Learning, Macroeconomic Dynamics and the Term Structure of Interest Rates

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Abstract

We present a macroeconomic model in which agents learn about the central bank's inflation target and the output-neutral real interest rate. We use this framework to explain the joint dynamics of the macroeconomy, and the term structures of interest rates and inflation expectations. Introducing learning in the macro model generates endogenous stochastic endpoints which act as level factors for the yield curve. These endpoints are sufficiently volatile to account for most of the variation in long-term yields and inflation expectations. As such, this paper complements the current macro-finance literature in explaining long-term movements in the term structure without reference to additional latent factors.

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1 Introduction

Since the seminal papers by Vasicek (1977) and Cox, Ingersoll, and Ross (1985), there is a consensus in the finance literature that term structure models should respond to three requirements: absence of arbitrage opportunities and both econometric and numerical tractability. Models designed to meet these criteria can be useful, for instance, in the pricing of fixed income derivatives and in the assessment of the risks implied by fixed income portfolios. More recently, however, a number of requirements have been added to the modeling of the yield curve dynamics. Satisfactory models should also (*i*) be able to identify the economic forces behind movements in the yield curve, (*ii*) take into account the way central banks implement their monetary policies, and (*iii*) have a macroeconomic framework consistent with the stochastic discount factor implied by the model. In this paper, we present a model that fulfills all of the above requirements and, in addition, integrates learning dynamics within this macro-finance framework.

The model presented in this paper builds on recent developments (phases) in the *affine term structure* literature. The first phase is characterized by the use of latent or unobservable factors, as defined in Duffie and Kan (1996) and summarized in Dai and Singleton (2000).[1] Although this framework excludes arbitrage opportunities and is reasonably tractable, the factors derived from such models do not have a direct economic meaning and are simply labeled according to their effect on the yield curve (i.e. as a "level", a "slope", and a "curvature" factor).

The second phase involves the inclusion of macroeconomic variables as factors in the standard affine term structure model. Ang and Piazzesi (2003) show that such inclusion improves the forecasting performance of vector autoregression (VAR) models in which no-arbitrage restrictions are imposed.[2] Their model, nevertheless, still includes unobservable factors without an explicit macroeconomic interpretation. Kozicki and Tinsley (2001, 2002) indicate the importance of long-run inflation expectations in modeling the yield curve and connect the level factor in the affine term structure models to these long-run inflation expectations. This interpretation of the level factor is confirmed by Dewachter and Lyrio (2006), who estimate an affine term structure model based only on factors with a well-specified macroeconomic interpretation.[3] The mentioned papers do not attempt, however, to propose a macroeconomic framework consistent with the pricing kernel implied by their models.

The third and most recent phase in this line of research is marked by the use of structural macro relations together with the standard affine term structure model. The structural macro model replaces the unrestricted VAR set-up adopted in previous research[4], and has commonly been based on a New-Keynesian framework. Hördahl, Tristani and Vestin (2003) find that the forecasting performance of such model is comparable to that of standard latent factor models. They are also able to explain part of the empirical failure of the expectations hypothesis. A similar approach is adopted by Rudebusch and Wu (2003). Bekaert, Cho and Moreno (2006) go one step further and estimate a similar model based on deep parameters. They ensure that the pricing kernel they formulate is consistent with their proposed macro model.

The success of the macro-finance models is remarkable given the well-documented dynamic

[1] Duffee (2002) and Duarte (2004) propose more flexible specifications for the market prices of risk.

[2] Other papers following this approach include Diebold, Rudebusch and Aruoba (2003).

[3] A related approach can be found in Berardi (2004).

[4] For instance, the models presented in Ang and Piazzesi (2003), Dewachter and Lyrio (2006) and Dewachter, Lyrio and Maes (2006).

inconsistencies between the long-run implications of the macroeconomic models and the term structure of interest rates.[5] In particular, standard macroeconomic models fail to generate sufficient persistence to account for the time variation at the long end of the yield curve. The success of macro-finance models in fitting jointly the term structure and the macroeconomic dynamics in fact crucially hinges on the introduction of additional inert and independent factors with a macroeconomic interpretation. For instance, Bekaert *et al.* (2005), Dewachter and Lyrio (2006) and Hördahl *et al.* (2006), among others, introduce a time-varying (partly) exogenous implicit inflation target of the central bank and show that it accounts for the time variation in long-maturity yields.

The main goal of this paper is to build and estimate macro-finance models that generate these additional factors endogenously from a macroeconomic framework. To this end, we introduce learning into the framework of standard macro-finance models.[6] Extending macro-finance models with learning dynamics seems a promising route to model jointly the macroeconomic and term structure dynamics for two reasons. First, learning generates endogenously additional and potentially persistent factors in the form of subjective expectations.[7] Second, learning, especially constant gain learning, introduces sufficient persistence in the perceived macroeconomic dynamics to generate a level factor in the term structure of interest rates. Such a level factor is crucial to account for the time variation in the long end of the yield curve.[8]

Our approach connects the macro-finance models of the term structure to the learning literature.[9] Links between learning and the term structure of interest rates are also actively analyzed in the learning literature. For example, Cogley (2005) uses a time-varying Bayesian VAR to account for the joint dynamics of macroeconomic variables and the term structure of interest rates. Kozicki and Tinsley (2005) use a reduced form VAR in macroeconomic and term structure variables and assume agents have imperfect information with respect to the inflation target. They find that subjective long-run inflation expectations are crucial in fitting movements in long-maturity yields and inflation expectations and report a substantial difference between the central bank's inflation target and the subjective expectations of the inflation target. Orphanides and Williams (2005a) introduce long-run inflation expectations in the structural macroeconomic models by substituting expectations by and calibrating the learning parameters on observed survey data.[10] This paper complements this recent and rapidly growing literature. First, we do not rely on reduced form VAR dynamics. Instead, we use a standard New-Keynesian model to describe the macroeconomic dimension and impose consistency of the pricing kernel for the term structure and the macroeconomic

[5] For instance, Gürkaynak, Sack and Swanson (2005) and Ellingesen and Söderstrom (2004) show that standard macroeconomic models cannot account for the sensitivity of long-run forward rates to standard macroeconomic shocks. Also, Kozicki and Tinsley (2001) note that long-run inflation expectations need to evolve sluggishly over time relative to actual inflation rates to account for the variability at the long end of the term structure.

[6] More specifically, we assume imperfect information with respect to the (long-run) values of the inflation target and the output-neutral real interest rate.

[7] For instance, Milani (2005) finds that the persistence in the learning dynamics is sufficiently strong to capture much of the inertia of the macroeconomic series.

[8] Orphanides and Williams (2005a, b) using a calibrated learning model show that learning affects the long end of the term structure.

[9] As an alternative to macro-finance models including learning, some authors have augmented pure finance models of the yield curve with survey data. Studies using this approach include Kim and Orphanides (2005) and Chun (2005).

[10] Other papers using survey expectations as proxies for the theoretical expectations include Roberts (1997) and Rudebusch (2002).

dynamics. Second, following Sargent and Williams (2005), we generate the subjective expectations based on a learning technology that is optimal given the structural equations and the priors of the agents. Third, we estimate jointly the deep parameters of the structural equations and the learning parameters. The term structure of interest rates and surveys of inflation expectations are included as additional information variables in the measurement equation. We find that the proposed model generates sufficiently volatile subjective long-run expectations of macroeconomic variables to account for most of the time variation in long-maturity yields and surveys of inflation expectations. This is achieved without reference to additional latent factors and hence offers an alternative approach to the current macro-finance literature.

The remainder of the paper is divided in four sections. In Section 2, we present the macroeconomic framework, which is based on a standard New-Keynesian macro model. We introduce imperfect information with respect to the long-run targets, the respective priors, and derive the optimal learning rule. The perceived and actual laws of motion are derived together with the conditions for stability of the macroeconomic dynamics. The perceived law of motion forms the basis to generate the implied term structures of interest rates and inflation expectations. The yield curve model is generated by imposing the standard no-arbitrage conditions with respect to the perceived law of motion. A model for inflation expectations is generated by working through the implications of the perceived law of motion. It is shown that both the yield curve and the inflation expectations can be modeled as affine functions of an extended state space. The estimation methodology is presented in Section 3. Both the yield curve and surveys of inflation expectations are used as additional information variables to identify subjective expectations. In Section 4, we present the estimation results and compare the performance of the estimated models in fitting the term structure of interest rates. We show that macro-finance models, built on structural equations and learning explain a substantial part of the time variation of long-maturity yields and inflation expectations. Subsequently, we apply the model to identify the historical record of the policy stance. Finally, we conclude in Section 5 by summarizing the main findings of the paper.

2 Macroeconomic dynamics

We use the standard monetary three-equation New-Keynesian framework as presented in, for instance, Hördahl *et al.* (2006), Bekaert *et al.* (2005) and Cho and Moreno (2006). These models can be considered as minimal versions of a fully structural model (e.g. Christiano, Eichenbaum and Evans 2005, Smets and Wouters 2003). We adopt this version of the model as it can be considered the benchmark model in the literature linking macroeconomic dynamics and the term structure. We follow the standard procedure employed in the learning literature and replace the rational expectations operator by a subjective expectations operator. This subjective expectations operator is denoted by E^P and is explained in detail in Section 2.2. In the model presented below, subjective expectations differ from rational expectations in that we assume that agents do not observe the inflation target of the central bank nor the equilibrium output-neutral real interest rate. Finally, in Section 2.3 we solve for the macroeconomic dynamics, i.e. the actual law of motion. The solution is given in the form of a reduced VAR(I) model in an extended state space.

2.1 Structural equations

The structural model used in this paper is a standard version of the New-Keynesian monetary model often used in the literature linking term structure to macroeconomic dynamics (see, for instance, Bekaert *et al.* 2005 and Hördahl *et al.* 2006). The model is a parsimonious three-equation representation of the underlying macroeconomic structure, containing aggregate supply and IS equations and a monetary policy rule identifying the riskless nominal interest rate. To account for the persistence in inflation, the output gap and the policy rate, we add indexation, habit formation and interest rate smoothing to the standard model.

The aggregate supply (AS) equation is motivated by the sticky-price models based on Calvo (1983). In line with the standard Calvo price-setting theory, we assume a world where only a fraction of the firms updates prices at any given date, while the non-optimizing firms are assumed to use some rule of thumb (indexation scheme) in adjusting their prices (e.g. Galí and Gertler 1999). This setting leads to a positive relation between (transitory) inflation on the one hand and real marginal costs on the other. Additional assumptions are made with respect to the marginal costs and the indexation scheme of the non-optimizing agents. First, we assume that marginal costs are proportional to the output gap and an additional cost-push shock, ε_π. Second, non-optimizing firms are assumed to adjust prices according to an indexation scheme based on past inflation rates. The degree of indexation is measured by the parameter δ_π and the indexation scheme at time t is given by $\pi^* + \delta_\pi(\pi_{t-1} - \pi^*)$ with $\pi^*$the inflation target and π_{t-1} the previous period inflation rate. Following these assumptions, the standard AS curve is given by:

$$\pi_t = c_\pi + \mu_{\pi,1} E_t \pi_{t+1} + \mu_{\pi,2} \pi_{t-1} + \kappa_\pi y_t + \sigma_\pi \varepsilon_{\pi,t} \tag{1}$$

$$\begin{aligned} c_\pi &= (1 - \tfrac{\delta_\pi}{(1+\psi\delta_\pi)} - \tfrac{\psi}{(1+\psi\delta_\pi)})\pi^* \\ \mu_{\pi,1} &= \tfrac{\psi}{(1+\psi\delta_\pi)},\ \mu_{\pi,2} = \tfrac{\delta_\pi}{(1+\psi\delta_\pi)} \end{aligned} \tag{2}$$

where ψ represents the discount factor, and κ_π measures the sensitivity of inflation to the output gap. Given the assumed proportionality of marginal costs and the output gap, κ_π is a rescaled parameter of the sensitivity of inflation to the real marginal cost. Endogenous inflation persistence, $\mu_{\pi,2} > 0$, arises as a consequence of the assumption that non-optimizing agents use past inflation in their indexation scheme. Finally, we impose long-run neutrality of output with respect to inflation. Given the set-up of the model, this amounts to setting the discount factor (ψ) to one. Long-run neutrality is characterized by inflation parameters in the AS equation adding up to one, implying that $\mu_{\pi,1} = (1 - \mu_{\pi,2})$.

The IS curve is recovered from the Euler equation on private consumption. Following the recent strand of literature incorporating external habit formation in the utility function (e.g. Cho and Moreno 2006), and imposing the standard market clearing condition, we obtain the following IS equation:

$$y_t = \mu_y E_t y_{t+1} + (1 - \mu_y) y_{t-1} + \phi(i_t - E_t \pi_{t+1} - r) + \sigma_y \varepsilon_{y,t} \tag{3}$$

where the parameters μ_y and ϕ are functions of the utility parameters related to the agent's level of risk aversion, σ, and (external) habit formation, h:[11]

$$\mu_y = \tfrac{\sigma}{\sigma+h(\sigma-1)},\ \phi = -\tfrac{1}{\sigma+h(\sigma-1)}. \tag{4}$$

Habit formation is introduced as a means to generate additional output gap persistence. Without consumption smoothing, i.e. $h = 0$, the purely forward-looking IS curve is recovered. The demand shock $\varepsilon_{y,t}$ refers to (independent) shocks in preferences.[12] Equation (3) clarifies the interpretation of r as an output-neutral real interest rate. Other things equal, *ex ante* real interest rate levels $(i_t - E_t\pi_{t+1})$ above r reduce output (and inflation), while for *ex ante* real interest rates below r output (and inflation) increases. Although we could allow for time variation in this output-neutral real interest rate, we restrain from doing so in order to avoid additional complexities in the estimation arising from the fact that this variable is unobservable.

We close the model by specifying a monetary policy in terms of a Taylor rule. Following Clarida, Galí and Gertler (1999), we use a policy rule accounting both for policy inertia and imperfect policy control. Policy inertia is modeled through an interest rate smoothing term and imperfect policy control is modeled by means of an idiosyncratic interest rate shock, $\varepsilon_{i,t}$. The monetary policy rate equation used in this paper is given by:

$$i_t = \left(1 - \gamma_{i-1}\right) i_t^T + \gamma_{i-1} i_{t-1} + \sigma_i \varepsilon_{i,t}. \tag{5}$$

We model the central bank's targeted interest rate, i_t^T, by means of a Taylor rule in the output gap, y_t, and inflation gap, $\pi_t - \pi^*$:

$$i_t^T = r + E_t\pi_{t+1} + \gamma_\pi(\pi_t - \pi^*) + \gamma_y y_t \tag{6}$$

where π^* denotes the inflation target of the central bank. This policy rule differs in its appearance from the standard formulation of Taylor rules as we assign a weight of one to the expected inflation term. By imposing this condition, we model explicitly the idea that the central bank is actually targeting an *ex ante* real interest rate in function of the macroeconomic state, i.e. $\pi_t - \pi^*$ and y_t. Modeling monetary policy in terms of an *ex ante* real interest rate has the advantage that the policy rule is active ($\gamma_\pi > 0$, $\gamma_y > 0$) and stabilizing ($\kappa_\pi > 0$, $\phi < 0$), independent of the expectations formation process.

The model can be summarized in a standard matrix notation by defining the state space by a vector of macroeconomic variables, $X_t = [\pi_t,\ y_t,\ i_t]'$, and a vector of structural shocks, $\varepsilon_t = [\varepsilon_{\pi,t},\ \varepsilon_{y,t},\ \varepsilon_{i,t}]'$. Using a vector C and matrices A, B, D and S of appropriate dimensions, we write the structural equations as:

$$AX_t = C + BE_tX_{t+1} + DX_{t-1} + S\varepsilon_t. \tag{7}$$

[11] We assume the following utility function:

$$U(C_t) = (1-\sigma)^{-1} G_t \left(\frac{C_t}{H_t}\right)^{1-\sigma}$$

with G_t an independent stochastic preference factor and an external habit level, H_t, specified as $H_t = C_{t-1}^h$. Note that in order to have a well-defined steady state, the habit persistence needs to be restricted, $0 \leq h \leq 1$, as explained in Fuhrer (2000).

[12] Note that only by linearly detrending output we obtain a one-to-one relation between the shock in the IS equation and preference (demand) shocks. In general, the interpretation of ε_y as a demand shock is at least partially flawed, given the fact that it might also contain shocks to permanent output.

We assume that the rational expectations model generates a unique and determinate solution. This solution is given in terms of a structural VAR, where C^{re}, Φ^{re} and Σ^{re} contain the structural restrictions imposed in the rational expectations model:

$$X_t = C^{re} + \Phi^{re} X_{t-1} + \Sigma^{re} \varepsilon_t. \tag{8}$$

Endpoints, ξ_t, refer to long-run expectations of observable macroeconomic variables, X_t (see, for instance, Kozicki and Tinsley 2001):

$$\xi_t = \lim_{s \to \infty} E_t X_{t+s}. \tag{9}$$

Within the context of the New-Keynesian framework, endpoints are deterministic and are identified by solving the rational expectations model for the steady state. Under the restriction that in the long-run no trade-off exists between the output gap and the monetary policy, i.e. $\mu_{\pi,1} = (1 - \mu_{\pi,2})$, the steady state of the model is determined by the level of the inflation target, π^*, the steady state of the output gap, y^* (fixed to zero), and the output-neutral real interest rate level, $r = r^*$:

$$\xi_t = \lim_{s \to \infty} E_t \begin{bmatrix} \pi_{t+s} \\ y_{t+s} \\ i_{t+s} \end{bmatrix} = V \begin{bmatrix} \pi^* \\ y^* \\ r^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \pi^* \\ y^* \\ r^* \end{bmatrix}. \tag{10}$$

The mapping V is determined within the rational expectations model. In a rational expectations framework, the inflation target determines the long-run inflation expectations. The long-run expectations for the output gap are fixed at $y^* = 0$ and the long-run expectations concerning the nominal interest rate are determined by the Fisher hypothesis, linking the endpoint of the interest rate to the sum of the real interest rate and the inflation expectations. Finally, we can rewrite the rational expectations solution in an extended state space, consisting both of the observable macroeconomic variables X_t and their respective endpoints ξ, $\tilde{X}_t = [X_t', \xi']$ as:

$$\begin{bmatrix} X_t \\ \xi \end{bmatrix} = \begin{bmatrix} \Phi^{re} & (I - \Phi^{re}) \\ 0 & I \end{bmatrix} \begin{bmatrix} X_{t-1} \\ \xi \end{bmatrix} + \begin{bmatrix} \Sigma^{re} \\ 0 \end{bmatrix} \varepsilon_t. \tag{11}$$

2.2 Perceived law of motion

In this section, we introduce the perceived law of motion (PLM). We discuss the specific priors of agents and the optimal learning rule implied by these priors. We assume that agents believe *a priori* in structural changes in the economy. As in Sargent and Williams (2005), we derive an optimal learning rule given such priors and the structural equations. Subsequently, we discuss the implications of the perceived law of motion for the term structure of interest rates and the term structure of inflation expectations.

