EFRW
7/22 2:00

Bargaining over Prices*

JULIO DÁVILA[†] AND JAN EECKHOUT[‡]
University of Pennsylvania[§]

May, 2003

Abstract

We consider the problem of two agents bargaining over the relative price of two goods they are endowed with. They alternatingly exchange price offers. The recipient of an offer can either accept it and choose the quantities to be traded, or reject and counter-offer a different relative price. The utilities are discounted until an agreement is reached. We show that stationary subgame perfect (SSP) equilibria of the bargaining over prices may exist that converge to the Walrasian allocation as discounting frictions vanish. When such SSP equilibria do exist, then there is always a multiplicity of them. We also show that, in addition, asymptotically inefficient SSP equilibria generically exist.

Keywords. Foundations of Walrasian equilibrium. Alternating-offer bargaining. Price theory.
JEL. C78. D40.

1 Introduction

Bargaining between two agents over the terms of trade of an exchange is, both historically and conceptually, at the heart of what trade (or by the same token economics) is about. Beyond the interesting issue of its positive import, bargaining over prices has undoubtedly a strong appeal to the theorist as a minimal first approach to providing a foundation for the Walrasian equilibrium in small markets.[1] For the sake of focusing on the fundamentals, in this paper we thoroughly explore the equilibrium outcomes of bargaining over prices in the simplest possible setup, that of a two-person exchange economy.[2] When all the dust has settled, we will have shown that bargaining over prices has promising virtues as a foundation for the Walrasian equilibrium, but important caveats will have been discovered on the way. We

*This paper folows from an earlier working paper version, Dávila-Eeckhout (2002). We have benefitted from (and we are very grateful for) valuable comments from Ken Burdett, Pierre-André Chiappori, George Mailath, Andy Postlewaite, Paolo Siconolfi, and Randall Wright.

†davilaj@econ.upenn.edu

‡eeckhout@ssc.upenn.edu

§Department of Economics, 3718 Locust Walk, Philadelphia PA 19104.

[1] Bargaining over prices is certainly not the only possible way to provide a foundation of the Walrasian equilibrium that is independent of a price-taking behavior assumption. Consider, for instance, the long literature on market games starting with Shubik (1973), Shapley-Shubik (1977).

[2] For a bargaining foundation of the Walrasian equilibrium in large economies see Gale (1986).

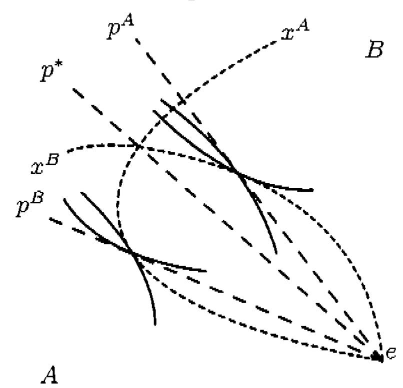
show that bargaining over prices as a limit outcome of increasingly patient agents can provide a foundation for the Walrasian equilibrium. But it is not by any means guaranteed, nor is the uniqueness of the outcome of this type of bargaining. We also show that bargaining over prices can lead even infinitely patient agents to allocations inside the Pareto frontier (and therefore non-Walrasian allocations).

Specifically, we consider an alternating-offers bargaining game (à la Rubinstein (1982) and Ståhl (1972)) in the context of a two-goods, two-persons exchange economy. Since, typically, the initial allocation of resources is not efficient, the agents will be interested in trading goods in order to benefit from the gains from trade and, if possible, attain an efficient allocation. Obviously, they have conflicting interests about the final allocation of resources and hence, in the absence of equilibrium prices announced by a Walrasian auctioneer to which they would react as price-takers and that would lead them to an efficient allocation, they will be led to bargain in order to improve their welfare with respect to their initial endowments. It is assumed that the agents bargain over prices. The player who makes an offer announces a price p (specifically, a price of good 1 in terms of good 2) that she is bound to honor, and the player who receives the price offer p can accept the offer and choose the quantities to be traded at price p, or reject the offer. If an offer is rejected, the roles are switched and the bargaining continues until an agreement is reached. The discounting of utilities reflects the cost of delay in reaching an agreement.

In this process of bargaining over prices, the determination of the price and the choice of the quantities to be traded are separated. Only once the price offered by one agent is accepted can the quantities to be traded be determined by the other agent. Such separation of price and quantity decisions has important applications that have received attention earlier in the theoretical literature. A common buy-out provision in two-person partnerships is studied in Cramton, Gibbons and Klemperer (1987) (more recently see also Moldovanu (2001)): one partner chooses the price of the shares, and the other partner chooses the quantity traded, i.e. whether to buy or sell. While that literature focusses on issues of incomplete information in a relatively stylized economy, the current paper analyzes a general economy, thus providing a complete information benchmark. Other applications include union-wage bargaining where once wages are agreed upon, employment levels are chosen. There is an extensive literature in labor economics (see, among others, Solow and MacDonald (1981), Farber (1986)) documenting the prevalence of such bargaining over wages. Moreover, the early axiomatic bargaining models pre-dating Nash's solution all consider bargaining over wages in the context of firm-union relations (see Harsanyi (1956)). Those models of axiomatic bargaining point to an outcome on the Pareto frontier. The model we consider in this paper can be interpreted as a formalization of such bargaining over wages by means of a particular extensive form bargaining game (i.e. alternating-offers bargaining), and hence as an alternative to the axiomatic bargaining approach.

Intuition may lead to expect that, as the frictions from discounting vanish, bargaining over prices will lead to a Walrasian outcome and hence to efficiency. In effect, consider first a unilateral monopoly in a two-goods, two-persons exchange economy. The monopolist (say agent A) announces a take-it-or-leave-it price offer p^A, and the customer (agent B) chooses a desired consumption $x^B(p^A)$ that maximizes her utility given the monopoly price (i.e. she chooses an allocation on her offer curve). In this two-stage game, a subgame perfect equilibrium then requires the monopolist to announce a price p^A that maximizes his utility, taking into account agent B will choose an allocation on her offer curve, i.e. the monopolist chooses a price that allows him to attain an indifference curve that is tangent to B's offer curve (see Figure 1 below).[3] Note that a take-it-or-leave-it offer is a particular instance of the alternating-offers protocol with complete myopia, i.e. with extreme discount factors equal to 0. There is no point in rejecting an offer because it is as if there were no second stage in the negotiation and, as a consequence, the proposer has all the market power. In Figure 1 the take-it-or-leave-it offer p^A of a monopolist A (similarly for p^B when B is the monopolist) corresponds to this extreme case in which the discount factors are 0.

Figure 1



As discount factors are made positive, there appears room for true negotiation, and as they converge to 1, the market power becomes evenly distributed between the two agents (given their endowments and preferences). In such negotiation, every agent knows from sequential rationality that any outcome, upon acceptance of an offer, will be on the offer curve of the agent accepting the offer. It may seem intuitive that as the discount factors approach 1, the extreme imbalance of market power that makes the outcome to be on one offer curve but not on the other vanishes and, therefore, the outcome be an allocation on *both* offer curves x^A and x^B. In other words, as the agents become infinitely patient, the outcome of the bargaining over prices seems to have to be an intersection point of the offer curves, i.e. a Walrasian allocation. Thus, when the agents play stationary strategies in the alternating-offers bargaining over prices, the Walrasian allocation is a natural candidate to be an equilibrium as discounting frictions vanish.

The argument above hints at the conjecture that bargaining over prices can provide a foundation for achieving the Walrasian allocation in small economies without price-taking agents. In this paper we analyze the existence of stationary subgame perfect (SSP) equilibria of the bargaining over prices. We are interested in the *asymptotic efficiency* of SSP equilibria, i.e. their efficiency as the discounting frictions vanish, and in particular their convergence to Walrasian equilibria.

Our main first contribution is to characterize a non-degenerate set[4] of economies for which there exist SSP equilibria of the bargaining over prices converging to a Walrasian equilibrium (Theorem 7). There are however two important caveats to this result. The first caveat is that whenever there exist

[3] In Figure 1 the point e represents the initial endowments in this exchange economy, the dotted curves are A's and B's offer curves x^A amd x^B, and the solid curves are indifference curves of A and B. The price supporting the Walrasian allocation is p^*.

[4] More specifically, with non-empty interior.

such SSP equilibria converging to a Walrasian equilibrium, there is a multiplicity of them (Corollary 9). Therefore, asymptotic efficiency and uniqueness do not obtain simultaneously. The second caveat is that there may not exist SSP equilibria converging to a Walrasian equilibrium at all (Theorem 7). The existence or non-existence of such asymptotically efficient SSP equilibria depends on how the discount factors converge to 1.

In addition to the possible existence of asymptotically efficient SSP equilibria, we show the generic existence of asymptotically inefficient SSP equilibria, i.e. equilibria that remain bounded away from the Pareto frontier as the discounting frictions vanish (Theorems 5 and 6). The inefficiency of these equilibria is not due to any delay in reaching an agreement, but rather to being characterized by prices that result in one of the agents accepting an offer and choosing an allocation off the contract curve. This turns out to be an equilibrium because the agent making the offer is indifferent between, on the one hand, consuming an allocation on the opponent's offer curve, and on the other hand, accepting an offer next period and consuming an allocation on his own offer curve. The interesting feature of these SSP equilibria is that on the stationary equilibrium path. both agents offer different prices when called to offer, but accept the other agent's (less favorable) price when called to accept. Still, both agents are indifferent because for each price, different quantities are traded.

All the results above rely heavily on another, this time technical, contribution of this paper. In order to analyze the existence of SSP equilibria of the bargaining over prices, we are led to characterize within the Pareto set the behavior of the curves of profiles of utilities for both agents that are attainable along their offer curves. We establish in Lemma 3 that for any generic exchange economy,[5] these curves of utility profiles intersect tangently *without crossing* at any Walrasian profile.

The conjecture described above that alternating offers on the agents' offer curves leads to the Walrasian allocation has independently been analyzed in Yildiz (2003). There, it is proven to hold under very special assumptions. As a consequence of our analysis this paper, it turns out that the assumptions in Yildiz (2003) under which the conjecture is found to hold are degenerate.

The remainder of the paper is as follows. In Section 2 we illustrate, in a simple Cobb-Douglas setup, examples of both asymptotically efficient and asymptotically inefficient SSP equilibria. In Section 3 we present the general model. In Section 4 we show the generic existence of asymptotically inefficient SSP equilibria (Theorems 5 and 6). In Section 5 we characterize the existence of asymptotically efficient SSP equilibria. Section 6 provides some further examples, and Sections 7 and 8 conclude with a general discussion and concluding remarks.

2 A Simple Example

In this section we illustrate the existence of both asymptotically efficient and asymptotically inefficient SSP equilibria of the bargaining over prices within a simple Cobb-Douglas setup. In effect, consider an economy with two agents A and B with preferences over bundles of two goods $x = (x_1, x_2)$ that are represented by the Cobb-Douglas utility functions $u^A = \sqrt{x_1^A x_2^A}$ and $u^B = \sqrt{x_1^B x_2^B}$. The total resources are $e = (1, 1)$ and the distribution of initial endowments between A and B is $e^A = (0.9, 0.3)$

[5] In lemma 3 this is established for two-goods, two-persons economics, but it is a general property that can be generalized to any n goods (see Section 7).

and $e^B = (0.1, 0.7)$. Since the initial allocation of resources is not efficient,[6] there are gains from trade to be realized.

Consider the bargaining game that lets one agent i offer terms of trade between the two goods, represented by the vector of prices $(p^i, 1)$, good 2 acting as the numeraire. Given the p^i offered by agent i, the other agent $-i$ reacts either announcing the desired quantities to be traded at the proposed terms of trade (that the proposer is bound to honor), or making a counter-offer of terms of trade, and so on until a trade takes place. The utility obtained by agents A and B from the consumption of the two goods is discounted by $\delta^A, \delta^B \in (0,1)$ respectively for each iteration in the bargaining prior to reaching an agreement.

