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ABSTRACT

It is now well known that the Sharpe ratio and other related reward-to-risk measures may be manipulated with option-like strategies. In this paper we derive the general conditions for achieving the maximum expected Sharpe ratio. We derive static rules for achieving the maximum Sharpe ratio with two or more options, as well as a continuum of derivative contracts. The optimal strategy rules for increasing the Sharpe ratio.

Our results have implications for performance measurement in any setting in which managers may use derivative contracts. In a performance measurement setting, we suggest that the distribution of high Sharpe ratio managers should be compared with that of the optimal Sharpe ratio strategy. This has particular application in the hedge fund industry where use of derivatives is unconstrained and manager compensation itself induces a non-linear payoff.

The shape of the optimal Sharpe ratio leads to further conjectures. Expected returns being held constant, high Sharpe ratio strategies are, by definition, strategies that generate regular modest profits punctunated by occasional crashes. Our evidence suggests that the "peso problem" may be ubiquitous in any investment management industry that rewards high Sharpe ratio managers.

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1 Introduction

The Sharpe ratio is one of the most common measures of portfolio performance. William Sharpe developed it in 1966 as a tool for evaluating and predicting the performance of mutual fund managers. Since then the Sharpe ratio, and its close analogues the Information ratio, the squared Sharpe ratio and M-squared, have become widely used in practice to rank investment managers and to evaluate the attractiveness of investment strategies in general. The appeal of the Sharpe measure is clear. It is an affine transformation of a simple t-test for equality in means of two variables, the first variable being the manager's time series of returns and the second being a benchmark.[1] The Sharpe ratio is also ubiquitous in academic research as a metric for bounding asset prices.[2]

Unfortunately, the Sharpe ratio is prone to manipulation – particularly by strategies that can change the shape of probability distribution of returns. For example, Henriksson and Merton (1981) and Dybvig and Ingersoll (1982) show that non-linear payoffs limit the applicability of the Sharpe ratio to the problem of performance evaluation. More recently, Bernardo and Ledoit (2000) show that Sharpe ratios are particularly misleading when the shape of the return distribution is far from normal.[3] Spurgin (2001) shows that managers can improve their expected Sharpe ratio by selling off the upper end of the potential return distribution. Other researchers, recognizing the limitations of the Sharpe ratio and its relatives, have sought alternatives to the reward-to-variability approach. These include stochastic-discount factor based performance measures (c.f. Chen and Knez (1996)) and more direct measures of active management skill (c.f. Grinblatt and Titman (1992)). The literature on performance evaluation is a large one (c.f. Brown, 2000 reference website), and much of it has focused on the limitations of standard measures. However, de-

[1]For a review of its history and use, see Sharpe (1994). For a current textbook discussion and applications of the Sharpe ratio, see for example, Bodie, Kane and Marcus (1999) p. 754-758, and back endsheet. For applications in the mutual fund industry, see Morningstar (1993) p.24

[2]See Cochrane and Saa- Requeno (1999) for a discussion of the application of Sharpe ratios to current asset pricing research.

[3]To address this problem they propose a semi-parametric alternative biased on the gain-loss ratio that, in effect discards the information in the tails of the distribution.

spite twenty years of academic understanding of the problems of benchmarking and performance measurement, the Sharpe ratio and its relatives remain fundamental tools in research and practice.

In this paper we take a different approach to the limitations of reward-to-variability measures. Rather than pointing out their limitations and proposing alternatives, we identify a class of strategies that maximize these performance measures, without requiring any manager skill. We derive rules for achieving the maximum Sharpe ratio when the manager has the freedom to take positions in derivative securities, and when the manager has a given history of returns. Our analysis shows that the best static manipulated strategy has a truncated right tail and a fat left tail. The optimal strategy involves selling out-of-the-money calls and selling out-of-the-money puts in an uneven ratio that insures a regular return from writing options and a large exposure to extreme negative events. We also show that the best dynamic strategy for maximizing the Sharpe ratio involves leverage conditional upon underperformance.

The results have a number of implications for investment management. Interest in alternative investments has grown dramatically in the past decade. Hedge funds in particular have attracted interest by institutional managers and high net worth individuals. Hedge funds have broad latitude to invest in a range of instruments including derivative securities. Mitchell and Pulvino(2001) documents that merger arbitrage, a common hedge fund strategy, generates returns that resemble a short put-short call payoff. Recent research by Agarawal and Naik(2001) shows that hedge fund managers in general follow a number of different styles that are nonlinear in the returns to relevant indices. In a manner similar to Henriksson and Merton, Agarawal and Naik use option-like payoffs as regressors to capture these non-linearities. In fact, option-like payoffs are inherent in the compensation-structure of the typical hedge fund contract. Goetzmann,Ingersoll and Ross (2001) show that the high water mark contract - the most common in the hedge fund industry – effectively leaves the investor short 20% of an at the money call at inception, and if the fund fares poorly, this becomes an out-of-the money call position.

Although the hedge fund industry is predicted on manager skill, the non-linear nature of the payoffs to some of its most popular strategies, its lack of restrictions on use of derivatives, and its asymmetric compensation structure all make performance measurement problematic. While some have proposed advanced solutions to these problems, in this paper we identify a set of strategies that, given the freedom to invest in derivatives or dynamically rebalance, can dramatically increase most types of reward-to-risk-based performance measures. While others have shown that this class of measures is not robust to manipulation, we show how to optimally game them. As such, our analysis provides guidelines for identifying the strategies that are most subject to failures in standard performance measurement. Interestingly, they happen to conform to some well-known hedge fund strategies – M &A arbitrage being one.

A recent hedge fund scandal highlights the relevance of understanding option-based techniques for maximizing the Sharpe ratio. According to a Wall Street Journal account of 2/01/2002, Integral Investment Management, a Dallas-based hedge fund run by biologist-turned-money manager, Conrad Seghers boasted "the highest Sharpe ratio in the industry" in 1998. The secret to Seghers success appears to have been in part, a short position in out-of-the-money puts on U.S. equity indices. Quoting from the Journal account: "Mr. Seghers and museum officials recall that he said Integral would combine the investments in such a way that he could guarantee profits of 1 percent to 2 percent a month in flat or rising markets. The fund,he said, could suffer losses only if the stocks to which the options were tied dropped more than 30 percent, providing a striking degree of investor protection." As we will show in this paper, the apparent short put position of Integral Investment Management, coupled with the implicit short call position of the hedge fund incentive compensation contract fairly closely resembles the optimal Sharpe ratio contract we derive in this paper. Had the Chicago Art Institute known ex ante the basis for fund's high historical Sharpe ratio, they might not have lost nearly $43 million.