2.2.1 Priors and learning. We deviate from the standard rational expectations framework by introducing a set of priors describing agents' subjective beliefs. Agents are assumed to believe in stochastic endpoints, ξ_t^P, for the macroeconomic variables. We differentiate between the deterministic endpoints of the structural equations, ξ, and the perceived stochastic endpoints, ξ_t^P. The priors of the agents are modeled in terms of a vector error-correction model (VECM) specification for the macroeconomic variables:

$$\begin{aligned} X_t &= \xi_t^P + \Phi^P (X_{t-1} - \xi_t^P) + \Sigma^P \varepsilon_t \\ \xi_t^P &= V^P \zeta_t^P \\ \zeta_t^P &= \zeta_{t-1}^P + \Sigma_\zeta v_{\zeta,t}. \end{aligned} \tag{12}$$

The time variation in the stochastic endpoints ξ_t^P is due to time variation in the underlying stochastic trends in the economy, $\zeta_t^P = [\pi_t^{*P}, y_t^{*P}, r_t^{*P}]$, representing the vector containing the perceived inflation target, π_t^{*P}, the perceived long-run output gap, y_t^{*P} (fixed to zero), and the perceived long-run output-neutral real interest rate, r_t^{*P}. Agents thus believe in time variation in either the inflation target or the output-neutral real interest rate and hence use a misspecified model. Imperfect information and/or imperfect credibility of monetary policy (with respect to the implicit inflation target) can be used to justify intuitively this modeling assumption. The priors on the standard deviations of the shocks to the stochastic trends, representing a measure of the uncertainty of agents, is given by Σ_ζ:

$$\Sigma_\zeta = \begin{bmatrix} \sigma_{\zeta,\pi} & 0 & 0 \\ 0 & \sigma_{\zeta,y} & 0 \\ 0 & 0 & \sigma_{\zeta,r} \end{bmatrix}. \tag{13}$$

The matrix V^P, describing the cointegration relations between the macroeconomic variables X_t and the stochastic trends ζ_t^P, maps the stochastic trends into stochastic endpoints, $\xi_t^P = V^P \zeta_t^P$. We assume that agents know the structure of the economy such that we identify V^P by its rational expectations equivalent, $V^P = V$. Equation (12) can be used to show that the stochastic endpoints also determine the long-run subjective expectations of the agents. Denoting the expectations operator consistent with the agents' priors by E_t^P, it can be verified that

$$\lim_{s\to\infty} E_t^P X_{t+s} = \xi_t^P = V\zeta_t^P \tag{14}$$

or equivalently

$$\lim_{s\to\infty} E_t^P \begin{bmatrix} \pi_{t+s} \\ y_{t+s} \\ i_{t+s} \end{bmatrix} = V \begin{bmatrix} \pi_t^{*P} \\ y_t^{*P} \\ r_t^{*P} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \pi_t^{*P} \\ y_t^{*P} \\ r_t^{*P} \end{bmatrix}. \tag{15}$$

The priors about the transitory dynamics, i.e. the dynamics relative to the stochastic endpoints, are assumed to coincide with the ones implied by the rational expectations model. This implies that the matrices Φ^P and Σ^P are identical to their rational expectations equivalents: $\Phi^P = \Phi^{re}$ and $\Sigma^P = \Sigma^{re}$. By equating the perceived transitory dynamics to those implied by the rational expectations model, we obtain that the perceived law of motion differs from the rational expectations solution only due to the introduction of stochastic endpoints.[13] As a consequence, the rational expectations solution is nested in the perceived law of motion. By imposing the rational expectations prior, i.e. $\Sigma_\zeta = 0$, $\pi_t^{*P} = \pi^*$, $y_t^{*P} = 0$ and $r_t^{*P} = r$, the perceived law of motion coincides with the rational expectations solution (eq. (8)).

The stochastic trends ζ_t^P and the structural shocks ε_t are assumed to be unobservable. Agents, therefore, face an inference problem for the stochastic endpoints ξ_t^P, which is solved by means of a mean squared error (MSE) optimal Kalman filter learning rule. Denoting the inferred values for the stochastic endpoints by $\xi_{t|t}^P$, the learning algorithm becomes:

$$\xi_{t|t}^P = \xi_{t-1|t-1}^P + K(X_t - E_{t-1}^P X_t) \tag{16}$$

[13] Note that the analysis can be extended by allowing for differences between Φ^P and Φ^{re} or between Σ^P and Σ^{re}. We refrain from this extension due to the unnecessary additional complexity in the learning rules. Since the main goal of this paper is to explain time variation in long-run yields, allowing for stochastic endpoints seems more appropriate.

where K is obtained as the steady-state solution to the Kalman filtering equations:

$$\begin{aligned} K_t &= P_{t|t-1}(I-\Phi^P)'F_t^{-1} \\ F_t &= (I-\Phi^P)P_{t|t-1}(I-\Phi^P)' + \Sigma^P\Sigma^{P\prime} \\ P_{t+1|t} &= P_{t|t-1} - P_{t|t-1}(I-\Phi^P)'F_t^{-1}(I-\Phi^P)P_{t|t-1} + \Sigma_\zeta\Sigma_\zeta'. \end{aligned} \tag{17}$$

The final PLM can be written in extended state space, with $\eta_t = [\varepsilon_t', v_{\zeta,t}']'$, as:

$$\tilde{X}_t = \tilde{\Phi}^P \tilde{X}_{t-1} + \tilde{\Sigma}^P \eta_t \tag{18}$$

or equivalently:

$$\begin{bmatrix} X_t \\ \xi_t^P \end{bmatrix} = \begin{bmatrix} \Phi^P & (I-\Phi^P) \\ 0 & I \end{bmatrix} \begin{bmatrix} X_{t-1} \\ \xi_{t-1}^P \end{bmatrix} + \begin{bmatrix} \Sigma^P & (I-\Phi^P)V\Sigma_\zeta \\ 0 & V\Sigma_\zeta \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ v_{\zeta,t} \end{bmatrix}. \tag{19}$$

Comparing the perceived law of motion, eq. (19), with the dynamics in extended state space of the rational expectations solution, eq. (11), brings forward the differences and congruencies between the two models. Assuming $\Phi^P = \Phi^{re}$ and $\Sigma^P = \Sigma^{re}$, the main difference lies in the dynamics of the endpoints. Under the perceived law of motion endpoints are stochastic while in the rational expectations model they are deterministic.

2.2.2 The term structure of interest rates. Standard no-arbitrage conditions are used to generate bond prices consistent with the perceived law of motion. Imposing no-arbitrage under the PLM reflects the view that bond prices are set by the private sector and should, therefore, be consistent with the perceived dynamics and information set of these agents. Within the context of default-free, zero-coupon bonds, no-arbitrage implies a pricing equation of the form:

$$P_t(\tau) = E_t^P(M_{t+1}P_{t+1}(\tau-1)) \tag{20}$$

where E^P denotes the subjective expectations operator generated by the PLM (see eq. (19)), $P(\tau)$ denotes the price of a default-free, zero-coupon bond with maturity τ, and M_t denotes the pricing kernel consistent with the PLM. We follow Bekaert *et al.* (2005) in using the utility function implied by the macroeconomic framework to identify the prices of risk. While this approach has the advantage of guaranteeing consistency of the pricing kernel, it comes at the cost of loss of flexibility in modeling the prices of risk.[14] The (log) pricing kernel, consistent with the PLM is the homoskedastic (log) pricing kernel:

$$m_{t+1} = -i_t - \frac{1}{2}\sigma_m^2 - \Lambda\eta_{t+1} \tag{21}$$

where the prices of risk, Λ, are determined by the structural parameters

$$\Lambda = \sigma e_y \tilde{\Sigma}^P + e_\pi \tilde{\Sigma}^P - \sigma_y e_y \tag{22}$$

where e_x denotes a vector selecting the elements of the x-equation, i.e. e_y selects the row of $\tilde{\Sigma}^P$ related to the y-equation. No-arbitrage restrictions imposed on conditional Gaussian and linear

[14] The standard approach in modeling the term strcture is to assume a generally affine term structure representation. As shown by Duffee(2002) and Dai and Singleton (2000), general affine representations do not restrict the prices of risk to be constant.

state space dynamics generate exponentially affine bond pricing models (see, for instance, Ang and Piazzesi 2003):

$$P(\tau) = \exp(a(\tau) + b(\tau)\tilde{X}_{t|t}) \tag{23}$$

where $\tilde{X}_{t|t}$ denotes the inferred state vector, obtained by replacing ξ_t^P by its inferred value $\xi_{t|t}^P$, $\tilde{X}_{t|t} = [X_t', \xi_{t|t}^{P\prime}]'$. The factor loadings $a(\tau)$ and $b(\tau)$ can be obtained by solving difference equations representing the set of non-linear restrictions imposed by the no-arbitrage conditions:

$$\begin{aligned} a(\tau) &= -\delta_0 + a(\tau-1) - (b(\tau-1))\tilde{\Sigma}^P\Lambda' + \frac{1}{2}b(\tau-1)\tilde{\Sigma}^P\tilde{\Sigma}^{P\prime}b(\tau-1)' \\ b(\tau) &= b(\tau-1)\tilde{\Phi}^P - \delta_1' \end{aligned} \tag{24}$$

with $\delta_0 = 0$, and δ_1 implicitly defined by the identity $i_t = \delta_1'\tilde{X}_{t|t}$. The system has a particular solution given the initial conditions $a(0) = 0$ and $b(0) = 0$.

Exponentially affine bond price models lead to affine yield curve models. Defining the yield of a bond with maturity τ_1 by $y(\tau_1) = -ln(P_t(\tau_1))/\tau_1$ and the vector of yields spanning the term structure by $Y_t = [y_t(\tau_1), ..., y_t(\tau_n)]'$, the term structure can be written as an affine function of the extended state space variables:

$$Y_t = A_y + B_y\tilde{X}_{t|t} + v_{y,t} \tag{25}$$

where A_y and B_y denote matrices containing the maturity-specific factor loadings for the yield curve ($A_y = [-a(\tau_1)/\tau_1, ..., -a(\tau_n)/\tau_n]'$ and $B_y = [-b(\tau_1)'/\tau_1, ..., -b(\tau_n)'/\tau_n]'$), and $v_{y,t}$ contains maturity-specific measurement errors.

2.2.3 The term structure of inflation expectations. The representation of the term structure of inflation expectations is obtained from the PLM by iterating the model forward. It is straightforward to show that the linearity of the PLM generates an affine representation for the term structure of inflation expectations in the extended state space, $\tilde{X}_{t|t}$. The term structure of average inflation expectations is described by

$$E_t^P\bar{\pi}(\tau) = \frac{1}{\tau}\sum_{i=0}^{\tau-1} E_t^P(\pi_{t+i}) = e_\pi(a_s(\tau) + b_s(\tau)\tilde{X}_{t|t}) \tag{26}$$

where $E_t^P\bar{\pi}(\tau)$ denotes the time t average inflation expectation over the horizon τ, e_π denotes a vector selecting π_t out of the vector $\tilde{X}_{t|t}$, and $a_e(\tau)$, $b_e(\tau)$, $a_s(\tau)$ and $b_s(\tau)$ are maturity-dependent functions generated by the system:

$$\begin{aligned} &a_e(\tau) = 0, \; b_e(\tau) = b_e(\tau-1)\tilde{\Phi}^P \\ &a_s(\tau) = 0, \text{ and } \; b_s(\tau) = \tfrac{1}{\tau}\sum_{i=0}^{\tau-1} b_e(i) \end{aligned} \tag{27}$$

solved under the initial conditions $a_e(0) = 0$ and $b_e(0) = I$. Equation (26), applied over varying horizons τ, forms the model-implied term structure of average inflation expectations. The term structure of inflation expectations, unlike the term structure of interest rates, is not observable. We use surveys of average inflation expectations for different maturities as a proxy for the term structure of inflation expectations. We relate these surveys, $s(\tau)$, to the model-implied average inflation expectations by allowing for idiosyncratic measurement errors, $v_{s,t}$, in the survey responses:

$$s_t(\tau) = e_\pi a_s(\tau) + e_\pi b_s(\tau)\tilde{X}_{t|t} + v_{s,t} \tag{28}$$

where $s_t(\tau)$ denotes the time t survey response concerning the average inflation expectations over the horizon τ. Finally, denoting the vector containing a set of surveys of inflation expectations for different horizons by $S_t = [s_t(\tau_1), ..., s_t(\tau_m)]'$, and defining $A_s = 0$ and $B_s = [(e_\pi b_s(\tau_1))', ..., (e_\pi b_s(\tau_m))']'$, equation (28) can be restated as:

$$S_t = A_s + B_s \tilde{X}_{t|t} + v_t. \tag{29}$$

2.3 Actual law of motion

The actual law of motion (ALM), describing the observed dynamics of macroeconomic variables, is obtained by substituting the subjective expectations (19) into the structural equations (7). Since the subjective expectations are formed on the basis of the inferred stochastic endpoints, $\xi^P_{t|t}$, and on observable macroeconomic data, the relevant space of the ALM coincides with that of the PLM, i.e. $\tilde{X}_{t|t}$. Due to the simplicity of the learning algorithm, the ALM can be solved in closed form. In Appendix A, we show that the ALM reduces to a standard VAR(I) in the extended state space:

$$\tilde{X}_{t|t} = \tilde{C}^A + \tilde{\Phi}^A \tilde{X}_{t-1|t-1} + \tilde{\Sigma}^A \varepsilon_t \tag{30}$$

with

$$\begin{aligned}
\tilde{C}^A &= \begin{bmatrix} (A - B(\Phi^P + K_\Phi))^{-1} C \\ K(A - B(\Phi^P + K_\Phi))^{-1} C \end{bmatrix} \\
\tilde{\Phi}^A &= \begin{bmatrix} \Phi^P & (A - B(\Phi^P + K_\Phi))^{-1} B(I - K_\Phi)(I - \Phi^P) \\ 0 & I - K(I - (A - B(\Phi^P + K_\Phi))^{-1} B(I - K_\Phi))(I - \Phi^P) \end{bmatrix} \\
\tilde{\Sigma}^A &= \begin{bmatrix} (A - B(\Phi^P + K_\Phi))^{-1} S \\ K(A - B(\Phi^P + K_\Phi))^{-1} S \end{bmatrix}
\end{aligned} \tag{31}$$

and $K_\Phi = (I - \Phi^P)K$, A, B and S and Φ^P determined by the parameters of the structural equations, and K the constant gain matrix implied by the agents' priors. The closed form solution can be used to highlight some of the properties of the ALM. First, subjective beliefs about the stochastic endpoints are only relevant for the actual macroeconomic dynamics to the extent that an expectation channel exists, i.e. $B \neq 0$. If an expectations channel exists, the extension of the state space becomes relevant and perceived stochastic trends affect macroeconomic outcomes. One aspect in which macroeconomic dynamics may be affected concerns the modeling of persistence. Under rational expectations, persistence is driven by inflation indexation, habit persistence, and interest rate smoothing affecting the roots of the $\Phi^{re} = \Phi^P$ matrix. Learning introduces an additional source of persistence in the form of the persistence in the subjective expectations, $\xi^P_{t|t}$. Persistence in the beliefs follows itself from the inertia in the learning rule, i.e. the updating procedure. Milani (2005) shows in a different context that persistence due to learning is important and (partly) takes over the role of inflation indexation and habit formation. In the empirical section we find similar results, especially for inflation persistence and interest rate smoothing.

Second, the rational expectations model is nested within the learning framework. By imposing the priors consistent with rational expectations, i.e. $\Sigma_\zeta = 0$ and $\xi^P_{t|t} = V[\pi^*, 0, r]'$, it can be verified that the ALM simplifies (as $K = 0$) to the rational expectations reduced form, equation (8). Third, the nonstationarity of the PLM does not necessarily carry over to the ALM. The eigenvalues of the matrix $\tilde{\Phi}^A$ depend both on the structural parameters contained in A, B, Φ^P and on the learning parameters K. Finally, if the ALM is stationary, the unconditional distribution of the extended state space vector $\tilde{X}_{t|t}$ is identified. Conditional on the maintained assumption of normality of the structural shocks, ε_t, this distribution is given by:

$$\tilde{X}_{t|t} \sim N(E\tilde{X}_{t|t}, \Omega_{\tilde{X}}) \tag{32}$$

with:

$$E\tilde{X}_{t|t} = \begin{bmatrix} (I - \Phi^{re})^{-1}C^{re} \\ (I - \Phi^{re})^{-1}C^{re} \end{bmatrix}$$

$$vec(\Omega_X) = (I - \tilde{\Phi}^A \otimes \tilde{\Phi}^A)^{-1} vec(\tilde{\Sigma}^A \tilde{\Sigma}^{A\prime}).$$

Equation (32) represents the unconditional distribution for the extended state under learning. This distribution is characterized by two properties. First, as far as unconditional means are concerned, the ALM and the rational expectations model are observationally equivalent. The unconditional mean of the rational expectations model, i.e. $(I - \Phi^{re})^{-1}C^{re}$, coincides with the unconditional mean under the ALM for both the observable macroeconomic variables (inflation, output gap, and policy rate) and the perceived long-run expectations of the agents. The rational expectations model thus serves as a benchmark in mean for the model under learning. Second, in line with the literature on constant gain learning (e.g. Evans and Honkapohja 2001), the unconditional variance of the stochastic endpoints, $\xi^P_{t|t}$, is in general positive, implying non-convergence of the stochastic endpoints to the true values implied by the rational expectations equilibrium, $[\pi^*, 0, r + \pi^*]'$.