Whenever an agent i decides to accept an offer at terms of trade p^{-i} by agent $-i$, individual rationality guarantees that he will choose to demand quantities $x^i(p^{-i})$ on his offer curve (i..e. that maximize his utility given p^{-i}). The resulting instantaneous utilities are $u^{-i}(e - x^i(p^{-i}))$ to the agent $-i$ who makes the offer, and $u^i(x^i(p^{-i}))$ to the agent i that receives it. Necessary conditions for a SSP equilibrium (p^A, p^B) of this bargaining game are given by[7]

$$\begin{aligned} u^A(x^A(p^B)) &= \delta^A u^A(e - x^B(p^A)) \\ u^B(x^B(p^A)) &= \delta^B u^B(e - x^A(p^B)). \end{aligned} \tag{1}$$

In effect, in each subgame, the agent accepting the offer should be not worse off than waiting one period and having his counter-offer accepted. For expositional clarity and for the remainder of this example section, let us consider the case in which δ^A and δ^B are close to one.[8] In the limit, when δ^A and δ^B are equal to 1, the first equation requires that the bundles of agent A resulting from accepting an offer $x^A(p^B)$, and from having an offer accepted $e - x^B(p^A)$, are on the same indifference curve for agent A. The second equation has a similar interpretation for agent B. Moreover, note that individual rationality implies that $x^A(p^B)$ and $x^B(p^A)$ are on the offer curves of A and B respectively.

When there is no discounting, the Walrasian allocation $x^{A*} = (0.6, 0.6)$, $x^{B*} = (0.4, 0.4)$ supported by $p^A = p^B = 1$ is a solution to equations (1).[9] With discount rates smaller than 1, exhibiting a SSP equilibrium close to the Walrasian allocation is quite a bit more involved. In Section 5, it is shown that for some paths of convergence of the discount factors to 1, SSP equilibria converging to the Walrasian allocation exist, but in general neither their existence nor uniqueness are guaranteed. In general, there exist an even number (possibly zero) of SSP equilibria converging to the Walrasian equilibrium. In Section 6 we return to this example and verify the conditions of existence of these equilibria.

More easy to see in Figure 2 below is another solution to the necessary conditions above, with prices

$$(p^A, p^B) = (1.750, 1.333) \tag{2}$$

[6] In this example, the contract curve is the diagonal of the Edgeworth box, while the initial endowments are off the diagonal.

[7] In the next section, the complete optimization program will be presented and solved.

[8] Of course, discounting is crucial for the interpretation of costly bargaining.

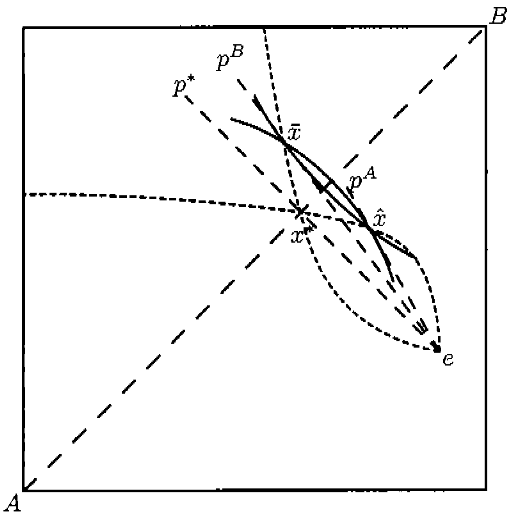
[9] There is a trivial solution to the previous system of equations when $\delta^A = \delta^B = 1$, which is the initial endowment allocation e (supported by prices equal to the marginal rates of substitution at this point). Nevertheless, it will never generate a SSP equilibrium as the discounting frictions vanish, because it never satisfies the conditions for subgame perfection of the equilibrium.

and two allocations $\bar{x}$ and $\hat{x}$ on A's and B's offer curves respectively, with

$$\begin{aligned} \bar{x}^A &= (\bar{x}_1^A(p^B), \bar{x}_2^A(p^B)) = (0.5625, 0.75) \\ \hat{x}^A &= (e - \hat{x}_1^B(p^A), e - \hat{x}_2^B(p^A)) = (0.75, 0.5625) \end{aligned} \tag{3}$$

and the complementary bundles for agent B. Note that, unlike the Walrasian solution, this other solution is not Pareto-efficient.

Figure 2



By a simple continuity argument, for δ^A and δ^B close to 1, there exists a solution $(p^A_{\delta^A\delta^B}, p^B_{\delta^A\delta^B})$ to the system of equations (1) close to the solution (1.750, 1.333). The solution $(p^A_{\delta^A\delta^B}, p^B_{\delta^A\delta^B})$ is a SSP equilibrium. Note that as $\delta^A, \delta^B \to 1$, it converges to $(p^A, p^B) = (1.750, 1.333)$ and hence remains bounded away from efficiency.

We illustrate now, using the same example, that under certain circumstances, a unique solution[10] to the system of equations (1) may exist that converges to the Walrasian allocation as $\delta^A, \delta^B \to 1$. Consider the same economy, but with initial endowments $e^A = (0.9, 0.1)$ and $e^B = (0.1, 0.9)$. It can be readily verified that for these initial endowments, the only solutions to the system of equations with $\delta^A = \delta^B = 1$ is $x^{A*} = x^{B*} = (0.5, 0.5)$ supported by $p^A = p^B = 1$. For this economy, if $\delta^A, \delta^B < 1$, there exists a unique SSP equilibrium that moreover converges to the Walrasian allocation as $\delta^A, \delta^B \to 1$. Unfortunately, this unique convergence result is only true under very special assumptions. Specifically, in our example only the economies with initial endowments on the anti-diagonal will have this property.[11] However, any small deviation away from the anti-diagonal gives rise to asymptotically inefficient SSP equilibria like the one shown in the previous example.

These Cobb-Douglas examples intend only to illustrate the possible SSP equilibria of the bargaining over prices. In the next sections we consider the general set-up and characterize their existence and asymptotic efficiency or inefficiency.

[10] In addition to the trivial one corresponding to the initial endowments.

[11] These economies satisfy the assumptions made in Yildiz (2003).

3 The Model

Consider an exchange economy with two agents $i \in \{A, B\}$ with standard preferences over nonnegative consumptions of two goods given by the utility functions u^A, u^B satisfying

Assumption 1 *For all $i \in \{A, B\}$, u^i is $\mathbb{R}_+$-valued,*[12] *continuous in $\mathbb{R}^2_+$, differentiable in $\mathbb{R}^2_{++}$, monotone,*[13] *differentiably strictly concave,*[14] *well-behaved at the boundary,*[15] *and such that i's demand is never simultaneously upward-sloped for both goods.*[16]

The agents are endowed with the nonnegative amounts $e^A = (e_1^A, e_2^A)$ and $e^B = (e_1^B, e_2^B)$ of the goods respectively. The total resources of the economy are $e = e^A + e^B$. Let us denote by $x^i = (x_1^i, x_2^i)$ the vector of goods consumed by $i \in \{A, B\}$, and an exchange economy by $\{u^i, e^i\}_{i \in \{A,B\}}$.

In the absence of a Walrasian auctioneer, the agents bargain over the terms of trade letting one agent make and offer of terms of trade by means of a price p of good 1 in terms of good 2, to which the other agent reacts either announcing a desired trade at the proposed terms of trade, or making a counter-offer of terms of trade, and so on until a trade takes place. The cost of the bargaining process itself is captured by the discount of the utility obtained from consumption by a factor $\delta^A, \delta^B \in (0, 1)$ for each offer rejected by A and B respectively. Not reaching an agreement amounts to consuming the initial endowments.

In any subgame agent A's best response to an offer p^B from agent B, will in general depend on A's beliefs about B's strategy. We will restrict ourselves to stationary strategies profiles and hence stationary beliefs. Then A's best response to B's offer p^B is

1. to accept B's offer, if the discounted utility A can obtain from making an optimal counter-offer p^A solving[17]

$$\max_{p^A} u^A(e - x^B(p^A)) \qquad (4)$$
$$u^B(x^B(p^A)) \geq \delta^B u^B(e - x^A(p^B))$$

   is, if accepted, not more than what A would obtain accepting B's offer immediately, i.e. $\delta^A u^A(e - x^B(p^A)) \leq u^A(x^A(p^B))$.

2. to reject B's offer otherwise, and make the optimal counter-offer p^A described above.

[12] For a given set of preferences, there is no reason to assume positive utility. However, in the bargaining game, negative utility would render delay desirable, rather than costly.

[13] In the sense that $Du^i(x) \in \mathbb{R}^2_{++}$ for all $x \in \mathbb{R}^2_{++}$.

[14] As a matter of fact, differentiably strictly quasi-concave (in the sense that $D^2u^i(x)$ is definite negative on the space orthogonal to $Du^i(x)$ for all $x \in \mathbb{R}^2_{++}$) suffices.

[15] In the sense that every indifference curve going through a point in $\mathbb{R}^2_{++}$ is completely contained in $\mathbb{R}^2_{++}$.

[16] That is to say, there is no vector of prices $(\bar{p}, 1)$ such that $\frac{dx_1^i}{dp}(\bar{p}) > 0$ and $\frac{dx_2^i}{dp^{-1}}(\bar{p}) > 0$. Note that this assumption does allow for backward-bending offer curves, and hence for any good being inferior for some range of prices. It only makes sure that the income effect is not so strong as to offset the substitution effect on the demand for *both* goods simultaneously.

[17] The constraint is a necessary condition for B to accept A's counter-offer.

Similar conditions characterize B's optimal behavior.

A SSP equilibrium of the bargaining problem described above consists of a pair of two prices (p^A, p^B), corresponding to each player's price offer, such that both B and A respectively would accept in any subgame if confronted with these or better offers. If confronted with worse offers, B and A reject and offer p^B and p^A respectively in the next period. Since at an SSP equilibrium (p^A, p^B), the offers are accepted, they must satisfy $\delta^A u^A(e - x^B(p^A)) \leq u^A(x^A(p^B))$, and similarly for B. Moreover, sequential rationality requires p^A and p^B to be solutions to the maximization problems above for A and B. The definition of a SSP equilibrium follows.

Definition 2 *A SSP Equilibrium of the bargaining problem above is a pair p^A, p^B such that*[18]

$$\begin{aligned} p^A &\in \arg\max_{\tilde{p}^A} u^A(e - x^B(\tilde{p}^A)) \\ u^B(x^B(\tilde{p}^A)) &\geq \delta^B u^B(e - x^A(p^B)) \end{aligned} \tag{5}$$

$$\begin{aligned} p^B &\in \arg\max_{\tilde{p}^B} u^B(e - x^A(\tilde{p}^B)) \\ u^A(x^A(\tilde{p}^B)) &\geq \delta^A u^A(e - x^B(p^A)) \end{aligned} \tag{6}$$

The first-order conditions the problem of agent A characterize its solution, and similarly for agent B. Therefore, necessary and sufficient[19] conditions for (p^A, p^B) to be an SSP equilibrium are

$$\begin{aligned} Du^A(e - x^B(p^A))Dx^B(p^A) - \lambda^A Du^B(x^B(p^A))Dx^B(p^A) &= 0 \\ Du^B(e - x^A(p^B))Dx^A(p^B) - \lambda^B Du^A(x^A(p^B))Dx^A(p^B) &= 0 \\ u^A(x^A(p^B)) - \delta^A u^A(e - x^B(p^A)) &= 0 \\ u^B(x^B(p^A)) - \delta^B u^B(e - x^A(p^B)) &= 0 \end{aligned} \tag{7}$$

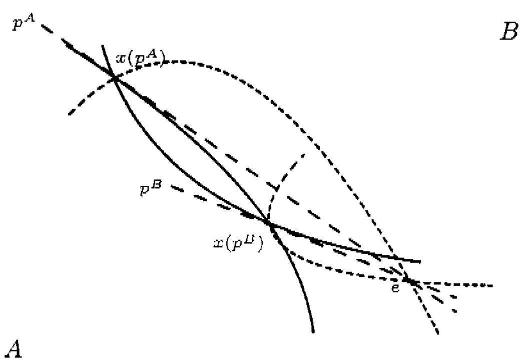
for some $\lambda^A, \lambda^B > 0$.

The two last equations in (7) have a convenient interpretation when $\delta^A, \delta^B = 1$. In effect (see Figure 3), $x^A(p^B)$ and $x^B(p^A)$ must be points on the offer curves (short-dashed curves) of A and B respectively, and leading to allocations of resources within the Edgeworth box between which both A and B are indifferent (the indifferent curves are represented in solid lines)

[18] Note that the conditions $\delta^A u^A(e - x^B(p^A)) \leq u^A(x^A(p^B))$ and $\delta^B u^B(e - x^A(p^B)) \leq u^B(x^B(p^A))$ are redundant with (5) and (6).