Although hedge funds are a natural industry in which to apply our analysis, the results are also relevant to more conventional asset classes. For example, Glosten and Jagannathan (1994) and Low (1999) show that small stocks returns have option-like characteristics – in particular, when measured against a large stock index, small stocks are effectively short some fraction of a put. Our analysis shows that this feature of small stock returns may enhance their apparent risk-adjusted performance compared to large stocks.

The intuition behind the solution identified in this paper is that managers can sell insurance for extreme states of nature that occur infrequently. As a result, in small sample, these insurance premia provide steady positive performance that enhances return without adding risk Mr. Seghers' "1 percent to 2 percent per month," if you will. This form of distribution may be especially susceptible to small-sample problems – it will depend crucially on whether an extreme event has or has not occurred in the sample period.

Although we derive conditions for maximizing the expected Sharpe ratio, the small sample properties may well indicate that managers with limited histories following this strategy have extraordinary high relative risk-adjusted performance. In addition, it implies that small sample might be measured in years, rather than months, because very infrequent events matter a lot to the measured performance of a manager pursuing this strategy. While most hedge fund managers have relatively short track records, our study shows that the data demands for performance evaluation are higher for such managers than for mutual fund managers or others restricted from derivatives use by regulation or charter.

More broadly, this analysis has implications for the emergence and survival of asset markets. Markets that provide fat-tailed, left-skew returns will look relatively attractive under a reward-to-risk metric, and thus may attract disproportionate interest and investment. This is true even in large sample. However, in a setting in which the existence of the asset market is conditional upon a return threshold, the attractiveness of a maximal Sharpe ratio distributions is even greater. For example, consider a market with an MSR distribution that disappears whenever the returns hit a very low lower bound. Such a market will tend to display highly positive Sharpe ratios as long as it is in existence. The documented positive Sharpe ratios of the hedge fund industry, and the short history of the industry are consistent with providing MSR-distributed returns in small sample.

The paper is structured as follows. Section 2 provides an example of how superior investment strategies may actually yield low Sharpe rations. Section 3 provides an example of how poor strategies might actually yield high Sharpe ratios. Section 4 derives the maximal Sharpe ratio under a range of conditions from complete markets to strategies constrained to a pair of strike prices and standard puts and calls. Section 5 discusses the further implications of the results and Section 6 concludes.

2 Type I Error: Low Sharpe Ratios of Great Strategies

In order to identify the optimal strategy for gaming the Sharpe ratio it helps to know what elements of a "good" strategy may lead to lower values. Why focus on this element? Because it is easy to eliminate or sell such returns and thereby artificially inflate a fund manager's apparent performance.

The Sharpe ratio contains two elements. The numerator, which is the realized return and the denominator which is the standard deviation. Thus, it is possible to trade off high returns for a lower standard deviation and potentially improve the quotient. In fact, this trade off implies that a manager may produce a remarkably small Sharpe ratio even with an undeniably winning strategy.

To see this, consider a simple, perfect-foresight equity investment strategy. Imagine an analyst who can perfectly pick firms *within* each industry, but is unable to identify which industry will perform best. Thus, for example, the analyst can tell you if General Motors will outperform Ford, but not if they will over or under-perform Intel.

The resulting strategy is quite simple: it shorts the bottom half of the firms within each industry and takes a long position in the other half. Net performance is the difference between the average buy-and-hold performance of the long portfolio minus the average buy-and-hold performance of the short portfolio over a period of one year. Being a zero-investment strategy, performance statistics are computed from the spread between the return of the long leg minus the return of the short side. Given the assumption of perfect within-industry foresight, there is no doubt that the returns of this strategy will always be positive – the only uncertainty is the variation in the positive return spread. The Sharpe ratio is the time-series mean of the net returns divided by the time-series standard deviation.[4] Because of this, the Sharpe ratio turns out to be a particularly poor measure of performance, since all the variation, in some sense, is positive. Clearly, no human being can produce such returns. However, the point here is that even somebody with such supernatural abilities will not fare very well when compared with others via the Sharpe ratio. The inability of this strategy to produce superior numbers will then help to identify ways ordinary managers can improve their numbers without improving their actual predictive abilities.

The Sharpe ratios of these "perfect" portfolios are influenced by variance across firms ("heterogeneity") and variance across years. A higher variance within industry means that the sorting ability of an analyst is more valuable. Thus, heterogeneity produces a higher mean performance. In contrast, time-series variation in heterogeneity produces greater time-series return variance, which lowers the Sharpe ratio.

Insert Table 1 Here

Table 1 presents the base case. Analysts are assumed capable of perfect sorting within 1-digit SIC codes. To qualify for inclusion in the portfolio for a given year, in December, a firm must have market capitalization in millions of dollars equal to the prevailing S&P level divided by 5.0 (this means that a firm as of the year 2000 must have a market cap above $250 million) and a stock market price of $5 or more. Each firm must have a valid return in January, available in *Research Insight* (from which all data is drawn), traded on a U.S. exchange, and not an ADR. Firms in SIC 9 were also excluded since due to the paucity of firms with this coding. The data set begins in 1981 and ends with 1999.

Of course, such returns would be an analyst's dream. The typical annual return spread for one analyst is between about 20% and 140%. The typical mean return for an individual industry is about 64%. However, the return volatilities are not

[4] When the portfolio is zero-investment, the risk-free rate is not subtracted off in computing the Sharpe ratio. A zero-investment portfolio can always be combined with a position in bonds. This increases the expected rate of return by the interest rate and does not alter the standard deviation. The Sharpe ratio can then be computed in the usual fashion. Computing Sharpe ratios in this fashion also permits us to ignore changes in the interest rate over the sample period

negligible, ranging from a low of 9% for SIC 2 (construction) to 31% for SIC 7 (finance). Thus, due to the time-series variation in sorting effectiveness, even though returns are guaranteed to be positive by the experiment, the Sharpe ratio is still a figure in the single digits. For some industries (e.g., SIC 4: utilities), even perfect foresight is not enough to have offered the analyst much more than a Sharpe ratio of 2.5! Five of our ten industry portfolios are hard-pressed to achieve a Sharpe ratio significantly above 3. If a fund were to have access to perfect analysts within each industry and allocated capital equally to each, its performance would have achieved a Sharpe ratio of only 4.26.