3 Estimation methodology

The actual law of motion for both macroeconomic variables and the inferred stochastic endpoints is used to estimate both the structural and the learning parameters. In order to identify the subjective beliefs, we use information variables directly related to the PLM, i.e. the term structure of interest rates and inflation expectations. In Section 3.1, we discuss the details of the estimation procedure. Subsequently, in Section 3.2, we explain the different versions of the model that are estimated.

3.1 Maximum likelihood estimation

The model is estimated by means of loglikelihood in the extended state space with the ALM dynamics serving as the transition equation:

$$\tilde{X}_{t|t} = \tilde{C}^A + \tilde{\Phi}^A \tilde{X}_{t-1|t-1} + \tilde{\Sigma}^A \varepsilon_t \tag{33}$$

and a measurement equation, relating the extended state to observable economic variables. The observable variables included in the measurement equation consist of macroeconomic variables, X_t (inflation, output gap, and policy rate), a sample of yields spanning the term structure of interest rates, Y_t (1, 2, 3, 4, 5 and 10 year yields), and a sample of the term structure of inflation expectations, proxied by survey data on inflation expectations, S_t (1 and 10 year average inflation expectations).[15]

[15] Survey expectations are increasingly used in the empirical literature. One of the earlier papers using survey expectations is Roberts (1997), showing that models including survey expectations can account for some of the (unexplained) inflation inertia. Orphanides and Williams (2005a) use survey expectations to calibrate the learning parameter in their macro model.

Survey expectations are also starting to be used in the bond pricing literature. For instance, Kim and Orphanides (2005) use survey expectations on short-term interest rate movements as an additional input in a otherwise standard Vasicek model. Also, Chun (2005) uses several survey expectations as additional inputs in a two-factor term structure model.

Finally, Bekaert *et al.* (2005) show the empirical relevance of surveys (on inflation) by showing that surveys help to forecast inflation better than any rational expectations model.

The observable variables are collected in the vector $Z_t = [X_t', Y_t', S_t']'$. Using the affine representation of each of these variables in the extended state space, as discussed in Section 2.2, the measurement equation becomes:

$$Z_t = A_m + B_m \tilde{X}_{t|t} + v_{z,t} \tag{34}$$

where $v_{z,t}$ denotes idiosyncratic measurement errors with variance-covariance matrix Ψ and A_m and B_m represent the derived affine representations of the respective subsets of observable variables X_t, Y_t and S_t (B_X is defined as: $X_t = B_X \tilde{X}_{t|t}$, i.e. $B_X = [I_{3\times3}, 0_{3\times3}]$, A_y, A_s, B_y and B_s are defined in equations (25) and (29), respectively):

$$A_m = \begin{bmatrix} 0 \\ A_y \\ A_s \end{bmatrix}, B_m = \begin{bmatrix} B_X \\ B_y \\ B_s \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Psi_y & 0 \\ 0 & 0 & \Psi_s \end{bmatrix}.$$

Prediction errors, $Z_t - E_{t-1}^A Z_t$, and their corresponding loglikelihood value $l(Z_t - E_{t-1}^A Z_t; \theta)$, where E_{t-1}^A denotes the expectations operator based on the ALM, are functions of both the structural macroeconomic shocks and the measurement errors:

$$\begin{aligned} & Z_t - E_{t-1}^A Z_t = B_m(\tilde{X}_{t|t} - E_{t-1}^A \tilde{X}_{t|t}) + v_{z,t} = B_m(\tilde{\Sigma}^A \varepsilon_t) + v_{z,t} \\ & l(Z_t - E_{t-1}^A Z_t; \theta) = -\tfrac{1}{2}\,|\Omega_Z| - \tfrac{1}{2}(Z_t - E_{t-1}^A Z_t)' \Omega_Z^{-1} (Z_t - E_{t-1}^A Z_t) \\ & \Omega_Z = B_m \tilde{\Sigma}^A \tilde{\Sigma}^{A\prime} B_m' + \Psi. \end{aligned} \tag{35}$$

One contribution of this paper is that the deep parameters of the structural equations and the parameters of the learning procedure are estimated jointly based on a wide variety of information variables, i.e. macroeconomic variables, term structure variables, and surveys of inflation expectations.[16] The parameters to be estimated are collected in the parameter vector θ, containing the deep parameters of the structural equations ($\delta_\pi, \kappa_\pi, \sigma, h, r, \pi^*, \gamma_\pi, \gamma_y, \gamma_{i-1}, \sigma_\pi, \sigma_y, \sigma_i$), the learning parameters (priors on the volatility of the stochastic trends $\sigma_{\zeta\pi}, \sigma_{\zeta r}$, and initial values $\zeta_{0|0}$), and the variances of the measurement errors ($diag(\Psi)$):

$$\theta = \left\{ \delta_\pi, \kappa_\pi, \sigma, h, r, \pi^*, \gamma_\pi, \gamma_y, \gamma_{i-1}, \sigma_\pi, \sigma_y, \sigma_i, \sigma_{\zeta\pi}, \sigma_{\zeta r}, \zeta_{0|0}, diag(\Psi) \right\}. \tag{36}$$

Not all deep parameters and learning parameters are estimated. We follow Hördahl *et al.* (2006) and Bekaert *et al.* (2005) by fixing the discount factor to one, $\psi = 1$. Also, throughout the estimation the prior on the uncertainty of the long-run value for the output gap is restricted to zero, $\sigma_{\zeta,y} = 0$. This restriction guarantees that the long-run expected output gap is fixed to zero under the PLM. Furthermore, we impose the theoretical constraints $\sigma_{\zeta,\pi}, \sigma_{\zeta,r} \geq 0$ and $0 \leq h \leq 1$. Finally, parameter estimates are constrained to satisfy two conditions. First, parameter estimates must be consistent with the existence of a unique rational expectations solution. Second, under learning, parameter estimates should imply eigenvalues of $\tilde{\Phi}^A$ strictly smaller than one in absolute value in order to guarantee stability of the ALM. The model is estimated using a Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm.

[16] Other research estimating the learning parameters include Orphanides and Williams (2005a) and Milani (2005). Orphanides and Williams (2005a) estimate the constant gain by minimizing the distance between the model-implied inflation expectations and those reported in the survey of professional forecasters. Milani (2005) estimates jointly, using Bayesian methods, the constant gain and the deep parameters of a structural macroeconomic model. We complement their analyses by including more information in the measurement equation, notably term structure of interest rates.

3.2 Estimated versions of the model

We estimate a total of six models. Model versions differ depending on (i) the type of information included in the measurement equation, and (ii) the assumptions made concerning the learning procedure. Four versions are based on the baseline model presented in the previous section, and two versions are extensions allowing for heterogeneity in the monetary policy.

Regarding the information included in the measurement equation, we distinguish between the Macro and the general versions of the model. In the Macro version, we restrict the measurement equation to incorporate only macroeconomic information, while in the general models we include all available information. The Macro version of the model is motivated by the concern that including term structure and survey information in the measurement equation may bias the estimates of the deep and learning parameters in order to fit the term structure and the survey expectations. To avoid such problem, a two-step procedure is employed. In the first step, the deep and learning parameters are estimated while restricting the measurement equation to contain only macroeconomic variables. In the second step, we fix the parameter estimates for the deep and learning parameters obtained in the first step and optimize the likelihood, based on the full measurement equation, over the remaining parameters, $diag(\Psi)$. This procedure ensures that deep and learning parameters are only based on the macroeconomic information included in the measurement equation, while the remaining parameters adjust to optimize the fit of the term structure and survey expectation part of the model. In the general version of the model, the estimation of all parameters is performed in one step, on the basis of the most general measurement equation.

We estimate both rational expectations and learning versions of the model. The learning versions of the model include four additional parameters, $\sigma_{\zeta,\pi}$, $\sigma_{\zeta,r}$ and the starting values for the stochastic trends, $\zeta_{0|0}$, describing the priors of the agents. The differentiation between rational expectations and learning models identifies the contribution of learning to the overall fit of the respective series. The four baseline models can be summarized as follows:

- Rational Expectations Macro: the rational expectations version is estimated using a two-step approach ensuring that the deep parameters are based only on macroeconomic information.
- Rational Expectations I: the rational expectations version is estimated using a one-step approach based on the general measurement equation.
- Learning Macro: the learning version is estimated using a two-step approach ensuring that the deep parameters are based only on macroeconomic information.
- Learning I: the learning version is estimated using a one-step approach based on the general measurement equation.

In addition to the four baseline models, we estimate two extensions to allow for heterogeneity in the monetary policy rule and in the agents' priors. The heterogeneity is modeled by means of chairman-specific policy rules and priors.[17] Specifically, the time-invariant policy rule parameters

[17] This procedure differs from other research that allows for time variation in the inflation target. For instance, Dewachter and Lyrio (2006), Kozicki and Tinsley (2005), and Hördahl *et al.* (2005) allow for variation in the inflation target of the central bank by modeling the inflation target as a inert autoregressive process. This approach results in quite variable inflation target dynamics. In contrast, this paper allows for discrete jumps in the inflation target at pre-specified dates. Beyond these dates, the inflation target is constant.

π^*, γ_π, γ_y and γ_{i-1} of the baseline models are replaced by chairman-specific parameters π^*_j, $\gamma_{\pi,j}$, $\gamma_{y,j}$ and $\gamma_{i-1,j}$, where j denotes the presiding chairman.[18] The heterogeneity in priors is modeled analogously by replacing the learning parameters $\sigma_{\zeta,\pi}$ and $\sigma_{\zeta,r}$ by their chairman-specific equivalents, $\sigma_{\zeta,\pi,j}$ and $\sigma_{\zeta,r,j}$. We estimate both the rational expectations version of this model, labeled Rational Expectations II and the learning version of the model, labeled Learning II. The model versions Rational Expectations I and Learning I, implying time-invariant policy rules and beliefs, are nested in the respective extensions and hence identify the contribution of allowing for policy heterogeneity in the overall fit.[19]

4 Estimation results

4.1 Data

We estimate the proposed models using quarterly data for the USA. The data covers the period from 1963:Q4 until 2003:Q4 (161 quarterly observations). The data set contains three series of macroeconomic observations obtained from Datastream: the quarter–by-quarter inflation (based on the Consumer Price Index - CPI, and collected from the Bureau of Labour Statistics), the output gap (constructed as the log of GDP minus the log of the natural output level, based on Congressional Budget Office data), and the Federal funds rate, representing the policy rate. Next to the macroeconomic variables, the data set includes six yields, spanning the yield curve, with maturities of 1, 2, 3, 4, 5 and 10 years. The data for yields up to five years are from the CRSP database.[20] The ten-year yields were obtained from the Federal Reserve. Finally, we also use survey data on short- and long-run inflation expectations. More specifically, we include the one- and ten-year average inflation forecast, as reported by the Federal Reserve Bank of Philadelphia in the Survey of Professional Forecasters.

Table 1 presents some descriptive statistics on the data set described above. These statistics point to the usual observations: the average term structure is upward sloping; the volatility of yields is decreasing with the maturity; normality is rejected for all series (based on JB statistics); and all variables display significant inertia, with a first order autocorrelation coefficient typically higher than 0.90. Inflation displays a somewhat lower inertia, i.e. an autocorrelation coefficient of 0.76.

Insert Table 1 and Figure 1

Table 1 also presents the correlation structure of the data. Three data features can be highlighted. First, the yields are extremely correlated across the maturity spectrum. This points to the well-known fact that a limited number of factors account for the comovement of the yields. Second, there is a strong correlation between the term structure and the macroeconomic variables,

[18] The chairmen included in the analysis are Martin (1951-1970), Burns (1970-1978), Miller(1978-1979), Volcker (1979-1987) and Greenspan (1987-2006). One exception is made to this rule. We divide the Volcker period in two sub-periods in order to account for the well-documented change in monetary policy that took place during this term, i.e. the change from monetary targeting to a more convential monetary policy. The first Volcker period ends in 1982Q3.

[19] For an analysis of regime changes on monetary policy, see Schorfheide (2005) or Sims and Zha (2004). Schorfheide finds evidence in favor of regime switches in monetary policy, while the evidence is less pronounced in Sims and Zha (2004). Both papers make use of Markov switching techniques identifying the regime breaks endogeneously. We, in contrast, fix the dates of the breaks to the moments of a change in the Fed chairman.

[20] We thank Geert Bekaert, Seonghoon Cho and Antonio Moreno for sharing the data set.

with significant positive correlations between inflation and the term structure and significant negative correlations between the term structure and the output gap. These correlation patterns are an indication of common factors driving macroeconomic and yield curve dynamics. Finally, we observe a substantial and positive correlation between the surveys of inflation expectations and both the macroeconomic variables (especially inflation and the Federal funds rate) and the yield curve. Again, this suggests that the factors affecting the yield curve and macroeconomic variables also drive movements in the surveys of inflation expectations.

4.2 Parameter estimates

Tables 2 and 3 report the estimation results for the rational expectations versions of the model. Our estimates for the Rational Expectations Macro model (Table 2) are broadly in line with the literature. We observe a mild domination of the forward looking terms for both the AS and IS curves ($\mu_{\pi,1} = 0.524$ and $\mu_y = 0.509$, respectively). The deviation from the purely forward-looking model ($\mu_{\pi,1} = 1$ and $\mu_y = 1$) is explained by the relatively high values for the inflation indexation parameter, δ_π, and the habit persistence, h, estimated at 0.91 and 1, respectively. Both estimates for the inflation sensitivity to the output gap, κ_π, and the sensitivity of the output gap with respect to the real interest rate, ϕ, are quite small, 0.00055 and -0.019, respectively. These values are much smaller than the ones typically used in calibration-based studies. However, they are commonly found in empirical studies using GMM or FIML methods. Our estimates imply an active monetary policy rule. The *ex ante* real interest rate reacts positively to both the inflation and the output gap, $\gamma_\pi = 0.674$ and $\gamma_y = 0.569$. Significant smoothing is also observed in the policy rule. As often found in the literature, some of the estimated parameters are not statistically significant. Similar results have been reported by, for instance, Cho and Moreno (2006).

Including the yield curve and the inflation survey data in the measurement equation tends to affect the parameters significantly. First, the estimated persistence decreases, as shown by the decrease in the indexation parameter δ_π, which takes a value of 0.67 in the Rational Expectations II model, and by the decrease in the habit persistence, h, both for the Rational Expectations I and II models to respectively 0.738 and 0.721. As a result of the drop in the indexation and/or the habit persistence, the forward-looking components ($\mu_{\pi,1}$ and μ_y) in the AS and IS equation increase. The estimates of monetary policy rule indicate for all versions of the model that (i) monetary policy is relatively inert, and (ii) the Taylor principle is satisfied since the *ex ante* real interest rate tends to increase with both the inflation gap and the output gap. The estimated inflation and output gap responses do vary across the alternative versions. Based on the results in Table 3, we find, as Clarida *et al.* (2000), a strong increase in the responsiveness to the inflation gap during the Volcker and Greenspan periods.

Insert Tables 2 and 3

Tables 4 and 5 report the estimation results for the versions where learning is introduced. The central parameters in the analysis, distinguishing learning models from rational expectations models, are the standard deviations of the perceived stochastic trends ζ_t^P, $\sigma_{\zeta,\pi}$ and $\sigma_{\zeta,r}$.[21] Our estimates for these parameters are statistically significant, indicating a rejection of rational expectations models. This finding holds irrespective of the version of the learning model and indicates the importance

[21] Note that the parameter $\sigma_{\zeta,y}$ was fixed to zero to be consistent with the assumption of long-run neutrality of money, see Section 2.

of the learning specification in modeling the joint dynamics of the macroeconomic variables, the yield curve and the survey expectations. Also, the estimated standard deviations, $\sigma_{\zeta\pi}$ and $\sigma_{\zeta r}$, are quite large and substantial variation in the estimates is observed across versions. The time series of subjective expectations, i.e. the stochastic trends $\xi^P_{t|t}$ implied by learning specification, are discussed below. The introduction of learning dynamics affects significantly the estimates of the deep parameters relative to those obtained for the rational expectations models. First, learning lowers significantly the degree of persistence derived from both the inflation indexation, δ_π, and the interest rate smoothing, γ_{i-1}. Across learning models, we find that the forward-looking component in the AS equation ($\mu_{\pi,1}$) increases substantially and significantly relative to the rational expectations versions of the model. This increase is explained by the decrease in the inflation indexation parameter.[22] The interest rate smoothing parameter drops significantly to values on average around 0.7 in the learning cases, which are more in line with Rudebusch (2002). A second effect of learning is that the inflation sensitivity to the output gap increases (for the Learning I and II models). We estimate κ_π levels of 0.05 and 0.012 in the Learning I and II models. Finally, note that one problematic feature of the estimation in the learning models is the identification of the inflation targets and the real interest rate r. Although these parameters are identified in the ALM, the standard errors of these parameters are very large, pointing to a high degree of uncertainty with respect to these parameters. This drop in significance can be attributed to the fact that the stochastic endpoints take over the role of these parameters in the expectation formation process.