[19] By the local nature of these conditions, we will assume that conditions guaranteeing that a local maximum is a global maximum hold.

Figure 3

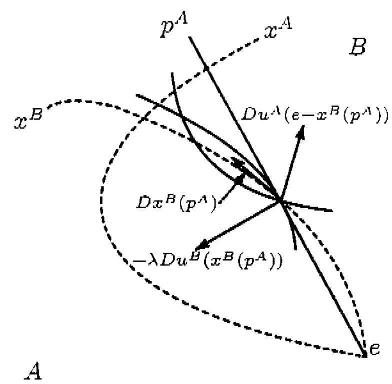


The first two equations in (7) can also be interpreted in this figure as follows. Consider, for instance, the first equation. Its equivalent rewriting in (8) below amounts to requiring that the tangent to B's offer curve at $x^B(p^A)$ — whose direction is given by $Dx^B(p^A)$ — be normal to a positive linear combination of the gradients of u^A and u^B at that point

$$\left(Du^A(e - x^B(p^A)) - \lambda^A Du^B(x^B(p^A))\right)Dx^B(p^A) = 0. \tag{8}$$

As illustrated in Figure 4, this is equivalent to saying that the slope of B's offer curve is at $x^B(p^A)$ not simultaneously smaller that A's marginal rate of substitution (with its negative sign) and bigger than B's. In more graphical terms: A's offer curve does not enter at $x(p^B)$ into the lens formed by the agents' indifference curves going through this point.[20] The second equation has a similar interpretation about the behavior of A's offer curve at $x^A(p^B)$.

Figure 4



[20]Obviously, in general the conditions above are not sufficient (see footnote 19) since the equation 1 may be satisfied locally, but the offer curve may still re-enter the lens away from this point. The same caveat of footnote 19 then applies here.

4 Generic existence of asymptotically inefficient SSP equilibria

The existence of SSP equilibria of the bargaining over prices can be conveniently analyzed as follows. Note first that the solutions to the necessary conditions

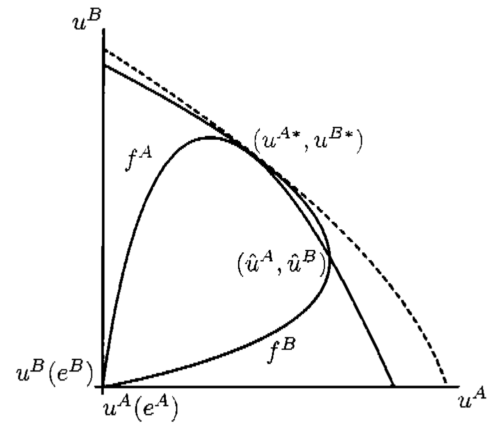
$$\begin{aligned} u^A(x^A(p^B)) - \delta^A u^A(e - x^B(p^A)) &= 0 \\ u^B(x^B(p^A)) - \delta^B u^B(e - x^A(p^B)) &= 0 \end{aligned} \tag{9}$$

correspond to intersections of the following two curves parametrized by p

$$\begin{aligned} f^A_{\delta^B}(p) &= (u^A(x^A(p)), \delta^B u^B(e - x^A(p))) \\ f^B_{\delta^A}(p) &= (\delta^A u^A(e - x^B(p)), u^B(x^B(p))). \end{aligned} \tag{10}$$

These curves are, for discount factors δ^A and δ^B close to 1, slight deformations of their counterparts f^A, f^B for $\delta^A, \delta^B = 1$ represented in Figure 5 below (more specifically $f^A_{\delta^B}$ is a proportional vertical shrink of f^A by a factor δ^B, and $f^B_{\delta^A}$ is a proportional horizontal shrink of f^B by a factor δ^A)

Figure 5



The curves f^A and f^B represent the utilities attained by each agent along the offer curves of A and B respectively in the Edgeworth box. Consider for example, a typical pattern of f^A: along A 's offer curve, A's utility increases, whereas B 's utility initially increases (starting from the utilities at the initial endowments) and eventually decreases beyond B's monopoly allocation. And similarly for f^B with the axes reversed. Of course, the two curves f^A and f^B intersect at the point $(u^{A*}, u^{B*}) = (u^A(x^A(p^*)), u^B(x^B(p^*)))$ shown in Figure 5, for a Walrasian equilibrium price p^*. The Walrasian intersection (u^{A*}, u^{B*}) is moreover on the Pareto frontier (in dashes in Figure 5), since it corresponds to an efficient allocation. Also it is easy to see from this and the differentiability of both f^A and f^B, and of the Pareto frontier itself that all the three curves are tangent at (u^{A*}, u^{B*}).[21]

Another general property of the intersection of f^A and f^B at the Walrasian intersection (u^{A*}, u^{B*}) will play a crucial role in what follows for showing the existence of SSP equilibria, namely that generically f^A and f^B do not cross at (u^{A*}, u^{B*}). This is a very useful property that is worth to be stated separately in the next Lemma. The proof of this Lemma can be found in the Appendix.

[21] A proof of this property can be found in our working paper Dávila-Eeckhout (2002).

Lemma 3 *For any generic*[22] *exchange economy* $\{u^i, e^i\}_{i\in\{A,B\}}$ *satisfying Assumption 1, the curves* f^A *and* f^B *intersect without crossing at the Walrasian intersection* (u^{A*}, u^{B*}).

A short discussion is now in order. If one assumes on the contrary that f^A and f^B do actually cross at the Walrasian intersection (u^{A*}, u^{B*}), then it can be readily proved that the bargaining over prices has only one SSP equilibrium that moreover necessarily converges to the Walrasian equilibrium as δ^A and δ^B converge to 1 (see Yildiz (2003)).[23] Unfortunately, what Lemma 3 above proves is that such assumption is only satisfied by degenerate economies.

Now we can address in general the issue of the existence and the asymptotic efficiency or inefficiency of SSP equilibria of the bargaining over prices. It turns out that asymptotically inefficient SSP equilibria of the bargaining over prices can be generically found both near a Walrasian equilibrium and away from it. We will refer to the two types of SSP equilibria as *local* and *global* SSP equilibria respectively. A different argument applies for the generic existence of each type of equilibrium. Let us consider first the generic existence of asymptotically inefficient global SSP equilibria, and then the generic existence of asymptotically inefficient local SSP equilibria.

4.1 Asymptotically inefficient global SSP equilibria

The existence of asymptotically inefficient global SSP equilibria of the bargaining over prices depends on the existence of intersections of f^A and f^B like $(\widehat{u}^A, \widehat{u}^B)$ in Figure 5. This intersection does not correspond to a Walrasian equilibrium, since it is inefficient. Note also that, since f^A and f^B intersect transversally at $(\widehat{u}^A, \widehat{u}^B)$, then by continuity a nearby intersection $(\widehat{u}^A, \widehat{u}^B)_{\delta^A\delta^B}$ of $f^A_{\delta^B}$ and $f^B_{\delta^A}$ persists as the discount factors δ^A and δ^B depart slightly from 1. This intersection $(\widehat{u}^A, \widehat{u}^B)_{\delta^A\delta^B}$ hence satisfies the necessary conditions for a SSP equilibrium of the bargaining over prices. It actually corresponds to such an equilibrium if both $f^A_{\delta^B}$ and $f^B_{\delta^A}$ have a negative slope there.[24] This will hold, by continuity, as long as both f^A and f^B have a negative slope at $(\widehat{u}^A, \widehat{u}^B)$. Finally, note that the intersection $(\widehat{u}^A, \widehat{u}^B)_{\delta^A\delta^B}$ of $f^A_{\delta^B}$ and $f^B_{\delta^A}$ converges to $(\widehat{u}^A, \widehat{u}^B)$ as δ^A and δ^B converge to 1. As a consequence, the corresponding SSP equilibrium of the bargaining over prices is not only inefficient for every δ^A and δ^B close to 1, but it also remains asymptotically inefficient as δ^A and δ^B converge to 1.

The generic existence of at least one inefficient intersection $(\widehat{u}^A, \widehat{u}^B)$ of f^A and f^B is stated in Lemma 4 below. Theorem 5 then establishes the existence of global asymptotically inefficient SSP equilibria of the bargaining over prices for a non-empty open set of economies showing that for these economies f^A and f^B have a negative slope at $(\widehat{u}^A, \widehat{u}^B)$. Its proof can be found in the Appendix as well.

[22] That is to say within an open and dense set of economies with respect to the usual topology in the space of endowments, and the topology of C^1 uniform convergence on compacts in the space of utility functions (actually for any C^n convergence as well).

[23] While each of the two assumptions A3 (both monopolistic outcomes are dominated by some allocation attainable along an offer curve)and A4 (there is a unique crossing of f^A and f^B within the interval defined by the profiles of utilities attained at the monopolistic outcomes) in Yildiz (2003) are not degenerate on their own, nonetheless the requirement of both of them holding simultaneously amounts to having a crossing of f^A and f^B at a Walrasian profile of utilities that, according to Lemma 3, is a degenerate property in the space of economies.

[24] This corresponds to the satisfaction of the other first two necessary and sufficient conditions for a SSP equilibrium in 7

Lemma 4 *For any generic*[25] *exchange economy* $\{u^i, e^i\}_{i\in\{A,B\}}$ *satisfying Assumption 1, there exist an inefficient intersection* $(\widehat{u}^A, \widehat{u}^B)$ *of the curves* f^A *and* f^B.

This is almost an immediate corollary of Lemma 3. In effect, assume that every intersection of f^A and f^B is efficient. Then the behavior of f^A and f^B at the boundaries implies that at some intersection these curves must cross. Nevertheless Lemma 3 establishes that this is generically not the case at an efficient intersection, and the conclusion follows.

Theorem 5 *For any economy* $\{u^i, e^i\}_{i\in\{A,B\}}$ *of a non-empty open set of economies satisfying Assumption 1, and for any discount factors* δ^A, δ^B *close enough to* 1*, there exist a SSP equilibrium of the bargaining over prices. Moreover, that equilibrium remains bounded away from efficiency as* $\delta^A, \delta^B \to 1$.

Intuitively, in the degenerate case in which there is no other intersection of f^A and f^B than the Walrasian one (u^{A*}, u^{B*}), f^A and f^B have clearly negative slopes. By Lemma 3, as the economy is slightly perturbed, the Walrasian intersection bifurcates into a new inefficient intersection $(\widehat{u}^A, \widehat{u}^B)$ in addition to the Walrasian one (u^{A*}, u^{B*}), which still exists. Again, the new inefficient intersection $(\widehat{u}^A, \widehat{u}^B)$ of f^A and f^B will persist at $(\widehat{u}^A, \widehat{u}^B)_{\delta^A\delta^B}$ as δ^A and δ^B depart slightly from 1. By continuity, the slopes of $f^A_{\delta^B}$ and $f^B_{\delta^A}$ at $(\widehat{u}^A, \widehat{u}^B)_{\delta^A\delta^B}$ will still be negative after a small enough perturbation of the economy and for δ^A and δ^B close enough to 1. This guarantees that it corresponds indeed to a SSP equilibrium of the bargaining over prices.

The payoffs $(\widehat{u}^A, \widehat{u}^B)_{\delta^A\delta^B}$ of the SSP equilibrium established in Theorem 5 converge to $(\widehat{u}^A, \widehat{u}^B)$, and not to (u^{A*}, u^{B*}), but the continuity argument used in Theorem 5 makes $(\widehat{u}^A, \widehat{u}^B)$ to be close to (u^{A*}, u^{B*}). Nevertheless, Theorem 5 intends only to establish that the set of economies for which there exist SSP equilibria of the bargaining over prices whose payoffs converge to $(\widehat{u}^A, \widehat{u}^B)$ is not empty and has moreover a non-empty interior. In general, $(\widehat{u}^A, \widehat{u}^B)$ needs not be close to (u^{A*}, u^{B*}), and hence the equilibrium whose payoffs converge to $(\widehat{u}^A, \widehat{u}^B)$ is a global SSP equilibrium.

