

If the entrepreneur drops a project immediately, she ends up with $w - C$. Since q is increasing in π, $\pi_{\min}(w)$ solves

$$q(\pi, w) = w - C. \tag{8}$$

Differentiating (7), the solo entrepreneur's FOC in the region where $T < \tau(w)$ is

$$\begin{aligned} 0 &= \left(\pi + w_T - \frac{1-k}{h(T)}\right) f(T) - f(T)\, w_T - (1 - F[T])\, r w_T + (1 - F[T])\,(r w_T - k) \\ &= \left(\pi - \frac{1-k}{h(T)}\right) f(T) - (1 - F[T])\, k \\ &= \left(\pi - \frac{1}{h(T)}\right) f(T), \end{aligned}$$

i.e.,

$$\pi = \frac{1}{h(T^S)}.$$

Therefore, if the value of this problem is positive, then the solution is $\min\left(\tau[w], T^S(\pi)\right)$, otherwise it is zero. Let $\pi_{\min}(w)$ denote the smallest realization of π for which $q(\pi, w) \geq w - C$. That is, $\pi_{\min}(w)$ is the lowest-quality project that an entrepreneur with initial wealth w will be willing to pursue further. Any project quality below $\pi_{\min}(w)$ she would terminate at once. Then the optimal stopping time, T^S, of a solo entrepreneur with initial wealth w is defined as follows

$$T^S(\pi) = \begin{cases} h^{-1}\left(\frac{1}{\pi}\right) & \text{if } \pi > \pi_{\min}(w), \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Although w enters its definition, we suppress it in the notation.

Since q is increasing in both arguments, $\pi_{\min}(w)$ is decreasing in w. That is, richer entrepreneurs will be willing to pursue lower quality projects. We refer now Figure 2 which is the solo entrepreneur's counterpart to the planner's version of the same thing in Figure 1. The decision rule in (9) is similar to the socially optimal rule in (4). One point about (9) should be clarified with the help of the figure: Since $1/\pi_{\min}$ is higher for wealthier entrepreneurs, and so terminations at youngest *strictly positive* ages will be observed among the richest entrepreneurs. But this does not mean that the rich entrepreneurs are less patient. The interval $[\pi_{\min}(w_1), \pi_{\min}(w_2)]$ consists of projects that entrepreneur 2 would terminate right away, but that entrepreneur 1 begins to terminate only at date $T^S(\pi_{\min}[w_1]) > 0$. Conditional on π, however, termination dates are not affected by w, as illustrated by the point $T^S(\pi)$ which does not depend on w.

The Socially Optimal Financing Mode

Equations (3) and (9) characterize the optimal decisions on individual projects given the decisions regarding the financing mode. It remained to determine whether an entrepreneur should go solo, seek VC-backing, or invest with a bank. Next, we restrict attention to the question whether an entrepreneur with wealth w should go solo or invest with a bank if VC-backing was not an option. We shall show that

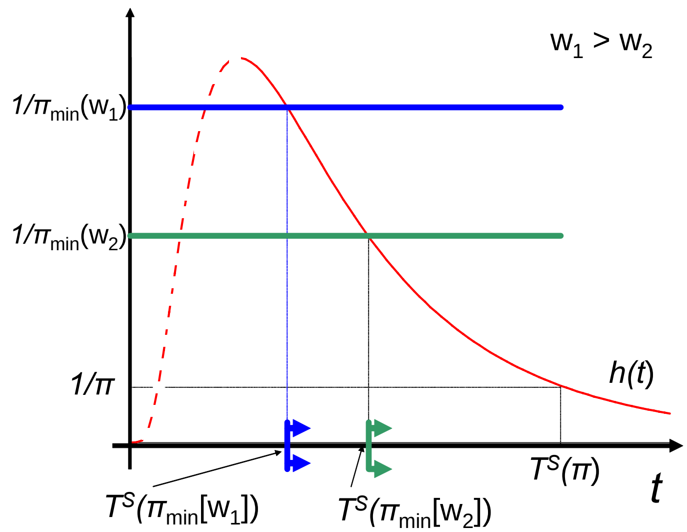

Figure 2: The determination of $T_w^S(\pi)$ for two different wealth levels

there is a cutoff level of wealth, w^*, above which the entrepreneur should go solo and otherwise should invest with a bank. Finally, we characterize those entrepreneurs who should get VC-backing.

The problem of an entrepreneur with limited wealth is that if she goes solo she might run out of money. That is, although it is socially optimal to support a project, a solo entrepreneur is unable to do so because of her liquidity constraint. Indeed, the social value of a VC in our model comes from his ability to finance *poor* entrepreneurs. Hence, those entrepreneurs should be matched with VCs who do not have enough liquidity to finance their own projects for long enough time. We shall assume that there are many poor entrepreneurs, with wealth below w^*, who would invest with a bank instead of going solo in the absence of VCs. Then, in the socially optimal outcome VCs are backing only (some of the) entrepreneurs that have wealth less than w^*.

Going Solo vs. Investing with a Bank. —From (6), $\tau'(w) = k - r[w-C]/k$. Differentiating the function q, defined by (7), with respect to w yields

$$\frac{\partial q(\pi,w)}{\partial w} = \begin{cases} 1 + \left(\frac{k-r[w-C]}{k}\right)\left(\pi - \frac{1}{h(\tau[w])}\right)e^{-rt}f(\tau[w]) & \text{if } w < \left(\frac{k}{r}+C\right)\left(1-e^{-rT^S(\pi)}\right) \\ 1 \quad \text{otherwise.} \end{cases} \tag{10}$$

The expected social value of a solo entrepreneur with wealth w is

$$Q^S(w) = \int q(\pi,w)\,dG(\pi).$$

Lemma 1 *For $w < k/r + C$,*

$$\frac{\partial Q^S}{\partial w} > 1.$$

The intuition behind the statement of this lemma is the following. A budget-constrained entrepreneur can use an additional dollar to prolong the time of supporting her project, instead of using it for consumption. The marginal value of consumption would be exactly one. Since sometimes it is socially efficient to finance the project longer than the budget-constrained entrepreneur can afford, her marginal value for a dollar exceeds one.

Proof. By (10), $dQ^S/dw \geq 1$, and it is strictly greater than unity whenever there are at least some realizations of π such that w is not enough to support the project up to the socially optimal time. But $T^S(\pi)$ is unbounded if π is. Therefore, $dQ^S/dw > 1$ whenever $w < k/r + C$. ■

On the other hand, if $w \geq k/r + C$, the entrepreneur can finance her project indefinitely if she wants. Since by assumption the expected social value of a project is positive, $Q^S(w) > w$ whenever $w > k/r + C$. Indeed, we have

Lemma 2 *For $w \geq \frac{k}{r} + C$,*

$$Q^S(w) = w + \sigma \quad \text{where} \quad \sigma \geq W\left(1 - E_{\pi,t}e^{-r\min(t,T^*(\pi))}\right) \geq 0. \tag{11}$$

An entrepreneur with $w > k/r + C$ can already support her project as long as it is socially optimal. She would use an additional dollar for consumption. Hence, her value for an additional dollar is exactly one, explaining why $Q^S(w) = w + \sigma$.

Proof. A rich-enough entrepreneur generates surplus

$$\begin{aligned}
\sigma &= -C + \int_0^{T^S(\pi)} \left(\pi - \frac{1}{h(t)}\right) e^{-rt} f(t)\, dt \\
&= -C + E_\pi \left\{ \int_0^{T^S(\pi)} e^{-rt} \left(\pi + W - \frac{1}{h(t)}\right) f(t)\, dt - W \int_0^{T^S(\pi)} e^{-rt} f(t)\, dt \right\} \\
&\geq -C + E_\pi \left\{ \int_0^{T^*(\pi)} e^{-rt} \left(\pi + W - \frac{1}{h(t)}\right) f(t)\, dt - W \int_0^{T^*(\pi)} e^{-rt} f(t)\, dt \right\}.
\end{aligned}$$

The second equality holds because we just added and subtracted $W \int_0^{T^S(\pi)} e^{-rt} f(t)\, dt$. The inequality holds because although $T^*(\pi)$ is a feasible policy for the entrepreneur, $T^S(\pi)$ is the optimal one. But, as we show later in (13), W is defined by the following equation

$$W = -C + E_\pi \left\{ \int_0^{T^*(\pi)} \left(\pi + W - \frac{1}{h(t)}\right) e^{-rt} f(t)\, dt + W e^{-rT^*(\pi)} \left(1 - F\left[T^*(\pi)\right]\right) \right\}.$$

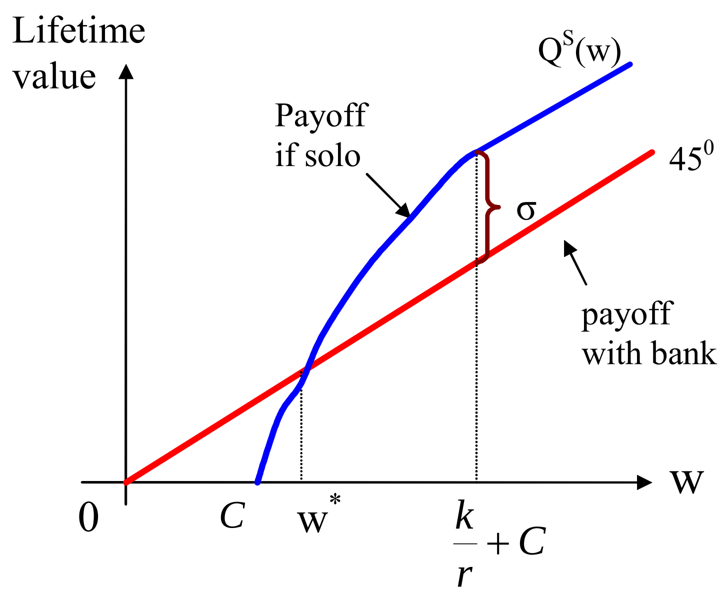

Figure 3: THE DETERMINATION OF w^*

Therefore

$$\begin{aligned}\sigma &\geq W - WE_\pi \left\{ e^{-rT^*(\pi)} \left(1 - F\left[T^*(\pi)\right]\right) + \int_0^{T^*(\pi)} e^{-rt} f(t)\, dt \right\} \\ & W - WE_\pi \left\{ \int_{T^*(\pi)}^{\infty} e^{-rT^*(\pi)} f(t)\, dt + \int_0^{T^*(\pi)} e^{-rt} f(t)\, dt \right\} \\ &= W - WE_\pi \int_0^{\infty} e^{-r\min(t,T^*(\pi))} f(t)\, dt = W\left(1 - E_{\pi,t} e^{-r\min(t,T^*(\pi))}\right).\end{aligned}$$

■

The marginal solo entrepreneur.—Thus we have shown that $Q^S(w)$ must look as drawn in Figure 3. It starts from zero when $w = C$ because at $w = C$, right after paying C, the entrepreneur would have no money left to continue supporting the project; thus $Q^S(C) = 0$. As w reaches $k/r + C$, $Q^S(w)$ reaches $w + \sigma$ which, in the case where $T^*(\pi) > 0$ for some π is strictly above the 45^0 line. We shall argue that from the Intermediate Value Theorem it follows that there exists a unique value of wealth, denoted by w^*, that solves the equation

$$Q^S(w) = w. \tag{12}$$

Thus w^* is the wealth of the poorest solo entrepreneur. Figure 3 depicts the choice between going solo and investing with a bank and the determination of w^*. The payoff, $Q^S(w)$ is continuous in w and is not defined if $w < C$ because the entrepreneur cannot pay the cost C. That w^* is unique follows because by Lemma 1 $\partial Q^S/\partial w > 1$ for $w < k/r + C$, and because $Q^S\left(\frac{k}{r} + C\right) > k/r + C$. This latter inequality holds, because the social value of a project is strictly positive. Therefore, at the point where the Q^S curve intersect with the 45-degree line, the slope of Q^S strictly exceeds unity. (Recall from Lemma 1 and Lemma 2 that the slope of Q^S turns into one only at $k/r + C$.)

Who Should get VC-backing and the Value of a free VC—We turn to the determination of W. We maintain the assumption that VCs finance those entrepreneurs who would otherwise not go solo but with invest with a bank. (Later, we provide a condition on the wealth distribution of the entrepreneurs which guarantees that this is indeed socially optimal.) Hence, the social value of a free VC is determined by the following equation

$$\begin{aligned} W &= -C + \int_0^\infty \int_0^{T^*(\pi)} \left(\pi + W - \frac{1}{h(t)}\right) e^{-rt} f(t)\, dt dG(\pi) \\ &\quad + W \int e^{-rT^*(\pi)} \left(1 - F\left[T^*(\pi)\right]\right) dG(\pi). \end{aligned} \tag{13}$$

From this

$$W = \frac{-C + \int_0^\infty \int_0^{T^*(\pi)} \left(\pi - \frac{1}{h(t)}\right) e^{-rt} f(t)\, dt dG(\pi)}{1 - \int_0^\infty \int_0^\infty \max\left(e^{-rt}, e^{-rT^*(\pi)}\right) f(t)\, dt dG(\pi)}. \tag{14}$$

To see that (14) indeed uniquely defines W, note that when W is zero, the right-hand side is positive. This is because if $W = 0$, $T^* = T^S$ and the social value of a project is positive. If W goes to infinity the right-hand side becomes negative because T^* converges to zero, and hence the $-C$ part will dominate. Finally, since the right-hand side is decreasing and continuous in W, the existence of unique solution is guaranteed by the Intermediate Value Theorem.