Insert Tables 4 and 5

The size and significance of the standard deviations of the stochastic trends, $\sigma_{\zeta,\pi}$ and $\sigma_{\zeta,r}$, establish the importance of the learning dynamics in the models. Figures 2 to 4 present the time series for inflation, the real interest rate and the policy rate, together with their endpoints, according to either the rational expectations or the learning version of the model. Endpoints, representing long-run (subjective) expectations, are deterministic in the rational expectations model and stochastic in the learning versions of the model. Each figure also differentiates between Macro and full versions of the model. Figure 2 illustrates the effects of introducing learning on long-run inflation expectations. Without learning, long-run inflation expectations are anchored by the inflation target of the central bank. These inflation targets were estimated in the range of 3 to 4 percent per year. Allowing for chairman-specific policy rules in the rational expectations models results in significant differences of the inflation targets across chairmen, as depicted in the Rational Expectations II model. The estimates of the inflation targets show a gradual increase over the seventies until the ending of the Volcker experiment. Subsequently, according to the estimates, inflation targets gradually fell over time (around 5.2% in the second Volcker period and around 3.2% during the Greenspan terms). Especially, the gradual decline in inflation targets seems unrealistic given the strong deflationary policy conducted by Volcker.[23]

Insert Figures 2, 3, and 4

[22] The decrease in the inflation indexation as a consequence of the introduction of learning is also found in other studies. For instance, Milani (2005), introducing constant gain learning in a New-Keynesian macroeconomic model, finds an even stronger effect, with the inflation indexation parameter close to zero after the introduction of learning.

[23] One explanation for the observed time series of inflation targets is that inflation targets adapt so as to fit the surveys of inflation expectations. Since under rational expectations long-run expectations coincide with the inflation targets, inflation targets need to track the survey of inflation expectations. Some evidence

The right-hand side panels of Figure 2 depict inflation together with the subjective long-run inflation expectations under learning. In contrast to the rational expectation versions, inflation targets and subjective long-run inflation expectations are no longer identical. Sizable and inert deviations of the subjective expectations from the inflation target are observed. This difference between inflation targets and long-run inflation expectations is observed irrespective of the specific modeling assumptions with respect to monetary policy (comparing Learning I and Learning II) and irrespective of the amount of information included in the measurement equation (comparing Learning Macro and Learning I and II). Moreover, the observed stochastic endpoints are remarkably similar across model specifications. Finally, note that the estimated chairman-specific inflation targets are more in line with the historical record of US monetary policy (although, as mentioned above, the inflation targets are very imprecisely estimated). Estimates of time-varying inflation targets, in line with our results, can be found in Kozicki and Tinsley (2005) and Milani (2005). Figure 3 depicts the real interest rate and the long-run expectations under rational expectations and learning. The estimates in the rational expectations models for the long-run real interest rate coincide with the estimated real interest rate (between 2.5% and 3%). In the learning specifications, we find again that subjective long-run expectations for the real interest rate deviate from its rational expectations value. However, unlike inflation expectations, the real rate expectations are closer to the values implied by the rational expectations models. Again, we observe strong similarities in the time series irrespective of the details of the specific learning models. Finally, Figure 4, using the Fisher parity in the long-run, adds inflation and real interest rate expectations to obtain long-run expectations for the short-run policy interest rate. Under learning, we observe again the sizable differences between implied rational expectations endpoints and the subjective long-run expectations. Given the differences in the variability of respectively inflation and real interest rate expectations under learning, expectations for the nominal rate are dominated by inflation expectations.

4.3 Comparing learning and rational expectations models

4.3.1 BIC and likelihood decomposition. We use two measures to compare the six models. An overall evaluation of the model fit across versions is based on the Schwarz Bayesian Information Criterion (BIC) criterion. The BIC criterion, though not a formal statistical test, is used to take into account the fact that (*i*) we use different procedures to estimate some of the versions (i.e. Macro and general versions) and (*ii*) that, although the rational expectations and learning models are nested, standard likelihood ratio tests are not appropriate since the parameter restrictions of the rational expectations models are on the boundary of the admissible parameter space, i.e. $\sigma_{\zeta\pi} = 0$ and $\sigma_{\zeta\pi} = 0$.

A second informal measure used to compare the alternative versions of the model is based on a decomposition of the likelihood. The overall likelihood is decomposed according to the contributions of the macroeconomic shocks and the respective measurement errors in the term structure and survey of inflation expectations and can be used to analyze the fit of the alternative models across each of

in favor of this interpretation can be found in Table 6. Comparing the macro part of the likelihood, one observes a drop from the Rational Expectations I to the Rational Expectations II model, indicating that allowing for chairman-specific inflation targets worsened the macroeconomic fit. This drop in likelihood is more than compensated by the increase in likelihood in the term structure of interest rates and the survey parts of the likelihood.

these dimensions. The decomposition of the loglikelihood is given by:

$$l(Z_t - E^A_{t-1}Z_t;\theta) = l_1(X_t - E^A_{t-1}X_t;\theta) + l_2(Y_t - E^A_{t-1}Y_t;\theta, X_t - E^A_{t-1}X_t) + l_3(S_t - E^A_{t-1}S_t;\theta, X_t - E^A_{t-1}X_t) \tag{37}$$

$$l_1 \propto -\tfrac{1}{2}\left|B_X\tilde{\Sigma}^A\tilde{\Sigma}^{A\prime}B_X^{\prime}\right| - \tfrac{1}{2}(X_t - E^A_{t-1}X_t)^{\prime}(B_X\tilde{\Sigma}^A\tilde{\Sigma}^{A\prime}B_X^{\prime})^{-1}(X_t - E^A_{t-1}X_t)$$

$$l_2 \propto -\tfrac{1}{2}\left|\Psi_y\right| - \tfrac{1}{2}(Y_t - A_y + B_y\tilde{X}_{t|t})^{\prime}(\Psi_y)^{-1}(Y_t - A_y + B_y\tilde{X}_{t|t})$$

$$l_3 \propto -\tfrac{1}{2}\left|\Psi_s\right| - \tfrac{1}{2}(S_t - A_s + B_s\tilde{X}_{t|t})^{\prime}(\Psi_s)^{-1}(S_t - A_s + B_s\tilde{X}_{t|t})$$

where we interpret the components l_1, l_2 and l_3 as the likelihood of the macroeconomic prediction errors, and of the measurement errors in the term structure and survey of inflation expectations, respectively.

The results for the BIC and the likelihood decomposition are presented in Table 6. From this table, we conclude that learning is important in the overall fit of the model. According to the BIC, learning models in general outperform their rational expectations counterparts. This finding is independent of the information included in the measurement equation. Moreover, learning does not only improve on the specific rational expectations counterpart, but leads to an overall improvement, irrespective of the specific model specification. For instance, even though the parameterization of the Learning I model is tighter, it still outperforms the Rational Expectations II model, i.e. a rational expectations model including chairman-specific policy rules. The overall improvement of learning over rational expectations models cannot be entirely attributed to a better fit in the survey and the yield curve dimensions. Excluding both types of series from the measurement equation does not overturn the finding of the better performance of the learning models. In particular, comparing the two models estimated on measurement equations only incorporating macroeconomic information (Rational Expectations Macro and Learning Macro), one observes an improvement in the fit (according to the BIC). Finally, selecting on the BIC, the preferred model is the Learning II model, incorporating both learning dynamics and heterogeneity in monetary policy rules and priors. [24] The likelihood decomposition shows the contribution of the macroeconomic, yield curve, and inflation expectations parts in the total average likelihood. The higher likelihood of learning models is found in each of the respective parts. Introducing learning thus increases the likelihood in all dimensions. A trade-off exists between on the one hand the macroeconomic part and the yield curve and inflation expectations on the other with respect to the type of information included in the measurement equation. Including term structure and survey expectations in the measurement equation slightly biases the model towards fitting yield curve and survey data at the expense of the macroeconomic part. From the macroeconomic perspective, the best model is the Learning Macro version (12.08 as average loglikelihood). This model does not only improve on the likelihood of the macroeconomic part relative to its rational expectations counterpart, but also outperforms all other

[24] The findings of the BIC are confirmed by likelihood ratio tests comparing the learning models to proxies for the rational expectations versions. More in particular, we reestimated the learning models, fixing the learning parameters to small but positive numbers, $\sigma_{\zeta,\pi} = \sigma_{\zeta,r} = 0.0001$, and $\zeta_{0|0} = [\pi^*,\ 0,\ r]$. Likelihood ratio (LR) tests performed using the latter models as the null hypothesis reject the proxy models at 1% significance levels.

Also, note that the Rational Expectations I and II and Learning I and II are nested. Likelihood ratio tests indicate that both the Learning I and Rational Expectations I models are rejected against the alternatives, Rational Expectations II and Learning II, with test statistics 747, p-value 0.000, and 692, p-value 0.000, respectively.

models in terms of likelihood of the macroeconomic dynamics.[25]

Insert Table 6

4.3.2 Prediction errors. Table 7 presents summary statistics for the prediction errors of all variables. That includes the R^2, the mean, the standard deviation, and the autocorrelation of the prediction errors for the alternative model specifications. These are used to evaluate the overall performance of each model and to compare the relative performance among models. In terms of the overall evaluation of the models, we do find evidence of model misspecification. This misspecification manifests itself in the significant entries for the mean and the autocorrelation coefficients of the prediction errors. Therefore, none of the models is accepted as a completely satisfactory representation of the joint dynamics of the macroeconomic, yield curve and survey expectations variables. We find significant but in most cases small means for the forecast errors in many of the series and models. Also, depending on the model and the specific series, some autocorrelation remains in the prediction errors.[26]

In terms of relative performance, there is a clear distinction between the learning and the rational expectations models. Introducing learning typically leads to (i) an increase in the in-sample predictive power for both the term structure of interest rates and the survey expectations, and (ii) a decrease in the inertia in the prediction errors. Comparing counterparts, e.g. the Rational Expectations I with the Learning I, we observe an outperformance of the learning model relative to its rational expectations counterpart. The explained variation of the 10-year maturity yields increases from 50% in the Rational Expectations I model to 88% in the Learning I model. Also, autocorrelation coefficients and the standard deviation of prediction errors decrease substantially. Finally, the contribution of allowing both for chairman-specific policies and learning is considerable when comparing the summary statistics. Comparing the Rational Expectations II model with the Learning II, it can be observed that both models perform equally in the macroeconomic dimension. The Learning II model, however, performs better in the yield curve and survey expectations by decreasing the standard deviation of the prediction errors by more than 20%.

Insert Table 7

4.4 Learning dynamics, inflation expectations and bond markets

Do macroeconomic models including learning fit the term structure of interest rates and inflation expectations? To answer this question we analyze the fitting errors of the respective models. Table 8 presents summary statistics for the fit of the yield curve and the survey expectations. Figures 5 to 8 present the fitted values for the term structure of interest rates and the surveys of inflation

[25] One could argue that survey expectations constitute a part of the macroeconomic dimension. Summing columns I and III leads to the following contributions of the (extended) macroeconomic fit: 17.68, 17.30, 17.78, 18.15, 18.31, 18.77 for the Rational Expectations Macro, I, and II, and Learning Macro, I and II models, respectively. In this interpretation, the Learing II model to outperform all other variants of the model.

[26] The rejection of the overall model is common in the macro-finance literature. For instance, Bekaert *et al.* (2005) estimating a set of models, including micro-founded models, find significant remaining autocorrelation in the prediction errors. Also, in standard estimation of New-Keynesian models, various authors report remaining autocorrelation in the prediction errors, e.g. Cho and Moreno (2006). Finally, also in the pure finance literature it has been shown to be extremely difficult to observe affine term structure representation that are not rejected by the data.

expectations. Figure 5 shows the fitted values for the one-year yield while Figure 6 depicts the fitted values for the ten-year yields. We focus on the Learning II model. This model explains about 95% of the variation of the yield curve and more than 85% of the variation in the long-run inflation expectations. Furthermore, the mean prediction errors for the yield curve are low, ranging from 6 to 20 basis points, which is in the order of magnitude of studies using latent factor models (e.g. de Jong 2000). The fit of the Learning II model for the one-year and the ten-year yields are depicted in the lower-right panels of Figures 5 and 6, respectively. The fits of the survey expectations are presented in Figures 7 and 8. We conclude that in general terms, the Learning II model fits relatively well.

The success in fitting the yield curve and the survey expectations is due to both the inclusion of learning dynamics and heterogeneity in the monetary policy rule and priors. To identify the contribution of learning, we compare the fit of the Rational Expectations II and Learning II models. We observe an increase in fit due to learning of in between 4% (one-year yield) and 14% (ten-year yield). Furthermore, the remaining autocorrelation in the fitting errors is brought down significantly. Comparing the fits of the models, Figures 5 and 6 for the yield curve and Figures 7 and 8 for the survey expectations, clearly shows the difference between the rational expectations and learning model. This difference is especially pronounced for the ten-year yield and the ten-year average inflation expectation. The effect of allowing for heterogeneity in monetary policy rules and priors can be identified by comparing the models Learning I and II. Heterogeneity in monetary policy rules and priors increases the explained variation by about 2 to 4 percent and decreases significantly the remaining autocorrelation in the fitting errors. Overall, we conclude that the inclusion of learning has a significant impact on the performance of macroeconomic models in fitting the yield curve and the surveys of inflation expectations. Note however, that none of the learning models is fully satisfactory given the remaining autocorrelation in the fitting errors.

Insert Table 8 and Figures 5, 6, 7 and 8

Why do learning models outperform their rational expectations counterparts? To answer this question, we analyze in detail the affine term structure representations of rational expectations and learning models. More specifically, we look at the affine representations for the term structure of interest rates and inflation expectations in a transformed state space, decomposing the observed macroeconomic variables in perceived permanent and temporary components. This decomposition is achieved by the rotation matrix T:

$$T = \begin{bmatrix} I_3 & -I_3 \\ 0 & I_3 \end{bmatrix} \tag{38}$$

which generates the targeted decomposition:

$$\tilde{X}_{t|t}^{T} = \begin{bmatrix} X_t - \xi_{t|t}^{P} \\ \xi_{t|t}^{P} \end{bmatrix} = T \begin{bmatrix} X_t \\ \xi_{t|t}^{P} \end{bmatrix}. \tag{39}$$

Affine representation of the term structure of interest rates and inflation expectations can be restated in this state space:

$$Y_t = A_y + B_y \tilde{X}_{t|t} + v_{y,t} = A_y + B_y T^{-1} T \tilde{X}_{t|t} + v_{y,t} = A_y + B_y^T \tilde{X}_{t|t}^T + v_{y,t} \tag{40}$$

and

$$S_t = A_s + B_s \tilde{X}_{t|t} + v_{s,t} = A_s + B_s T^{-1} T \tilde{X}_{t|t} + v_{s,t} = A_s + B_s^T \tilde{X}_{t|t}^T + v_{s,t}. \tag{41}$$

Insert Figures 9 and 10

Figures 9 and 10 show this transformed yield curve and inflation expectations loadings both for the respective rational expectations and learning models.[27] The bond market loadings in Figure 9 are typically classified according to their impact on the term structure. Based on this figure, we identify one slope factor driving the yield spread, represented by the perceived transitory interest rate component, and two curvature factors, i.e. the perceived output gap and the perceived inflation gap. The curvature factors affect primarily the intermediate maturity yields. We also obtain a level factor. The level factor is driven only by changes in the perceived stochastic endpoints and, more specifically, the stochastic endpoint for the policy rate. The important feature of the level factor is that it exerts its influence equally over the entire yield curve.

Figure 10 depicts the loadings for the term structure of inflation expectations. Analogously to the yield curve loadings, we find a slope factor for inflation expectations in terms of inflation deviations from the perceived target and a level factor driven by the stochastic endpoint for inflation. This structure in factor loadings is recovered independently of the model specifications. While both rational expectations and learning models share a level factor in the transformed state space, the implications of this factor differ across models. The level factors are driven by the endpoints of the policy rate and inflation, $\xi^P_{i,t}$ and $\xi^P_{\pi,t}$, respectively. Rational expectations models imply deterministic endpoints fixed at the levels implied by the rational expectations model, i.e. $\xi_{i,t} = r + \pi^*$ and $\xi_{\pi,t} = \pi^*$.[28] Therefore, the level factor is constant and hence cannot explain time variation in the long-maturity yields or inflation expectations. Learning models differ from the rational expectations models since they generate endogenously stochastic endpoints for the policy rate and inflation. These stochastic endpoints are sufficiently volatile to account for the time variation in the long-end of the yield curve and the inflation expectations.

4.5 Monetary policy regimes

Finally, as an application of the model, we estimate the historical record of the monetary policy stance as implied by the Learning II model. From the IS equation, the policy stance of the central bank can be measured by comparing the *ex ante* real interest rate, $i_t - E^P_t \pi_{t+1}$, with the equilibrium real rate r. According to the policy stance measure, tough (loose) monetary policy stance is characterized by an *ex ante* real interest rate above (below) the neutral rate r. Figure 11 sets out the policy stance against both the inflation gap, $\pi_t - \pi^*$, and the output gap y. We also show the estimated targets for each of the chairmen to identify the different periods. The chairmen included in the analysis are Martin (1951-1970), Burns (1970-1978), Miller(1978-1979),

[27] Note that in the Rational Expectations II and Learning II models, yield curve and inflation expectations loadings also depend on the policy rule parameters. Given that we identify six policy regimes, we have six sets of loadings. For reasons of brevity, we only present the loadings implied by the Greenspan policy rules.

[28] Note that to the extent that one allows for time-varying inflation targets within the rational expectations framework, one can generate exogenously volatility in the endpoints. This is the approach followed in the standard macro-finance literature. The Rational Expectations II panels in Figures 2 and 4 are examples of this approach. The main advantage of learning is that there is no need to refer to exogenous shocks (i.e. in the inflation target) to account for the time variation in the long-end of the yield curve. The stochastic endpoints are generated endogenously in the model.

Volcker (1979-1987) and Greenspan (1987-2006).

Insert Figure 11

Figure 11 depicts the estimates of the policy stances based on the subjective expectations implied by the Learning II model. First, the Martin term, at least since the beginning of the sample in 1963, is characterized by a relatively neutral monetary policy stance. The estimates of the policy stance variable are close to zero. The second regime, i.e. the Burns term, is characterized by a much more active policy responding quite fast to changes in the output and inflation gap. Monetary policy responded strongly to both recessions which resulted in a loose monetary policy stance over most of the period. The policy stance variable captures the Volcker disinflation policy in the early eighties. The positive correlation with the inflation gap clearly demonstrates the fact that policy was aimed at fighting inflation despite the deep recession following the increase in nominal and real interest rates. Finally, and maybe surprisingly, our estimates indicate some variation in the policy stance during the Greenspan term. Early in the term and with the inflation gap positive, we find a positive value for the policy stance, indicating a tight monetary policy. Further in the term, we find that the policy stance is more correlated with the output gap than with the inflation gap. Especially the downturn (in terms of the negative output gap) in the early nineties gave rise to a switch in policy. Following the downturn of the economy after 2001, our estimates of the policy stance decrease considerably.