4.2 Asymptotically inefficient local SSP equilibria

Asymptotically inefficient local SSP equilibria of the bargaining over prices exist generically around the Walrasian equilibrium as well. More specifically, arbitrarily close to the Walrasian equilibrium of a generic economy there exist actually several such equilibria of a nearby economy. In other words, one can always substitute any generic economy with an arbitrarily similar economy with SSP equilibria of the bargaining over prices within any given distance of the Walrasian equilibrium, no matter how small this distance is. These SSP equilibria will moreover still be asymptotically inefficient as δ^A and δ^B converge to 1, i.e. their payoffs will remain bounded away from the Pareto frontier. Of course, the closer we require these equilibria to be to the Walrasian equilibrium, the smaller the distance from the Pareto frontier at which they will be kept at bay as δ^A and δ^B converge to 1. But since anyway such equilibria can be found arbitrarily close to the Walrasian equilibrium by adequately perturbing the original generic economy, we refer to these equilibria as local SSP equilibria of the bargaining over prices. The next theorem states the existence of these equilibria in precise way. Its proof is provided in the Appendix.

[25] That is to say, within an open and dense set of economies with respect to the usual topology in the space of endowments, and the topology of C^1 uniform convergence on compacts in the space of utility functions (actually for any C^n convergence as well).

Theorem 6 *Within any neighborhood of any generic*[26] *exchange economy* $\{u^i, e^i\}_{i\in\{A,B\}}$ *satisfying Assumption 1 there exists an open set of economies* $\{\widetilde{u}^i, \widetilde{e}^i\}_{i\in\{A,B\}}$ *with multiple asymptotically inefficient SSP equilibria of the bargaining over prices.*

5 Asymptotically efficient SSP equilibria

We will now address the existence of SSP equilibria of the bargaining over prices that do converge to the Walrasian allocation as the discount factor δ^A and δ^B approach 1. It turns out that this depends on how δ^A and δ^B approach 1. More specifically, for some paths for δ^A and δ^B approaching 1 there will exist asymptotically efficient SSP equilibria, while for other paths there will not be any such equilibrium. None of the two cases is a special or degenerate case when compared to the other, which leaves no room for claiming any of the two phenomenons as the typical one.

In order to see this recall that, by Lemma 3, the curves f^A and f^B intersect tangently at the profile of utilities (u^{A*}, u^{B*}) of a Walrasian equilibrium but do not cross each other. A consequence of this property is that the intersection at (u^{A*}, u^{B*}) is not robust to the departure of δ^A and δ^B from 1. As a matter of fact, two different possibilities arise: (i) either the intersection of f^A and f^B vanishes as any of δ^A and δ^B becomes distinct from 1, or (ii) the tangent intersection of f^A and f^B bifurcates into two new transversal intersections.

In the first case (when the intersection vanishes) there cannot be SSP equilibria converging to the Walrasian equilibrium as the agents become infinitely patient. In the second case (when the intersection bifurcates) two distinct SSP equilibria appear. Moreover, these equilibria do converge to the Walrasian outcome as the agents become infinitely patient, and they are therefore asymptotically efficient. This section addresses in detail the conditions under which each of the two cases above arises.

Recall that f^A and f^B denote the paths of utilities attainable along the A's and B's offer curves

$$\begin{aligned} f^A(p) &= (u^A(x^A(p)), u^B(e - x^A(p))) \\ f^B(p) &= (u^A(e - x^B(p)), u^B(x^B(p))). \end{aligned} \tag{11}$$

Abusing notation only slightly, let f^A denote also the function associating $u^B(e - x^A(p))$ to $u^A(x^A(p))$ for all p, and similarly let f^B denote the function associating $u^A(e - x^B(p))$ to $u^B(x^B(p))$. If p^* is a Walrasian price of the economy and $(u^{A*}, u^{B*}) = (u^A(x^A(p^*)), u^B(x^B(p^*)))$, then clearly $u^{B*} = f^A(u^{A*})$ and $u^{A*} = f^B(u^{B*})$. We are interested in seeing what happens to the Walrasian intersection (u^{A*}, u^{B*}) of f^A and f^B when δ^A and δ^B depart slightly from 1. More specifically, we want to know the conditions under which there exists an intersection of the graphs of the functions $\delta^A f^A$ and $\delta^B f^B$ around (u^{A*}, u^{B*}) for δ^A and δ^B close to 1, since such an intersection would correspond to a SSP equilibrium. The next theorem establishes that the existence of such an intersection of $\delta^A f^A$ and $\delta^B f^B$ depends on how δ^A and δ^B approach 1. Its proof is provided in the Appendix.

[26] That is to say, for an open and dense set of economies with respect to the usual topology in the space of endowments, and the topology of C^1 uniform convergence on compacts in the space of utility functions (actually for any C^n convergence as well).

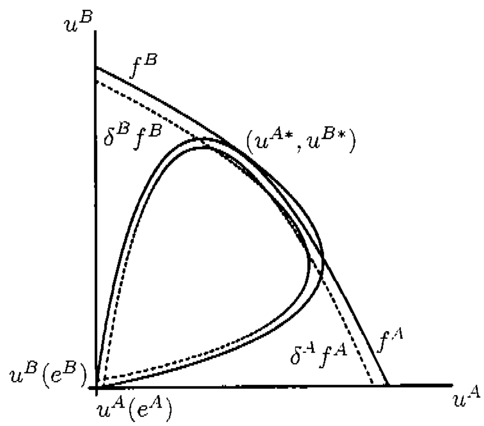
Theorem 7 *For any generic*[27] *economy* $\{u^i, e^i\}_{i\in\{A,B\}}$ *satisfying Assumption 1*

1. *there exist paths to* 1 *for the discount factors* δ^A, δ^B *for which multiple SSP equilibria converge to the Walrasian equilibrium, and*

2. *there also exist paths to* 1 *for the discount factors* δ^A, δ^B *for which no SSP equilibrium converges to the Walrasian equilibrium.*

The argument behind the proof consists of showing first that the discount factors can be made to depart from 1 in such a way that $\delta^A f^A$ is shifted vertically downwards from f^A less than the inverse function of $\delta^B f^B$, i.e. $(\delta^B f^B)^{-1}$, from f^B around the Walrasian intersection (u^{A*}, u^{B*}). In this case the Walrasian intersection (u^{A*}, u^{B*}) bifurcates (as shown in Figure 6 below) into two new intersections that correspond to SSP equilibria.

Figure 6

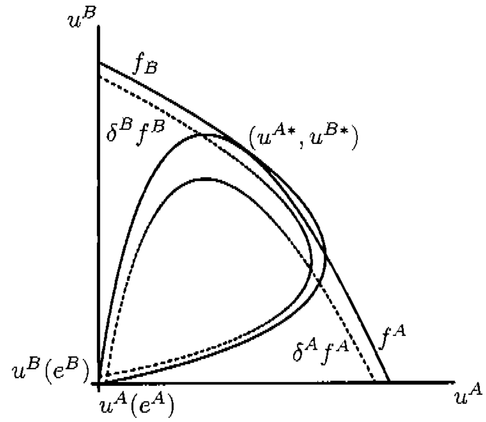


Similarly, the discount factors can also be made to depart from 1 in such a way that $\delta^A f^A$ drops actually more than $(\delta^B f^B)^{-1}$ around the Walrasian intersection (u^{A*}, u^{B*}), and in this case the Walrasian intersection (u^{A*}, u^{B*}) vanishes (as shown in Figure 7) and no SSP equilibrium approaches the Walrasian

[27] That is to say, for an open and dense set of economies with respect to the usual topology in the space of endowments, and the topology of C^1 uniform convergence on compacts in the space of utility functions (actually for any C^n convergence as well).

equilibrium.

Figure 7



Clearly, neither $(\delta^B f^B)^{-1}(u^{A*}) < \delta^A f^A(u^{A*})$ nor $(\delta^B f^B)^{-1}(u^{A*}) > \delta^A f^A(u^{A*})$ is a condition that is necessarily satisfied by any given economy, and hence one may wonder whether the paths of discount factors guaranteeing or preventing the existence of SSP equilibria approaching the Walrasian equilibrium of Theorem 7 are either very general or too special. It turns out that robust examples can be found of paths of discount factors both guaranteeing or preventing the existence of SSP equilibria approaching the Walrasian equilibrium, as established by the next theorem, the proof of which is given in the Appendix. As a consequence, none of these possibilities can be considered degenerate with respect to the other.

Theorem 8 *The set of sequences $\left\{(\delta_n^A, \delta_n^B)\right\}_n$ of discount factors for which there exist SSP equilibria converging to a Walrasian equilibrium as well as the set of sequences for which there is no such equilibria are not degenerate.*

Theorem 7 provides a link between the multiplicity and the inefficiency of SSP equilibria of the bargaining over prices. In effect, from Theorem 7 it follows that, for a generic economy, either there is no SSP equilibrium converging to the Walrasian equilibrium as the agents become infinitely patient, or else an even number of them must exist. As a consequence, the uniqueness of a SSP equilibrium can only obtain if this equilibrium remains inefficient in the limit as discounting frictions vanish. Hence the following corollaries.[28]

Corollary 9 (*Multiplicity*) *Any generic*[29] *economy $\{u^i, e^i\}_{i\in\{A,B\}}$ satisfying Assumption 1 with a SSP equilibrium of the bargaining over prices converging to the Walrasian allocation as $\delta^A, \delta^B \to 1$, has a multiplicity of such equilibria.*

Corollary 10 (*Inefficiency*) *If a generic*[30] *economy $\{u^i, e^i\}_{i\in\{A,B\}}$ satisfying Assumption 1 has a unique SSP equilibrium of the bargaining over prices for all δ^A and δ^B close to 1, then this equilibrium remains inefficient in the limit as $\delta^A, \delta^B \to 1$.*

[28] Note that Corollary 9 implies Corollary 10, but not the other way around.

[29] That is to say, for an open and dense set of economies with respect to the usual topology in the space of endowments, and the topology of C^1 uniform convergence on compacts in the space of utility functions (actually for any C^n convergence as well).

[30] The previous footnote applies also here.

6 Some Further Examples: uniqueness and non-existence

The examples in this section serve the purpose of illustrating some of the results derived above. First, we derive a unique SSP equilibrium that is bounded away from efficiency. We show for a given path of discount factors that there exists no SSP equilibria converging to the Walrasian allocation. The second example illustrates that no SSP equilibrium exists. Again, there are no SSP equilibria converging to the Walrasian allocation, but in addition, the only candidate equilibrium that satisfies the necessary conditions does not satisfy the conditions for perfections, i.e. the profile of utility along the offer curves do not cross with negative slope.

1. Consider the Cobb-Douglas example of Section 2 again. Both agents have now as utility functions $u^A = \sqrt{x_1^A x_2^A}$ and $u_B = \sqrt{x_1^B x_2^B}$, the total resources are $e = (1,1)$, and now let $(\delta^A, \delta^B) \to (1,1)$ at a rate $r = \frac{\log \delta^A}{\log \delta^B} = 2$. It can be seen in the proof of Theorem 7 in the Appendix that the nonexistence of SSP equilibria converging to the Walrasian allocation is the result of the fulfillment of the two inequalities (51) and (53), i.e.

$$\begin{aligned} \frac{u^{B*}}{u^{A*}} &> -\frac{1}{f^{B\prime}(u^{B*})}\frac{1-\delta^B}{1-\delta^A} \\ f^A(u^A) &\geq (f^B)^{-1}(u^{A*}). \end{aligned} \tag{12}$$

It is easy to see now that this is the case in this example. In effect, since

$$\lim_{(\delta^A,\delta^B)\to(1,1)} \frac{1-\delta^B}{1-\delta^A} = \frac{1}{r} = \frac{1}{2} \tag{13}$$

from de l'Hôpital's rule. Now consider an economy with initial endowments $e^A = (0.9, 0.3)$ and $e^B = (0.1, 0.7)$. Then the corresponding Walrasian equilibrium price is $p^* = 1$ and the corresponding equilibrium allocation is $x^{A*} = (0.6, 0.6)$ and $x^{B*} = (0.4, 0.4)$, so that $\frac{u^{B*}}{u^{A*}} = \frac{2}{3}$. It is easily verified that the utility possibility frontier (where $(\overline{u}^A, \overline{u}^B)$ denotes any element on the frontier) is given by $\overline{u}^B = 1 - \overline{u}^A$ and that as a result, $\left|\frac{d\overline{u}^B}{d\overline{u}^A}\right| = 1$, so that $\lim_{(\delta^A,\delta^B)\to(1,1)} \left|\frac{d\overline{u}^B}{d\overline{u}^A}\right| \frac{1-\delta^B}{1-\delta^A} = \frac{1}{2}$, which is strictly smaller than $\frac{u^{B*}}{u^{A*}} = \frac{2}{3}$. As a result, condition (51) in the Appendix is satisfied. On the other hand, the condition $f^A(u^A) \geq f^B(u^{A*})$ is satisfied, since it is equivalent to[31]

$$\frac{(1-p^{*2})\nabla_1^A(x^{A*}) - 2p^*\nabla_2^A(x^{A*})}{(1-p^{*2})\nabla_2^A(x^{A*}) + 2p^*\nabla_1^A(x^{A*})} < -\frac{\nabla_1^B(x^{B*})}{\nabla_2^B(x^{B*})} \tag{14}$$

where the left-hand side of this condition at the Walrasian allocation, with $p^* = 1$, reduces to,

$$-\frac{\nabla_2^A(x^{A*})}{\nabla_1^A(x^{A*})} = -\frac{2x_2^{A*} - e_2^A}{2x_1^{A*} - e_1^A} = -3, \tag{15}$$

which is the inverse of the slope of A's offer curve in the Walrasian equilibrium allocation $(0.6, 0.6)$, and the right hand side of condition (14) in the Appendix is the slope of B's offer curve in the Walrasian allocation

$$-\frac{\nabla_1^B(x^{B*})}{\nabla_2^B(x^{B*})} = -\frac{2x_1^{B*} - e_1^B - 1}{2x_2^{B*} - e_2^B - 1} = -\frac{1}{3}. \tag{16}$$

[31] Note that $c^* = 1$ in this example (see proof of Lemma 4).