The average duration of the average venture-backed project, $\bar{t}$, can be computed as follows

$$\bar{t} = \int \int_0^\infty \min\left(t, T^*[\pi]\right) f(t)\, dt dG(\pi).$$

At any point in time, there is a measure $x/\bar{t}$ of free VCs. Recall that at any instance of time there is an inflow of λ of new entrepreneurs. Among them there is a measure of $\lambda\Psi(w^*)$ who has so little wealth that, in the absence of VCs, would choose to invest with a bank. If $x/\bar{t} \leq \lambda\Psi(w^*)$ then it is indeed socially optimal to match the VCs with these entrepreneurs for whom it would otherwise not be socially optimal to go solo.

The following proposition summarizes our findings:

Proposition 1 *If $\lambda\Psi(w^*) > x/\bar{t}$, the socially optimal outcome is defined as follows:*

(i) An entrepreneur with initial wealth $w > w^*$ goes solo. A measure of $x/\bar{t}$ entrepreneur gets VC-backing at every instance of time, each of them with wealth less than w^*. The rest of them invest with a bank.

(ii) The termination decision of a venture-backed project is determined by (3), and that of the solo project is by (9).

3.2 Competitive Outcome

In what follows we show that under some conditions on the distribution, Ψ, of entrepreneurs' wealth, the socially optimal outcome is implemented as a competitive equilibrium. Recall, that the social value of a free VC, W, plays an important role in determining the socially optimal decisions. In order that the VC makes optimal decisions, it is essential that his market value (when he is free) should be exactly W. But this implies that the VC must have enough market power to be able to extract all the surplus from individual projects. We guarantee this market power to the VCs by assuming that there are more entrepreneurs who is willing to seek VC-backing than VCs. In other words, we assume that $\lambda\Psi(w^*)$ is large enough compared to the available free VCs, $x/\bar{t}$.

But this is not the whole story. Recall, that in our model there is a double-sided moral hazard problem at work. Neither the effort of an entrepreneur, nor the investment of a VC is contractible. Hence, the VC must be able to provide a contract to the entrepreneur which induces the socially efficient termination decisions by both parties and, in addition, enables the VC to extract the whole surplus.

Recall, a contract consists of two numbers (p, s), where p is paid by the entrepreneur before π is realized, and s is the sharing rule upon the realization of π. We shall show that if the sharing rule is

$$s^* = \frac{a}{1 + rW},$$

the termination rules of both parties are indeed the socially optimal ones. But how can the VC extract the whole surplus from the entrepreneur?

Let $Q^{VC}(s)$ denote the continuation value to the entrepreneur from a contract specifying sharing rule s, conditional on both parties supporting the project up to $T^*(\pi)$. Then

$$Q^{VC}(s) = \int \int_0^{T^*(\pi)} \left(s\pi - \frac{a}{h(t)} \right) e^{-rt} f(t)\, dt dG(\pi).$$

Hence, the contract that enables the VC able to extract the total surplus from a project must specify an up-front payment

$$p^* = Q^{VC}(s^*). \tag{15}$$

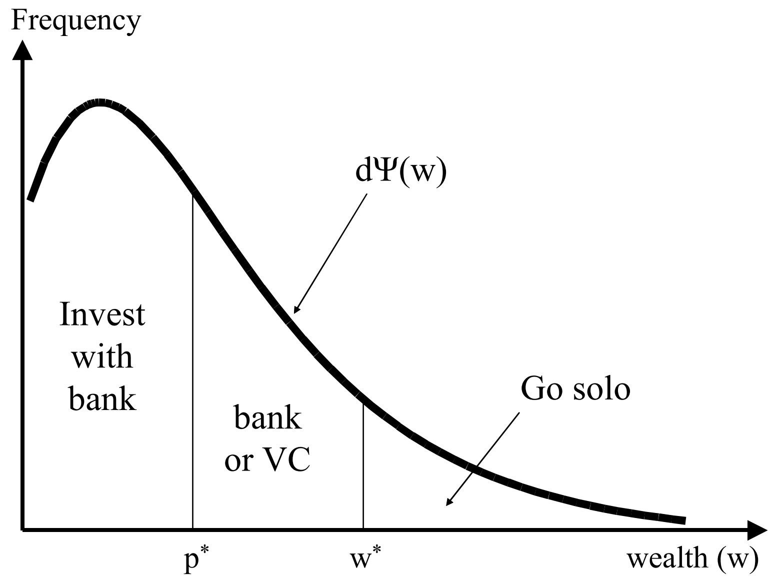

Figure 4: THE EQUILIBRIUM ALLOCATION OF ENTREPRENEURS TO ACTIVITIES

The selection of entrepreneurs into activities.—Entrepreneurs' choices of the mode of investment are described in Figure 4. The fraction of entrepreneurs that wishes to get VC backing is $\Psi(w^*)$. But of these, the fraction that can also afford to pay p^* is just $\Psi(w^*) - \Psi(p^*)$. This is the area "bank or VC" in Figure 4. Hence the distributional assumption we need is

$$\Psi(w^*) - \Psi[p^*] > \frac{x}{\lambda \bar{t}}. \tag{16}$$

Theorem 1 *If (16) holds, the socially optimal outcome is also a competitive equilibrium outcome supported by the following strategies:*

(i) A VC always offers the contract (p^*, s^*). *If the contract is accepted, he follows the socially optimal decisions, defined by (3).*

(ii) An entrepreneur with wealth $w \geq w^*$ *goes solo, and follows the socially optimal termination rule defined by (9).*

(iii) An entrepreneur with wealth $w \in (p^*, w^*)$ *seeks VC-backing with probability* $x/(\bar{t}\lambda(\Psi(w^*) - \Psi(p^*)))$ *and invests with a bank otherwise. Entrepreneurs seeking VC backing accept the contract offered by the VC, and follow the socially optimal decisions defined by (3).*

(iv) An entrepreneur with wealth $w \leq p^*$, *invests her money with a bank.*

Notice that (16) requires that w^* be larger than p^*. This turns out to be so because we have the following two Lemmas:

Lemma 3

$$p^* = \frac{(1-k)}{1+rW}C \tag{17}$$

Proof. The proof is contained in the seven lines preceding eq. (51) of the Appendix. ■

Lemma 4

$$w^* > p^*. \tag{18}$$

Proof. Using (12) and the fact that $Q^S(C) = -C$ (after paying C, the entrepreneur would have no money left to continue supporting the project), we have $w^* > C$. But from (17) $C = \left(\frac{1+rW}{1-k}\right)p^* > p^*$. These two inequalities imply (18) ■

Notice that in the equilibrium described above, the VCs extracts all the surplus from individual projects.[4] The condition $\lambda(\Psi(w^*) - \Psi(p^*)) \geq x/\bar{t}$ guarantees that (i) there are enough *poor* entrepreneurs who prefers not to go solo, but (ii) among these entrepreneurs, there are enough who has enough cash in hand to pay the VC up front the expected surplus of the project, p^*. Since the VCs extracts all the social surplus from the projects, their market value will be exactly the social value of a VC, W.

The equilibrium is further described in Figure 5. The Figure takes the equilibrium features of the contract as given, except for p. That is, as p varies, s is held fixed at s^*. Figure 5 may be explained as follows:

1. If there were no VCs, a total of $\Psi(w^*)$ entrepreneurs would simply abandon their projects and invest their wealth with banks, and the remaining $1 - \Psi(w^*)$ would go solo as shown in Figure 3.

2. Since investing with a bank offers the entrepreneur zero rents, the entrepreneurs' demand for VCs is infinitely elastic at p^* up to the point $\Psi(w^*) - \Psi(p^*)$; the poorest $\Psi(p^*)$ entrepreneurs could not afford the fee.

3. At any p higher than p^*, no one would demand VC services. At any p below this value, the payoff to going with a VC would strictly dominate that of going to a bank. But not all $\Psi(w^*)$ of the entrepreneurs could afford to sign with a VC; an entrepreneur must have w at least as large as p and there would develop a demand for VC's of at least[5] $\Psi(w^*) - \Psi(p)$. This is also where the demand curve has a kink because a reduction in p below p^* raises continuously the number of entrepreneurs that can afford the up-front fee p *and* are willing to sign with a VC.

[4] Moreover, once terminated by a VC, the entrepreneur would not wish to continue the project alone (either through self finance or bank finance) because the VC retains his equity in the project even after ceasing to invest in it. Thus the entrepreneur's reward would not rise, but her costs would, and so she would strictly prefer to stoip right away.

[5] We say "at least" because once $p < p^*$, some entrepreneurs with $w > w^*$ (i.e., some of those that would choose the solo option if p were equal p^*) will demand VC backing.

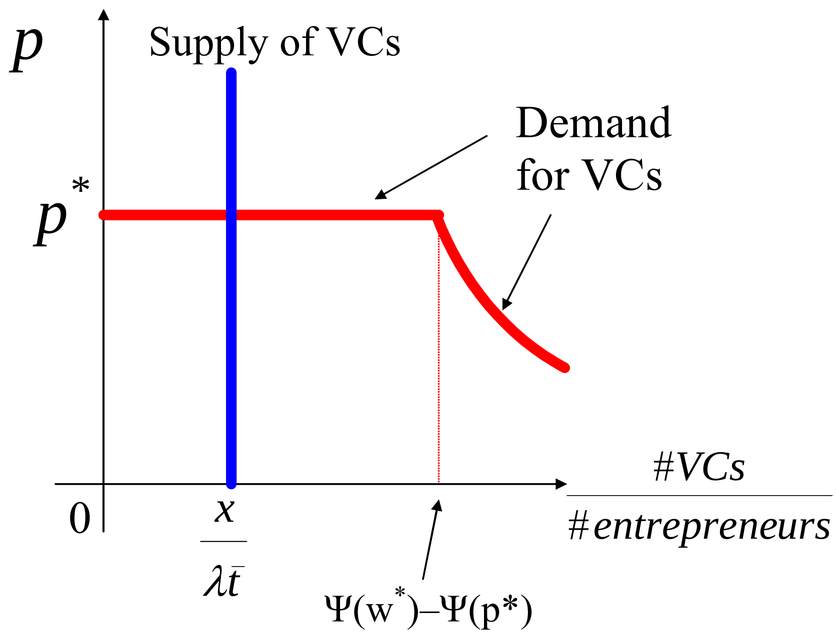

Figure 5: THE DETERMINATION OF p

3.3 Proof of Theorem 1

First, we prove that given the decision about the financing mode, the entrepreneurs' as well as the VCs' decisions regarding the termination time of a project are indeed socially optimal. That is, we prove the second part of claims (i), (ii), and (iii) of Theorem 1. If an entrepreneur decides to go solo, then she is the one who incurs all the costs related to the project, but she also enjoys all the potential benefits. In other words her costs and benefits are identical to the social costs and benefits, hence she obviously follows the socially optimal decision rules describe by (9). Therefore, we only have to show that if a project is venture backed, the entrepreneur and the VC both follow the socially optimal decision rule defined by (3).

Second, we show that given the decisions regarding the individual projects, the decisions regarding the financing mode are as described in the first parts of claims (ii), (iii), and (iv) of the theorem.

Since the VCs extracts all the surplus they obviously have no incentive to offer different contracts.

Incentive Compatibility of the Contract (p^*, s^*)

We analyze the incentives of the agents to support the project after a contract (p, s) is signed and both parties learn the value of π.

Entrepreneur.—Suppose first, that the entrepreneur trusts that the project is always financed by the VC, and that she will get $s\pi$ if the project is successful. Since the project has no salvage value, if it is terminated the entrepreneur gets zero as her terminal payoff. Recall, new ideas occur only to new entrepreneurs. Therefore she solves

$$V^{E}(\pi) \equiv \max_{T} \int_{0}^{T} \left(s\pi - \frac{a}{h(t)} \right) e^{-rt} f(t)\, dt. \tag{19}$$

If the solution, $T^{e}(\pi)$, is interior it is defined by the corresponding first-order condition:

$$h\left(T^{E}(\pi)\right) = \frac{a}{s\pi}. \tag{20}$$

The local second-order condition, which is also is also the sufficient condition, is again $h'(T^{e}(\pi)) < 0$. Finally, if the value of the maximization problem is negative, she does not start to exert effort.

VC.— Recall, the market value of a free VC, that is the expected payoff of a VC who is not yet in a contractual relationship with an entrepreneur is just W. Suppose now, that the VC trusts that the project is always supported by the entrepreneur, and he gets $(1-s)\pi$, if the project succeeds. The VC's maximization problem after signing the contract is

$$V^{VC}(\pi) = \max_{T} \int_{0}^{T} \left((1-s)\pi + W - \frac{k}{h(t)} \right) e^{-rt} f(t)\, dt + e^{-rT}(1 - F[T]) W. \tag{21}$$

In other words, the VC can find a new project immediately after one is over (whether it was terminated or whether it succeeded). If the solution, $T^{VC}(\pi)$, is interior, it must solves the first-order condition

$$h\left(T^{VC}(\pi)\right) = \frac{k + rW}{(1-s)\pi}. \tag{22}$$

The sufficient condition is again $h'\left(T^{VC}(\pi)\right) < 0$. If the value of the maximization problem in (21) is less than W, the VC does not start to invest into the project.

Incentive compatibility.— The agents stop supporting the project at the same moment if and only if $h\left(T^{VC}(\pi)\right) = h\left(T^{E}(\pi)\right)$. From (20) and (22) it follows that this equality holds if and only if

$$\frac{a}{s\pi} = \frac{k + rW}{(1-s)\pi}.$$

But this requires that

$$s = \frac{a}{(a + k + rW)}. \tag{23}$$

Optimality.— Recall from (3) that the socially optimal termination decision, $T^*(\pi)$, satisfies $h(T^*(\pi)) = (1 + rW)/\pi$. Hence, in order to achieve the socially optimal rule, we need that

$$\frac{a}{s\pi} = \frac{1 + rW}{\pi}.$$

Given the incentive-compatible s in (23), we need that

$$a + k + rW = 1 + rW.$$

But this is true since $a + k = 1$.