5 Conclusions

In this paper we built and estimated a macroeconomic model including learning. Learning was introduced in the model by assuming that agents do not believe in constant equilibrium real rates nor in time-invariant inflation targets. Given these priors, the optimal learning rule was derived in terms of a Kalman gain updating rule. We estimated the model including, next to the standard macroeconomic variables, yields and surveys of inflation expectations in the measurement equation. The structural parameters and learning parameters were estimated jointly. The findings of the paper can be summarized as follows. First, according to several measures, including learning improves the fit of the model, independent of the type of information included in the measurement equation. Although learning models improve on the rational expectations models, they are not fully satisfactory. Autocorrelation in the errors was found to be significant. Finally, we found that introducing learning in a standard New-Keynesian model generated sufficiently volatile stochastic endpoints to fit the variation in the long-maturity yields and the surveys of long-run inflation expectations. Our learning model, therefore, complements the current macro-finance literature linking macroeconomic and term structure dynamics.

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6 Appendix A: ALM dynamics

In this appendix, we derive a closed form solution for the actual law of motion (ALM). The derivation of the actual law of motion follows the standard approach in the learning literature by substituting subjective expectations, i.e. the PLM, into the structural equations. The structural equations are described in equation (7), which is repeated here as:

$$AX_t = C + BE_tX_{t+1} + DX_{t-1} + S\varepsilon_t \tag{42}$$

while the PLM is described by means of a VECM in the inferred stochastic endpoints:

$$X_t = (I - \Phi^P)\xi^P_{t|t} + \Phi^P X_{t-1} + \Sigma^P \varepsilon_t \tag{43}$$

and a learning rule based on the Kalman filter updating rule:

$$\xi^P_{t|t} = \xi^P_{t-1|t-1} + K(X_t - E^P_{t-1}X_t). \tag{44}$$

6.1 Deriving the Actual Law of Motion

A first step in obtaining the actual law of motion (ALM) consists of deriving the expectations implied by the PLM, equations (43) and (44). Under the PLM, the one-step ahead prediction, $E^P_t X_{t+1}$ is given by:

$$E^P_t X_{t+1} = (I - \Phi^P)E^P_t \xi^P_{t+1|t+1} + \Phi^P X_t. \tag{45}$$

Under the PLM dynamics, the stochastic endpoints $\xi^P_{t|t}$ are random walks, i.e. $E^P_{t-1}(X_t - E^P_{t-1}X_t) = 0$, such that $E^P_t \xi^P_{t+1|t+1} = \xi^P_{t|t}$. The one-step ahead expectations are given by

$$E^P_t X_{t+1} = (I - \Phi^P)\xi^P_{t|t} + \Phi^P X_t. \tag{46}$$

Substituting the learning rule, eq. (44), for $\xi^P_{t|t}$ we obtain a description for the expectations as:

$$E^P_t X_{t+1} = (I - \Phi^P)(\xi^P_{t-1|t-1} + K(X_t - E^P_{t-1}X_t)) + \Phi^P X_t \tag{47}$$

or equivalently, by lagging equation (46) one period giving a closed form expression for $E^P_{t-1}X_t = (I - \Phi^P)\xi^P_{t-1|t-1} + \Phi^P X_{t-1}$:

$$E^P_t X_{t+1} = (I - \Phi^P)(\xi^P_{t-1|t-1} + K(X_t - (I - \Phi^P)\xi^P_{t-1|t-1} - \Phi^P X_{t-1})) + \Phi^P X_t. \tag{48}$$

This expression can also be written as:

$$E^P_t X_{t+1} = (I - (I - \Phi^P)K)(I - \Phi^P)\xi^P_{t-1|t-1} + (\Phi^P + (I - \Phi^P)K)X_t - (I - \Phi^P)K\Phi^P X_{t-1}. \tag{49}$$

By denoting the matrix $(I - \Phi^P)K$ by K_Φ we obtain the final expression for the one-step ahead expectation as:

$$E^P_t X_{t+1} = (I - K_\Phi)(I - \Phi^P)\xi^P_{t-1|t-1} + (\Phi^P + K_\Phi)X_t - K_\Phi \Phi^P X_{t-1}. \tag{50}$$

The second step in deriving the ALM dynamics consists of inserting the subjective expectations, eq. (50), into the structural equations, i.e. eq. (42):

$$AX_t = C + B((I - K_\Phi)(I - \Phi^P)\xi^P_{t-1|t-1} + (\Phi^P + K_\Phi)X_t - K_\Phi \Phi^P X_{t-1}) + DX_{t-1} + S\varepsilon_t. \tag{51}$$

Solving for X_t we obtain:

$$\begin{aligned} X_t &= (A-B(\Phi^P+K_\Phi))^{-1}C+(A-B(\Phi^P+K_\Phi))^{-1}B(I-K_\Phi)(I-\Phi^P)\xi^P_{t-1|t-1} \\ &\quad (A-B(\Phi^P+K_\Phi))^{-1}(D-BK_\Phi\Phi^P)X_{t-1}+(A-B(\Phi^P+K_\Phi))^{-1}S\varepsilon_t. \end{aligned} \tag{52}$$

Note that if the rational expectations solution is unique, and if $\Phi^P = \Phi^{re}$, the expression $(A-B(\Phi^P+K_\Phi))^{-1}(D-BK_\Phi\Phi^P)$ equals Φ^P which allows us to rewrite the above dynamics as:

$$\begin{aligned} X_t &= (A-B(\Phi^P+K_\Phi))^{-1}C+(A-B(\Phi^P+K_\Phi))^{-1}B(I-K_\Phi)(I-\Phi^P)\xi^P_{t-1|t-1} \\ &\quad \Phi^P X_{t-1}+(A-B(\Phi^P+K_\Phi))^{-1}S\varepsilon_t. \end{aligned} \tag{53}$$

Equation (53) describes the actual law of motion for the observable macroeconomic variables as a function of the previous state, X_{t-1}, the inferred stochastic endpoints, $\xi^P_{t-1|t-1}$ and the structural shocks, ε_t. This description is only a partial description of the ALM, since the dynamics of the stochastic endpoints is not taken into account. In order to obtain a complete characterization of the ALM, we add the learning rule, i.e. equation (44). The joint dynamics of the observable macroeconomic variables, X_t, and the inferred stochastic endpoints, $\xi^P_{t|t}$ is given by:

$$\begin{aligned} \begin{bmatrix} I & 0 \\ -K & I \end{bmatrix}\begin{bmatrix} X_t \\ \xi^P_{t|t} \end{bmatrix} &= \begin{bmatrix} (A-B(\Phi^P+K_\Phi))^{-1}C \\ 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} \Phi^P & (A-B(\Phi^P+K_\Phi))^{-1}B(I-K_\Phi)(I-\Phi^P) \\ -K\Phi^P & (I-K(I-\Phi^P)) \end{bmatrix}\begin{bmatrix} X_{t-1} \\ \xi^P_{t-1|t-1} \end{bmatrix} \\ &\quad + \begin{bmatrix} (A-B(\Phi^P+K_\Phi))^{-1}S \\ 0 \end{bmatrix}\varepsilon_t \end{aligned} \tag{54}$$

where the dynamics for $\xi^P_{t|t}$ are given by equation (44). Finally, pre-multiplying by

$$\begin{bmatrix} I & 0 \\ -K & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ K & I \end{bmatrix} \tag{55}$$

yields a complete description of the ALM:

$$\begin{aligned} \begin{bmatrix} X_t \\ \xi^P_{t|t} \end{bmatrix} &= \begin{bmatrix} (A-B(\Phi^P+K_\Phi))^{-1}C \\ K(A-B(\Phi^P+K_\Phi))^{-1}C \end{bmatrix} \\ &\quad + \begin{bmatrix} \Phi^P & (A-B(\Phi^P+K_\Phi))^{-1}B(I-K_\Phi)(I-\Phi^P) \\ 0 & I-K(I-(A-B(\Phi^P+K_\Phi))^{-1}B(I-K_\Phi))(I-\Phi^P) \end{bmatrix}\begin{bmatrix} X_{t-1} \\ \xi^P_{t-1|t-1} \end{bmatrix} \\ &\quad + \begin{bmatrix} (A-B(\Phi^P+K_\Phi))^{-1}S \\ K(A-B(\Phi^P+K_\Phi))^{-1}S \end{bmatrix}\varepsilon_t. \end{aligned} \tag{56}$$

This ALM is represented in extended state space, $\tilde{X}_{t|t} = [X_t',\ \xi^{P\prime}_{t|t}]'$ by

$$\tilde{X}_{t|t} = \tilde{C}^A + \tilde{\Phi}^A\tilde{X}_{t-1|t-1} + \tilde{\Sigma}^A\varepsilon_t \tag{57}$$

with

$$\tilde{C}^A = \begin{bmatrix} (A - B(\Phi^P + K_\Phi))^{-1}C \\ K(A - B(\Phi^P + K_\Phi))^{-1}C \end{bmatrix}$$

$$\tilde{\Phi}^A = \begin{bmatrix} \Phi^P & (A - B(\Phi^P + K_\Phi))^{-1}B(I - K_\Phi)(I - \Phi^P) \\ 0 & I - K(I - (A - B(\Phi^P + K_\Phi))^{-1}B(I - K_\Phi))(I - \Phi^P) \end{bmatrix} \tag{58}$$

$$\tilde{\Sigma}^A = \begin{bmatrix} (A - B(\Phi^P + K_\Phi))^{-1}S \\ K(A - B(\Phi^P + K_\Phi))^{-1}S \end{bmatrix}.$$

6.2 Properties of the Actual Law of Motion

Based on the final representation of the ALM as stated in equation (57), some properties of the ALM can be described in more detail. A first property is that the unconditional mean of the ALM coincides with the unconditional mean of the rational expectations model. Denoting the expectations operators under rational expectations and under the ALM by respectively E^{re} and E^A, the equivalence between unconditional expectations can be formalized as:

$$\begin{aligned} E^A X_t &= E^{re} X_t = (I - \Phi^{re})^{-1} C^{re} \\ E^A \xi^P_{t|t} &= E^{re} X_t = (I - \Phi^{re})^{-1} C^{re}. \end{aligned} \tag{59}$$

We show this property by showing that $X_t = (I - \Phi^{re})^{-1} C^{re} = \xi_{t|t}$ is a steady state under the ALM. In the derivation we make extensive use of the properties of the rational expectations solution. More specifically, the unconditional mean for X_t based on the rational expectations model is given by:

$$E^{re}(X_t) = (I - \Phi^{re})^{-1} C^{re} \tag{60}$$

where the values for Φ^{re} and C^{re} satisfy the rational expectations conditions:

$$\begin{aligned} C^{re} &= (A - B\Phi^{re})^{-1}C + (A - B\Phi^{re})^{-1} B C^{re} \\ \Phi^{re} &= (A - B\Phi^{re})^{-1} D \\ \Sigma^{re} &= (A - B\Phi^{re})^{-1} S. \end{aligned} \tag{61}$$

We now show that the unconditional mean of X_t under the ALM, denoted by $E^A_t X_t$ coincides with the unconditional mean of the rational expectations model:

$$E^A_t X_t = E^{re} X_t = (I - \Phi^{re})^{-1} C^{re}. \tag{62}$$

In order to show this equivalence, we show that the point $X_t = (I - \Phi^{re})^{-1} C^{re}$ and $\xi_{t|t} = (I - \Phi^{re})^{-1} C^{re}$ are a steady state for the ALM. Substituting this particular point in the ALM, we obtain that this point is a steady state if it solves:

$$\begin{aligned} (I - \Phi^{re})^{-1} C^{re} = {} & (A - B(\Phi^P + K_\Phi))^{-1} C + \Phi^P (I - \Phi^{re})^{-1} C^{re} + \\ & (A - B(\Phi^P + K_\Phi))^{-1} B(I - K_\Phi)(I - \Phi^P)(I - \Phi^{re})^{-1} C^{re}. \end{aligned}$$

Noting that $\Phi^{re} = \Phi^P$ we can rewrite the equation by subtracting from both sides $\Phi^P (I - \Phi^{re})^{-1} C^{re}$, resulting in the equality:

$$(I - \Phi^P)(I - \Phi^{re})^{-1} C^{re} = (A - B(\Phi^P + K_\Phi))^{-1} C + (A - B(\Phi^P + K_\Phi))^{-1} B(I - K_\Phi)(I - \Phi^P)(I - \Phi^{re})^{-1} C^{re}.$$

Pre-multiplying by $(A - B(\Phi^P + K_\Phi))^{-1}$,

$$(A - B(\Phi^P + K_\Phi))C^{re} = C + B(I - K_\Phi)C^{re}.$$

Finally, this condition holds whenever a rational expectations equilibrium exists, i.e. adding $BK_\Phi C^{re}$ to both sides, the above condition reduces to the rational expectations condition for C^{re}

$$(A - B\Phi^P)C^{re} = C + BC^{re}.$$

The above derivation thus implies that if a rational expectations equilibrium exists, then the unconditional expectations of the rational expectations equilibrium coincides with the steady state of the ALM. If we assume, moreover, that all of the eigenvalues of $\tilde{\Phi}^A$ are strictly smaller than 1 in absolute value, the steady state of the ALM is attracting and defines the unconditional mean of the observable variables X_t. The second equality, i.e. $E^A \xi^P_{t|t} = (I - \Phi^{re})^{-1}C^{re}$ can be shown analogously.

A second property is the unconditional normality of the extended state vector $\tilde{X}_{t|t}$ under the ALM. Assuming a standard normal distribution for the structural shocks, ε_t, it is well known that the linearity of the state space dynamics and the assumed stability of the ALM (all eigenvalues of $\tilde{\Phi}^A$ are assumed to be strictly smaller than 1) implies that the unconditional distribution for $\tilde{X}_{t|t}$ is:

$$\tilde{X}_{t|t} \sim N(E^A \tilde{X}_{t|t}, \Omega_{\tilde{X}})$$

with:

$$E^A \tilde{X}_{t|t} = \iota_{2\times 1} \otimes (I - \Phi^{re})^{-1} C^{re}$$

$$vec(\Omega_{\tilde{X}}) = (I - \tilde{\Phi}^A \otimes \tilde{\Phi}^A)^{-1} vec(\tilde{\Sigma}^A \tilde{\Sigma}^{A\prime}).$$

7 Appendix B: The level factor in the affine term structure representation under the PLM

As claimed in Section 4.4, the presence of a set of stochastic endpoints under the PLM generates a level factor in the factor loadings. In order to show the appearance of a level factor in the term structure representation, we use a rotation on the state space vector $\tilde{X}_{t|t}$ decomposing the state vector into (perceived) temporary and permanent components. The rotation, generating the permanent-transitory decomposition, is given by T:

$$\tilde{X}_t^T = \begin{bmatrix} \pi_t - \xi^P_{t|t,\pi} \\ y_t - \xi^P_{t|t,y} \\ i_t - \xi^P_{t|t,i} \\ \xi^P_{t|t,\pi} \\ \xi^P_{t|t,y} \\ \xi^P_{t|t,i} \end{bmatrix} = T \begin{bmatrix} \pi_t \\ y_t \\ i_t \\ \xi^P_{t|t,\pi} \\ \xi^P_{t|t,y} \\ \xi^P_{t|t,i} \end{bmatrix} \quad \text{with } T = \begin{bmatrix} I_{3\times 3} & -I_{3\times 3} \\ 0 & I_{3\times 3} \end{bmatrix}.$$

Applying the rotation to the state space implies a rotation of the yield curve loadings from B_y to B_y^T where $B_y^T = B_y T^{-1}$. The affine term structure representation is rotated into:

$$Y_t = A_y + B_y T^{-1} T \tilde{X}_{t|t} = A_y + B_y^T \tilde{X}_{t|t}^T$$

The matrix of factor loadings B_y^T are generated by transforming the ODE of the original $b(\tau) = [b_X(\tau), b_\xi(\tau)]$ into the ODE generating $b^T(\tau) = [b_X^T(\tau), b_\xi^T(\tau)]$ with $b^T(\tau) = b(\tau)T^{-1}$. Taking the system of difference equations for $b(\tau)$:

$$\begin{aligned} a(\tau) &= -\delta_0 + a(\tau-1) - (b(\tau-1))\tilde{\Sigma}^P \Lambda' + \frac{1}{2} b(\tau-1)\tilde{\Sigma}^P \tilde{\Sigma}^{P\prime} b(\tau-1)' \\ b(\tau) &= b(\tau-1)\tilde{\Phi}^P - \delta_1' \end{aligned}$$

we obtain:

$$b^T(\tau) = [b_X^T(\tau), b_\xi^T(\tau)] = [b_X(\tau), b_\xi(\tau)]T^{-1} = [b_X(\tau-1), b_\xi(\tau-1)]T^{-1} T \tilde{\Phi}^P T^{-1} - \delta_1' T^{-1}.$$

Finally, from the PLM, equation (19), $\tilde{\Phi}^P$ is identified as

$$\tilde{\Phi}^P = \begin{bmatrix} \Phi^P & (I - \Phi^P) \\ 0 & I_{3\times 3} \end{bmatrix} \tag{63}$$

such that:

$$T\tilde{\Phi}^P T^{-1} = \begin{bmatrix} I_{3\times 3} & -I_{3\times 3} \\ 0 & I_{3\times 3} \end{bmatrix} \begin{bmatrix} \Phi^P & (I - \Phi^P) \\ 0 & I_{3\times 3} \end{bmatrix} \begin{bmatrix} I_{3\times 3} & I_{3\times 3} \\ 0 & I_{3\times 3} \end{bmatrix} = \begin{bmatrix} \Phi^P & 0 \\ 0 & I_{3\times 3} \end{bmatrix}$$

and

$$\delta_1' T^{-1} = [0,0,1,0,0,0] \begin{bmatrix} I_{3\times 3} & I_{3\times 3} \\ 0 & I_{3\times 3} \end{bmatrix} = [0,0,1,0,0,1]$$

the ODE for $b^T(\tau) = [b_X^T(\tau), b_\xi^T(\tau)]$ can be decoupled into:

$$\begin{aligned} b_X^T(\tau) &= b_X^T(\tau-1)\Phi^P - [0,0,1] \\ b_\xi^T(\tau) &= b_\xi^T(\tau-1) I_{3\times 3} - [0,0,1]. \end{aligned}$$

Note that the latter function $b_\xi^T(\tau)$, given the initial condition $b_\xi^T(\tau) = 0$, has as solution $b_\xi^T(\tau) = -[0,0,\tau]$ such that the loading for the permanent components, $\xi^P_{t|t}$, i.e. $B_y^T = -b_\xi^T(\tau)/\tau = [0,0,1]$ for all maturities. The fact that the factor loadings are identical across maturities identifies the level factor.