These "perfect portfolios" display smooth performance over the sample period, because they contain a lot of firms. Thus, it makes little difference if we compute Sharpe ratios with annual returns or with monthly returns and then annualize them.[5] For smaller portfolios with more month-to-month variation (but assuming perfect annual sorting ability), this effect would further lower the Sharpe ratio computed from monthly returns.[6]

Although the time-series variation problem is familiar to practitioners, its magnitude may not have been. In the realm of equity buy-and-hold strategies, hypothetical strategies often return Sharpe ratios between 1 and 3. Our strategy puts these numbers into perspective: the scale upon which such unmanipulated equity buy-and-hold strategies should be judged is not really minus infinity to plus infinity, but, say, -5 to $+5$.

Despite the fact that no human being can hope to replicate the perfect foresight returns from the portfolios analyzed in this section, the resulting Sharpe ratios are less than spectacular. However, the problem lies not with the strategy or its high returns, but its highest returns. If the manager could ex post discard all re-

[5]The more volatile the within-year returns, the lower are the monthly annualized Sharpe ratios relative to those computed directly from annual returns. To illustrate the point, consider a portfolio that shows (100%,−40%, 200%, −40%) forever. Thus, the mean is 55%, the standard deviation is 101.4%. At lower frequency, the portfolio would perform at (20%,80%) forever. The mean is just slightly lower (55%), the *2 period* standard deviation is 30%.

[6]Variations permitting more frequent perfect foresight, more extreme portfolios, more detailed industry abilities, or different capitalization requirements can expand the range slightly, but typically yield similar conclusions.

turns above the minimum realized annual return, then the all returns would be the same and positive so the Sharpe ratio would be infinite. Such a strategy would not work in general since, without perfect foresight, the minimum return will usually be negative, but the same general principle applies. The Sharpe ratio usually can be improved by eliminating the highest returns. This is the subject of the next section.

3 Type II Error: Using Derivatives To Maximize the Sharpe Ratio

One efficient way to truncate high returns to is through the use of derivatives. As we noted in the Chicago Art Institute example above, a strategy tied to a market for which options exist is particularly susceptible to maximizing Sharpe ratio strategies. This is particularly easy if the fund's primary purpose is index enhancement, as index options are readily available.

3.1 The Maximal Sharpe Ratio in a Complete Market

Consider first a market which is complete over all price outcomes or can be made so with dynamic trading. The standard single-period portfolio problem would be to maximize expected utility, $\sum p_i u(z_i)$, where p_i is the probability at time 0 for state i realized at time T, $u(\cdot)$ is the utility function of the investor, and z_i is the total return (not excess return) in state i. The optimal portfolio, z_i°, is characterized by

$$u'(z_i^\circ) = \theta \hat{p}_i / p_i \tag{1}$$

where $\theta = \mathbb{E}[u'(z^\circ)]$ is the Lagrange multiplier from the budget constraint and $\hat{p}_i$ is the risk-neutral probability of state i.[7]

[7]We express the budget constraint and hence the optimal portfolio in terms of the risk-neutral probabilities rather than the state prices for ease of comparison with later results. The risk-neutral probability of a state is equal to the state price multiplied by the risk-free discount factor.

Suppose instead that the investor wishes to form a portfolio with the largest possible Sharpe ratio. Any portfolio can be decomposed into a risk-free asset plus a risky zero-investment portfolio, $\tilde{z} = e^{rT} + \tilde{x}$. Alternatively, $\tilde{x}$ is the excess return on the portfolio in question. The Sharpe ratio S of the portfolio is the ratio of the expected return in excess of interest earned to the standard deviation. In terms of the zero-investment portfolio (or excess return) the Sharpe ratio is[8]

$$S = \frac{\mathbb{E}[\tilde{x}]}{\sqrt{\mathrm{Var}[\tilde{x}]}} = \frac{\sum p_i x_i}{\left[\sum p_i x_i^2 - \left(\sum p_i x_i\right)^2\right]^{1/2}} . \tag{2}$$

The Sharpe-ratio-maximizing portfolio is not uniquely defined. In fact, the Sharpe ratio is invariant to scaling by leverage so the same maximum Sharpe ratio can be achieved at any positive expected excess return $\sum p_i x_i = \bar{x} \geq 0$. Since $\bar{x} = 0$ is trivially achieved, the mean must not be negative or the Sharpe measure is clearly not maximized.

As shown in the Appendix, the maximal-Sharpe-ratio excess return in state i for a portfolio with mean excess return $\bar{x}$ is

$$\begin{aligned} x_i^* &= \bar{x} + \gamma(1 - \hat{p}_i/p_i) \\ \text{where} \qquad \gamma &= \frac{\bar{x}}{\sum \hat{p}_i^2/p_i - 1} . \end{aligned} \tag{3}$$

So the Sharpe-ratio-maximizing payoff is linear in the likelihood ratio of the risk-neutral probability to the true probability (or state price per unit probability).

The variance of the "optimized" portfolio is $\sum p_i (x_i^* - \bar{x})^2 = \gamma \bar{x}$. Therefore, the square of the maximal Sharpe ratio is

$$S_*^2 = \frac{\bar{x}^2}{\gamma \bar{x}} = \sum \hat{p}_i^2/p_i - 1 . \tag{4}$$

[8]Equation (2) and the following results give the true Sharpe ratio of the portfolio. In any period, the sample Sharpe ratio may differ from this population value. Its distribution and small sample properties will depend on the probability distribution of the states.

This final sum can also be expressed as $\sum p_i(\hat{p}_i/p_i)^2$ so the maximal Sharpe ratio is one less than the expectation of the square of the realized likelihood ratio considered as a random variable on the states.