It remained to show that, if s is defined by (23), the minimum value of π which makes the RHS of (1) at least W, i.e., $\pi_{\min}$, is the same as the value of π which makes the RHS (21) at least W, and that the RHS of (19) is nonnegative. That is

Lemma 5 *Let $\pi_{\min}$ solve (2). If s is defined by (23), then (i) $V^{VC}(\pi_{\min}) = 0$ and (ii) $V^E(\pi) = 0$*

(proved in the Appendix).

Since V^{VC} and V^E are increasing in π, the Lemma implies that both are nonnegative for all $\pi \geq \pi_{\min}$. Thus we have shown that if $s = a/(a + k + rW)$, then both (20) and (22) become just (3). That is, for all π, $T^E(\pi) = T^{VC}(\pi) = T^*(\pi)$. This implies that the VC as well as the entrepreneur support the project up until it is socially optimal to support it. This shows the second parts of claims (*i*) and (*iii*) of the Theorem, to the effect that both parties follow the socially-optimal termination decisions defined in (3).

The Choice of Financing Mode of an Entrepreneur

We now show the first parts of claims (*ii*) and (*iii*) and claim (*iv*) of Theorem 1.

Suppose first, that an entrepreneur has initial wealth $w \leq p^*$. Since $w < w^*$, she is better off putting her money into the bank instead of going solo. Furthermore, she cannot contract with a VC, because she does not have enough liquidity to pay the VC p up-front. Hence, she invests with the bank.

Suppose that $w \in (p, w^*)$. Since $w < w^*$, entrepreneur is still better off by investing with a bank instead of going solo. However, she has enough wealth to pay p^* to the VC. Since the VCs extract all the surplus from the projects, these entrepreneurs are indifferent between seeking VC-backing or investing with a bank. So they can randomize according the claim (iii) of Theorem 1.

If $w > w^*$, the entrepreneur will go solo, since $Q^S(w) > w$, and her other options all provide her with a payoff of w.

Discussion of the efficiency result

The fact that the equilibrium contract induces the socially optimal stopping time may seem surprising at first, because there is a two-sided moral hazard problem in our model. Notice however, that if $s = a/(a+k+rW)$, then the objective function of the entrepreneur is simply a times the objective function of the social planner. True enough, when the entrepreneur solves her maximization problem, she only cares about her own cost, a, instead of the social cost $a+k$. However, if $s = a/(a+k+rW)$, then the entrepreneur cares only about her own benefit, $[a/(a+k+rW)]\pi$, instead of the social benefit π. Both the cost and the benefit in the maximization problem of the entrepreneur are down-scaled by a compared to the social surplus function. Therefore they are maximized at the same value of T.

Two assumption to guarantee that the socially optimal outcome is also supported as a competitive equilibrium. First, there must be few VCs relative to the number of entrepreneurs that seek VC-backing. And, second, among these entrepreneurs there must be sufficiently many that have enough liquidity to pay $Q^{VC}(s^*)$ up-front.

The first assumption is crucial to our result. This assumption provides the VCs with market power. They are able to offer contracts that enable them to extract the full social surplus. That is why the market value and the social value of a free VC are the same.

The assumption regarding the number of rich entrepreneurs among those who would not go solo is far less important. More complicated contracts would make it possible to extract surplus from entrepreneurs who do not have enough cash in hand to start with. Recall, that we have restricted attention to contracts which specify a time-independent sharing rule, that is s cannot depend on the time when the project succeed. The VCs could extract surplus from a more liquidity constrained entrepreneur by offering contracts when this sharing rule is increasing. Recall that with the fixed s the entrepreneur was only indifferent between exerting effort and shirking at the time of termination, but strictly preferred to exert effort anytime before. If s was allowed to change over time, the entrepreneur could have been made indifferent between working and shirking at any time before the termination of the project, and by such contracts surplus could have been extracted from *poor* entrepreneurs too.

Discussion of work on efficiency in the market for venture capital

Bergemann and Hege (1998, forthcoming) argue that dynamic contracts between entrepreneurs and VCs is inefficient relative to first best. In their model the project succeeds with some probability in each period, and the payoff is proportional to the invested funds. As in our setup, as time passes without success, agents down-date their prior, that is, they become more and more pessimistic. The main difference is that Bergemann and Hege (1998) focus on the following moral hazard problem: The entrepreneur can divert the invested funds to private consumption (with or) without the VC observing it. The trade-off the entrepreneur then faces is the following: On the downside, if she diverts the funds, she reduces the probability that the project succeeds. On the upside, if she diverts the funds: (1) she benefits directly by consuming them, (2) she potentially prolongs the time that she gets the stream of funds.

The main result of Bergemann and Hege (1998) is that the optimal contract specifies a decreasing stream of funds. The project is supported for a time that is shorter than would be socially efficient, and the project gets less funds. The reason stems from the trade-off described above. The investment stream should be specified such that the entrepreneur has no incentive to divert it. If it is decreasing, the entrepreneur understands that if she diverts it, then: (1) the payoff upon success decreases (recall it is proportional to the size of the funds), and (2) the future stream of funds is less attractive, because it is decreasing [6]

Inderst and Muller (2004) and Michelacci and Suarez (2004) build search models and assume that Nash bargaining divides the rents between the VC and the entrepreneur. As is the case in search models with a matching function, the "Hosios condition" (which states factor shares in the constant-returns-to-scale matching function should equal the factors' relative bargaining strength) must hold in order that the equilibrium be efficient. It is pure coincidence if that equality should obtain, and so generically these models imply inefficiency of equilibrium – policies that change incentives for entry by one side or the other can generally improve the sum of the payoffs. In our model, by contrast, efficiency holds on an open set of all parameter values; although (16) is not in terms of primitives, it is seen that w^*, p^*, and $\bar{t}$ are continuous in the parameters of the model, and, hence, that there is a range of all parameters for which the condition holds.

Our model takes the relative numbers of entrepreneurs and VCs as exogenous. Since VCs get the full social value of their capital, if we endogenized venture capital we would expect that an optimal amount of it would be created. Entrepreneurs receive a zero return on their ideas. Indeed, the marginal social value of another entrepreneur with an idea is zero – because society does not have a free VC to finance her idea, the entrepreneur would generate as much social benefit if she were invest her wealth with a bank. Figure 5 shows, however, that the poorest $\Psi(w^*)$ entrepreneurs receive no value from their ideas and, in a model in which getting ideas took resources, would have no incentive to devote any effort to invention. In that case, optimality would survive only if entrepreneurs received ideas about projects incidentally, say through learning by doing, or if they were born with ideas.

4 Empirical implications

This section lists some qualitative implications of the model and compares them with evidence from the Corporate Finance literature. The next section presents estimates of the model.

[6] Lerner (1998) has argued that Bergemann and Hege's assumption that the manager can divert funds is unrealistic because the VC usually monitors activities in his firms on a weekly and sometimes on a daily basis, attend monthly board meetings and so forth.

4.0.1 Value at IPO

Hochberg (2004) finds that venture-backed firms are worth more at IPO than non-venture-backed firms. We shall show that this is true if the non-venture-backed firm is managed by a wealthy solo entrepreneur. Consistent with us, the minimum value in Table 1 is smaller for the non-venture-backed sample: proceeds were smaller by a factor of two thirds, and size by a factor of almost four

The average value of a venture-backed company that succeeds at age t is $E\left(\pi \middle| T^{VC}(\pi) \geq t\right)$. Similarly, the average value of a firm a wealthy solo entrepreneurs which succeeds at age t is $E\left(\pi \middle| T^{S}(\pi) \geq t\right)$. Notice that both T^{VC} and T^{S} are increasing. Hence, in order to conclude $E\left(\pi \middle| T^{VC}(\pi) \geq t\right) \geq E\left(\pi \middle| T^{S}(\pi) \geq t\right)$ it is enough to show that $T^{VC}(\pi) \leq T^{S}(\pi)$, which is what the next Proposition does.

Proposition 2

$$T^{VC}(\pi) \leq T^{S}(\pi). \tag{24}$$

The intuition behind this proposition is the following. The number of VCs is smaller than the number of entrepreneurs. This provides VCs with market power and a high equilibrium return on their investment, which in turn, makes the VCs' opportunity cost of supporting projects high. As a result, VCs are impatient with projects that have not yet succeeded.

Proof. Whenever π is such that $T^{S}(\pi) > 0$ (i.e., so that T^{S} has an interior solution), then h is decreasing at T^{S}. If $T^{VC}(\pi) > 0$, then h is also decreasing at T^{VC} and a comparison of (4) and (3) implies $T^{S}(\pi) > T^{VC}(\pi)$. Then (24) holds if $T^{S}(\pi) = 0 \Longrightarrow T^{VC}(\pi) = 0$, i.e., if

$$\pi_{\min} \geq \pi_{\min}^{S}(w) \quad \text{for } w \geq C + k/r \tag{25}$$

We now prove that (25) holds.

Recall, $\pi_{\min}^{S}$ solves

$$\int_{0}^{T^{S}(\pi)} \left(\pi - \frac{1}{h(t)}\right) e^{-rt} f(t)\, dt = 0. \tag{26}$$

Also recall that $\pi_{\min}$ solves $V(\pi_{\min}) = W$. Since V is increasing, it is enough to show that $V\left(\pi_{\min}^{S}\right) \leq W$.

$$\begin{aligned}
V\left(\pi_{\min}^{S}\right) &= \int_{0}^{T^{VC}\left(\pi_{\min}^{S}\right)} \left(\pi_{\min}^{S} + W - \frac{1}{h(t)}\right) e^{-rt} f(t)\, dt + e^{-rT^{VC}} \left(1 - F\left[T^{VC}\right]\right) W \\
&= \int_{0}^{T^{VC}\left(\pi_{\min}^{S}\right)} \left(\pi_{\min}^{S} - \frac{1}{h(t)}\right) e^{-rt} f(t)\, dt + \int_{0}^{\infty} W e^{-r\min\left\{t, T^{VC}\left(\pi_{\min}^{s}\right)\right\}} f(t)\, dt \\
&\leq \int_{0}^{T^{S}\left(\pi_{\min}^{S}\right)} \left(\pi_{\min}^{S} - \frac{1}{h(t)}\right) e^{-rt} f(t)\, dt + \int_{0}^{\infty} W e^{-r\min\left\{t, T^{VC}\left(\pi_{\min}^{S}\right)\right\}} f(t)\, dt \\
&= \int_{0}^{\infty} W e^{-r\min\left\{t, T^{VC}\left(\pi_{\min}^{S}\right)\right\}} f(t)\, dt \leq W,
\end{aligned}$$

where the first equality follows from T^S being the solo's optimal termination rule (and not T^{VC}), and the last equality from (26). ■

Note that this proposition compares the terminations of venture-backed firms with the terminations of solo firms run by wealthy entrepreneurs. We cannot rank terminations of venture-backed firms with those of entrepreneurs with $w < C + k/r$, because the latter are sometimes forced into terminations for lack of money.[7]

4.0.2 Terminations

Proposition 2 showed that for each π, venture-backed firms are terminated faster than a solo one. But we wish to also compare the hazard rates for the two kinds of firms because this is how much of the evidence is presented. We now turn to the definition of these hazards.

The C.D.F.s of terminations.—Let $\Phi(t)$ be the CDF of terminations, and let the density be $\phi(t)$. Now $\pi \sim G(\pi)$ and the low-π projects are terminated first – see Figure 6. By (3), $\pi = (1 + rW)/h\left(T^{VC}(\pi)\right)$, and so the fraction of projects terminated by date t (conditional on no success until date t) is $\Pr(T \le t \mid \tau \ge t) \equiv \Phi(t)$. Thus the C.D.F. of terminations for the venture-backed sample is

$$\Phi^{VC}(t) = \begin{cases} 0 & \text{for } t = 0 \\ G(\pi_{\min}) & \text{for } t \in \left(0, h^{-1}\left[\frac{1+rW}{\pi_{\min}}\right]\right) \\ G\left(\frac{1+rW}{h(t)}\right) & \text{for } t \ge h^{-1}\left(\frac{1+rW}{\pi_{\min}}\right). \end{cases} \tag{27}$$

whereas for the solo sample it is

$$\Phi^{S}(t) = \begin{cases} 0 & \text{for } t = 0 \\ G(\pi_{\min}[w]) & \text{for } t \in \left(0, h^{-1}\left[\frac{1}{\pi_{\min}(w)}\right]\right) \\ G\left(\frac{1}{h(t)}\right) & \text{for } t \ge h^{-1}\left(\frac{1}{\pi_{\min}(w)}\right). \end{cases} \tag{28}$$

Note that it is conditional on no success. The *termination hazard* for the venture-backed firms therefore is

$$\psi^{VC}(t) \equiv \frac{\phi^{VC}(t)}{1 - \Phi^{VC}(t)} = \begin{cases} \infty & \text{for } t = 0 \\ 0 & \text{for } t \in \left(0, h^{-1}\left[\frac{1+rW}{\pi_{\min}}\right]\right) \\ (1 + rW)\,\gamma\left(\frac{1+rW}{h(t)}\right)\left(\frac{-h'(t)}{[h(t)]^2}\right) & \text{for } t \ge h^{-1}\left(\frac{1+rW}{\pi_{\min}}\right). \end{cases} \tag{29}$$

[7] The quality of venture capital is homogeneous in our model, hence there is no reason for the VC to try to signal higher ability by taking actions that to outsiders seem successful. "Grandstanding" is said to occur when VC sends companies to an IPO before their time in the hope of establishing a reputation for being able to quickly guide companies to success. A reputable VC can more easily open new funds. Our model does not explain grandstanding, but the finding that $W > 0$ is certainly consistent with it.

where

$$\gamma(\pi) \equiv \frac{g(\pi)}{1 - G(\pi)}$$

is the hazard rate of the profit distribution. For solo firms the hazard $\psi^S \equiv \frac{\phi^S}{1-\Phi^S}$is

$$\psi^S(t,w) = \begin{cases} \infty & \text{for } t = 0 \\ 0 & \text{for } t \in \left(0, h^{-1}\left[\frac{1}{\pi_{\min}(w)}\right]\right) \\ \gamma\left(\frac{1}{h(t)}\right)\left(\frac{-h'(t)}{[h(t)]^2}\right) & \text{for } t \geq h^{-1}\left(\frac{1}{\pi_{\min}(w)}\right). \end{cases} \tag{30}$$

Thus the wealthy solo entrepreneur will tolerate projects of quality lower than venture-backed entities will. On the other hand, we cannot say if the poor solo entrepreneurs would also tolerate lower-quality projects than VC's because $\pi^S_{\min}(w)$ is decreasing in w, so that the wealthy entrepreneur makes fewer immediate terminations than the poor entrepreneur. In other words, we don't know whether $\pi^S_{\min}(w) < \pi_{\min}$ for $w < k/r + C$, and so we cannot say if immediate terminations are higher for venture-backed or for solo-run projects. In addition, poorer entrepreneurs might run out of money and must terminate sooner.