It follows from Theorem 7 that there is no SSP equilibrium converging to the Walrasian allocation for this economy.

As for the existence of SSP equilibria that remain bounded away from efficiency, it is sufficient to show that the slope of B's offer curve at $\hat{x}$ in Figure 2 (see section 2) is flatter than the slope of A's indifference curve through the same point, i.e.

$$\frac{2x_1^B - e_1^B - 1}{2x_2^B - e_2^B - 1} < \frac{x_2^A}{x_1^A} \tag{17}$$

where x^A and x^B are evaluated at $\hat{x}$, i.e. $x^A = (0.75, 0.5625)$ and $x^B = (0.25, 0.4375)$. It can be immediately verified that this is satisfied, since $0.727 < 0.75$. As a result, by Theorem 5 there will exist an SSP equilibrium of this economy converging to this inefficient outcome as δ^A and δ^B converge to 1. Moreover, by Theorem 7 this is the only SSP equilibrium of this economy.

2. One can repeat the same exercise for an economy with initial endowments $e^A = (E, 0.3)$ and $e^B = (1 - E, 0.7)$ where $0.95 < E \leq 1$. It can be easily verified that there is still no SSP equilibrium converging to the Walrasian allocation. However, now the only candidate to be a SSP equilibrium delivered by Theorem 5 ceases to be one. To see this, note that for, say, $E = 0.96$ at the new candidate solution $\hat{x}$, with corresponding allocations $\hat{x}^A = (0.882, 0.578)$ and $\hat{x}^B = (0.118, 0.421)$, the slope of B's offer curve is steeper than of A's indifference curve (more specifically $0.939 > 0.655$). As a result, for this economy, there does not exist any SSP equilibrium.

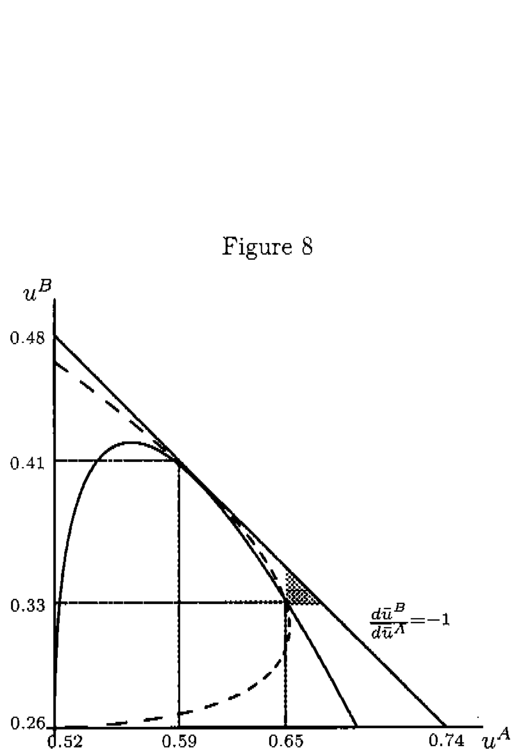
Finally, compare the outcome of the SSP equilibrium of the bargaining game over prices with the bargaining equilibrium over allocations. Binmore (1987) shows that for discounting frictions disappearing, bargaining over allocations leads to an allocation on the contract curve and hence efficiency (but in general different from the Walrasian allocation). In the example we considered earlier, the players' outside options are equal to the utility obtained in the initial endowment, $(u^A, u^B) = (0.52, 0.26)$. Since the total surplus on the contract curve is constant and equal to 1, the net surplus is 0.22 (i.e. 1 minus the sum of the outside options). Then given $\frac{\log \delta^A}{\log \delta^B} = r = 2$, bargaining over allocations converges to

$$\left(0.52 + 0.22\frac{1}{1+r}, 0.26 + 0.22\frac{r}{1+r}\right) = (0.59, 0.41), \tag{18}$$

which is different from the utility profile at the Walrasian allocation $(0.6, 0.4)$. On the other hand, bargaining over prices leads to the unique, inefficient SSP equilibrium in this example with payoffs

(0.65, 0.33). See Figure 8.

Figure 8



7 Discussion

7.1 Comparison with the Nash Axiomatic Bargaining-over-prices Solution

Does there exist a Nash Bargaining-over-prices problem that obtains the same outcome as the SSP equilibrium of the bargaining game over prices? The answer is yes. To see this, consider the Nash bargaining problem where one player (say player A) chooses the quantity of trade, and the Nash bargaining program selects a price, given bargaining power α, that solves the following maximization program

$$p \in \arg\max_p \left(u^A\left(x^A(p)\right)\right)^{\alpha}\left(u^B\left(e - x^A(p)\right)\right)^{1-\alpha} \tag{19}$$

where $x^A(p)$ denotes the allocation on A's offer curve given price p. Note that this is a modified version of Nash's static bargaining problem. Here, once p is determined, player A chooses the quantity. As a result, Nash's second axiom – that the solution be Pareto optimal – is not satisfied. However, in this modified version, conditional on optimal behavior in the ensuing subgame by player A (i.e. conditional on the equilibrium being subgame perfect), the bargaining solution is required to be "constrained" Pareto optimal.

The first order condition for this problem is:

$$\alpha u^A\left(x^A(p)\right)^{\alpha-1} u^B\left(e - x^A(p)\right)^{1-\alpha} Du^A\left(x^A(p)\right) Dx^A(p)$$
$$- (1-\alpha) u^A\left(x^A(p)\right)^{\alpha} u^B\left(e - x^A(p)\right)^{-\alpha} Du^B\left(e - x^A(p)\right) Dx^A(p) = 0$$

which is satisfied if, and only if,[32]

$$\left[\left(\frac{u^B\left(e - x^A(p)\right)}{u^A\left(x^A(p)\right)}\right) \alpha Du^A\left(x^A(p)\right) - (1-\alpha) Du^B\left(e - x^A(p)\right)\right] Dx^A(p) = 0. \tag{20}$$

[32] Note that $\left(\frac{u^B(e-x^A(p))}{u^A(x^A(p))}\right)^{-\alpha} \neq 0$ because the utility functions take only positive values.

This holds if either the first vector in brackets is the null vector, or both the vector in brackets and $Dx^A(p)$ are non-null but orthogonal.[33] This second case implies[34]

$$\frac{Du^B\left(e-x^A(p)\right)Dx^A(p)}{Du^A\left(x^A(p)\right)Dx^A(p)}=\frac{\alpha}{1-\alpha}\left(\frac{u^B\left(e-x^A(p)\right)}{u^A\left(x^A(p)\right)}\right). \tag{21}$$

Remind that $\left(u^A\left(x^A(p)\right), u^B\left(e-x^A(p)\right)\right)$ is the profile of utilities along A's offer curve f^A, i.e. $u^B\left(e-x^A(p)\right)=f^A\left(u^A\left(x^A(p)\right)\right)$. Then the derivative is of f^A is given by

$$f^{A\prime}\left(u^A\left(x^A(p)\right)\right)=-\frac{Du^B\left(e-x^A(p)\right)Dx^A(p)}{Du^A\left(x^A(p)\right)Dx^A(p)} \tag{22}$$

and as a result, the solution p of this Nash bargaining problem has to satisfy

$$f^{A\prime}\left(u^A\left(x^A(p)\right)\right)=-\frac{\alpha}{1-\alpha}\left(\frac{u^B\left(e-x^A(p)\right)}{u^A\left(x^A(p)\right)}\right). \tag{23}$$

Note that since $\frac{\alpha}{1-\alpha}$ takes any positive value for some $\alpha\in[0,1]$, any negative slope $f^{A\prime}\left(u^A\left(x^A(p)\right)\right)$ of f^A can be made equal to the right-hand side for some $\alpha\in[0,1]$. Therefore any point on the negatively sloped portion of f^A, the constrained Pareto frontier, is a Nash bargaining solution for some choice of α within $[0,1]$.

7.2 More than two goods

It is worth to note that the existence of global asymptotically inefficient SSP equilibria for a nonempty open set of economies shown in Theorem 5 can be generalized to a setup with more than two goods, since it follows from Lemma 3, i.e. the fact that generically f^A and f^B intersect tangently but do not cross at any profile of utilities (u^{A*}, u^{B*}) corresponding to a locally unique Walrasian allocation.

In effect, on the one hand, the tangent intersection follows from the fact that any profile of utilities (u^{A*}, u^{B*}) corresponding to a Walrasian allocation must be both on f^A and f^B, and on the (smooth) Pareto frontier, while f^A and f^B must be within set of attainable utilities. This tangency means that the slopes of f^A and f^B at (u^{A*}, u^{B*}), i.e. the derivatives $f^{A\prime}(u^{A*})$ and $(f^B)^{-1\prime}(u^{A*})$, coincide. On the other hand, assuming a (necessarily non-transversal) crossing of f^A and f^B at (u^{A*}, u^{B*}) would require the derivative $(f^B)^{-1\prime}(u^A)$ to be smaller than the derivative $f^{A\prime}(u^A)$ at every u^A distinct from u^{A*} in a neighborhood of u^{A*}. As a consequence, the graphs of the derivatives of f^A and $(f^B)^{-1}$ would necessarily have to intersect at $(u^{A*}, f^{A\prime}(u^{A*}))=(u^{A*}, (f^B)^{-1\prime}(u^{A*}))$ without crossing each other, and hence that intersection of the graphs of $f^{A\prime}$ and $(f^B)^{-1\prime}$ itself would have to be tangent, i.e. non-transversal, as well.

Now, while the non-transversality of the intersection of f^A and $(f^B)^{-1}$ at (u^{A*}, u^{B*}) follows, as explained above, from the fact that (u^{A*}, u^{B*}) is the profile of utilities corresponding to a Walrasian allocation, the non-transversality of the intersection of $f^{A\prime}$ and $(f^B)^{-1\prime}$ at $(u^{A*}, f^{A\prime}(u^{A*}))=(u^{A*}, (f^B)^{-1\prime}(u^{A*}))$ does not follow necessarily from any assumption on the economy or the properties of a Walrasian allocation. As a matter of fact, the non-transversality of the intersection of $f^{A\prime}$ and $(f^B)^{\prime}$

[33] The second vector $Dx^A(p)$ is non-null for any generic utility function u^A.

[34] Generically in utility functions, $Du^A\left(x^A(p)\right)$ and $Dx^A(p)$ are not orthogonal.

at $(u^{A*}, f^{A\prime}(u^{A*})) = (u^{A*}, (f^B)^{-1\prime}(u^{A*}))$ imposes a constraint on the derivatives *up to the order* 2 of the utility functions u^A and u^B at the Walrasian allocation x^{A*} and x^{B*} respectively. That condition will not be satisfied generically.