If the number of states is finite or the probability likelihood ratio, $\hat{p}_i/p_i$, is otherwise bounded above across states, then the maximal Sharpe ratio can be achieved with a limited liability portfolio. The smallest excess return is

$$x^*_{\min} \equiv \min_i x^*_i = \bar{x} + \gamma \left[1 - \max_i (\hat{p}_i/p_i) \right] . \tag{5}$$

Since γ is proportional to $\bar{x}$, so is this minimal excess return. Therefore, by setting the mean excess return at a sufficiently low level, the smallest total return, $x^*_{\min} + e^{-rT}$ can be made positive and limited liability is achieved. If there are infinitely many states and the probability likelihood ratio is unbounded, then a limited liability portfolio with a Sharpe ratio arbitrarily close to the maximal value can be formed by holding a portfolio with excess return $-e^{-rT}$ (i.e., total return of zero) in the states with the highest ratios and excess returns proportional to those given in (3) for the states with lower ratios. Again by setting the mean excess return to a sufficiently small number, the fraction of states with a zero return can be made as small as desired and the resulting Sharpe ratio will be arbitrarily close to that achieved with an unconstrained portfolio.

We now examine the properties of the maximal-Sharpe-ratio portfolio. From (3) and (4), the return on the maximal-Sharpe-ratio portfolio in state i is

$$x^*_i = \bar{x} + \frac{\bar{x}}{S^2_*} \left(1 - \hat{p}_i/p_i \right) . \tag{6}$$

So for the maximal Sharpe ratio portfolio, the return in each state deviates from the expected return by an amount proportional to the difference of the probability likelihood ratio $\hat{p}_i/p_i$ from 1. States with risk-neutral probability exceeding their true probability will have smaller than average returns and vice versa. The larger the deviation between the risk-neutral and true probabilities, the greater the difference from the mean return.

Comparing this portfolio to the solution to the standard problem in (1), we see that[9]

$$x_i^* = \bar{x} - \frac{\bar{x}}{S_*^2}\left(\frac{u'(z_i^\circ)}{\mathbb{E}[u'(z^\circ)]} - 1\right) . \tag{7}$$

Therefore, the excess return on the maximal Sharpe ratio portfolio differs from the average return by an amount proportional to the difference between the realized marginal utility and the expected marginal utility for an optimally invested portfolio. In particular, it is monotonically decreasing in the marginal utility. Since utility is concave, x_i^* is also monotonically increasing (but not linear) in z_i°. For typical utility functions with $u'''(\cdot) > 0$, x^* will be concave in z°. That is, the total return including interest on the maximal-Sharpe-ratio portfolio will exceed z_i° in the midportion of the outcomes and fall short of z_i° for very good or very bad outcomes or, usually, both.

All of these comparisons are meaningful only if the optimal portfolio for the standard problem is identified. Note, however, that we do not actually require a complete market in the Arrow Debreu sense to do this analysis. Take any portfolio or asset with a particular pattern of returns to use as a basis. Next, determine the maximal-Sharpe-ratio portfolio which can be constructed by trading a complete set of derivative claims contingent on it. By using a basis asset as a benchmark it now becomes possible to derive other standard performance measures.

Let B denote the excess return on a benchmark index. The covariance of the Sharpe-ratio-maximizing portfolio with the benchmark is

$$\begin{aligned} \mathrm{Cov}[x^*, B] = \mathbb{E}[(x^* - \bar{x})B] &= \sum p_i \left[-B_i \frac{\bar{x}}{S_*^2}\left(\hat{p}_i/p_i - 1\right)\right] \\ &= \frac{\bar{x}}{S_*^2} \sum \left(p_i B_i - \hat{p}_i B_i\right) = \frac{\bar{x}}{S_*^2}\bar{B} \end{aligned} \tag{8}$$

[9]The utility function used here can be any for which the standard problem has an "interior" solution.

so the beta and alpha of the maximal-Sharpe-ratio portfolio are

$$\begin{aligned}\beta_* &= \frac{\mathrm{Cov}[x^*, B]}{\mathrm{Var}[B]} = \frac{\bar{x}\bar{B}}{S_*^2 \mathrm{Var}[B]} = \frac{\bar{x}}{\bar{B}} \frac{S_B^2}{S_*^2} \\ \alpha_* &= \bar{x} - \beta_* \bar{B} = \bar{x}\left(1 - S_B^2/S_*^2\right)\end{aligned} \tag{9}$$

where S_B is the Sharpe ratio of the benchmark. Of course, the alpha is subject to severe manipulation. It can be made as large as desired by levering the portfolio to increase it's mean return.[10]

Other performance measures related to the Sharpe ratio have also been proposed. Modigliani and Modigliani's [1997] M-squared is measured relative to some benchmark, B, usually an index like the S&P 500. It is the expected rate of return that would be earned on a portfolio if it were levered so its standard deviation were equal to that on the benchmark.

$$M^2 \equiv \frac{\bar{x}}{\sqrt{\mathrm{Var}[x]}} \sqrt{\mathrm{Var}[B]} + e^{rT} = S_x \sqrt{\mathrm{Var}[B]} + e^{rT} \,. \tag{11}$$

Clearly maximizing the Sharpe ratio also maximizes the M-squared measure for any benchmark.

Sharpe's [1981] information ratio is the ratio of the excess return to the standard deviation measured relative to some risky benchmark in place of the risk-free asset. The information ratio for a portfolio with excess returns, x with respect to a benchmark with excess return B is defined as

$$I \equiv \frac{\mathbb{E}[x - B]}{\sqrt{\mathrm{Var}[x - B]}} \,. \tag{12}$$

[10]This is always true and not a particular problem with this structure. The Treynor measure, $T \equiv \alpha/\beta$, was introduced to avoid this problem. Like the Sharpe measure, the Treynor measure is unaffected by leverage. The Treynor measure for the maximal Sharpe ratio portfolio is

$$T_* = \alpha_*/\beta_* = \left(S_*^2/S_B^2 - 1\right)\bar{B}$$

which does not depend on the mean $\bar{x}$. Of course, the Treynor measure has its own manipulation problem. It can be made as large as desired by reducing the beta.