In words the previous proposition says that as long as they are rich enough, solo entrepreneurs will bring their firms to market later than VCs, and when they do, their companies will on average be less valuable. Some of the evidence on venture-backed firms vs. other startups is roughly in the form

$$\Pr(\text{Termination}) = a_0 + a_1 \cdot \text{Venture Dummy} + a_2 \cdot \text{Firm Age} + ... \tag{31}$$

The venture dummy is set to equal one if the company is venture backed and zero otherwise. Sometimes "Age" variable is replaced by investment round; the two are highly correlated. The inclusion of age gives the regression the interpretation of a hazard rate. Proposition 2 then tells us that at least when firm age is zero, a_1 is positive.

Only with further restrictions is (31) with $a_1 > 0$ is a good approximation globally to what our model implies.

Proposition 3 *When γ is non-decreasing, then for $w \geq C + k/r$,*

$$\psi^{VC}(t) \geq \psi^S(t,w)$$

for all t for which ψ^{VC} and ψ^S are both positive.

Proof. The relation in (25) prevents us from being able to rank $\frac{1+rW}{\pi_{\min}}$ and $\frac{1}{\pi_{\min}(w)}$. Therefore we cannot say which hazard becomes positive first. From (29) and (30),

$\psi^{VC} = \psi^{S} = 0$ for $t \in \left(0, h^{-1}\left[\max\left(\frac{1}{\pi_{\min}(w)}, \frac{1+rW}{\pi_{\min}}\right)\right]\right)$ and both are positive for $t \geq h^{-1}\left[\min\left(\frac{1}{\pi_{\min}(w)}, \frac{1+rW}{\pi_{\min}}\right)\right]$. On this region,

$$\frac{\psi^{VC}(t)}{\psi^{S}(t,w)} = \frac{1+rW}{\gamma\left(\frac{1}{h(t)}\right)}\gamma\left(\frac{1+rW}{h(t)}\right) \geq 1+rW$$

because γ is non-decreasing. ■

In the example that we estimate, $G(\pi)$ is assumed to be exponential so that γ is a constant. Therefore it will have the property that $\psi^{VC} > \psi^{S}$, i.e., that the hazard of terminations is higher for venture-backed firms. The population of solo firms being compared is, however, those with $w \geq C + k/r$.

4.0.3 Survival to IPO

Our model does not distinguish types of success; bringing a product to market and having an IPO are equivalent. Ber and Yafeh (2004) find that the probability of survival until the IPO stage is higher for venture-backed companies. which, in our notation(i) Hellmann and Puri (2000) find that venture-backed companies are quicker to market their products. Translated into our notation, (i) says that τ, the date of success, occurs earlier for venture-backed companies, and (ii) says that the probability that τ occurs before termination is higher for a venture-backed company. In our model, the outcome is ambiguous. For a given π, the VC's stopping time is $\min\left(\tau, T^{VC}[\pi]\right)$. For that same π, the entrepreneur's stopping time is $\min\left(\tau, \tau[w], T^{S}[\pi]\right)$. On the one hand, $T^{VC}(\pi) < T^{S}(\pi)$. But on the other, $T^{VC}(\pi)$ may exceed $\tau(w)$. The latter may happen for those entrepreneurs with $w \in (w^*, k/r + C)$; they may run out of money before reaching their optimal termination date $T^{S}(\pi)$. Thus, whether venture-backed projects are terminated earlier or later depends on the distribution of wealth. Given that in the population of entrepreneurs w is distributed according to the CDF $\Psi(w)$, the precise condition is as follows:

Proposition 4 *If*

$$1 - \Psi\left(\frac{k}{r} + C\right) > \Psi\left(\frac{k}{r} + C\right) - \Psi(w^*), \tag{32}$$

the median venture-backed firm is likely to (i) *be terminated more quickly,* (ii) *be worth more at IPO than the median solo firm and* (iii) *succeed more quickly conditional on not being terminated.*

Proof. (i) and (ii) follow because when (32) holds, the median w satisfies the conditions of Proposition 3. (iii): For the same reason, for the median solo firm, by

(24), $T^{VC}(\pi) \le T^S(\pi)$ for each π. If it is going to succeed, then, a venture-backed project π must do so earlier than a solo project. And since the ex-ante distribution of π, $G(\pi)$ is the same for venture-backed and solo firms, venture-backed successes are quicker than solo successes. ■

These claims all follow for the simple reason that when (32) holds, the median solo firm has its w above $k/r + C$ and will be more patient with any project than a venture-backed firm would be.

4.0.4 Good projects receive more investment rounds

Gompers (1995) finds that bad projects tend to be identified early and get dropped, and that it is the good projects that receive more investment. This happens in our model: The amount that the VC expects to invest is increasing in π. First of all, projects with $\pi \le \pi_{\min}$ receive no investment beyond the initial outlay C. For a project with $\pi > \pi_{\min}$, investment proceeds for $T^{VC}(\pi)$ rounds, a number that solves the equation

$$h(T) = \frac{1 + rW}{\pi}. \tag{33}$$

At the point of intersection $h' < 0$, as shown by the solid line in Figure 1. The dashed portion is not admissible because the second-order conditions fail. When the solution exists,

$$\frac{\partial T}{\partial \pi} = -\frac{h(T)}{h'(T)} > 0,$$

so that the maximum number of investment stages rises with the project's quality. We now illustrate this in Figure 6. Until date $t = h^{-1}([1 + rW]/\pi_{\min})$, no projects are being terminated, and successes are drawn from the distribution $G(\pi \mid \pi \ge \pi_{\min})$. Projects to the left of $\pi_{\min}$ are terminated right away. At $t = h^{-1}([1 + rW]/\pi_{\min})$, the truncation point, $(1 + rW)/h(t)$ starts to move to the right. Thus the conditional mean of the projects that are funded rises. Let $\Gamma_t(\pi)$ be the distribution of π among projects that bear fruit at date t. Then for t larger than the value at which the mode of h occurs (so that the condition $h(t) = \frac{1+rW}{\pi}$ represents a maximum),

$$\Gamma_t(\pi) = \frac{G(\pi) - G\left(\max\left\{\pi_{\min}, \frac{1+rW}{h(t)}\right\}\right)}{1 - G\left(\max\left\{\pi_{\min}, \frac{1+rW}{h(t)}\right\}\right)} \tag{34}$$

for $\pi \ge \max\left\{\pi_{\min}, \frac{1+rW}{h(t)}\right\}$. For t below the mode,

$$\Gamma_t(\pi) = \frac{G(\pi) - G(\pi_{\min})}{1 - G(\pi_{\min})} \tag{35}$$

for $\pi \ge \pi_{\min}$.

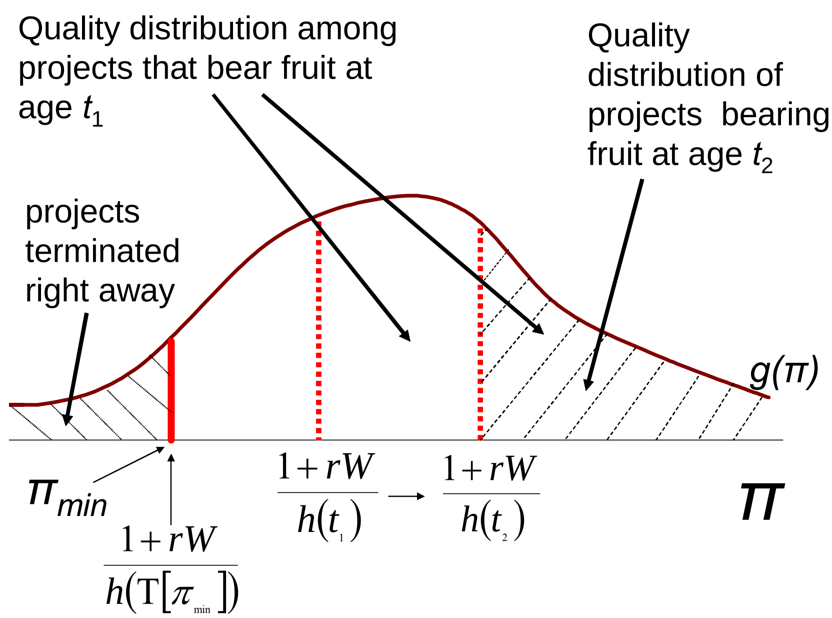

Figure 6: Good projects receive more investment rounds

This is where the assumption that π and τ are independent has bite. A sufficient negative correlation between the two would overturn the result. If high-π projects also had low-enough τ's, but if the low-π projects were still worth supporting for a while, it would be the *bad* projects that receive more investment rounds.

4.0.5 The VC's rate of return

Cochrane (2004) and Guler (2003) find that the rate of return VCs receive falls with the age of the project at completion. Cochrane computes the rate of return, and Guler computes the internal rate of return (IRR). These are slightly different concepts in the present model, so let us now derive them.

The rate of return on projects that succeed at age t.—Calculating rates if return is easy because the returns all come at the same date t. All that is needed, then, is to bring all costs (which are distributed over $[0, t]$) into date-zero dollars, i.e., to take their present value discounted at the rate r. The VC gets a fraction $(1 - s^*)$ of the payoff, The present value of all costs net of the transfer p^* would be $\int_0^t ke^{-ru}du + C - p^*$ where p^* is defined in (15). On a project of quality π, the realized rate of return, $R(t, \pi)$ would solve the equation

$$e^{R(t,\pi)t} = \frac{(1 - s^*)}{\int_0^t ke^{-ru}du + C - p^*}\pi \tag{36}$$

Now π differs over projects that succeed at t, and their distribution depends on t, being ever more truncated from the left as shown in Figure 6. Since the π's differ, so do the returns. When collapsing a distribution of returns Two concepts are used in the literature. The *geometric* rate of return, call it $R^G(t)$, given by the formula

$$R^G(t) = \int R(t,\pi)\, d\Gamma_t(\pi), \tag{37}$$

is just the average of the rates of return. The *arithmetic* rate of return satisfies the equation

$$R^A(t) = \ln \int e^{R(t,\pi)} d\Gamma_t(\pi).$$

Neither R^G nor R^A can be said to be correct or incorrect; each attempts to measure, in a single number, the properties of a distribution. because the function e^R is convex in R, Jensen's inequality implies that $R^A(t) \geq R^G(t)$, with strict inequality if Γ_t has positive variance. [8]

In a finite sample of projects with their π's drawn from Γ_t, the realized $R^G(t)$ and $R^A(t)$ would deviate from their theoretically-predicted values, but Cochrane has a fairly large sample, at least for the successes registered fairly early on in the firms' lives. Over sufficiently many projects that lasted t periods *and* succeeded, this would roughly be the realized rate of return.

The IRR on projects that succeed at age t.—Parallel to the definition of the rate of return in (36), we can define the internal rate of return on the quality-π project that matures at t; call it IRR(τ, π), as solving the equation

$$e^{\mathrm{IRR}(t,\pi)t} = \frac{(1-s^*)}{\int_0^t ke^{-\mathrm{IRR}(t,\pi)u} du + C - p^*}\pi. \tag{38}$$

Note the difference in the denominators of (36) and (38). Note that we can, once again, have an arithmetic and geometric concepts of the IRR. We shall not fit the IRR in this paper although Guler (2003) calculates it in her paper for the companies in her sample.