Note that this argument is independent of the number of goods in the economy and relies only on the properties of the curves f^A and f^B of profiles of utilities attainable along the agents offer curves.

8 Conclusion

We have analyzed in this paper a model of alternating-offers bargaining over prices in an exchange economy. Because the only allocations that arise in equilibrium must necessarily be on the offer curve of the agent accepting the offer, and the market power of infinitely patient agents is evenly distributed between both agents, a sensible conjecture about the equilibrium outcome of the bargaining over prices as discounting frictions vanish is an allocation on both offer curves, i.e. a Walrasian allocation.

We have indeed shown that the Walrasian allocation can be the outcome of the bargaining over prices in the limit, as the agents become infinitely patient. In effect, Theorem 7 establishes the convergence of SSP equilibria to the Walrasian allocation for some paths of the discount factors towards 1. Moreover, it establishes that whenever this convergence obtains, there is actually an even number of SSP equilibria converging to the same Walrasian allocation, in such a way that convergence towards the Walrasian outcome necessarily comes with the multiplicity of SSP equilibria. Nevertheless, Theorem 7 establishes as well that the convergence of the SSP equilibria of the bargaining over prices to Walrasian outcomes is not guaranteed. Moreover, and contrarily to what the initial intuition might tell us, Theorems 5 and 6 establish generically the convergence of some outcomes of bargaining over prices to outcomes that remain in the interior of the Pareto set.

The central conclusion of this analysis of bargaining over prices is that this procedure can lead to the Walrasian allocation, thereby offering a bargaining foundation for Walrasian Equilibrium that does not depend on price-taking behavior. At the same time, one recognizes that convergence is not unique and that other equilibria inside the Pareto frontier exist as well. The success – albeit partial – in providing a foundation for the Walrasian equilibrium in small economies suggests that bargaining over prices constitutes a crucial ingredient. More work is needed to uncover a negotiation procedure that provides a unqualified foundation. This paper intends to be a first step in that direction.

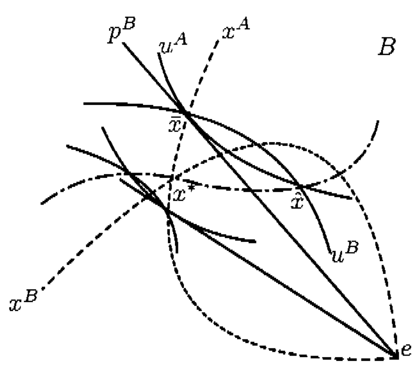
9 Appendix

Proof. (Lemma 3) For any allocation $\overline{x}$ on A's offer curve and close to the Walrasian allocation x^* there exist another allocation $\widehat{x}$ giving the agents the same utilities. Thus $\overline{x}$ and $\widehat{x}$ satisfy

$$\begin{aligned} u^A(\hat{x}^A) - u^A(\bar{x}^A) &= 0 \\ u^B(e^A + e^B - \hat{x}^A) - u^B(e^A + e^B - \bar{x}^A) &= 0 \\ Du^A(\bar{x}^A)(\bar{x}^A - e^A) &= 0 \\ (p, 1)(\bar{x}^A - e^A) &= 0 \end{aligned} \qquad (24)$$

that is to say (written in terms of A's variables): (i) A gets from $\bar{x}$ the same utility that he gets from $\hat{x}$, (ii) B gets from $\hat{x}$ the same utility that he gets from $\bar{x}$, (iii) $\bar{x}$ is on A's offer curve, and (iv) $\bar{x}$ is affordable to A at the price p of good 1 in terms of good 2. This system defines implicitly $\hat{x}$ as a differentiable function of p. Thus as p varies or, equivalently, as $\bar{x}$ runs along A's offer curve, $\hat{x}$ follows a smooth path that goes through x^* for $p = p^*$, for which $\bar{x} = \hat{x} = x^*$ (see Figure 9)

Figure 9



In effect, the function that determines A's bundle $(\hat{x}_1^A, \hat{x}_2^A)$ in the allocation $\hat{x}$ for each p in the system (24) is the composition of the function $\bar{\xi}^A$ associating A's bundle $(\bar{x}_1^A, \bar{x}_2^A)$ in the allocation $\overline{x}$ to each p that is implicitly defined by

$$\begin{aligned} Du^A(\bar{x}^A)(\bar{x}^A - e^A) &= 0 \\ (p, 1)(\bar{x}^A - e^A) &= 0 \end{aligned} \qquad (25)$$

and the function $\hat{\xi}^A$ associating $(\hat{x}_1^A, \hat{x}_2^A)$ to each $(\bar{x}_1^A, \bar{x}_2^A)$ that is implicitly defined by

$$\begin{aligned} u^A(\hat{x}^A) - u^A(\bar{x}^A) &= 0 \\ u^B(e^A + e^B - \hat{x}^A) - u^B(e^A + e^B - \bar{x}^A) &= 0. \end{aligned} \qquad (26)$$

On the one hand, the Jacobian of the left-hand side of (25) is

$$\begin{pmatrix} \nabla_1^A(\bar{x}^A) & \nabla_2^A(\bar{x}^A) & 0 \\ p & 1 & \bar{x}_1^A \end{pmatrix} \qquad (27)$$

where $\nabla^A(\bar{x}^A)$ denotes the gradient of A's offer curve at $\bar{x}^A$. For a utility function u^A generic with respect to the topology of C^1 uniform convergence on compacts,[35] the first two columns are always linearly independent, even at the Walrasian equilibrium allocation x^*. Therefore, the system (25) defines implicitly $(\bar{x}_1^A, \bar{x}_2^A)$ as a function $\bar{\xi}^A$ of p and

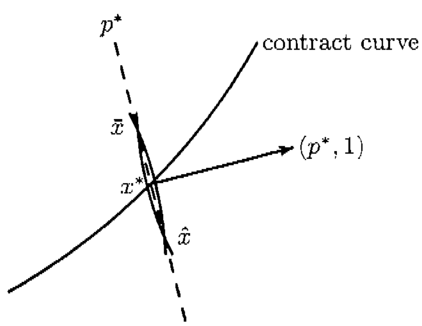
$$D\bar{\xi}^A(p) = -\begin{pmatrix} \nabla_1^A(\bar{x}^A) & \nabla_2^A(\bar{x}^A) \\ p & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \bar{x}_1^A \end{pmatrix}$$
$$= -\begin{vmatrix} \nabla_1^A(\bar{x}^A) & \nabla_2^A(\bar{x}^A) \\ p & 1 \end{vmatrix}^{-1} \begin{pmatrix} -\nabla_2^A(\bar{x}^A)\bar{x}_1^A \\ \nabla_1^A(\bar{x}^A)\bar{x}_1^A \end{pmatrix}. \qquad (28)$$

On the other hand, the Jacobian of the left-hand side of (26) is

$$\begin{pmatrix} D_1u^A(\hat{x}^A) & D_2u^A(\hat{x}^A) & -D_1u^A(\bar{x}^A) & -D_2u^A(\bar{x}^A) \\ -D_1u^B(\hat{x}^B) & -D_2u^B(\hat{x}^B) & D_1u^B(\bar{x}^B) & D_2u^B(\bar{x}^B) \end{pmatrix}. \qquad (29)$$

This Jacobian clearly drops rank at efficient allocations, and hence at the Walrasian allocation x^*. As a consequence, the theorem of the implicit function does not apply there. Nonetheless, the equations (26) still define $(\hat{x}_1^A, \hat{x}_2^A)$ as a function of $(\bar{x}_1^A, \bar{x}_2^A)$ since, for strictly convex preferences and any given point $\bar{x}$, there exists a unique $\hat{x}$ where the two indifference curves going through $\bar{x}$ cross each other again (if $\bar{x}$ happens to be efficient, then $\hat{x}$ actually coincides with $\bar{x}$). This function is clearly differentiable off the contract curve (where the implicit function theorem applies), but also on the contract curve (where the implicit function theorem does not apply). In effect, as $\bar{x}$ departs slightly from an efficient allocation x^* on the contract curve, the lens formed by the two indifference curves going through $\bar{x}$ will cross again (almost) at a point $\hat{x}$ across the contract curve in the direction of the line supporting x^* as a Walrasian equilibrium (see Figure 10).

Figure 10



The linear mapping approximating locally this behavior is

$$D\hat{\xi}^A(x^{A*}) = \begin{pmatrix} p^* & -1 \\ 1 & p^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -c^* \end{pmatrix} \begin{pmatrix} p^* & -1 \\ 1 & p^* \end{pmatrix}^{-1} \qquad (30)$$

[35] Actually for any topology of C^n uniform convergence on compacts.

for some $c^* > 0$ that in general will depend on the curvature of A's and B's indifference curves at the Walrasian allocation x^*. In words, $D\hat{\xi}^A(x^{A*})$ consists of the composition of (i) a change to an orthogonal basis containing the price vector $(p^*, 1)$, (ii) a jump across the first axis of that basis, and (iii) the undoing of the change of basis. Therefore,

$$\begin{pmatrix} \frac{d\hat{x}_1^A}{dp}(p^*) \\ \frac{d\hat{x}_2^A}{dp}(p^*) \end{pmatrix} = D\hat{\xi}^A(x^{A*})D\bar{\xi}^A(p^*) \tag{31}$$

$$= -\frac{x_1^{A*}}{p^{*2}+1}\begin{vmatrix} \nabla_1^A(x^{A*}) & \nabla_2^A(x^{A*}) \\ p^* & 1 \end{vmatrix}^{-1} \begin{pmatrix} (c^* - p^{*2})\nabla_2^A(x^{A*}) + (1+c^*)p^*\nabla_1^A(x^{A*}) \\ (1 - c^*p^{*2})\nabla_1^A(x^{A*}) - (1+c^*)p^*\nabla_2^A(x^{A*}) \end{pmatrix}$$

and as a result,

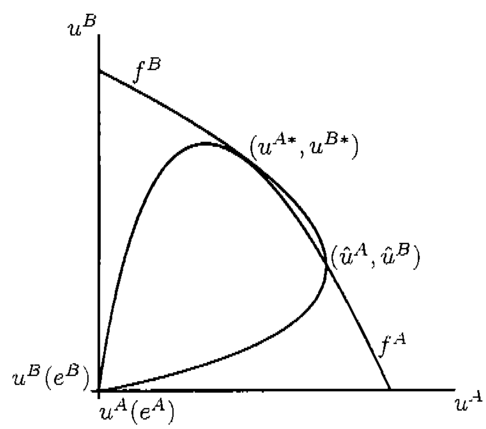
$$\frac{d\hat{x}_2^A}{d\hat{x}_1^A}(x_1^{A*}) = \frac{\frac{d\hat{x}_2^A}{dp}(p^*)}{\frac{d\hat{x}_1^A}{dp}(p^*)} = \frac{(1 - c^*p^{*2})\nabla_1^A(x^{A*}) - (1+c^*)p^*\nabla_2^A(x^{A*})}{(c^* - p^{*2})\nabla_2^A(x^{A*}) + (1+c^*)p^*\nabla_1^A(x^{A*})}. \tag{32}$$

If, as shown in Figure 9, the slope of the path followed by $\widehat{x}$ is at x^* smaller than the slope of B's offer curve, i.e.

$$\frac{d\hat{x}_2^A}{d\hat{x}_1^A}(x_1^{A*}) < -\frac{\nabla_1^B(x^{B*})}{\nabla_2^B(x^{B*})}, \tag{33}$$

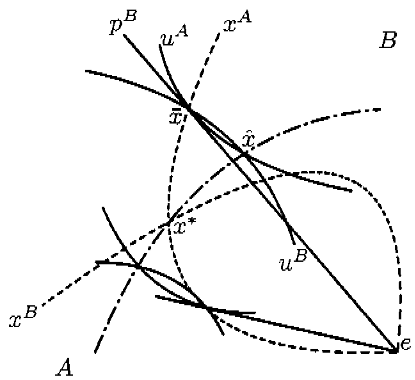
then around the Walrasian allocation, for any given level of utility u^A close to , agent B attains on B's offer curve a higher utility than on A's. In other words, around (u^{A*}, u^{B*}) the curve f^B is above the curve f^A, as shown in Figure 11.