This is just the Sharpe ratio for the quantity $x - B$ instead of the excess return x alone. Since both the excess return x^* and the benchmark excess return B can be purchased at a zero cost, so can the portfolio with excess return $x^{**} = x^* + B$. Clearly this latter portfolio has the maximal information ratio which is equal in value to the maximal Sharpe ratio derived earlier, $I_* = S_*$.

The next section analyzes this problem in a exponential-normal model where the mean-variance analysis is usually justified.

3.2 The Maximal Sharpe Ratio for a Normal Benchmark

In this section, we analyze the maximal Sharpe ratio problem for a benchmark (market) portfolio with a normally distributed return. Let ξ denote the return on a portfolio which is to be used as a comparison basis. This portfolio might be taken to be the market portfolio, but it need not be. We assume that any derivative asset based on ξ can be traded, so the market can be completed with respect to states defined over outcomes of ξ.

For a continuous state space, analysis similar to that leading to (4) yields a maximal squared Sharpe ratio of

$$S_*^2 = \int \hat{p}^2(\xi)/p(\xi)\mathrm{d}\xi - 1 = \mathbb{E}\left[\hat{p}^2(\xi)/p^2(\xi)\right] - 1 \tag{13}$$

which is achieved by a portfolio with an excess return of

$$x^*(\xi) = \bar{x} + y[1 - \hat{p}(\xi)/p(\xi)] \tag{14}$$

where $y = \bar{x}/S_*^2$ and p and $\hat{p}(\xi)$ are the true and risk-neutral probability densities for the state space.

The likelihood ratio $\hat{p}(\xi)/p(\xi)$ can be determined as in (1) from the utility function of the representative investor who holds the benchmark

$$\hat{p}(\xi)/p(\xi) = u'(\xi)/\mathbb{E}[u'(\xi)]\,. \tag{15}$$

The typical conjugate assumption for a normal distribution is exponential utility. In this case the probability likelihood ratio is[11]

$$\hat{p}(\xi)/p(\xi) = \exp\left[-\xi\frac{\mu - R}{\sigma^2} + \frac{\mu^2 - R^2}{2\sigma^2}\right]. \tag{16}$$

The square of the maximal Sharpe ratio is[12]

$$S_*^2 = \exp\left[\left(\frac{\mu - R}{\sigma}\right)^2\right] - 1 = \exp(S_\xi^2) - 1 \tag{17}$$

where S_ξ is the Sharpe ratio of the benchmark index. The maximal Sharpe ratio clearly exceeds the benchmark's Sharpe ratio, and the larger the benchmark's Sharpe ratio, the larger is the difference. For example, the annual Sharpe ratio of the S&P 500 index was 0.450 from 1926 – 2000. Assuming normality, the maximal Sharpe ratio is 0.474, which is 5% higher. For a benchmark Sharpe of 0.6, the maximal Sharpe ratio is 0.658, almost 10% higher.

The final column in Table **??** shows the maximal Sharpe ratio that can be obtained for normally distributed returns. For example, for an index Sharpe ratio of 1.00, the maximal Sharpe ratio is 1.31.

The return on the Sharpe-ratio-maximizing portfolio deviates substantially from normal. The return is

$$x^*(\xi) = \bar{x}\left[1 + \frac{1 - \exp\left(-\xi\frac{\mu-R}{\sigma^2} + \frac{\mu^2-R^2}{2\sigma^2}\right)}{1 + \exp\left[(\mu - R)^2/\sigma^2\right]}\right]. \tag{18}$$

This is increasing in the index, but it is bounded above by $x^*_{\max} \equiv \bar{x}(1 + [1 + \exp(S_*^2)]^{-1})$ and unbounded below. The return has a "reflected" lognormal distribution; i.e., $x^*_{\max} - x^*$ is lognormally distributed.

[11]For exponential utility, $u'(\xi) = e^{-a\xi}$, the representative investor will hold the benchmark portfolio unlevered for $a = (\mu - R)/\sigma^2$. For this investor, $\mathbb{E}[u'(\xi)] = \exp[-(\mu^2 - R^2)/\sigma^2]$.

[12]Recall that if z is normally distributed with mean $\bar{z}$ and standard deviation σ_z, then $\mathbb{E}[e^{\alpha z}] = \exp(\alpha\bar{z} + \frac{1}{2}\alpha^2\sigma_z^2)$.

3.3 The Maximal Sharpe Ratio for a Lognormal Basis

This section analyzes the maximal Sharpe ratio problem in a continuous-state lognormal environment. This permits the use Black-Scholes option-pricing techniques to determine the risk-neutral probabilities which yields specific numerical results.

Assume that ξ has a lognormal distribution with an instantaneous expected rate of return of μ and a logarithmic variance of σ^2 per unit time. The risk-neutral probability distribution is the same with μ replaced by r, the continuously-compounded interest rate. As shown in the Appendix, the maximal Sharpe ratio for a lognormal basis is[13]

$$S_* = \left(\exp\left[\frac{(\mu - r)^2}{\sigma^2} T \right] - 1 \right)^{1/2} . \tag{19}$$

The maximal Sharpe ratio is increasing in the absolute value of the instantaneous risk premium of the basis, $\mu - r$, and decreasing in its standard deviation. Increasing both the risk premium and the standard deviation proportionally leaves the maximal Sharpe ratio unchanged. The maximal Sharpe ratio is also increasing in the investment horizon, T, because lengthening the horizon is equivalent to increasing the risk premium and the variance (not the standard deviation) proportionally.[14]

The Sharpe ratio of the basis alone is

$$S_\xi = \frac{e^{\mu T} - e^{rT}}{e^{\mu T}\sqrt{e^{\sigma^2 T} - 1}} = \frac{1 - e^{-(\mu - r)T}}{\sqrt{e^{\sigma^2 T} - 1}} . \tag{20}$$

This ratio is also increasing in the risk premium and decreasing in the standard deviation of the index. For realistic parameter values, it is increasing in the investment horizon for T up to about ten years: a longer horizon than will be used in most conceivable circumstances.[15]

[13] This formula is very similar to (17). However, $(\mu - r)/\sigma$ is not the Sharpe ratio of the benchmark in this case since μ, r, and σ are the instantaneous parameters. The Sharpe ratio of the index is given below.