Proposition 5 *$R(t,\pi)$ and IRR(t,π) are strictly increasing in π and strictly decreasing in t*

[8] For instance (and this is Cochrane's assumption, but it is not consistent with our model, as is evident from Figure 6), if $\Gamma_t(\pi)$ were the normal distribution, then $R(t,\pi)$ would be normally distributed with mean $R^G(t)$, and with a variance the we shall denote by $Var_t(R)$. Then $\int e^{R(t,\pi)} d\Gamma_t(\pi) = e^{R^{G(t)} + \frac{1}{2}Var_t(R)}$, and we would then have

$$R^A(t) = R^G(t) + \frac{1}{2} Var_t(R).$$

Proof. From (36), $\ln R(t,\pi) = \frac{1}{t}\left(\ln\left[(1-s^*)\pi\right] - \ln\left[\int_0^t ke^{-ru}du + C - p^*\right]\right)$ and the claim for R follows. Multiplying both sides of (38) by the denominator of its RHS leads to

$$\int_0^t ke^{\mathrm{IRR}(t,\pi)(t-u)}du + e^{\mathrm{IRR}(t,\pi)t}(C-p^*) = (1-s^*)\pi.$$

By Lemma 3,[$C - p^* = \frac{kC}{1+rW} > 0$. Therefore The LHS is increasing in IRR(τ,π) and in t. The RHS is increasing in π, and so the claim for IRR follows as well. ■

Unfortunately, while for π fixed, $R(t,\pi)$ is declining in t, we cannot prove in general that $R^G(t)$ and $R^A(t)$ decline in t. The reason is the selection effect on π that Figure 6 portrays. Projects that last longer are subject to more stringent selection – the truncation point $(1+rW)/h(t)$ moves to the right as the products age. This positive selection effect may offset the fact that older projects have higher cumulative costs. In the estimated model the denominator effect easily dominates, and R^G and R^A both decline rapidly with t.

The relation between the rate of return and the IRR.—For completeness, we shall add a tangential result. In the (empirically relevant) parameter range for which the IRR exceeds the outside rate of interest, r, the IRR also exceeds the rate of return:

Lemma 6 *For each* (τ,π),

$$IRR(t,\pi) > R(t,\pi) \quad \textit{if and only if} \quad IRR(t,\pi) > r.$$

Proof. Since IRR$(t,\pi) > r$, $e^{-\mathrm{IRR}(t,\pi)u} < e^{-ru}$ for all $u \geq 0$. Then the denominator in (38) is smaller than the denominator in (36) and the claim follows. ■

The rate of return on all projects.—When calculating the VC's rate of return, we are concerned with the rate of return on all projects. To do so, first we hold t fixed: Of all projects that last exactly t periods, a fraction $\frac{h(t)}{h(t)+\psi(t)}$ succeed, and the rest fail. To compute the rate of return on *all* projects that end at date t, we simply would multiply the RHS of (36) and (38) and then proceed as before. This would lower the estimated rate of return on a project that ends at t roughly by a factor of $\frac{h(t)}{h(t)+\psi(t)}$.

The excess rate of return on venture capital.—Denote by C_{PV} the EPV of the costs of all the projects that the VC will bear in his lifetime; it satisfies the equation

$$C_{PV} = C + \frac{1 - k\int e^{-rt}|dS(t)|}{r} + C_{PV}\int e^{-rt}|dS(t)|$$

Therefore the VC's lifetime reward as a fraction of the lifetime costs of *all* the projects that he will oversee is $\frac{W}{C_{PV}}$, and its flow value is

$$\varepsilon^{VC} \equiv r\frac{W}{C_{PV}}, \tag{39}$$

having the dimensions of the excess rate of an return.

The excess rate of return on solo projects.—Although we do not observe returns on solo projects, they are implied by the estimated model. These depend on w; the model implies that wealthier entrepreneurs should receive higher returns Let C_{PV}^{w} be the PV of costs on an entrepreneurial project:

$$C_{PV}^{w} = C + \frac{1 - \int e^{-r \min(t, \tau[w])} \left| dS\left(t\right) \right|}{r},$$

where $S^{S}\left(t\right) = \left(1 - F\left[t\right]\right)\left(1 - \Phi^{S}\left[t\right]\right)$, where $\tau\left(w\right)$ is defined in (6) and where Φ^{S} is defined in (28). The rate of return of the entrepreneur in excess of r is

$$\varepsilon\left(w\right) = r\frac{Q^{S}\left(w\right) - w}{C_{PV}^{w}}. \tag{40}$$

The denominator is always strictly positive, because $C_{PV}^{w} \geq C$. At the point w^{*}, where the entrepreneur is indifferent between going solo and investing with a bank or VC, the excess return is zero, i.e., $\varepsilon\left(w^{*}\right) = 0$. Since $\frac{\partial Q^{S}(w)}{\partial w} > 1$, the numerator rises with w, but so does the denominator. It rises with the entrepreneur's level of wealth. The excess return becomes flat at the point $C + k/r$, i.e., the point where the solo entrepreneur ceases to be liquidity constrained in any state of the world, i.e., for any realization of π.

4.0.6 The entrepreneur's stake

When fitting the entrepreneur's share $s^{*} = \frac{1-k}{1+rW}$ in the firm's equity, we shall use Kaplan and Stromberg's (2003, Table 2) numbers for cash flow rights, i.e., the fraction of a portfolio company's equity value that different investors and management have a claim to. Pooling over all rounds, the mean claim of founders 31.1%, that of VCs is 46.7%, and that of other non-VC investors is 22.2%. Since our model does not include non-VC investors, we constrain s^{*} to the share of founders in claims other than those of the outside investors. That is we should have s^{*}

$$s^{*} \approx \frac{31.1}{31.1 + 46.7} = 0.40. \tag{41}$$

In the model, once the entrepreneur signs the contract with the VC, her share of the project drops from unity to s^{*}, where it remains until the end. Lerner (1994, Table 5) and Kaplan and Stromberg (2003, Table 8) find that the greatest dilution of the entrepreneur's equity stake occurs in the first financing round. Contrary to the model, however, it appears that s continues to fall as the project ages, though at a decelerating rate. The fall is accompanied by a rise in the number of VCs in the syndicate.

5 Estimating the model

The estimation uses only a venture-backed sample and therefore henceforth we drop the superscript VC when possible. Before getting to the example, we derive the distribution of termination times and the distribution of contract durations or the "Survivor Function."

Contract duration.—The survival of contracts requires that neither a success nor a termination has taken place. Let t denote the date of the "event" that the firm experiences. The event is either a success or a failure, but not both. Only one event per firm can occur. For some firms no event occurs and these are called the "survivors." That is, if τ is date of success and $T(\pi)$ is the date of termination, then $t = \min(\tau, T[\pi])$. Since τ and π are independent random variables, the CDF of t is $1 - S(t)$, where

$$S(t) = (1 - F[t])(1 - \Phi[t]) \tag{42}$$

is the *Survivor function* – the fraction of firms surviving past age t.

The data.—Our data (described in the Appendix) include a distribution of T (terminations) and the distribution of τ (successes) for about 1400 firms, and data on internal rates of return for VC's by age of project completion. We also use information on the VCs' rate of return by age of completed project.

5.0.7 Pareto-exponential example

We now estimate a five-parameter example which leads to simple formulas, some of which are derived in the Appendix. One object is to estimate W and based on that estimate we shall derive the excess rate of return on venture capital. We note that we shall *not* truncate the waiting time distribution at 10 or 12 years when the venture fund closes. We shall assume that the VC maintains his interest in a company beyond the fund-closing date, and that he can continue to fund it and collect on any return that it generates. This is a important part of the portfolio, containing more than a quarter ($\frac{365}{1355} = 0.27$) of all the firms.

Fitting investment flows.—We shall fit C and k to the data on the investment profile, i.e., the sequences of investment rounds, but converted to flows of investment as a function of time. The sizes of the rounds are reported in Appendix Table A2. The conversion procedure is also described there. But the parameters C and k affect other variables such as termination rates as well, which is why the fits to the investment profiles do not look as close as they would be if C and k were chosen to fit the investment series alone. Also, we fit the model to the series of age-t investments relative to first-year investments.

Fitting the success hazard, h.—For the waiting-time to success we choose a distribution the hazard of which peaks at 4.5 years, as does the empirical hazard $\hat{h}$ (see Appendix Table A1) and which is continuous. A two-parameter distribution that achieves this is the mixture, with weights $\frac{\rho}{2+\rho}$ and $\frac{2}{2+\rho}$ respectively, of a Beta distribution on $[0, t_{\min}]$ and a Pareto distribution on $[t_{\min}, \infty)$:

$$F(t) = \frac{\rho}{2+\rho}\left(\frac{\min(t, t_{\min})}{t_{\min}}\right)^2 + I_{[t_{\min},\infty)}\frac{2}{2+\rho}F^P(t),$$

where

$$F^P(t) = 1 - \left(\frac{t}{t_{\min}}\right)^{-\rho}, \qquad \text{for } t \geq t_{\min}. \tag{43}$$

Its hazard rate is continuous and has the essential features of the bell shape in Figures (1) and (2). Then for $t < t_{\min}$, the density is $f(t) = \frac{1}{t_{\min}}\frac{2\rho}{2+\rho}\frac{t}{t_{\min}}$, and therefore (see the Appendix 3 for the derivations),

$$h(t) = \begin{cases} \frac{1}{t_{\min}}\frac{2\rho t_{\min}^{-1}t}{2+\rho-\rho t_{\min}^{-2}t^2} & \text{for } t < t_{\min} \\ \frac{\rho}{t} & \text{for } t \geq t_{\min}. \end{cases} \tag{44}$$

Fitting the termination hazard, ψ.—We assume that $G(\pi) = 1 - e^{-\lambda\pi}$ to reflect the well-known tendency for payoffs to be right-skewed. Our assumption that the social planner wants to pay C and use VC's to support projects will be met if λ is small enough so that the mean π, which equals $\frac{1}{\lambda}$, is large enough compared to C and k. Thus the model's five parameters are $\rho, t_{\min}, \lambda, k$, and C. With these functional forms for f and G, the terminations hazard in (30) now reads

$$\psi(T) = \begin{cases} \infty & \text{for } T = 0 \\ 0 & \text{for } T \in (0, \tilde{\rho}\pi_{\min}) \\ \frac{\lambda}{\tilde{\rho}} & \text{for } T \geq \tilde{\rho}\pi_{\min} \end{cases}, \tag{45}$$

where

$$\tilde{\rho} = \frac{\rho}{1+rW}.$$

Thus, ψ assumes only three values. Initially, the hazard is infinite: The mass-point of immediate terminations is $\left(1 - e^{-\lambda\pi_{\min}}\right)$. When plotting, we spread this mass over the first period. Thereafter, no terminations occur for a while, and then the terminations hazard becomes $\lambda(1+rW)/\rho$, for ever. The parameter ρ raises h thereby making a project more attractive at any age and, for that reason, ρ lowers ψ. The estimation procedure actually fits $S(t)$ and not $\psi(t)$, because together with $h(t)$, $S(t)$ implies $\psi(t)$. In any case the fits of both variables are reported visually in Figure 7.

Fitting the rates of return.—We shall fit only $R^G(t)$ as given in (37). Substituting the functional forms for h and G into (34) and (35), we compute Γ_t and then substitute that into the expression for R^G (and, if needed, for R^A).

Fitting s^* as in (41).—We fit $s^* = \frac{1-k}{1+rW}$ to the Kaplan-Stromberg numbers as described in (41)

Altogether, then we fit five series, and plot six of them. We assume that $r = 0.07$ – roughly the return on equity. We maintain the assumption that $a + k = 1$, which means that our estimates are in units of total marginal costs per year. We divide the empirical investment flows by the first-year investment flow, and fit the model to that series. This makes all the implications homogeneous of degree zero in the vector $\left(a, k, \lambda^{-1}, C, W\right)$.[9] The parameters are ρ, $t_{\min}$, λ, k and C. The estimation algorithm is described in the Appendix.

5.0.8 Estimates

*T*he estimation procedure and the data sources are described in the Appendix. We present two sets of estimates which deal differently with the investment series which was especially hard for the model to fit. The first set of estimates places a high penalty on fitting the investment numbers whereas the second does not and, as a result, the fit of the remaining series improves dramatically.

Estimate 1.—The first set of parameter estimates are reported in Table 1A, and some statistics of interest are reported in Table 1B. The fit is described Visually in Figure 7.

ESTIMATE # 1: PARAMETERS AND STATISTICS

Param.	*Est.*
ρ	2.13
$t_{\min}$	1.8
λ	0.23
k	0.98
C	2.0

Table 1A: Estimate 1

Values of endogenous variables, etc.	
Lifetime value of VC $= W$	0.00
Payment to VC $= p^*$	0.03
Marginal project for the VC $= \pi_{\min}$	2.63
Immediate terminations $= 1 - e^{-\lambda \pi_{\min}}$	0.45
Average project quality $= E(\pi) = \lambda^{-1}$	4.37

Table 1B: Estimate 1

The fit is not great, but maybe not bad for a model that has only 5 parameters. The following points on Estimate 1 are of note:

1. In all panels but the third, age is measured as the number of years elapsed since the date of first investment. This is the concept that the model dictates, for

[9] For instance, the terminations hazard would really be $\lambda (a + k + W) / \rho$, and it is homogeneous of degree zero as claimed. The absolute values of these variables would then be obtained by multiplying them by

$$\begin{aligned} & \left(\frac{1}{\hat{k}}\right) \text{X (Average investment per unit of time in periods 2 and beyond)}. \\ = & \left(\frac{1}{0.86}\right)(4.8) = \$5.58 \text{ million} \end{aligned}$$

using the estimate $\hat{k} = 0.86$ from Table 2A.