Table 1

Summary of data statistics (USA, 1963:Q4-2003:Q4, 161 observations)

	π	y	i	$\bar{y}_{1y}$	$\bar{y}_{2y}$	$\bar{y}_{3y}$	$\bar{y}_{4y}$	$\bar{y}_{5y}$	$\bar{y}_{10y}$	S_{1y}	S_{10y}
Mean (%)	4.49**	-1.11**	6.63**	6.52**	6.73**	6.90**	7.03**	7.11**	7.42**	4.18**	4.01**
Stdev (%)	2.83**	2.59**	3.28**	2.73**	2.67**	2.58**	2.53**	2.48**	2.47**	1.98**	1.49**
Auto	0.76**	0.95**	0.91**	0.93**	0.94**	0.95**	0.95**	0.96**	0.96**	0.98**	0.96**
Skew	1.49**	-0.47**	1.21**	0.78**	0.81**	0.85**	0.90**	0.88**	0.99**	0.83**	1.14**
Kurt	5.51**	3.51**	5.20**	4.06**	3.96**	3.92**	3.95**	3.67**	3.66**	2.79**	3.72**
JB	101.85	7.61	71.59	23.84	23.88	24.84	27.92	23.63	29.07	15.88	23.02
	(0.00)	(0.02)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
						Correlation matrix					
π	1.00										
y	-0.25**	1.00									
i	0.72**	-0.25**	1.00								
$\bar{y}_{1y}$	0.67**	-0.31**	0.95**	1.00							
$\bar{y}_{2y}$	0.65**	-0.39**	0.93**	0.99**	1.00						
$\bar{y}_{3y}$	0.62**	-0.44**	0.90**	0.98**	0.99**	1.00					
$\bar{y}_{4y}$	0.61**	-0.48**	0.89**	0.96**	0.99**	0.99**	1.00				
$\bar{y}_{5y}$	0.59**	-0.51**	0.87**	0.95**	0.98**	0.99**	0.99**	1.00			
$\bar{y}_{10y}$	0.59	-0.57	0.84	0.92**	0.96**	0.98**	0.99**	0.99**	1.00		
S_{1y}	0.83**	-0.46**	0.77**	0.77**	0.76**	0.74**	0.75**	0.73**	0.73**	1.00	
S_{10y}	0.80**	-0.63**	0.86**	0.86**	0.87**	0.870**	0.88**	0.88**	0.90**	0.98**	1.00

The sample period is 1963:Q4 to 2003:Q4 (161 quarterly observations). Inflation (π) is expressed in annual terms and is constructed by taking the quarterly percentage change in the consumer price index (CPI), that is $\pi_t = 4\ln(CPI_t/CPI_{t-1})$. The series for the CPI index is obtained from the Bureau of Labor Statistics. The output gap (y) series is constructed from data provided by the Congressional Budget Office (CBO). The bond yield data are based on data from Piazzesi. This data set concerns month-end yields on zero-coupon U.S. Treasury bonds with maturities of 1, 2, 3, 4, 5 and 10 years, expressed in annual terms. The Fed rate is used as the short-term interest rate (i), or the policy rate. *Mean* denotes the sample arithmetic average in percentage p.a., *Stdev* standard deviation, *Auto* the first order quarterly autocorrelation, *Skew* and *Kurt* stand for skewness and kurtosis, respectively. *JB* stands for the Jarque-Bera normality test statistic with the significance level at which the null of normality may be rejected underneath it. ** indicates significance at the 5% confidence level.

Table 2

Parameter estimates - Rational Expectations Macro and Rational Expectations I

$$\pi_t = \mu_{\pi,1} E_t \pi_{t+1} + (1 - \mu_{\pi,1}) \pi_{t-1} + \kappa_\pi y_t + \sigma_\pi \varepsilon_{\pi,t}$$

$$y_t = \mu_y E_t y_{t+1} + (1 - \mu_y) y_{t-1} + \phi(i_t - E_t \pi_{t+1} - r) + \sigma_y \varepsilon_{y,t}$$

$$i_t = \left(1 - \gamma_{i-1}\right) \left[r + E\pi_{t+1} + \gamma_\pi(\pi_t - \pi^*) + \gamma_y y_t\right] + \gamma_{i-1} i_{t-1} + \sigma_i \varepsilon_{i,t}$$

		Rat. Exp. Macro		Rat. Exp. I	
π-eq.	$\mu_{\pi,1}$	0.524**	(0.019)	0.527**	(0.007)
	$\kappa_\pi(\times 10^2)$	0.055	(0.278)	0.582**	(0.236)
y-eq.	μ_y	0.509**	(0.013)	0.580**	(0.018)
	ϕ	-0.019*	(0.011)	-0.012**	(0.005)
i-eq.	γ_{i-1}	0.862**	(0.036)	0.934**	(0.004)
	γ_π	0.674*	(0.356)	0.100	(0.165)
	γ_y	0.569	(0.504)	0.010	(0.172)
	r	0.025**	(0.010)	0.028**	(0.002)
	π^*	0.032**	(0.011)	0.044**	(0.002)
Stdev	σ_π	0.0063**	(0.0004)	0.0069**	(0.0004)
	σ_y	0.0043**	(0.0003)	0.0070**	(0.0009)
	σ_i	0.0133**	(0.0004)	0.0134**	(0.0005)
Struct	δ_π	0.908**	(0.069)	0.895**	(0.025)
	h	1.000	—	0.738**	(0.055)
	$\sigma(\times 10^{-2})$	0.274	(0.170)	0.496**	(0.201)

Stdev denotes the standard deviation. *Struct* denotes structural parameters. Maximum likelihood estimates with standard errors between brackets. A ** denotes significantly different from zero at the 5% significance level, * denotes significance at the 10% level.

TABLE 3

Parameter estimates - Rational Expectations II

$$\pi_t = \mu_{\pi,1} E_t \pi_{t+1} + (1-\mu_{\pi,1})\pi_{t-1} + \kappa_\pi y_t + \sigma_\pi \varepsilon_{\pi,t}$$

$$y_t = \mu_{y,1} E_t y_{t+1} + (1-\mu_y) y_{t-1} + \phi(i_t - E_t \pi_{t+1} - r) + \sigma_y \varepsilon_{y,t}$$

$$i_t = \left(1-\gamma_{i-1}\right)\left[r + E\pi_{t+1} + \gamma_\pi(\pi_t - \pi^*) + \gamma_y y_t\right] + \gamma_{i-1} i_{t-1} + \sigma_i \varepsilon_{i,t}$$

		Rational Expectations II					
π-eq.	$\mu_{\pi,1}$	0.598**	(0.010)				
	$\kappa_\pi(\times 10^2)$	0.627**	(0.256)				
y-eq.	$\mu_{y,1}$	0.589**	(0.018)				
	ϕ	-0.020**	(0.008)				
	r	0.029**	(0.002)				
		Martins		Burns		Miller	
i-eq.	γ_{i-1}	0.863**	(0.033)	0.612**	(0.032)	0.782**	(0.515)
	γ_π	0.745	(0.537)	0.244**	(0.109)	0.374	(1.605)
	γ_y	0.397	(0.425)	0.735**	(0.181)	1.131	(2.044)
	π^*	0.018**	(0.003)	0.038**	(0.002)	0.055**	(0.012)
		Volcker (a)		Volcker (b)		Greenspan	
i-eq.	γ_{i-1}	0.840**	(0.009)	0.959**	(0.022)	0.951**	(0.011)
	γ_π	0.541**	(0.212)	2.301	(2.091)	1.676**	(0.847)
	γ_y	0.531**	(0.222)	-1.189**	(0.505)	1.674**	(0.717)
	π^*	0.082**	(0.002)	0.052**	(0.001)	0.032**	(0.001)
Stdev	σ_π	0.0071**	(0.0004)				
	σ_y	0.0069**	(0.0008)				
	σ_i	0.0194**	(0.0017)				
Struct	δ_π	0.673**	(0.029)				
	h	0.721**	(0.052)				
	$\sigma(\times 10^{-2})$	0.294**	(0.115)				

Stdev denotes the standard deviation. *Struct* denotes structural parameters. Maximum likelihood estimates with standard errors between brackets. A ** denotes significantly different from zero at the 5% significance level, * denotes significance at the 10% level.

Table 4

Parameter estimates- Learning Macro and Learning I

$$\pi_t = \mu_{\pi,1} E_t \pi_{t+1} + (1 - \mu_{\pi,1})\pi_{t-1} + \kappa_\pi y_t + \sigma_\pi \varepsilon_{\pi,t}$$

$$y_t = \mu_y E_t y_{t+1} + (1 - \mu_y) y_{t-1} + \phi(i_t - E_t \pi_{t+1} - r) + \sigma_y \varepsilon_{y,t}$$

$$i_t = \left(1 - \gamma_{i-1}\right)\left[r + E\pi_{t+1} + \gamma_\pi(\pi_t - \pi^*) + \gamma_y y_t\right] + \gamma_{i-1} i_{t-1} + \sigma_i \varepsilon_{i,t}$$

		Learning Macro		Learning I	
π-eq.	$\mu_{\pi,1}$	0.672^{**}	(0.056)	0.759^{**}	(0.023)
	$\kappa_\pi(\times 10^2)$	0.431	(0.504)	4.912	(3.184)
y-eq.	$\mu_{y,1}$	0.504^{**}	(0.028)	0.541^{**}	(0.007)
	ϕ	-0.008	(0.023)	-0.038^{**}	(0.016)
i-eq.	γ_{i-1}	0.833^{**}	(0.039)	0.671^{**}	(0.009)
	γ_π	0.401	(0.300)	0.149	(0.097)
	γ_y	0.504	(0.416)	0.363^{**}	(0.046)
	r	0.028	(0.265)	0.030	(0.053)
	π^*	0.031	(0.669)	0.036	(0.360)
Stdev	σ_π	0.0062^{**}	(0.0005)	0.0087^{**}	(0.0006)
	σ_y	0.0043^{**}	(0.0003)	0.0050^{**}	(0.0003)
	σ_i	0.0132^{**}	(0.0004)	0.0126^{**}	(0.0004)
Struct	δ_π	0.489^{**}	(0.123)	0.318^{**}	(0.039)
	h	1.000	—	0.913^{**}	(0.036)
	$\sigma(\times 10^{-2})$	0.618	(1.778)	0.142^{**}	(0.058)
Learning	$\sigma_{\zeta,\pi}$	0.044^{**}	(0.013)	0.015^{**}	(0.001)
	$\sigma_{\zeta,y}$	0.000	—	0.000	—
	$\sigma_{\zeta,r}$	0.043	(0.141)	0.023^{**}	(0.001)
Initial points	$\xi_{0,\pi}$	0.018	(0.022)	0.012^{**}	(0.005)
	$\xi_{0,y}$	0.000	—	0.000	—
	$\xi_{0,i}$	0.014	(0.022)	0.042^{**}	(0.004)

Stdev denotes the standard deviation. *Struct* denotes structural parameters. Maximum likelihood estimates with standard errors between brackets. A ** denotes significantly different from zero at the 5% significance level, * denotes significance at the 10% level.

TABLE 5

Parameter estimates - Learning II

$$\pi_t = \mu_{\pi,1} E_t \pi_{t+1} + (1 - \mu_{\pi,1})\pi_{t-1} + \kappa_\pi y_t + \sigma_\pi \varepsilon_{\pi,t}$$

$$y_t = \mu_y E_t y_{t+1} + (1 - \mu_y) y_{t-1} + \phi(i_t - E_t \pi_{t+1} - r) + \sigma_y \varepsilon_{y,t}$$

$$i_t = \left(1 - \gamma_{i-1}\right)\left[r + E\pi_{t+1} + \gamma_\pi(\pi_t - \pi^*) + \gamma_y y_t\right] + \gamma_{i-1} i_{t-1} + \sigma_i \varepsilon_{i,t}$$

		Learning II					
π-eq.	$\mu_{\pi,1}$	0.728**	(0.027)				
	$\kappa_\pi(\times 10^2)$	1.182**	(0.323)				
y-eq.	$\mu_{y,1}$	0.528**	(0.008)				
	ϕ	-0.022**	(0.008)				
	r	0.026	(0.100)				
		Martins		Burns		Miller	
i-eq.	γ_{i-1}	0.804**	(0.046)	0.268**	(0.047)	0.637**	(0.388)
	γ_π	0.406	(1.150)	0.161	(0.111)	0.244	(1.014)
	γ_y	0.012	(0.293)	0.513**	(0.060)	0.310	(0.315)
	π^*	0.028	(0.265)	0.087	(0.621)	0.053	(0.478)
Learning	$\sigma_{\zeta,\pi}$	0.018**	(0.003)	0.014**	(0.002)	0.019**	(0.006)
	$\sigma_{\zeta,y}$	0.000	—	0.000	—	0.000.	—
	$\sigma_{\zeta,r}$	0.007	(0.008)	0.016**	(0.002)	0.019	(0.040)
		Volcker (a)		Volcker (b)		Greenspan	
i-eq.	γ_{i-1}	0.185**	(0.049)	0.795**	(0.022)	0.850**	(0.018)
	γ_π	0.564**	(0.095)	0.353	(0.290)	0.405	(0.630)
	γ_y	0.109	(0.079)	0.369**	(0.122)	0.224	(0.210)
	π^*	0.003	(0.177)	0.010	(0.285)	0.025	(0.266)
Learning	$\sigma_{\zeta,\pi}$	0.004	(0.004)	0.018**	(0.002)	0.008**	(0.002)
	$\sigma_{\zeta,y}$	0.000	—	0.000	—	0.000	—
	$\sigma_{\zeta,r}$	0.031**	(0.002)	0.049**	(0.003)	0.016**	(0.002)
Stdev	σ_π	0.0088**	(0.0006)				
	σ_y	0.0048**	(0.0004)				
	σ_i	0.0105**	(0.0006)				
Struct	δ_π	0.374**	(0.051)				
	h	0.935**	(0.035)				
	$\sigma(\times 10^{-2})$	0.235**	(0.079)				
Initial points	$\xi_{0,\pi}$	0.009	(0.008)				
	$\xi_{0,y}$	0.000	—				
	$\xi_{0,i}$	0.033**	(0.004)				

Stdev denotes the standard deviation. *Struct* denotes structural parameters. Maximum likelihood estimates with standard errors between brackets. A ** denotes significantly different from zero at the 5% significance level, * denotes significance at the 10% level.

Table 6

Likelihood decomposition

	Components				
	Macroeconomy	Yield curve	Inf. Exp.	Total	BIC
Rat. Exp. Macro	12.07	20.98	5.61	38.66	-76.95
Rat. Exp. I	11.75	23.23	5.55	40.53	-80.67
Rat. Exp. II	11.57	25.07	6.21	42.85	-84.31
Learning Macro	12.08	22.08	6.07	40.23	-79.72
Learning I	11.81	26.05	6.50	44.36	-88.22
Learning II	12.03	27.74	6.74	46.51	-91.76

Table 7

Summary statistics of forecast errors of macroeconomic variables, yield curve, and survey of inflation expectations

	π	y	i	y_{1y}	y_{3y}	y_{5y}	y_{10y}	S_{1y}	S_{10y}
Panel A: Rational Expectations Macro									
R^2	0.77	0.91	0.83	0.71	0.49	0.35	0.21	0.75	0.36
Mean (%)	0.07	-0.04	0.05	0.02	0.66**	1.08**	1.64**	0.31**	0.51**
Stdev (%)	1.20	0.78	1.36	1.48	1.83	2.00	2.19	0.92	1.02
Auto	-0.25**	0.21**	-0.10	0.38**	0.75**	0.86**	0.93**	0.58**	0.97**
Panel B: Rational Expectations I	π	y	i	y_{1y}	y_{3y}	y_{5y}	y_{10y}	S_{1y}	S_{10y}
R^2	0.76	0.87	0.82	0.76	0.74	0.68	0.50	0.65	-0.08
Mean (%)	-0.04	-0.35**	-0.03	-0.25**	-0.08	0.00	0.14	0.03	-0.10
Stdev (%)	1.21	0.94	1.38	1.34	1.32	1.41	1.74	1.08	1.22
Auto	-0.20**	0.53**	-0.12	0.19**	0.51**	0.72**	0.89**	0.74**	0.98**
Panel C: Rational Expectations II	π	y	i	y_{1y}	y_{3y}	y_{5y}	y_{10y}	S_{1y}	S_{10y}
R^2	0.74	0.84	0.83	0.82	0.84	0.83	0.79	0.75	0.72
Mean (%)	0.05	-0.36**	-0.03	-0.25**	-0.03	0.07	0.25	0.01	-0.08
Stdev (%)	1.27	1.04	1.34	1.16	1.03	1.03	1.12	0.91	0.62
Auto	0.09	0.63**	-0.12	0.10	0.40**	0.59**	0.75**	0.80**	0.79**
Panel D: Learning Macro	π	y	i	y_{1y}	y_{3y}	y_{5y}	y_{10y}	S_{1y}	S_{10y}
R^2	0.79	0.90	0.83	0.77	0.75	0.74	0.79	0.83	0.66
Mean (%)	0.00	-0.07	-0.01	-0.03	0.63**	1.20**	2.83**	-0.02	0.29**
Stdev (%)	1.15	0.82	1.34	1.30	1.30	1.26	1.14	0.74	0.72
Auto	-0.08	0.26**	-0.10	0.22**	0.48**	0.57**	0.58**	0.72**	0.88**
Panel E: Learning I	π	y	i	y_{1y}	y_{3y}	y_{5y}	y_{10y}	S_{1y}	S_{10y}
R^2	0.70	0.89	0.83	0.84	0.86	0.88	0.88	0.88	0.66
Mean (%)	0.02	-0.22**	-0.03	-0.24**	-0.02	0.14**	0.52**	-0.11**	0.10
Stdev (%)	1.36	0.88	1.34	1.09	0.96	0.87	0.84	0.64	0.68
Auto	0.42**	0.43**	0.07	0.24**	0.46**	0.52**	0.59**	0.85**	0.97**
Panel F: Learning II	π	y	i	y_{1y}	y_{3y}	y_{5y}	y_{10y}	S_{1y}	S_{10y}
R^2	0.72	0.89	0.87	0.88	0.90	0.91	0.92	0.90	0.81
Mean (%)	0.03	-0.19	0.04	-0.18**	-0.14**	-0.09	0.19**	-0.09	0.04
Stdev (%)	1.32	0.84	1.18	0.96	0.82	0.75	0.70	0.59	0.51
Auto	0.36**	0.34**	0.11	0.21**	0.32**	0.38**	0.41**	0.83**	0.94**

Mean denotes the sample average in percentage per year, *Stdev* the standard deviation in percentage per year, and *Auto* the first order quarterly autocorrelation. A ** denotes significantly different from zero at the 5% significance level.