[14] The investment horizon T is the interval over which one return is measured not the entire period of data. For example, if five years of monthly returns are used to compute the average and standard deviation, then the horizon is one month and not five years.

[15] T represents the investment horizon or rebalancing interval of the decision maker. It is commonly assumed that the Sharpe measure is proportional to the square root of this horizon. How-

Insert Table 2 Here

Table 2 shows the Sharpe ratio for the basis and for the optimized portfolio for various parameter values. The maximum improvement beyond that available on the basis is larger the larger is the Sharpe ratio of the basis itself. So a higher risk premium or a smaller volatility on the basis allows for a greater percentage manipulation of the Sharpe ratio. Also substantially more improvement is possible over a one-year horizon than over a monthly horizon. This result follows because the maximal Sharpe ratio increases at a faster rate than the basis Sharpe ratio with the investment horizon.

To put these numbers into context, consider the expected rate of return required by the index for it to produce a Sharpe ratio equal to S_*. Set the index Sharpe ratio in (20) equal to the maximal Sharpe ratio and solve for μ to determine the "apparent" return on the maximal-Sharpe-ratio portfolio

$$\mu_{\text{app}} \equiv r - \frac{1}{T} \ln\left(1 - S_* \sqrt{e^{\sigma^2 T} - 1}\right) \tag{22}$$

For example, if the basis risk premium of $\mu - r$ equals 10% and the volatility 20%, the 13.1% improvement in the annual Sharpe ratio is apparently equivalent to an extra return of 146 basis points on the index. The other values for μ_{app} are given in the table. These are economically meaningful numbers; however, we have been careful to label them apparent out-performance because there is no actual out-performance implied.

ever, this is precisely true only if the expected return and variance are proportional to the interval. Because it is the logarithmic variance and the continuously-compounded expected rate of return which are proportional to the horizon, the index and maximal Sharpe ratios actually grow slower and faster than the square root of the time interval, respectively, as can be seen from a Taylor expansion of (20) and (19):

$$S_\xi \approx \frac{\mu - r}{\sigma}\sqrt{T}\left(1 - \frac{\mu - r + \sigma^2}{2} T + \cdots\right)$$

$$S_* \approx \frac{\mu - r}{\sigma}\sqrt{T}\left(1 + \frac{(\mu - r)^2}{2\sigma^2} T + \cdots\right).$$

The excess return x^* on the maximal Sharpe ratio portfolio is from (14) and (48)

$$x^*(\xi) = \bar{x} + \gamma \left(1 - \exp\left[\frac{\mu - r}{2}\left(\frac{\mu + r}{\sigma^2} - 1\right) T\right] \xi^{-(\mu - r)/\sigma^2}\right) . \tag{23}$$

This payoff is illustrated in Figure 1 for $\mu = 15\%$, $r = 5\%$, $\sigma = 15\%$, and $T = 1$. The return shown has the same expected value as that on the index, i.e., $\bar{x} = e^{\mu T} - e^{rT}$. Note that the maximal-Sharpe-ratio return is substantially less than that on the index in both tails. The maximal-Sharpe-ratio return exceeds that on the index only over the range from about -4% to 25%. Of course, this central range does account for nearly 60% of the probability distribution.

Insert Figure 1 Here

Because ξ is lognormally distributed, ξ to any power is as well, and $x^*(\xi)$ will have a translated and reflected lognormal distribution.

$$p_x\left(x^*(\xi)\right) = \frac{\delta}{\bar{x} + \gamma - x^*} \frac{1}{\sigma\sqrt{T}} \phi\left(\frac{-\delta \cdot \ell\text{n}\left(\frac{\bar{x}+\gamma-x^*}{\gamma}\right) + \frac{r-\mu}{2}T}{\sigma\sqrt{T}}\right)$$
$$\text{where} \qquad \delta \equiv \frac{\sigma^2}{\mu - r} . \tag{24}$$

The maximal Sharpe ratio return has a reflected lognormal distribution bounded above by $\bar{x} + \gamma$ and with an infinite left rather than right tail.

Insert Figure 2 Here

The distribution of returns on this portfolio is shown in Figure 2 along with that of the index. Again, they are constructed to have the same expected payoff, and the parameters used are $\mu = 15\%$, $r = 5\%$, $\sigma = 15\%$, $T = 1$. The variance of the return on the optimal portfolio will, of course, be smaller than that on the index by the factor $(S_\xi/S_*)^2$.

$$\text{Var}[x^*(\xi)] = \frac{S_\xi^2}{S_*^2}\text{Var}[\xi] = \frac{S_\xi^2}{S_*^2} e^{2\mu T}\left(e^{\sigma^2 T} - 1\right) . \tag{25}$$

The most obvious feature of the distribution is its long left tail giving rise to negative skewness and high kurtosis.

Because both distributions are lognormal, all other moments are determined by the means and variances. In particular the normalized third and fourth moments are

$$\frac{\mathbb{E}[(\xi-\bar{\xi})^3]}{(\mathrm{Var}[\xi])^{3/2}} = (\omega_\xi+2)\sqrt{\omega_\xi-1} \qquad \frac{\mathbb{E}[(\xi-\bar{\xi})^4]}{(\mathrm{Var}[\xi])^{2}} - 3 = \omega_\xi^4 + 2\omega_\xi^3 + 3\omega_\xi^2 - 3$$

$$\frac{\mathbb{E}[(x^*-\bar{x}^*)^3]}{(\mathrm{Var}[x^*])^{3/2}} = -(\omega_x+2)\sqrt{\omega_x-1} \quad \frac{\mathbb{E}[(x^*-\bar{x}^*)^4]}{(\mathrm{Var}[x^*])^{2}} - 3 = \omega_x^4 + 2\omega_x^3 + 3\omega_x^2 - 3 \tag{26}$$

where $\omega_\xi \equiv e^{\sigma^2 T}$ and $\omega_x \equiv e^{\sigma^2 T/\delta^2}$.

Table 3 shows the skewness and kurtosis of the maximal Sharpe ratio portfolio and the basis portfolio for basis volatilities of $\sigma = 15\%$, 20%, and 25% and horizons of one month and one year.