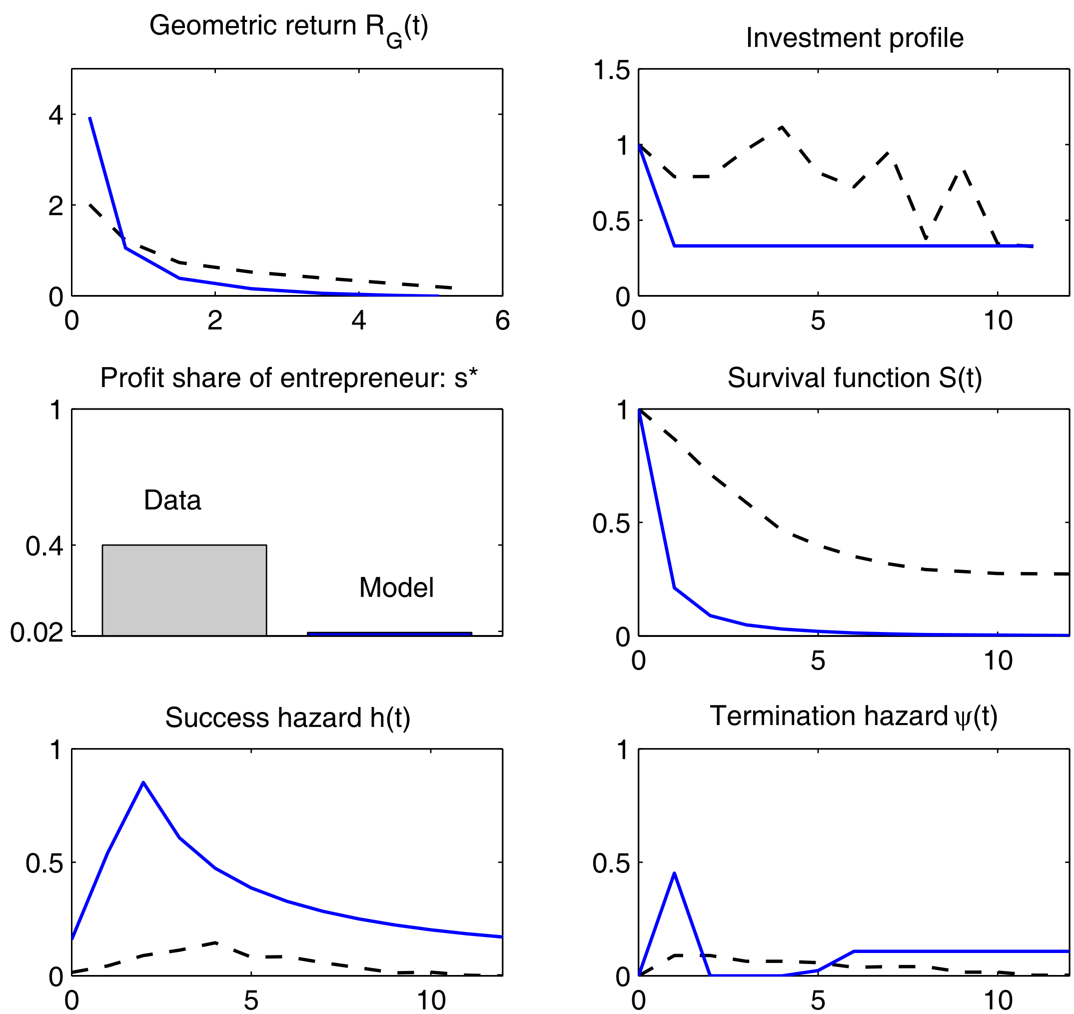

Figure 7: **Estimate #1**: Data (dashed line) and model (solid line) when the penalty on investment-profile deviations is relatively high; $\rho = 2.13$, $t_{\min} = 1.8$, $\lambda = 0.23$, $k = 0.98$, and $C = 2.0$

it assumes that a constant investment flow must be made in order to keep the firm alive.

2. *Panel* 1.—The vertical axis measures the geometric rate of return in hundreds of percentage points. The dashed line is Cochrane's the average log return by company age given in Table A3. The highest returns are on projects that succeed early. The model (solid line) overpredicts the early returns by a factor of almost two

3. *Panel* 2.—The predicted investment flow is 1 in the first year of life, and $\frac{k}{C+k}$ thereafter.

4. *Panel* 3.—This version of the model also badly underpredicts s^*, by a factor of 20

5. *Panels* 4-6.—The dashed lines in these three panels represent the data from the last three columns of Appendix Table A1 for the ratio #left/1355, for $\hat{h}$, and for $\hat{\psi}$. The model underpredicts survival (Panel 4). Too many exits happen early on. This is mainly because the model overpredicts early successes (Panel 5) and, to a lesser extent, because it also overpredicts early terminations (Panel 6): The date-zero termination hazard is undefined because a mass of $1-e^{-\lambda\pi_{\min}}$ firms are terminated immediately. We spread this mass, estimated at 0.45, evenly over the first year of the firm's life, and this is the spike in Panel 6.

Estimate 2.—The model has a hard time fitting the investment numbers as well as all the other series. The parameters of best fit for the other series involve a high estimate of C, so that the first year's investment flow, $C+k$, is very high relative to the investment flow in later years, k. We note the following:

1. *Panel* 1.—The model no longer overpredicts the early returns, but this comes at the cost of *underpredicting* the later returns by a factor of almost two

2. *Panel* 2.—The predicted investment flow is much too front-loaded and fits badly.

3. *Panel* 3.—Here there is substantial improvement, though s^* is still underpredicted by a factor of 3

4. *Panels* 4-6.—The fit here is not much more reasonable. The model still underpredicts survival (Panel 4), but by much less than before. Now the model overpredicts only the early successes (Panel 5) and fits much better the terminations (Panel 6): Now, a much smaller number of firms, 15 percent, are terminated immediately.

Estimate # 2: Parameters and Statistics

Param.	*Est.*
ρ	0.52
$t_{\min}$	1.8
λ	0.025
k	0.86
C	18.8

Table 2A: Estimate 2

Values of endogenous variables, etc.	
Lifetime value of VC = W	0.19
Payment to VC = p^*	2.52
Marginal project for the VC = $\pi_{\min}$	6.74
Immediate terminations = $1 - e^{-\lambda \pi_{\min}}$	0.15
Average project quality = $E(\pi) = \lambda^{-1}$	40.35

Table 2B: Estimate 2

The excess rate of return on venture capital.—One thing common to both sets of estimates is that VCs get very little excess return. Evaluating (39), the VC's excess return is negligible under both sets of estimates:

$$\varepsilon^{VC} \equiv r\frac{W}{C_{PV}} = \begin{cases} 0 & \text{under Estimate 1,} \\ (0.07)\,\frac{0.19}{21.8} = 0.001 & \text{under Estimate 2.} \end{cases},$$

The excess rate of return on solo projects.—Evaluating (40), TO BE COMPLETED.

5.0.9 Discussion of the empirical results

Our estimates should be interpreted as the return to a VC fund *before* VC compensation and portfolio-management charges are deducted. Once this is done, our estimates would be closer to the return on the S&P 500.Our estimates imply VC returns that are pretty much in line with estimates of returns to private equity. Kaplan and Schoar (2005) find that returns to private equity are, on average, similar to those on the S&P 500, but that there is a large variation among funds. At the high end of the distribution, Ljungqvist and Richardson find a return of 5% in excess of the S&P 500. Lerner, Schoar and Wong (2005) find that funds earn more if they employ experienced VCs. Thus the human capital of VCs acts to raise the returns of funds that have more such capital, and that the above-normal returns are partly explained by a scarcity of good fund managers, as our model stresses.

We do not explain the heterogeneity of the returns on the venture funds. In our model, venture capital is homogeneous, though differences could easily be introduced. For instance, an experienced VC would have better signals about a project's likely success, which would lead to a more favorable distribution of waiting times, F, and of payoffs, G. Panel 1 of Figures 7 and 8 shows that returns drop off quickly as the waiting time increases so that an ability to bring successes forward seems to have a very high return.

Venture funds usually run for 10-12 years, during which time capital is tied up. The illiquidity of the investment may also partly explain why returns are above normal. Our model does not include this force; the VC values earnings at the rate r and

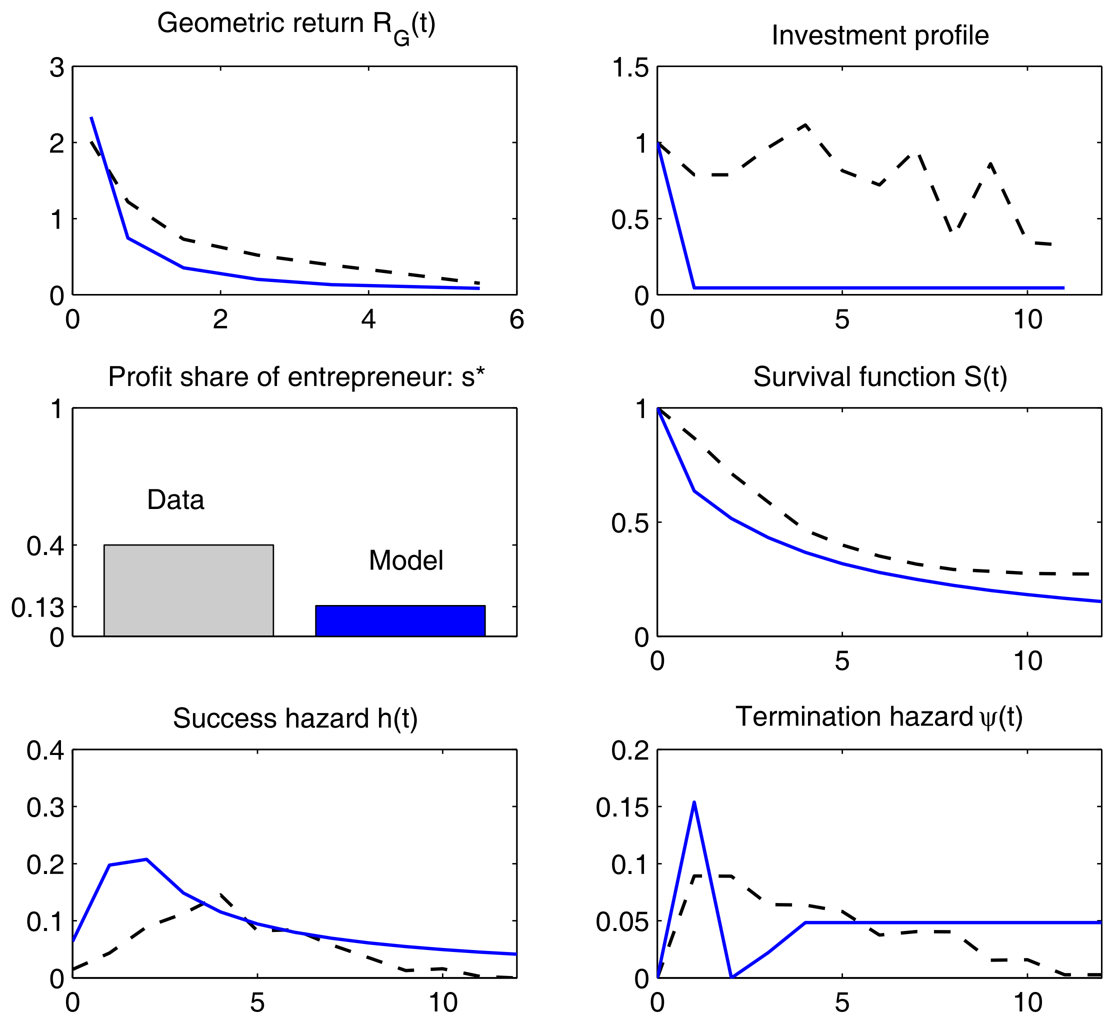

Figure 8: **Estimate #2**: Data (dashed line) and model (solid line) when the penalty on investment-profile deviations is relatively low; $\rho = 0.52$, $t_{\min} = 1.8$, $\lambda = 0.25$, $k = 0.86$, and $C = 18.8$

never needs to cash out early. The venture fund may close in its tenth year, but we assume that the VC retains his position in the 27 percent of firms that are still left in his portfolio.

Everyone is risk-neutral, so that the covariance of the venture funds with the market portfolio plays no role. Because of the high-tech nature of the portfolio firms, payoffs to their having IPOs or being acquired are correlated with the Nasdaq more than with the S&P 500. Gompers and Lerner (1997, Exhibit 1) report a correlation of .60 with the S&P 500.

One should be able to show that the wealthiest entrepreneurs earn a higher rate of return on their projects than a VC does. We are thus linking the VC returns to those of solo entrepreneurs, and find (TO BE COMPLETED, REMAINDER OF THE PARAGRAPH MAY NOT APPLY) that the VCs are in the right tail of the distribution, but not at the very top of it. equity. If one takes, as we have, r to be the return on the S&P 500, then the model implies that all solo entrepreneurs should earn at least that much of a return. If one allows for risk aversion, and for the fact that solo entrepreneurs' assets are highly concentrated in their own businesses and more risky than the S&P 500, then both solo and venture-backed entrepreneurs would be predicted to earn more than the S&P 500. This seemingly contradicts recent evidence in Moskowitz and Vissing-Jorgensen (2002) that returns to private equity are no higher than the returns to public equity. [10]On the other hand, ...an extrapolation of venture-backed returns onto the returns implied for the population of solo entrepreneurs. This exercise should be qualified by noting that the types of solo entrepreneurs whose projects may ever get on VCs' radar screens are not typical of all projects run by entrepreneurs.

6 Conclusion

We estimated a model of the market for venture capital in which VCs were scarce relative to the number of potential projects. This led a high equilibrium return on VC capital and a tendency for venture-backed companies reach IPOs earlier, and to be worth more at IPO than other start-ups. The equilibrium turned out to be socially optimal.

We used the estimated model to infer the rate of return on venture capital and on entrepreneurship, the latter rising with the entrepreneur's wealth. The VC is near the top of the distribution of project returns, but not at the very top; the wealthiest solo entrepreneurs expect to earn more (TO BE CHECKED).