Figure 11



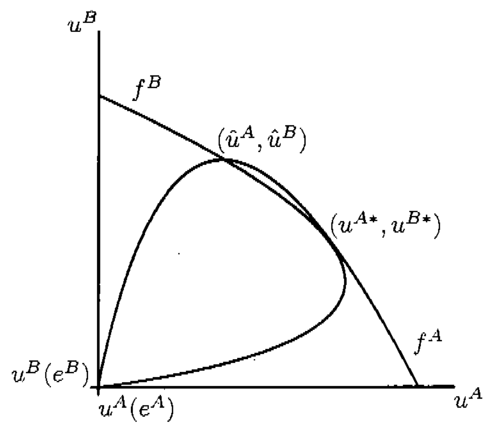
If, on the contrary, the path followed by $\widehat{x}$ had at x^* a slope bigger than B's offer curve, then around the Walrasian allocation, for any given level of utility u^A close to , agent B attains on A's offer curve a higher utility than on B's (see Figure 12).

Figure 12



In other words, around (u^{A*}, u^{B*}) the curve f^A is above the curve f^B.

Figure 13



Only in the case in which the path followed by $\widehat{x}$ had at x^* a slope exactly equal to that of B's offer curve, i.e.

$$\frac{(1-c^*p^{*2})\nabla_1^A(x^{A*}) - (1+c^*)p^*\nabla_2^A(x^{A*})}{(c^*-p^{*2})\nabla_2^A(x^{A*}) + (1+c^*)p^*\nabla_1^A(x^{A*})} = -\frac{\nabla_1^B(x^{B*})}{\nabla_2^B(x^{B*})} \tag{34}$$

could a crossing of f^A and f^B occur at (u^{A*}, u^{B*}). The equation (34) imposes a constraint on the partial derivatives of order two of the utility functions at the Walrasian allocation x^* that is degenerate in the space of utility functions with respect to the topology of C^1 uniform convergence on compacts.[36] Equivalently, it constrains the profile of slopes of A's and B's offer curves at every Walrasian allocation

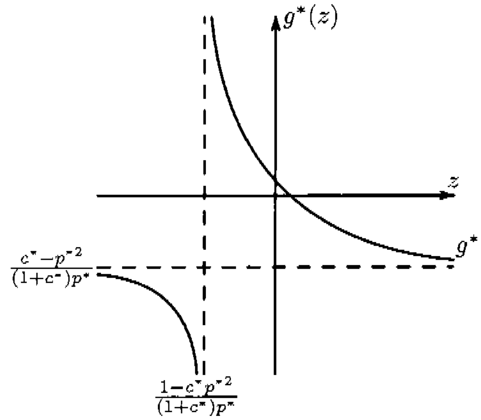
[36] Actually, with respect to any such C^n topology. Interestingly enough, the perturbation need not always be made in the space of utility functions. For instance, in the case of the Cobb-Douglas example introduced in section 2 (for which $c^* = 1$ always), this condition is satisfied only for initial endowments on the anti-diagonal of the Edgeworth box, i.e. in a closed and nowhere dense subset of endowments for the given utility functions.

x^*, $\left(-\frac{\nabla_1^A(x^{A*})}{\nabla_2^A(x^{A*})}, -\frac{\nabla_1^B(x^{B*})}{\nabla_2^B(x^{B*})}\right)$, to be on the graph of the function g^* such that[37]

$$g^*(z) = \frac{(1+c^*)p^* + (1-c^*p^{*2})z}{(1+c^*)p^*z + (c^* - p^{*2})} \tag{35}$$

which clearly holds for degenerate economies only (Figure 14 depicts g^* for $c^* = 1$ and $1 < p^{*2}$).[38] ■

Figure 14



Proof. (Theorem 5) By Lemma 4, there exists an inefficient intersection $(\widehat{u}^A, \widehat{u}^B)$ of f^A and f^B, and hence inefficient allocations $\hat{x}, \bar{x}$, that solve the necessary conditions (24) for a SSP equilibrium. By continuity, this inefficient intersection remains for discount factors δ^A and δ^B close to 1, and still corresponds to inefficient allocations $\hat{x}, \bar{x}$. This intersection corresponds to a SSP equilibrium whenever both curves f^A and f^B are downward-sloped at $(\widehat{u}^A, \widehat{u}^B) = (u^A(\hat{x}^A), u^B(\hat{x}^B)) = (u^A(\bar{x}^A), u^B(\bar{x}^B))$.[39]

[37] In effect,

$$-\frac{\nabla_1^B(x^{*B})}{\nabla_2^B(x^{*B})}) = \frac{(1+c^*)p^* + (1-c^*p^{*2})(-\frac{\nabla_1^A(x^{*A})}{\nabla_2^A(x^{*A})})}{(1+c^*)p^*(-\frac{\nabla_1^A(x^{*A})}{\nabla_2^A(x^{*A})}) + (c^* - p^{*2})}.$$

[38] Note that

$$\lim_{z\to\infty} g^*(z) = \frac{1-c^*p^{*2}}{(1+c^*)p^*}$$
$$\lim_{z\to\frac{c^*-p^{*2}}{(1+c^*)p^*}} g^*(z) = \infty$$

and, for all $z \in \mathbb{R}$,

$$\frac{dg^*}{dz}(z) = -\frac{(1-c^*p^{*2})(c^*-p^{*2}) + ((1+c^*)p^*)^2}{((1+c^*)p^*z + (c^*-p^{*2}))^2} \neq 0$$

for all $c^*, p^* > 0$.

[39] Note that the negative slope at $(u^A(\hat{x}^A), u^B(\hat{x}^B)) = (u^A(\bar{x}^A), u^B(\bar{x}^B))$ is equivalent to satisfying the first two equations of (7). To see this, note that the first two equations require that at a candidate equilibrium allocation, there exists no allocation on the offer curve of either agent that is inside the lens of Pareto-improving allocations, which means precisely that no profile of utilities exists where both are increasing along the offer curve. Thus, should both curves of profiles of utilities not be downward-sloped, then a mutually beneficial counter-offer could be made to the agent whose curve of profiles of utilities along his offer curve is upward-sloped.

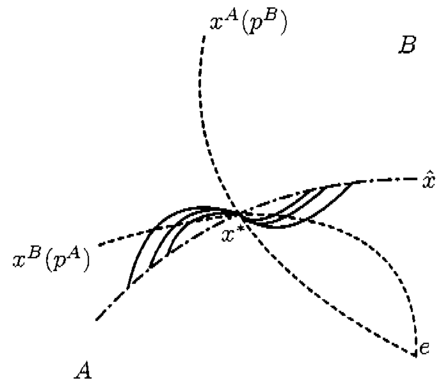
Note then that if the slope $\frac{d\hat{x}_2^A}{d\hat{x}_1^A}$ of the path followed by $\hat{x}$ in the modified system (25) is close to the slope $-\frac{\nabla_1^B(x^{B*})}{\nabla_2^B(x^{B*})}$ of B's offer curve at the Walrasian allocation x^*, then the slopes of f^A and f^B at $(\widehat{u}^A, \widehat{u}^B)$ would be close to their common negative slope at the Walrasian allocation profile (u^{A*}, u^{B*}), and hence they will be negative also. But

$$\frac{d\hat{x}_2^A}{d\hat{x}_1^A}(x_1^{A*}) = \frac{(1 - c^*p^{*2})\nabla_1^A(x^{A*}) - (1 + c^*)p^*\nabla_2^A(x^{A*})}{(c^* - p^{*2})\nabla_2^A(x^{A*}) + (1 + c^*)p^*\nabla_1^A(x^{A*})} \tag{36}$$

i.e. $\left(-\frac{\nabla_1^A(x^{A*})}{\nabla_2^A(x^{A*})}, \frac{d\hat{x}_2^A}{d\hat{x}_1^A}(x_1^{A*})\right)$ is on the graph of g^* in Figure 14 above. Hence $\frac{d\hat{x}_2^A}{d\hat{x}_1^A}(x_1^{A*})$ is close to $-\frac{\nabla_1^B(x^{B*})}{\nabla_2^B(x^{B*})}$ whenever $\left(-\frac{\nabla_1^A(x^{A*})}{\nabla_2^A(x^{A*})}, -\frac{\nabla_1^B(x^{B*})}{\nabla_2^B(x^{B*})}\right)$ is close to the graph of g^*, and the conclusion follows. ■

Proof. (Theorem 6) Consider a sequence $\{\hat{x}_n\}$ of paths followed $\hat{x}$ that differ from the path corresponding to any given offer curve $x^A(p)$ of A only in a compact neighborhood of x^*. Let $\{\hat{x}_n\}$ be such that it converges pointwise to the original path followed by $\hat{x}$ while keeping their slopes at x^* smaller than the slope of B's offer curve (see Figure 15).[40]

Figure 15



The pointwise convergence of $\{\hat{x}_n\}$ to $\hat{x}$ guarantees the pointwise convergence within a compact of the associated offer curves $\{x_n^A(p)\}$ to $x^A(p)$. Also the (piecewise) monotone and pointwise convergence of $\{x_n^A(p)\}$ within a compact guarantees that their convergence to $x^A(p)$ is uniform indeed. As a consequence, the utility functions u_n^A generating these offer curves $x_n^A(p)$ converge in the topology of C^1 convergence on compacts towards the utility function u^A that generates the offer curve $x^A(p)$.

It just remains to verify that the new intersection of each $\hat{x}_n$ with B's offer curve is not a new Walrasian equilibrium. For $\hat{x}_n$ to cross B's offer curve arbitrarily close to the Walrasian allocation x^*,

[40] Since at Walrasian equilibrium alocation x^*

$$\frac{d\hat{x}_2^A}{d\hat{x}_1^A}(x_1^{*A}) = \frac{(1 - c^*p^{*2})\nabla_1^A(x^{*A}) - (1 + c^*)p^*\nabla_2^A(x^{*A})}{(c^* - p^{*2})\nabla_2^A(x^{*A}) + (1 + c^*)p^*\nabla_1^A(x^{*A})}, \tag{37}$$

then $\frac{d\hat{x}_1^A}{d\hat{x}_2^A}(x_1^{*A})$ is an injective function of the slope of A's offer curve at x^* by the function g^* defined in (35). As a consequence, the slope at x^* of the path followed by $\hat{x}$, $\frac{d\hat{x}_2^A}{d\hat{x}_1^A}(x_1^{*A})$, can be made to take almost any value varying adequately the slope of A's offer curve at x^* (the range of g^* is $\mathbb{R}$ without $\frac{1-c^*p^{*2}}{(1+c^*)p^*}$).

we just need to be able to make $\frac{d\hat{x}_2^A}{d\hat{x}_1^A}(x_1^{A*})$ equal to the slope of B's offer curve at x^*, i.e.