Insert Table 3 Here

The equation for the maximal Sharpe ratio and the maximizing portfolio are completely general; however, many of the specific results here depend on the log-normal distribution assumption. For example, suppose instead the index has a log-Laplace distribution, $p(\xi) = (2\beta\xi)^{-1}\exp(-|\ln\xi - \alpha|/\beta)$. In this case, the maximal-Sharpe-ratio portfolio is

$$x^*(\xi) = \bar{x} + \frac{\bar{x}}{S_*^2}\left[1 - \exp\left(-\frac{|\ln\xi-\hat{\alpha}| - |\ln\xi-\alpha|}{\beta}\right)\right] \tag{27}$$
$$\text{where} \qquad S_*^2 = \tfrac{2}{3}\exp\left[(\alpha-\hat{\alpha})/\beta\right] + \tfrac{1}{3}\exp\left[-2(\alpha-\hat{\alpha})\beta)\right] - 1$$

where $\hat{\alpha}$ is the risk-neutral value of the location parameter α.[16]

[16]The logarithmic variance of the return is $\mathrm{Var}[\ln\xi] = 2\beta^2$. The continuously compounded expected rate of return is $\mu = \frac{1}{T}[\alpha - \ln(1-\beta^2))$. Therefore, the risk-neutral mean parameter is $\hat{\alpha} = rT + \ln(1-\beta^2)$ so $\alpha - \hat{\alpha} = (\mu - r)T$.

As with lognormality, the maximal-Sharpe-ratio is increasing in the risk-premium and decreasing in the logarithmic variance of the basis. Also the return on the maximal-Sharpe-ratio portfolio is increasing in the return on the index and has an upper bound. However, in this case, the upper bound is achieved and the returns are equal for all $\xi \geq e^{\alpha}$. A more striking difference is that this maximal-Sharpe-ratio portfolio also has a lower bound, and its payoff is again constant for all $\xi \leq e^{\hat{\alpha}}$. So for a log-Laplace distribution, the maximal-Sharpe-ratio portfolio eliminates both tails of the distribution.

3.4 Maximizing the Sharpe Ratio With One Call and One Put

In practice a money manager may not be able to construct the Sharpe-ratio-maximizing portfolio because a complete market in contingent claims does not exist or because it may be too expensive to trade or dynamically create too many options. However, even if the manager is allowed to trade only *one or two* ordinary put and call options, he can significantly enhance his Sharpe ratio. Furthermore, these two options will typically be liquid (near-the-money) options.

Suppose a money manager invests \$1 in the index, purchases κ European puts with a strike of K and sells η European calls with the strike of H ($H > K$) to create the following simple linear return pattern

$$P = \begin{cases} \xi + \kappa(K - \xi) & \xi \leq K \\ \xi & K < \xi < H \\ \xi - \eta(\xi - H) & H \leq \xi \,. \end{cases} \tag{28}$$

Many interesting patterns are included here. For example, writing covered calls is equivalent to $\kappa = 0$ and $\eta = 1$. Buying portfolio insurance is equivalent to $\kappa = 1$ and $\eta = 0$. Partial write programs, partial insurance, and combinations are also included.

What are the mean, variance, and Sharpe ratio of the portfolio P? The non-central moments of the distribution of the portfolio's payoff are

$$\mathbb{E}[P^s] = \int_0^K [\xi + \kappa(K-\xi)]^s p(\xi)\, d\xi + \int_K^H \xi^s p(\xi)\, d\xi + \int_H^\infty [\xi - \eta(\xi - H)]^s p(\xi)\, d\xi \,. \quad (29)$$

Using (53) and (54) for $z = (\mu - \frac{1}{2}\sigma^2)T$ and $v^2 = \sigma^2 T$, we have

$$\begin{aligned}\mathbb{E}[P] = {} & \kappa K \Phi(-h_K^-) + (1-\kappa)e^{\mu T} \cdot \Phi(-h_K) + e^{\mu T} \cdot [\Phi(h_K) - \Phi(h_H)] \\ & + H\eta\Phi(h_H^-) + (1-\eta)e^{\mu T} \cdot \Phi(h_H)\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}[P^2] = {} & \kappa^2 K^2 \Phi(-h_K^-) + 2\kappa(1-\kappa)Ke^{\mu T} \cdot \Phi(-h_K) + (1-\kappa)^2 e^{(2\mu+\sigma^2)T} \cdot \Phi(-h_K^+) \\ & + e^{(2\mu+\sigma^2)T} \cdot \left[\Phi(h_K^+) - \Phi(h_H^+)\right] + \eta^2 H^2 \Phi(h_H^-) \\ & + 2\eta(1-\eta)He^{\mu T} \cdot \Phi(h_H) + (1-\eta)^2 e^{(2\mu+\sigma^2)T} \cdot \Phi(h_H^+)\end{aligned}$$

where

$$h_Z \equiv \frac{-\ell\text{n}Z + \left(\mu + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \quad h_Z^{\pm} \equiv h_Z \pm \sigma\sqrt{T} \,. \quad (30)$$

The Sharpe ratio is

$$S = \frac{\mathbb{E}[P] - P_0 e^{rT}}{\sqrt{\mathbb{E}[P^2] - \mathbb{E}^2[P]}} \quad (31)$$

where P_0 is the initial value of the portfolio.

The initial value of the portfolio is

$$P_0 = 1 + \kappa\mathcal{P}(1, T; K) - \eta C(1, T; H) \quad (32)$$

where $\mathcal{P}(\cdot)$ and $C(\cdot)$ are the formulas for a put and a call; in this case the Black-Scholes formulas. The Sharpe ratios for portfolios holding just one option are given by setting κ or η to zero. The Sharpe ratios for portfolios holding options at more than two strikes can be computed similarly. However, as shown below, only one or two options are required to achieve most of the possible improvement in the Sharpe ratio.