Table 8

Summary statistics of fitting errors of yield curve and survey of inflation expectations

	y_{1y}	y_{3y}	y_{5y}	y_{10y}	S_{1y}	S_{10y}
Panel A: Rational Expectations Macro						
R^2	0.84	0.54	0.38	0.23	0.70	0.38
Mean (%)	-0.04	0.61**	1.04**	1.62**	0.26**	0.52**
Stdev (%)	1.10	1.75	1.96	2.16	1.00	1.01
Auto	0.64**	0.85**	0.91**	0.95**	0.57**	0.97**
Panel B: Rational Expectations I						
R^2	0.87	0.78	0.70	0.52	0.65	-0.05
Mean (%)	-0.22	-0.05	0.02	0.16	0.06	-0.08
Stdev (%)	0.97	1.22	1.37	1.70	1.07	1.20
Auto	0.51**	0.72**	0.82**	0.94**	0.66**	0.97**
Panel C: Rational Expectations II						
R^2	0.91	0.88	0.85	0.81	0.82	0.73
Mean (%)	-0.23**	0.01	0.11	0.27**	0.02	-0.08
Stdev (%)	0.83	0.90	0.95	1.07	0.78	0.61
Auto	0.43**	0.62**	0.72**	0.82**	0.75**	0.82**
Panel D: Learning Macro						
R^2	0.88	0.78	0.75	0.80	0.81	0.71
Mean (%)	-0.02	0.66**	1.22**	2.85**	0.01	0.33**
Stdev (%)	0.96	1.21	1.24	1.12	0.80	0.69
Auto	0.53**	0.70**	0.74**	0.79**	0.70**	0.85**
Panel E: Learning I						
R^2	0.93	0.91	0.91	0.91	0.91	0.71
Mean (%)	-0.20**	0.02	0.17**	0.53**	-0.07	0.14**
Stdev (%)	0.73	0.78	0.74	0.74	0.54	0.64
Auto	0.39**	0.56**	0.60**	0.71**	0.77**	0.96**
Panel F: Learning II						
R^2	0.95	0.95	0.95	0.95	0.92	0.85
Mean (%)	-0.16**	-0.10**	-0.06	0.20**	-0.05	0.09
Stdev (%)	0.59	0.59	0.57	0.55	0.51	0.46
Auto	0.23**	0.33**	0.39**	0.52**	0.71**	0.92**

Mean denotes the sample average in percentage per year, *Stdev* the standard deviation in percentage per year, and *Auto* the first order quarterly autocorrelation. A ** denotes significantly different from zero at the 5% significance level.

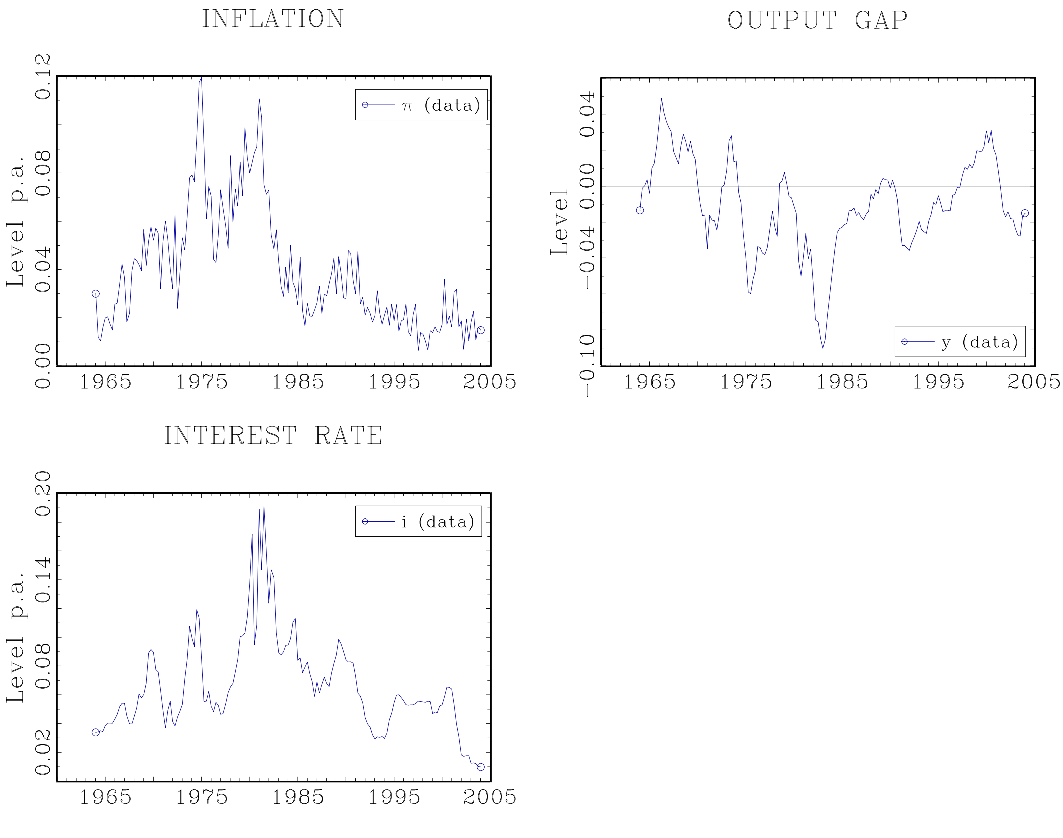

Figure 1: Data, USA, 1963:Q4-2003:Q4 (161 observations).

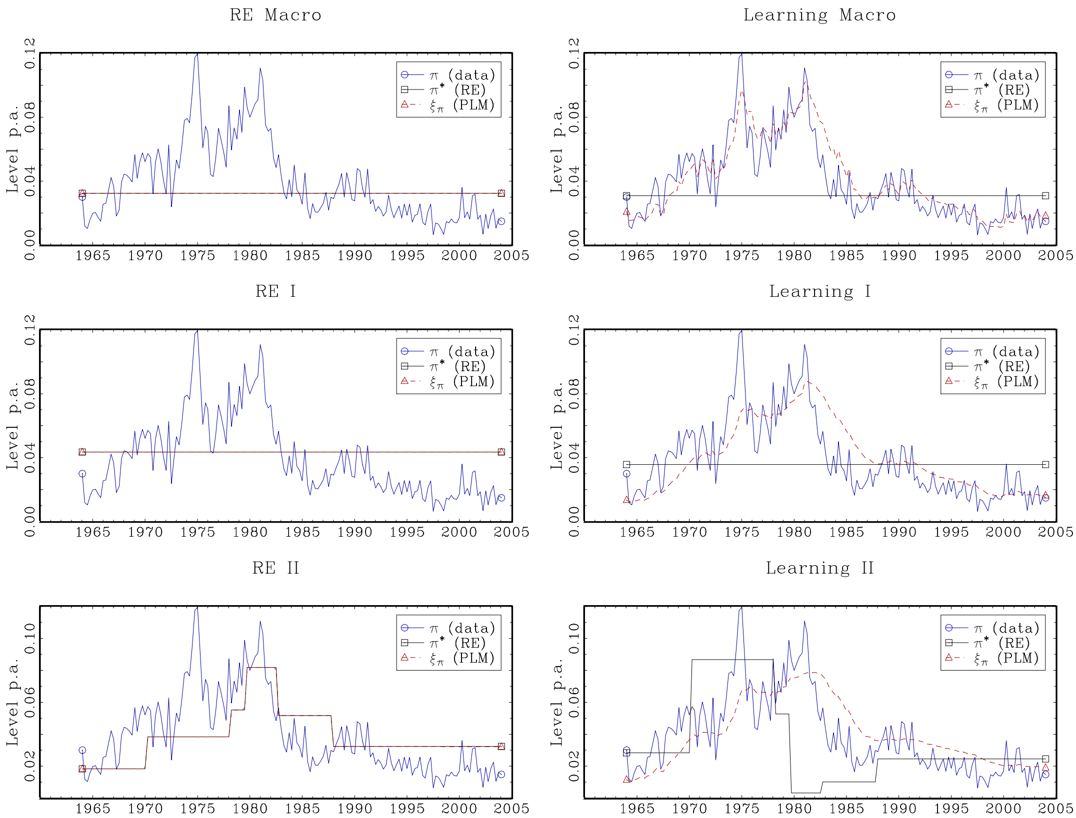

Figure 2: Inflation.

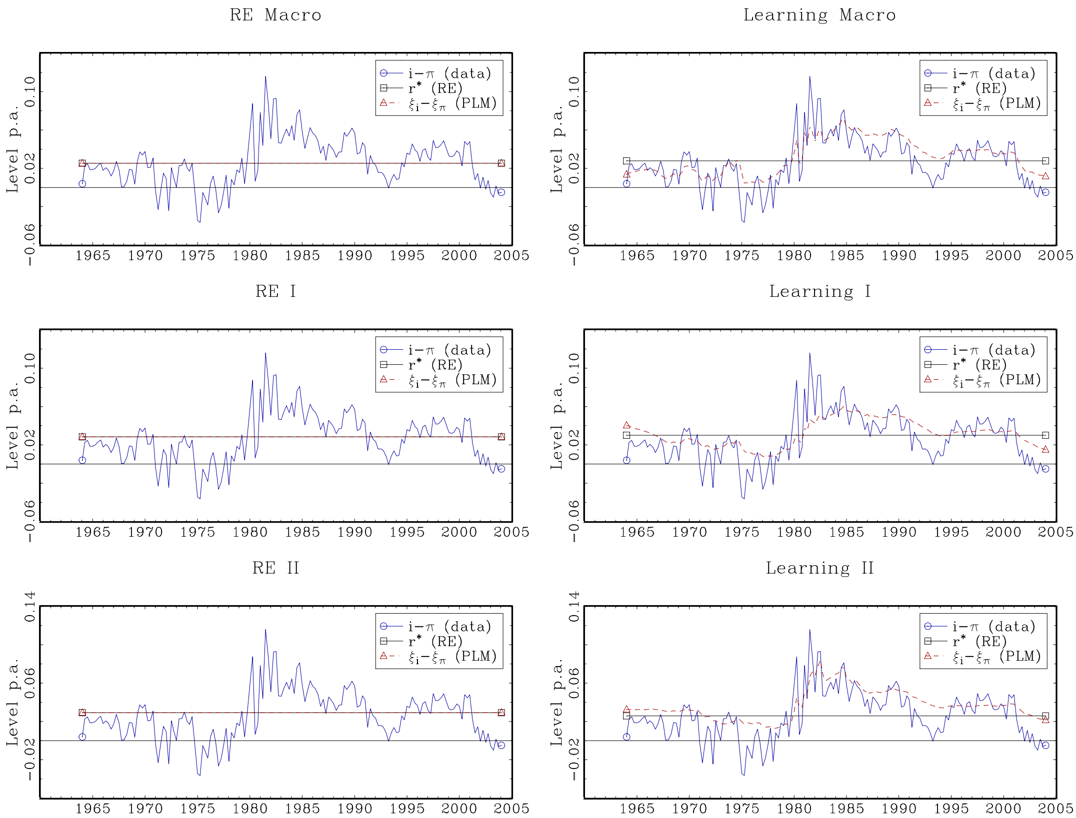

Figure 3: Real interest rate.

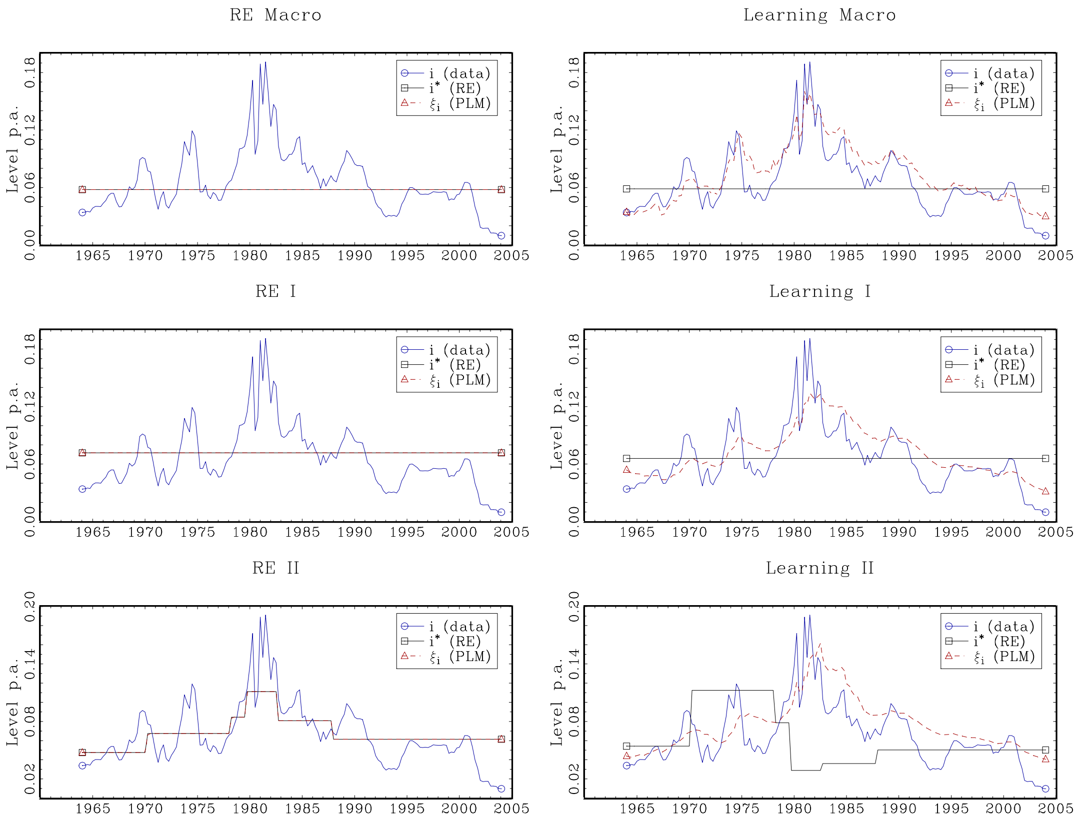

Figure 4: Policy interest rate.

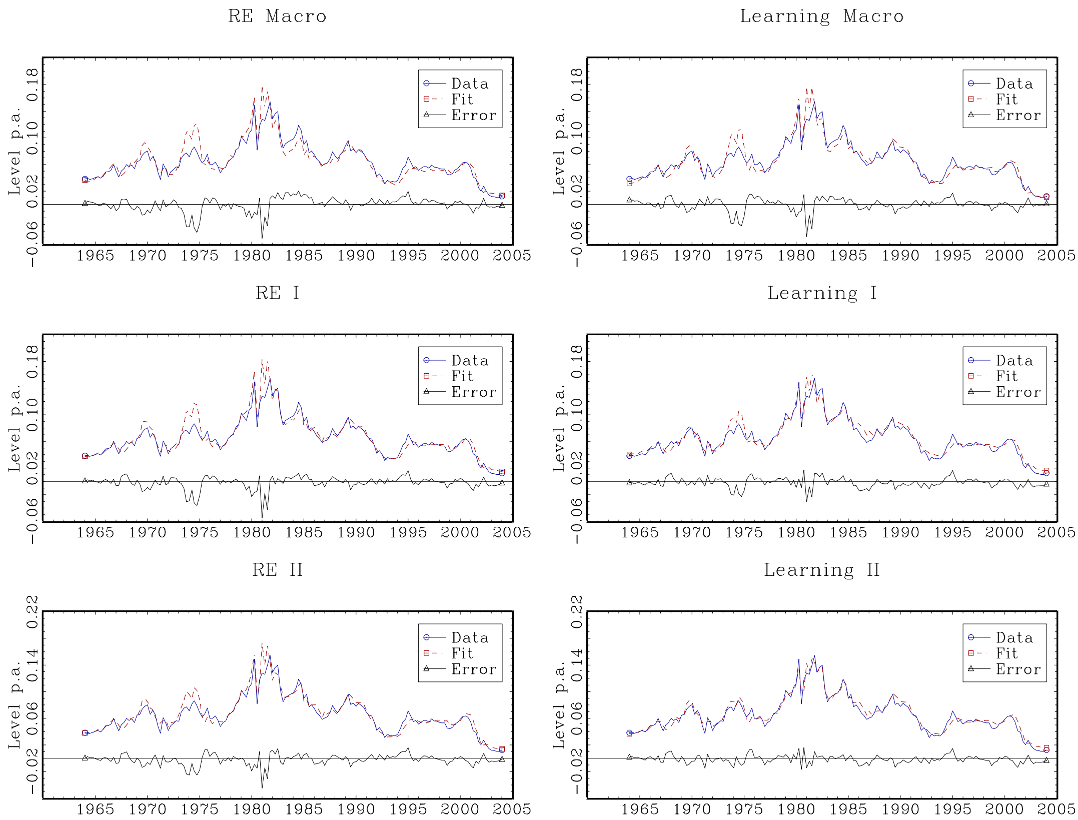

Figure 5: Term structure fit across models, one-year yield.