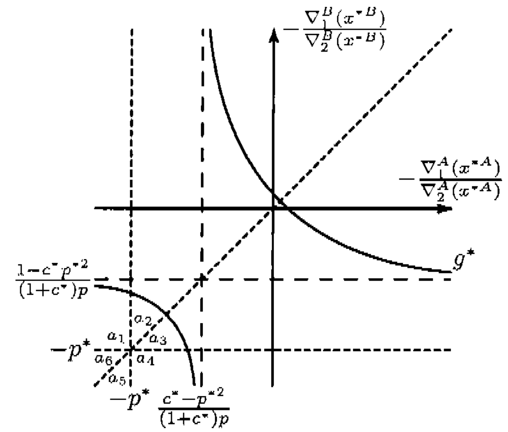
$$\frac{d\hat{x}_2^A}{d\hat{x}_1^A}(x_1^{A*}) = -\frac{\nabla_1^B(x^{B*})}{\nabla_2^B(x^{B*})}. \tag{38}$$

But since

$$\frac{d\hat{x}_2^A}{d\hat{x}_1^A}(p^*) = \frac{(1-c^*p^{*2})\nabla_1^A(x^{A*}) - (1+c^*)p^*\nabla_2^A(x^{A*})}{(c^*-p^{*2})\nabla_2^A(x^{A*}) + (1+c^*)p^*\nabla_1^A(x^{A*})}, \tag{39}$$

this is achieved making the profile $\left(-\frac{\nabla_1^A(x^{A*})}{\nabla_2^A(x^{A*})}, -\frac{\nabla_1^B(x^{B*})}{\nabla_2^B(x^{B*})}\right)$ of slopes of the offer curves at the Walrasian allocation x^* to be on the graph of the function g^* in (35) above. It is straightforward to check that the graph of g^* can be attained from any profile of slopes for the offer curves without making appear new intersections between them. This can be seen in Figure 16 below for the case $c^* = 1$ and $1 < p^{*2}$. There the whole plane of profiles of slopes of the offer curves of A and B at a Walrasian equilibrium is partitioned in the six areas a_i, $i = 1, \ldots, 6$ within which the number of Walrasian equilibria remains locally constant. The graph of g^* intersects every a_i, except for the areas a_5 and a_6 that correspond to preferences that do not satisfy the Assumption 1 (in particular, they violate the requirement of A's and B's demand not being simultaneously upward-sloped for both goods).[41]

Figure 16



Far enough in the sequence $\{\hat{x}_n\}$, the A's marginal rate of substitution at the intersection of $\hat{x}_n$ with B's offer curve is close to the relative price supporting x^*, and hence smaller than the slope of B's offer curve at x^*. By continuity, the same will be true for δ^A and δ^B close to 1. This guarantees that this intersection corresponds to a SSP equilibrium. A similar argument applies to the other intersection of $\hat{x}_n$ with B's offer curve. ∎

[41] The same holds true for any $c^*, p^* > 0$. In effect, the relevant property is that, since

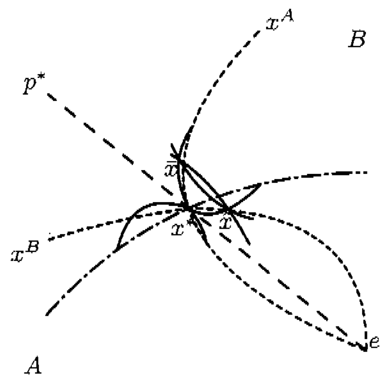
$$-p^* < \frac{c^* - p^{*2}}{(1+c^*)p^*}$$

and

$$-p^* < \frac{1 - c^*p^{*2}}{(1+c^*)p^*}$$

always, then the asymptotes of g^* (and hence g^* itself) intersect every a_i, but a_5, a_6.

Figure 17



Proof. (Theorem 7) Note that the efficiency of a Walrasian allocation implies that f^B is invertible in a neighborhood of the profile of utilities (u^{A*}, u^{B*}) corresponding to the Walrasian equilibrium x^*, and hence so is $\delta^B f^B$ around u^{A*} for δ^B close enough to 1. Thus we can wonder about whether, for given $\delta^A, \delta^B < 1$, $(\delta^B f^B)^{-1}(u^{A*})$ is smaller or bigger than $\delta^A f^A(u^{A*})$. This is useful for our purposes because

$$(\delta^B f^B)^{-1}(u^{A*}) < \delta^A f^A(u^{A*}) \tag{40}$$

along with

$$f^A(u^A) \leq (f^B)^{-1}(u^{A*}) \tag{41}$$

for every u^A close enough to u^{A*} implies the existence of two other intersections of $\delta^A f^A$ and $\delta^B f^B$. If, on the contrary, equation (40) holds with the opposite inequality (i.e. $(\delta^B f^B)^{-1}(u^{A*}) > \delta^A f^A(u^{A*})$ along with $f^A(u^A) \leq (f^B)^{-1}(u^{A*})$), then there exists no intersection of $\delta^A f^A$ and $\delta^B f^B$ in some neighborhood of (u^{A*}, u^{B*}) for δ^A, δ^B close enough to 1.[42]

Let us consider the first case. Clearly, since $f^A(u^{A*}) = u^{B*}$, then for any $\delta^A < 1$,

$$\delta^A f^A(u^{A*}) = \delta^A u^{B*}. \tag{43}$$

As for $(\delta^B f^B)^{-1}(u^{A*})$, let $\widetilde{f}^B(u^B, \delta^B) = \delta^B f^B(u^B)$. Then

$$\widetilde{f}^B(u^{B*}, 1) = f^B(u^{B*}) = u^{A*}. \tag{44}$$

Since the linear approximation of $\widetilde{f}^B$ at $(u^{B*}, 1)$ is

$$\widetilde{f}^B(u^B, \delta^B) \approx \widetilde{f}^B(u^{B*}, 1) + D_1\widetilde{f}^B(u^{B*}, 1)(u^B - u^{B*}) + D_2\widetilde{f}^B(u^{B*}, 1)(\delta^B - 1), \tag{45}$$

[42]Similarly $(\delta^B f_B)^{-1}(u^{A*}) > \delta^A f_A(u^{A*})$ along with

$$f_A(u^A) \geq f_B^{-1}(u^{A*}) \tag{42}$$

guarantees the existence of two other intersections as well, while $(\delta^B f_B)^{-1}(u^{A*}) < \delta^A f_A(u^{A*})$ along with $f_A(u^A) \geq f_B^{-1}(u^{A*})$ guarantees that there exists no intersection of $\delta^A f_A$ and $\delta^B f_B$ in some neighborhood of (u^{A*}, u^{B*}) for δ^A, δ^B close enough to 1.

then $(\delta^B f^B)^{-1}(u^{A*})$ is the level of utility u^B for B such that

$$u^{A*} \approx u^{A*} + f^{B\prime}(u^{B*})(u^B - u^{B*}) + f^B(u^{B*})(\delta^B - 1) \tag{46}$$

i.e.

$$(\delta^B f^B)^{-1}(u^{A*}) \approx u^{B*} + \frac{u^{A*}}{f^{B\prime}(u^{B*})}(1-\delta^B). \tag{47}$$

Therefore

$$(\delta^B f^B)^{-1}(u^{A*}) < \delta^A f^A(u^{A*}) \tag{48}$$

holds for δ^B smaller but close to 1 if, and only if,

$$u^{B*} + \frac{u^{A*}}{f^{B\prime}(u^{B*})}(1-\delta^B) < \delta^A u^{B*} \tag{49}$$

i.e. if, and only if,

$$\frac{u^{B*}}{u^{A*}} < -\frac{1}{f^{B\prime}(u^{B*})}\frac{1-\delta^B}{1-\delta^A}. \tag{50}$$

Note that the range of values taken by $\frac{1-\delta^B}{1-\delta^A}$ in every neighborhood of $(\delta^A, \delta^B u^{A*}) = (1,1)$ in $(0,1)\times(0,1)$ is $\mathbb{R}_{++}$. Therefore there always exist discount factors δ^A, δ^B arbitrarily close to 1 for which the condition (50) holds, as well as discount factors δ^A, δ^B arbitrarily close to 1 for which the reversed inequality

$$\frac{u^{B*}}{u^{A*}} > -\frac{1}{f^{B\prime}(u^{B*})}\frac{1-\delta^B}{1-\delta^A} \tag{51}$$

holds. Since, generically, either

$$f^A(u^A) \leq (f^B)^{-1}(u^{A*}) \tag{52}$$

or

$$f^A(u^A) \geq (f^B)^{-1}(u^{A*}) \tag{53}$$

holds for all u^A close enough to u^{A*}, the conclusion follows. ■

Proof. (Theorem 8) Letting $(\bar{u}^{A*}, \bar{u}^{B*})$ denote a profile of utilities on the Pareto frontier, and letting $\frac{d\bar{u}^{B*}}{d\bar{u}^{A*}}$ denote the common slope $\frac{1}{f^{B\prime}(u^{B*})} = f^{A\prime}(u^{A*})$ of f^A and f^A at (u^{A*}, u^{B*}), the condition

$$\frac{u^{B*}}{u^{A*}} = -\frac{1}{f^{B\prime}(u^{B*})}\frac{1-\delta^B}{1-\delta^A}. \tag{54}$$

that separates the two cases in Theorem 7 of existence and nonexistence of SSP equilibria converging to a Walrasian equilibrium, can be written equivalently as

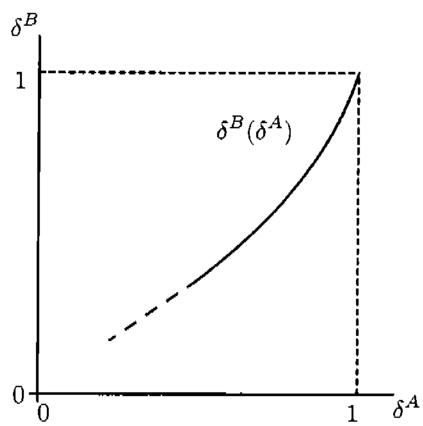
$$\delta^B = 1 + \frac{u^{B*}}{u^{A*}}\frac{d\bar{u}^{A*}}{d\bar{u}^{B*}}(1-\delta^A), \tag{55}$$

which implies that the slope of this separating curve at $(\delta^A, \delta^B) = (1,1)$ is

$$\frac{d\delta^B}{d\delta^A} = -\frac{u^{B*}}{u^{A*}}\frac{d\overline{u}^{A*}}{d\overline{u}^{B*}}. \tag{56}$$

As a consequence, $\frac{d\delta^B}{d\delta^A}$ is bounded above and bounded away from 0. In effect, because of the discounting in the bargaining game, utility functions are constrained to be positive. As a consequence, $0 < \frac{u^{B*}}{u^{A*}} < \infty$. Similarly, for any concave utility functions $0 < \frac{d\bar{u}^{A*}}{d\bar{u}^{B*}} < \infty$ anywhere on the Pareto frontier and away from the boundary. As a result, the curve of (δ^A, δ^B) separating the set of discount factors for which there is no SSP equilibrium converging to the Walrasian equilibrium from the set of those discount factors for which an even number of such equilibria exist, approaches $(1,1)$ with a slope bounded away from the horizontal and vertical slopes of the boundary of $[0,1] \times [0,1]$ at $(1,1)$ (see Figure 18 below). As a consequence, no degenerate convergence is required for either the existence or the nonexistence of SSP equilibria converging to the Walrasian allocation to obtain. ∎

Figure 18



References

[1] BINMORE, KENNETH, "Nash Bargaining Theory III", in Binmore K., and P. Dasgupta (eds) *The Economics of Bargaining*, 1987, Blackwell.

[2] BINMORE, KENNETH, ARIEL RUBINSTEIN, AND ASHER WOLINSKY, "The Nash Bargaining Solution in Economic Modelling", *Rand Journal of Economics*, **17**, 1986, 176-188.

[3] CRAMTON, PETER, ROBERT GIBBONS, AND PAUL KLEMPERER, "Dissolving Partnerships Efficiently", *Econometrica* 53(3), 1987, 615-632.

[4] DÁVILA, J. AND J. EECKHOUT, "Is Bargaining over Prices Efficient?", PIER Working Paper 01-045, Penn Institute for Economics Research (2002)

[5] FARBER, HENRY S., "The Analysis of Union Behavior", *Handbook of Labor Economics*, Vol. II, O. Ashenfelter and R. Layard (eds.), 1986, 1039-1089.

[6] GALE, DOUGLAS, "Bargaining and Competition: Part I", *Econometrica*, **54**, 1986, 785-806.

[7] HARSANYI, JOHN C., "Approaches to the Bargaining Problem Before and After the Theory of Games: a Critical Discussion of Zeuthen's, Hicks', and Nash's Theories", *Econometrica*, 24, 144-157, 1956.

[8] MOLDOVANU, BENNY, "How to Dissolve a Partnership", University of Mannheim mimeo, 2001.

[9] RUBINSTEIN, ARIEL, "Perfect Equilibrium in a Bargaining Model", *Econometrica*, **50 (1)**, 1982, 97-110.

[10] SHAPLEY, LL. AND M. SHUBIK, "Trade using one commodity as medium of payment", *Journal of Political Economy*, **85**, 1977, 937-968.

[11] SHUBIK, M., "A Theory of Money and Financial Institutions: Commodity Money, Oligopoly, Credit, and Bankruptcy in a General Equilibrium Model", *Western Economic Journal*, **11**, 1973, 24-38.

[12] SOLOW, ROBERT, AND IAN MACDONALD, "Wage Bargaining and Employment", *American Economic Review*, **71 (5)**, 1981, 896-908.

[13] STÅHL, I., *Bargaining Theory*, Stockholm: Stockholm School of Economics, 1972.

[14] YILDIZ, MUHAMET, "Walrasian Bargaining", MIT working paper 2003.