[10]One could allow non-pecumiary benefits to going solo, i.e., to being one's own boss, in the form of a smaller value of a for solo projects than for venture-backed projects. A satisfactory comparison to other activities would model a lower disutility of effort if it is devoted to entrepreneurship (whether venture-backed or solo) than if it is devoted to wage work.

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7 Appendix

7.1 Data and the estimation algorithm

The data on τ and T.—These data are from the VentureExpert database provided by Venture Economics, and are described in detail by Guler (2002). The following table summarizes the data on successes and terminations. Age is measured as the number of periods since the date of first investment.

age	ipo	acq	term	#evnts	#left	$\hat{h}$	$\hat{\psi}$
0	12	8	0	20	1355	0.01	0.00
1	39	19	119	177	1335	0.04	0.09
2	54	49	103	206	1158	0.09	0.09
3	65	42	61	168	952	0.11	0.06
4	67	47	50	164	784	0.15	0.06
5	27	24	36	87	620	0.08	0.06
6	22	23	20	65	533	0.08	0.04
7	16	11	19	46	468	0.06	0.04
8	5	10	17	32	422	0.04	0.04
9	0	5	6	11	390	0.01	0.02
10	2	4	6	12	379	0.02	0.02
11	0	1	1	2	367	0.00	0.00
12	0	0	1	1	365	0.00	0.00

Table A1

The last three columns are plotted as the dashed lines in panels four (there normalized by dividing by 1355), five, and six of Figures 7 and 8.

The data on average investment by round.—These are from Guler 2003 (Table 6, column 2) of are reported in Table A2

Investment round	1	2	3	4	5	6	7	8	9	10	11	12
Amount ($ millions)	6.5	5.3	5.5	7	8.4	6.4	5.9	8.2	3.4	8.1	3.4	3.4

Table A2

Because the model has k paid per unit of time and C at the outset, we need to convert these into spending per year. By comparing the speed of terminations we arrived at the conversion factor for converting rounds into flows. If I_j is the average amount invested in round j, we convert this into a flow $I_t = \theta_t I_t$, where[11]

$$\theta_t \equiv \frac{1}{1.25} - \frac{1}{5}\left(\frac{1}{1.25} - \frac{1}{1.5}\right)t.$$

[11] Between the first and the sixth round,the termination hazard falls from 0.12 to 0.08. On the other hand, between year 1 and year 6, the termination hazard falls from 0.09 to 0.04. Thus the ratio of the two hazards rises from $\frac{12}{9} = 1.25$ to $\frac{8}{4} = \frac{1}{2}$. As a rough calculation, then, initially, rounds are once every 1.25 years, and by year 6, they are once every 1.5 years.

Corrected in this way, and then normalized by dividing by year-1's investment flow, these numbers are plotted as the dashed lines in panels two of Figures 7 and 8.

The data on R.—These come from Cochrane (2004) Table 6d. They are annualized log returns and we reproduce them in Table A3:

Annualized log returns, percent

	Age bins						
	1-6mo	6-12mo	1-2yr	2-3yr	3-4yr	4-5yr	5yr-∞
Average R	2.01	1.22	0.73	0.52	0.39	0.27	0.15

Table A3

The data are also plotted as the dashed line in the first panels of Figures 7 and 8.

Composition of firms by sector.—The companies in the sample were in high-tech sectors. Table A4 reports their industry composition:

Industry Group	*# Companies*	*Percent*
Computer Related	496	36.6
Non-High-Technology	287	21.2
Communications and Media	202	14.9
Medical/Health/Life Science	183	13.5
Biotechnology	114	8.41
Semiconductors/Other Elect	73	5.39

Table A4

Computations.—Here is how we calculated the hazards:

1. Column 6, "# left" is the empirical counterpart of $(1 - F[t])(1 - \Phi[t])$, i.e., of $S(t)$ in (42).

2. Column 7, "$\hat{h}$" is the ratio $(\texttt{ipo} + \texttt{acq})/(\texttt{\#left})$. So, e.g., the value of this ratio at age 1 is $\frac{39+19}{1335} = 0.043$, its value at age 2 is $\frac{54+49}{1158} = 0.089$, and so on. We now show that this is the empirical counterpart of $h(t)$. The sum of the columns $(\texttt{ipo} + \texttt{acq})$ we interpret as the number of successes at date t among firms for whom $T > t$. The probability of $\tau = t$ *and* its surviving beyond t is $f(t)(1 - \Phi[t])$. Therefore we equate these two concepts:

$$(\texttt{ipo} + \texttt{acq}) = f(t)(1 - \Phi[t]).$$

   Therefore as calculated in Column 7, the "success hazard" is

$$\frac{\texttt{ipo} + \texttt{acq}}{\texttt{\# left}} = \frac{f(t)(1 - \Phi[t])}{(1 - F[t])(1 - \Phi[t])} = \frac{f(t)}{1 - F(t)} = h(t).$$

3. Column 8, "$\hat{\psi}$" is the ratio (`term`)/(`#left`). This is the empirical counterpart of $\psi(t)$; to see why, note that term is the number terminated at date t among firms for whom $T \geq t$ and $\tau \geq t$. The probability of $T = t$ *and* $\tau \geq t$ is $\Phi'(t)(1 - F[t])$. Therefore we equate the two concepts: (term) $= \Phi'(t)(1 - F[t])$ whereupon, as calculated in Column 8, the "termination hazard" is

$$\frac{\texttt{term}}{\texttt{\# left}} = \frac{\Phi'(t)(1 - F[t])}{(1 - F[t])(1 - \Phi[t])} = \frac{\Phi'(t)}{1 - \Phi[t]} = \psi(t).$$

Solving for the marginal project, $\pi_{\min}$.—As Figure 6 shows, projects for which $\pi < \pi_{\min}$ are terminated at once, whereas for the rest, T is determined by (33), so that

$$T(\pi) = \begin{cases} 0 & \text{for } \pi < \pi_{\min} \\ \tilde{\rho}\pi & \text{for } \pi \geq \pi_{\min}. \end{cases} \tag{46}$$

In (1), an integration by parts leads to

$$\begin{aligned} \int_0^T e^{-rt} f(t)\,dt &= -e^{-rt}(1 - F[t])\Big|_0^T + r\int_0^T e^{-rt}(1 - F[t])\,dt \\ &= 1 - e^{-rT}(1 - F[T]) + r\int_0^T e^{-rt}\frac{f(t)}{h(t)}dt, \end{aligned}$$

so that when substituted into (1) yields

$$V(\pi) \equiv W + \max_T \int_0^T \left(\pi - \frac{1 + rW}{h(t)}\right) e^{-rt} f(t)\,dt$$

Substituting the Pareto form for h, the social surplus that project π delivers is

$$\begin{aligned} V(\pi) - W &= -\int_0^{t_0} e^{-rt}dt + \int_{t_0}^{\infty} e^{-rt}\max\left(0, \pi - \frac{t}{\tilde{\rho}}\right)\rho t_0^{\rho} t^{-1-\rho}dt \\ &= -\frac{(1 - e^{-rt_0})}{r} + \rho t_0^{\rho}\int_{t_0}^{\infty} e^{-rt}\left(\pi - \frac{t}{\tilde{\rho}}\right) t^{-1-\rho}dt. \end{aligned} \tag{47}$$

The marginal project $\pi_{\min}$ therefore solves $V(\pi) = W$. Substituting $\pi = t/\tilde{\rho}$ on the RHS of (47), we see that $V(t_0/\tilde{\rho}) < W$; since V is increasing in π,

$$\pi_{\min} > \frac{t_0}{\tilde{\rho}}.$$

Therefore terminations will not start until some time *after* t_0. In other words, after the initial burst of terminations at $T = 0$ we should, for a while, see successes only, and only later should terminations begin.

The distribution of terminations.—Then (27) gives us

$$\Phi(T)=\begin{cases}0 & \text{for } T=0\\ 1-e^{-\lambda\pi_{\min}} & \text{for } T\in(0,\tilde{\rho}\pi_{\min})\\ 1-e^{-\left(\frac{\lambda}{\tilde{\rho}}\right)T} & \text{for } T\geq\tilde{\rho}\pi_{\min}\end{cases}$$

which, for $T\geq\tilde{\rho}\pi_{\min}$ is also exponential. The hazard of this distribution is (45). In the estimations, we assign the mass point $1-e^{-\lambda\pi_{\min}}$ of terminations to year 1. This is to account for the lag with which Guler's (2003) detects a termination; she declares a company as terminated 482 days (sixteen months) after it received its last financing round and no other event occurred after this round.

Contract duration.—From (43) and (42), $S(t)=(1-F[t])(1-\Phi[t])$

$$S(t)=\begin{cases}1 & \text{for } t=0\\ e^{-\lambda\pi_{\min}} & \text{for } 0<t<t_0\\ \left(\frac{t_0}{t}\right)^{\rho}e^{-\lambda\pi_{\min}} & \text{for } t\in(t_0,\tilde{\rho}\pi_{\min})\\ \left(\frac{t_0}{t}\right)^{\rho}e^{-\left(\frac{\lambda}{\tilde{\rho}}\right)t} & \text{for } t\geq\tilde{\rho}\pi_{\min}\end{cases}. \tag{48}$$

The geometric rate of return.—The mean geometric return of successes between two points in time a and b predicted by the model can be obtained analytically using (37) and the density of τ:

$$R_G^{[a,b]}=\int_a^b R_G(t)f(t)dt$$

The IRR.—Figuring out the IRR requires that we calculate p (the up-front payment the VC makes to the entrepreneur). Evaluating (**??**) and (15) we have the following implicit function for the IRR, defined as R, taking as parametrically given the age of success, t:

$$e^{-Rt}\left(\frac{\rho}{\rho+\psi(t)}\right)\frac{k+rW}{1+rW}E\left(\pi\mid\pi\geq\frac{t}{\tilde{\rho}}\right)-\int_0^t ke^{-Ru}du+p^*-C=0,$$

where

$$p^*=Q(s^*)=\int\int_0^{T(\pi)}\left(\frac{a\pi}{1+rW}-\frac{a}{h(t)}\right)e^{-rt}f(t)\,dtdG(\pi)$$

$$p^*=\frac{1-k}{1+rW}t_0^{\rho}\int_{\pi_{\min}}^{\infty}\int_0^{\tilde{\rho}\pi}\left(\pi-\frac{t}{\tilde{\rho}}\right)e^{-rt}\rho t^{-(1+\rho)}dt\lambda e^{-\lambda\pi}d\pi-\int_0^{t_0}(1-k)e^{-rt}dt. \tag{49}$$

(we must subtract separately the costs incurred over $(0,t_0)$ which the first integral does not capture). For $t\in[0,1)$, however, the formula must be adjusted to $\Phi(1)-\Phi(t_0)$

$$e^{-R}(1-\Phi_0)(1-t_0)\left(\frac{\rho}{\rho+\psi(t_0)}\right)\frac{k+rW}{1+rW}E\left(\pi\mid\pi\geq\frac{t_0}{\tilde{\rho}}\right)-\int_0^1 ke^{-Ru}du+p^*-C=0$$

where $\Phi_0 = e^{-\lambda\pi_{\min}}$ is the number terminated right away (and on these alone $R = -\infty$). But these are mixed with the successes that occur during the first year, and when averaged, they still produce a positive R. For $t > t_0$, the equation to be solved has two functional forms, depending on which formula in (45) applies We shall estimate ρ, λ, k, W, t_0 and C.

We program $V(\pi)$ as follows

$$V(\pi) = \max_{A,R}\left[\sup_{T \geqslant 0}\left\{\int_{t_0}^{T}(\pi + W + \frac{1}{r})e^{-rt}f(t)dt + (1 - F(T))e^{-rT}(W + \frac{1}{r}) - \frac{1}{r}\right\}, W\right] - C$$

VC Value and termination policy.—The value to the VC of an *accepted* project with payoff π and terminated at T is

$$\begin{aligned}
V_a(T|\pi, W) &= \int_{t_0}^{T}\left[(\pi + W)e^{-rt} - \int_0^t e^{-rs}ds\right]f(t)dt \\
&\quad + (1 - F(T))(e^{-rT}W - \int_0^T e^{-rt}dt) \\
&= \int_{t_0}^{T}\left[(\pi + W)e^{-rt} - (1 - e^{-rt})/r\right]f(t)dt \\
&\quad + (1 - F(T))(e^{-rT}W - (1 - e^{-rT})/r) \\
&= \int_{t_0}^{T}(\pi + W + \frac{1}{r})e^{-rt}f(t)dt + (1 - F(T))e^{-rT}(W + 1/r) - \frac{1}{r}
\end{aligned}$$

The optimal stopping time solves

$$\frac{dV_a(T|\pi, W)}{dT} = 0 = e^{-rT}f(T)\left(\pi - \frac{1 + rW}{h(T)}\right), \text{ if } T > t_0.$$

For the Pareto distribution $T = \pi\rho/(1 + rW)$, and so

$$V_a(\pi, W) = \rho t_0^{\rho}\int_{t_0}^{T}(\pi + W + \frac{1}{r})\frac{e^{-rt}}{t^{\rho+1}}dt + (1 - F(T))e^{-rT}(W + \frac{1}{r}) - \frac{1}{r}$$

Estimation Strategy.—The set of unknown parameters is $\{t_0, \rho, \lambda, W, k\}$. To estimate them we fix $t_0 = 1.8$[12] and then estimate the remaining unknowns by minimizing the residual sum of squares given in (50). To compute the RSS we follow the following steps:

1. Using $\{\rho, t_0, \lambda, W\}$ compute $\pi_{\min}$ from $V_a(\pi_{\min}; W) = W$;

[12] Note that the highest sensible value for this parameter is below 2 — if we choose a value above 2, then the model will predict zero terminations for projects of age 1 to 2. In subsequent versions of the paper, we also want to maximize with respect to this parameter within reasonable bounds.