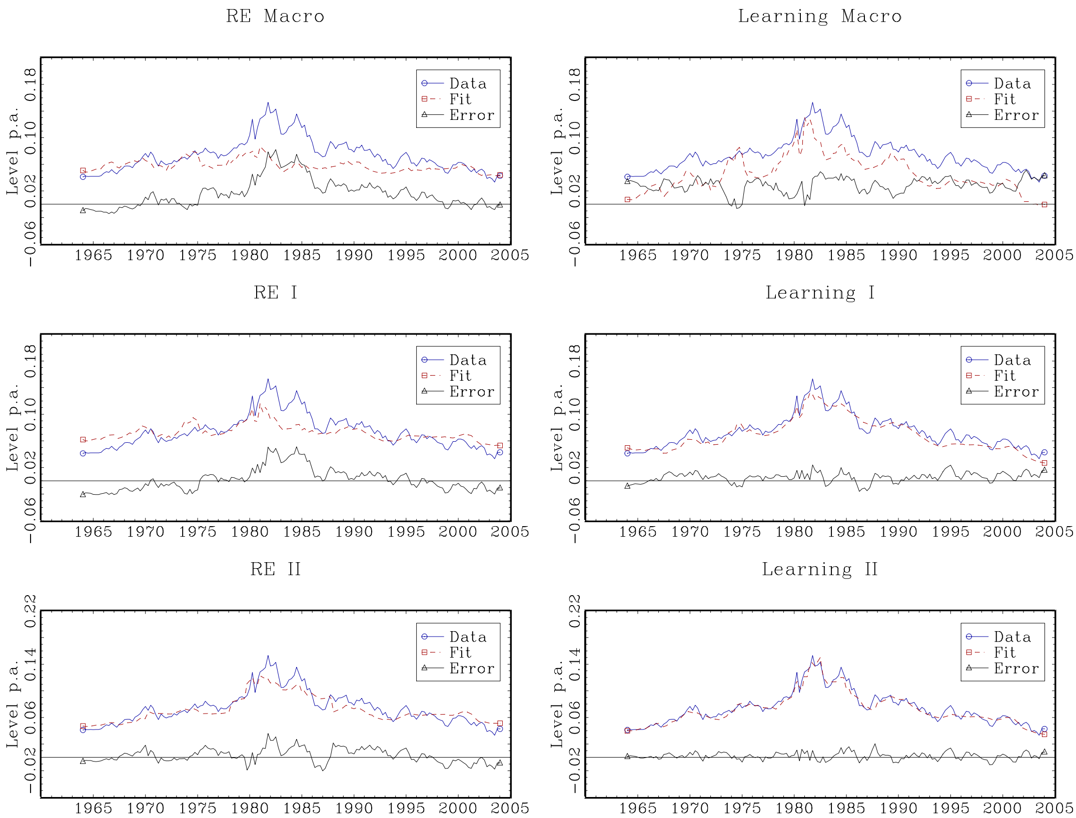

Figure 6: Term structure fit across models, ten-year yield.

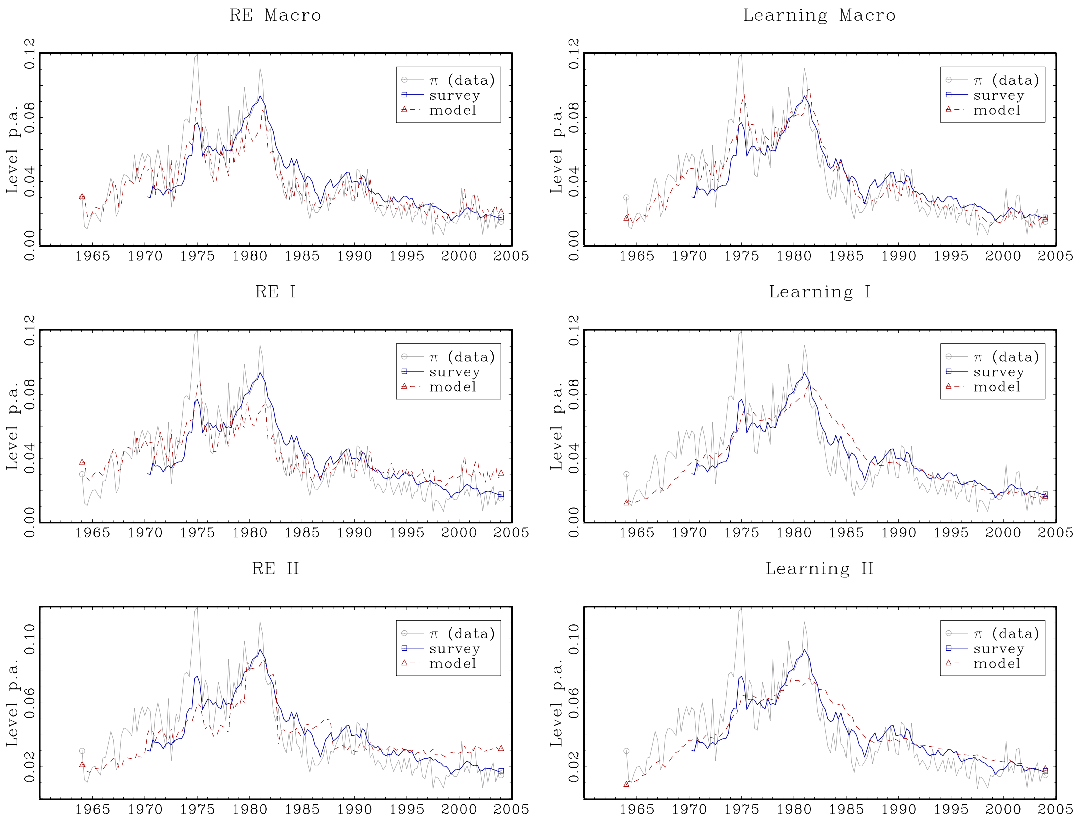

Figure 7: Fit of survey of one-year average inflation expectations across models.

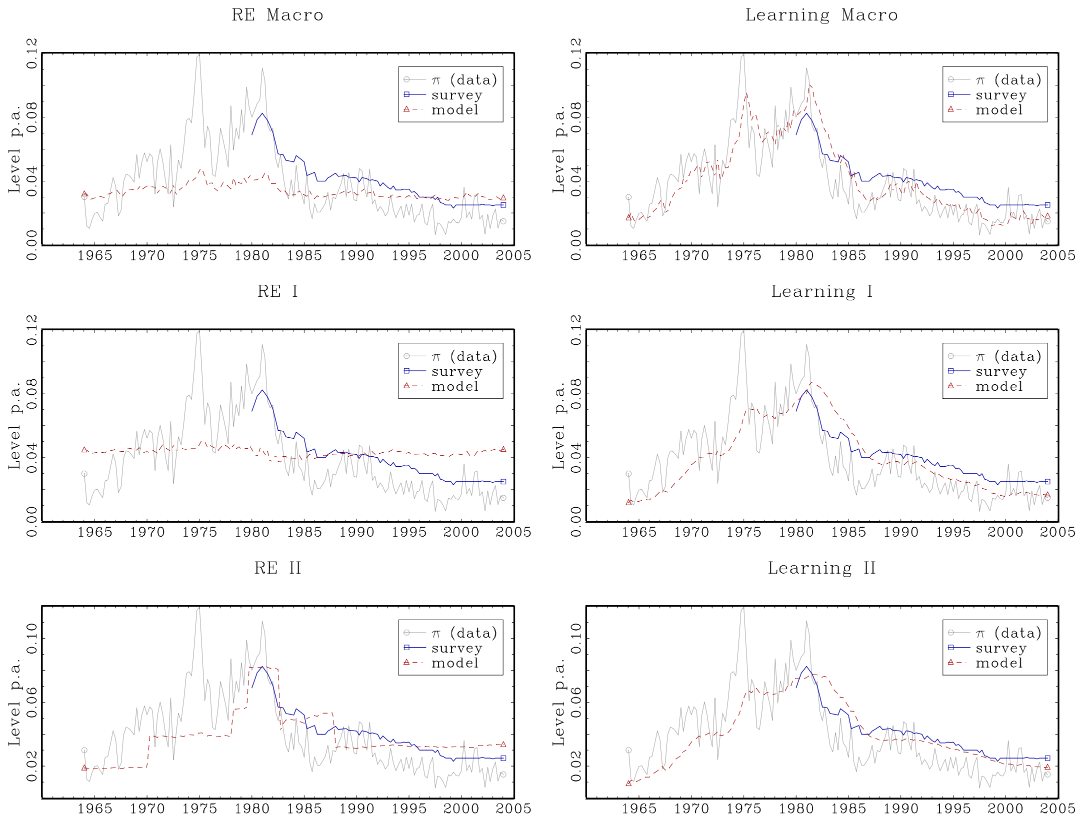

Figure 8: Fit of survey of ten-year average inflation expectations across models.

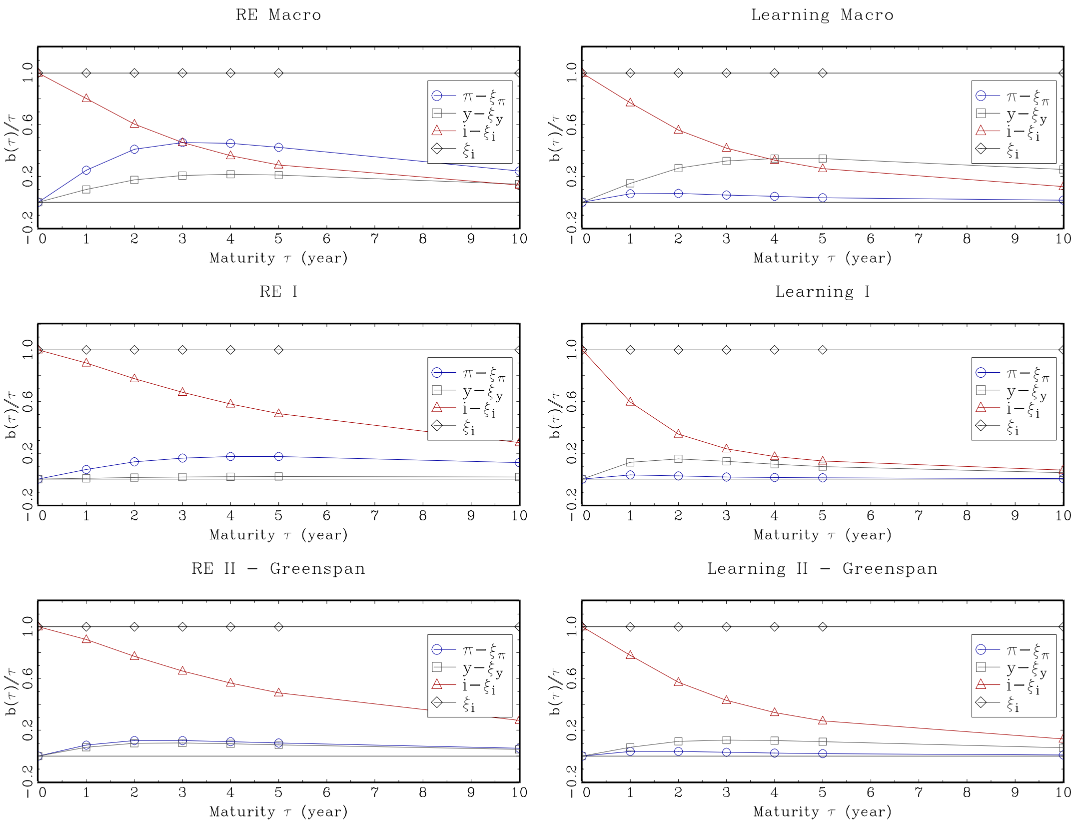

Figure 9: Loading - Term structure of interest rates across models.

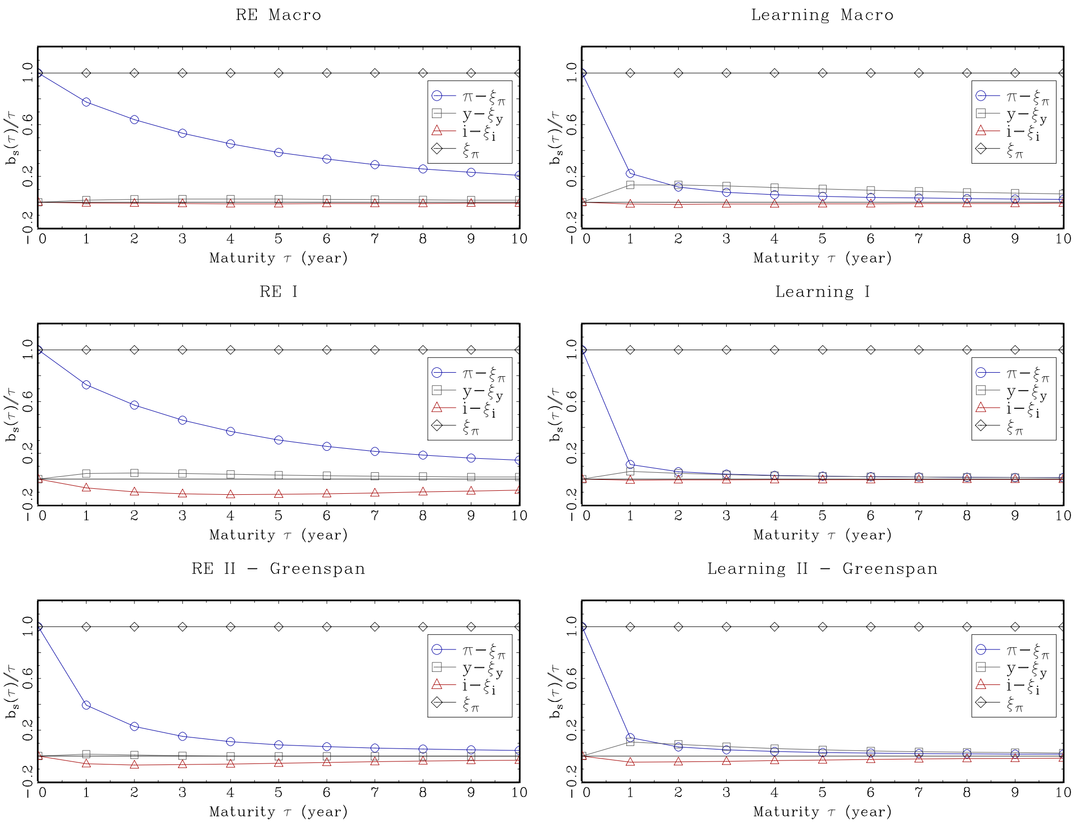

Figure 10: Loading - Inflation expectation across models.

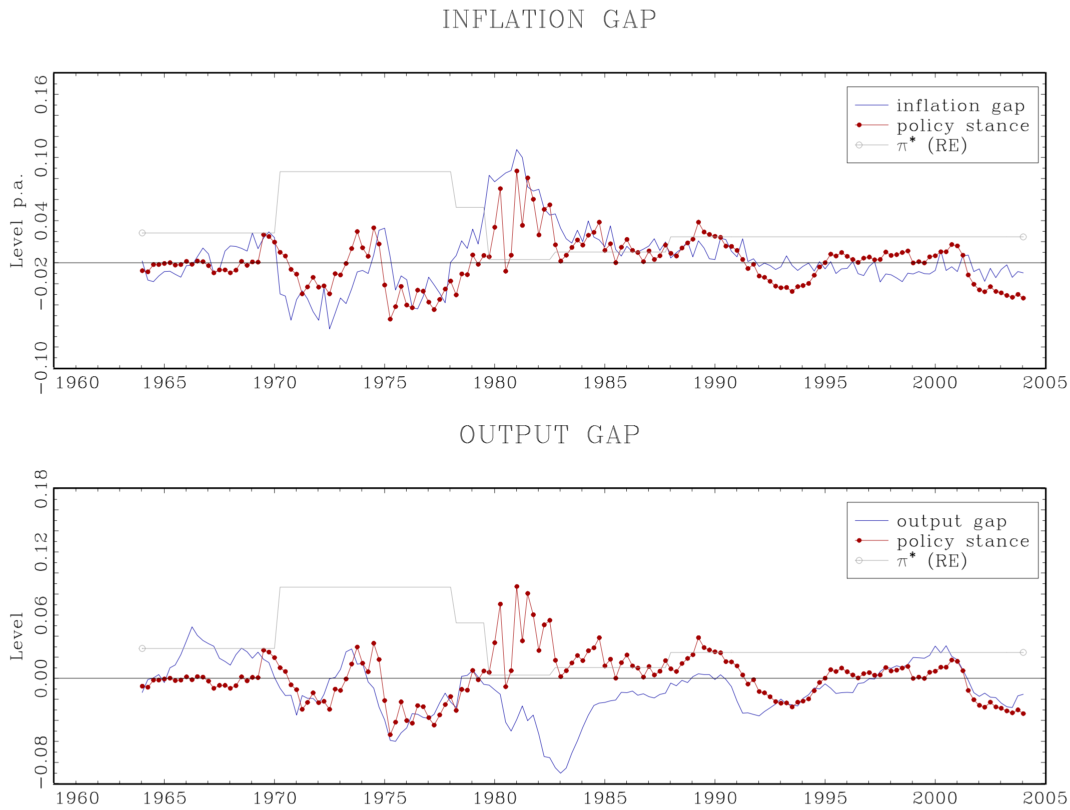

Figure 11: Learning II model - Policy stance against inflation gap and output gap.