2. Using $\{\rho, t_0, \lambda, W, \pi_{\min}\}$ compute C according to

$$C = \int_{\pi_{\min}}^{\infty} \Big[\int_{t_0}^{T} e^{-rt}(\pi + W + \frac{1}{r}) f(t) dt + (1 - F(T)) e^{-rT}(W + \frac{1}{r}) \Big] g(\pi) d\pi$$
$$- (1 - G(\pi_{\min}))(W + \frac{1}{r})$$

3. Using $\{\rho, t_0, \lambda, W, k, \pi_{\min}\}$ compute p^* via

$$p^* = (1-k) \int_{\pi_{\min}}^{\infty} \Big[\int_{t_0}^{T} e^{-rt}\Big(\frac{\pi}{1+rW} + \frac{1}{r}\Big) f(t) dt + (1 - F(T)) \frac{e^{-rT}}{r}) \Big] g(\pi) d\pi$$
$$- (1-k)(1 - G(\pi_{\min}))/r$$

4. Using $\{\rho, t_0\}$ compute $h_t, t = 0, 1, .., 12$ according to the formula given in section 7.3. For the first two periods, we calculate the hazard as follows to obtain a higher degree of precision:

$$h_t = \frac{F(t) - F(t-1)}{F(t-1)},$$

where $F(t)$ is the distribution function of successes obtained from the model.

5. Using $\{\rho, t_0, \lambda, W, \pi_{\min}\}$ compute the survival function, $S_t, t = 0, 1, .., 12$;

$$S(t) = \begin{cases} 1, & t = 0 \\ e^{-\lambda \pi_{\min}}, & t \in (0, t_0) \\ (t_0/t)^{\rho} e^{-\lambda \max(\pi_{\min}, t(1+rW)/\rho)}, & t \in [t_0, \infty) \end{cases}$$

6. Using $\{\rho, t_0, \lambda, W, k, \pi_{\min}, p^*, C\}$ compute the predicted average geometric rate of return $\hat{R}_t, t = 0.5, 1, 2, 3..5,.$ as described in (7.1).

7. The predicted investment profile is $I_1 = 1$ and $I_t = \frac{k}{C+k}$ for $t = 2, 3...13$.

8. The predicted share of the entrepreneur in the profit $\pi$is

$$s^* = \frac{1-k}{1+rW}.$$

9. Compute criterion function

$$\begin{aligned} \text{RSS} \quad = \quad & w_1 \sum_{t=0}^{12} (\hat{h}_t - h_t)^2 + w_2 \sum_{t=0}^{12} (\hat{S}_t - S_t)^2 + w_3 \sum_{t=1}^{13} (\hat{R}_t - R_t)^2 \qquad (50) \\ & + w_4 \sum_{t=1}^{13} (\hat{I}_t - I_t)^2 + w_5 (\ln(\hat{s}^*/s^*))^2 + w(\ln(\hat{W}/W))^2 \end{aligned}$$

For the estimation we choose $r = 0.07$. We use two different weighting schemes: one that penalizes deviations from the empirical cost profile rather heavily ($w_C = (4, 1, 5, 2, 0.2, 0)$[13]), and another one that puts a less heavy weight on costs but penalizes if W goes to close to zero ($w_W = (4, 1, 5, 0.2, 0.2, 0.1)$). This yielded the estimates reported in Table 2A.

Estimates of R, S, and h are plotted in Figures 7 and 8.

Details on how C and p^* were computed.—The formula for C is

$$
\begin{aligned}
C &= \int_{\pi_{\min}}^{\infty} V_a(\pi, W) g(\pi) d\pi - (1 - G(\pi_{\min}))W \\
&= \int_{\pi_{\min}}^{\infty} \left[\int_{t_0}^{T(\pi)} \left(\pi + W + \frac{1}{r}\right) e^{-rt} f(t) dt + (1 - F(T(\pi))) e^{-rT(\pi)} \left(W + \frac{1}{r}\right) - \frac{1}{r} \right] g(\pi) d\pi \\
&\quad - (1 - G(\pi_{\min}))W \\
&= S_2 + \left(W + \frac{1}{r}\right)(S_1 + S_3 - 1 + G(\pi_{\min})),
\end{aligned}
$$

where

$$
\begin{aligned}
S_1 &= \int_{\pi_{\min}}^{\infty} \int_{t_0}^{T(\pi)} e^{-rt} f(t) dt \; g(\pi) d\pi \\
S_2 &= \int_{\pi_{\min}}^{\infty} \pi \int_{t_0}^{T(\pi)} e^{-rt} f(t) dt \; g(\pi) d\pi \\
S_3 &= \int_{\pi_{\min}}^{\infty} (1 - F(T(\pi))) e^{-rT(\pi)} g(\pi) d\pi
\end{aligned}
$$

Then, similarly,

$$
\begin{aligned}
p^* &= (1 - k)(S_2/(1 + rW) + \frac{1}{r}(S_1 + S_3 - 1 + G(\pi_{\min}))) \\
&= (1 - k)(S_2 + \left(W + \frac{1}{r}\right)(S_1 + S_3 - 1 + G(\pi_{\min})))/(1 + rW) \\
&= (1 - k)C/(1 + rW) \qquad (51)
\end{aligned}
$$

Note that most of the integrals in these formulae do not have closed-form solutions. However, most of them involve integration against a function of type e^{-rt}. After a simple change of variable, *Laguerre Quadrature* can be used to calculate integrals

[13] Although it looks like w_3 is smaller than w_1 and w_2, it is, in its effect on the estimates, a larger than w_1 weight because the range of R exceeds the range of h by a factor of 10, and it is comparable to w_2 because the range of R exceeds the range of S by a factor of two.

of this type to a very high degree of precision. We use 30 nodes to calculate these integrals. Checks with Monte-Carlo integrals showed that the error in the calculations are of negligible order. To obtain the predicted geometric rate of return for projects succeeding between two points a and b, we evaluate the geometric return $R_G(t)$ for a number of equally spaced points x_i on $[a, b]$and sum the obtained values using weights $f(x_i)/(F(b) - F(a))$.

We use a line-search method to minimize RSS with respect to the parameters of the model. Due to the complexity of the model, we cannot use arguably superior optimization methods based on analytical gradients. However, the minimization process proved to be rather robust; the algorithm converged to the same solution for more than 10 randomly chosen starting points for each of the weighting schemes. Using numerical gradients and standard errors of the moments we try to match, it is possible to obtain (approximate) standard errors for our estimators. This is an important next step in this project.

7.2 Proof of Lemma 4

(i) Notice that, given that a project is supported up to time T,

$$\begin{aligned} V(\pi) - V^{VC}(\pi) &= \int_0^{T(\pi)} \left(\pi - \frac{1}{h(t)} - \left[(1-s)\pi - \frac{k}{h(t)} \right] \right) e^{-rt} f(t)\, dt \\ &= \int_0^{T(\pi)} \left(\pi - \frac{1}{h(t)} - \left[\left(1 - \frac{a}{1+rW}\pi \right) - \frac{k}{h(t)} \right] \right) e^{-rt} f(t)\, dt \\ &= a \int_0^{T(\pi)} \left(\frac{1}{1+rW}\pi - \frac{1}{h(t)} \right) e^{-rt} f(t)\, dt \qquad \text{(because } 1-k=a\text{)}. \end{aligned}$$

Multiplying through by $(1+rW)/a$,

$$\begin{aligned} 0 &= V(\pi) - V^{VC}(\pi) \qquad \Longleftrightarrow \\ 0 &= \int_0^{T(\pi)} \left(\pi - \frac{1+rW}{h(t)} \right) e^{-rt} f(t)\, dt \\ &= \int_0^{T(\pi)} \left(\pi - \frac{1}{h(t)} \right) e^{-rt} f(t)\, dt + W \int_0^{T(\pi)} -re^{-rt}(1-F[t])\, dt. \qquad (52) \end{aligned}$$

Now, integrating by parts, one can rewrite the last expression in the previous equality chain as

$$\begin{aligned} \int_0^{T(\pi)} -re^{-rt}(1-F[t])\, dt &= e^{-rt}(1-F[t])\Big|_0^{T(\pi)} + \int_0^{T(\pi)} e^{-rt} f(t)\, dt \\ &= e^{-rT(\pi)}(1-F[T(\pi)]) - 1 + \int_0^{T(\pi)} e^{-rt} f(t)\, dt \qquad (53) \end{aligned}$$

Substituting from (53) into (52) we see, that (52) reads

$$\begin{aligned}
0 &= \int_0^{T(\pi)} \left(\pi - \frac{1}{h(t)}\right) e^{-rt} f(t)\,dt + W\left(e^{-rT(\pi)}\left(1 - F\left[T(\pi)\right]\right) - 1 + \int_0^{T(\pi)} e^{-rt} f(t)\,dt\right) \\
&= \int_0^{T(\pi)} \left(\pi - \frac{1}{h(t)}\right) e^{-rt} f(t)\,dt + W\left(e^{-rT(\pi)}\left(1 - F\left[T(\pi)\right]\right) + \int_0^{T(\pi)} e^{-rt} f(t)\,dt\right) - W \\
&= \int_0^{T(\pi)} \left(\pi + W - \frac{1}{h(t)}\right) e^{-rt} f(t)\,dt + W e^{-rT(\pi)}\left(1 - F\left[T(\pi)\right]\right) - W \\
&= \frac{1}{a}\left(V(\pi) - W\right).
\end{aligned}$$

Therefore (52) and (53) imply that

$$0 = V(\pi) - V^{VC}(\pi) \qquad \Longleftrightarrow 0 = V(\pi) - W \tag{54}$$

Since $V(\pi_{\min}) = W$, this implies claim (i).

(ii) If $s = a/(a + k + rW)$ then

$$\begin{aligned}
&\int_0^{T} \left(\frac{a}{a + k + rW}\pi - \frac{a}{h(t)}\right) e^{-rt} f(t)\,dt \\
= \;& \frac{a}{1 + rW}\int_0^{T} \left(\pi - \frac{1 + rW}{h(t)}\right) e^{-rt} f(t)\,dt.
\end{aligned}$$

But this is exactly $V(\pi) - V^{VC}(\pi)$ (see (52)) which, by claim (i) of this Lemma was shown to be zero at $\pi_{\min}$. ■

7.3 Derivation of the hazard for the estimated example

We use n as the exponent in order to clarify the algebra, and write the mixing parameter as μ. Then

$$F(t) = (1 - \mu)\left(\frac{\min(t, t_{\min})}{t_{\min}}\right)^n + I_{[t_{\min},\infty)}\mu F^P(t),$$

Then For $t < t_{\min}$, $f(t) = \frac{1}{t_{\min}}(1 - \mu)\,n\left(\frac{t}{t_{\min}}\right)^{n-1}$, and therefore

$$\begin{aligned}
\frac{f(t)}{1 - F(t)} &= \frac{1}{t_{\min}} \frac{(1 - \mu)\,n\left(\frac{t}{t_{\min}}\right)^{n-1}}{1 - (1 - \mu)\left(\frac{t}{t_{\min}}\right)^n} = \frac{1}{t_{\min}} \frac{(1 - \mu)\,n t_{\min}^{1-n} t^{n-1}}{1 - (1 - \mu)\,t_{\min}^{-n} t^n} \\
&= \frac{(1 - \mu)\,n t_{\min}^{1-n} t^{n-1}}{t_{\min} - (1 - \mu)\,t_{\min}^{1-n} t^n}
\end{aligned}$$

and

$$\lim_{t\nearrow 1}\frac{f(t)}{1-F(t)}=\frac{1}{t_{\min}}\frac{1-\mu}{\mu}n$$

For $t\geq t_{\min}$

$$\begin{aligned}\frac{f(t)}{1-F(t)} &= \frac{\mu\rho t_{\min}^{\rho}t^{-\rho-1}}{1-\left[(1-\mu)+\mu\left(1-\left(\frac{t}{t_{\min}}\right)^{-\rho}\right)\right]}\\ &= \frac{\mu\rho t_{\min}^{\rho}t^{-\rho-1}}{1-1+\mu-\mu+\mu\left(\frac{t}{t_{\min}}\right)^{-\rho}}\\ &= \frac{\rho t_{\min}^{\rho}t^{-\rho-1}}{\mu-\mu\left(1-\left(\frac{t}{t_{\min}}\right)^{-\rho}\right)}=\frac{\rho}{t}\end{aligned}$$

Therefore the hazards are equal at $t_{\min}$ if

$$\frac{1}{t_{\min}}\frac{1-\mu}{\mu}n=\frac{\rho}{t_{\min}},$$

i.e., if

$$\mu=\frac{1}{1+\frac{\rho}{n}}$$

After setting $n=2$, this leads to $\mu=\frac{2}{2+\rho}$, $1-\mu=\frac{\rho}{2+\rho}$, and, hence for $t<t_{\min}$,

$$\frac{f}{1-F}=\frac{1}{t_{\min}}\frac{(1-\mu)\,nt_{\min}^{1-n}t^{n-1}}{1-(1-\mu)\,t_{\min}^{-n}t^{n}}=\frac{1}{t_{\min}}\frac{\frac{\rho}{2+\rho}nt_{\min}^{-1}t}{1-\frac{\rho}{2+\rho}t_{\min}^{-2}t^{2}}=\frac{1}{t_{\min}}\frac{2\rho t_{\min}^{-1}t}{2+\rho-\rho t_{\min}^{-2}t^{2}}$$

which leads to to (44).