Insert Figure 3 Here

Figure 3 shows the Sharpe ratio for using a single call in various combinations with the basis for parameter values $r = 5\%$, $\mu = 15\%$, $\sigma = 15\%$, $T = 1$. For these parameters the Sharpe ratio for the stock is 0.631. By selling 0.843 calls at a strike of 1.0098,[17] the Sharpe ratio can be pushed to 0.731. Using two strikes allows an improvement of the Sharpe ratio to 0.743. This portfolio is characterized by $\kappa = -2.58$, $K = 0.88$, $\eta = 0.77$, $H = 1.12$. The maximal Sharpe ratio is 0.748, so 86% of the total possible increase in the Sharpe ratio can be achieved with one option contract and 96% can be achieved with just two option contracts.

The improvement in the Sharpe ratio is not critically sensitive to the exact value of the strike price. For example, a Sharpe ratio of 0.716 or 0.694 can be achieved by using a call which is 5% in- or out-of-the-money in place of the best single call with a strike 1% in-the-money. A Sharpe ratio of 0.737 can be achieved using both of these options. Near-the-money options are very liquid and seldom is the strike price gap as large as 10%. Therefore, simple puts and calls should be able to provide most of the improvement possible in the Sharpe ratio.

Insert Figure 4 Here

Figure 4 plots the payoff on the put-call-stock portfolio and compares it to that on the maximal Sharpe ratio portfolio.[18] The distributions are similar in many respects

[17]Selling 5.36 puts at the same strike gives identical results. By put-call parity holding a share and buying κ puts is equivalent to holding a share and buying a portfolio long κ calls, short κ shares, and long bonds. The net position is κ calls and $1-\kappa$ shares. Eliminating the leverage which does not affect the Sharpe ratio gives $\kappa/(1-\kappa)$ calls for each share. Because $-5.36/(-5.36-1) = 0.843 = \eta$, the positions are equivalent.

[18]The illustrated best put and call portfolio does not appear to be a "best" fit for the curve describing the maximal Sharpe ratio portfolio for the simple reason it is not best fit for the illustrated curve.

though the returns on the option portfolio are larger for both very high and low returns on the basis. This means that the option portfolio has less negative skewness and typically more kurtosis than the maximal Sharpe ratio portfolio.

3.5 Dynamic Optimization (Existing Return History)

Once the measurement period has begun, leverage can be used to further enhance the recorded Sharpe ratio. Consider a money manager who has an existing history of returns with realized average excess return, $\bar{x}_h$ and Sharpe ratio, S_h. He wishes to maximize the Sharpe ratio measured from these and future returns. Since the variance can be expressed as a function of the Sharpe ratio and mean excess return, the total-period Sharpe ratio is

$$S = \frac{\alpha \bar{x}_h + (1-\alpha)\bar{x}_f}{\sqrt{\alpha \bar{x}_h^2 (1 + S_h^{-2}) + (1-\alpha)\bar{x}_f^2 (1 + S_f^{-2}) - [\alpha \bar{x}_h + (1-\alpha)\bar{x}_f]^2}} \tag{33}$$

where $\bar{x}_f$ and S_f are the expected excess return and Sharpe ratio in the future and α is the fraction of the total period which has passed.

From (33), the entire-period Sharpe ratio is monotonic in S_f so the maximal-Sharpe-ratio portfolio should always be employed, $S_f = S_*$. The only remaining question is how much leverage should be used.

Insert Figure 5 Here

Figure 5 illustrates the leverage problem for various conditions. The parameters are $\alpha = 0.4, S_* = 0.6, \sigma_h = 15\%$. The five curves show the entire-period Sharpe ratio plotted against the expected excess return in the future for historical average excess returns of $\bar{x}_h = -2\%$, 2%, 6%, 10%, and 14%.

As illustrated for $\bar{x}_h = -2\%$, extreme leverage ($\bar{x}_f \to \infty$) should be employed whenever the history has a negative average excess return. Fortunately, the Sharpe ratio does not depend strongly on the expected excess return for high values, so in practice the leverage need not be extreme to reap most of the benefits.

If the return history has a positive average excess return, the proper strategy is not to eliminate all excess returns in the future even if $\bar{x}_h$ and S_h are very large. Doing so would lower the entire-period mean and usually increase the variance. As shown in the Appendix, the portfolio should be levered so the expected excess return in the future is

$$\bar{x}_f^* = \bar{x}_h \frac{1 + S_h^{-2}}{1 + S_*^{-2}} \,. \tag{34}$$

This equation says the expected return in the future should be set near the historical mean. In fact, if the historical Sharpe ratio equals the maximal Sharpe ratio, the portfolio should be levered so its expected excess return is equal to the historical average excess return. To do otherwise increases the variance and thereby reduces the entire-period Sharpe ratio since the variance is measured as the squared deviations around the weighted mean.

Note that in Figure 5 the historical standard deviation is held constant at 15% across the different $\bar{x}_h$ curves rather than the Sharpe ratio. Therefore, as the historical average increases from 2% to 14%, the historical Sharpe ratio rises from 0.133 to 0.933. This accounts for the decrease in the optimal leverage. The historical returns have different weight in the entire period standard deviation and expected return so increasing the leverage increases the former at a faster rate. This means that with a higher past Sharpe ratio, using leverage to lower the entire period variance gives more benefit than increasing the expected excess return.

This strategy achieves an entire-period Sharpe ratio of

$$S^2 = \begin{cases} \dfrac{(1-\alpha)S_*^2}{1+\alpha S_*^2} & \text{for} \quad S_h \le 0 \\[2ex] \dfrac{\alpha S_h^2(1+S_*^2) + (1-\alpha)(1+S_h^2)S_*^2}{1+(1-\alpha)S_h^2 + \alpha S_*^2} & \text{for} \quad S_h > 0 \,. \end{cases} \tag{35}$$

Insert Figure 6 Here

Figure 6 illustrates the Sharpe ratio that can be achieved with the proper strategy. The maximal Sharpe ratio in the future is 0.6. The entire-period Sharpe ratio is, of course, increasing in the historical Sharpe ratio. If the historical Sharpe ratio exceeds the maximal Sharpe ratio, then the entire-period Sharpe ratio is decreasing in α since this good result will have more of an impact on the entire-period Sharpe ratio. The converse also holds.

The entire-period Sharpe ratio is constant for $S_h < 0$ and nearly linear for Sharpe ratios above 0.3 In fact, the plot is very much like the price of an option. Furthermore, by analogy with option pricing, a strategy which produces a volatile Sharpe measure early on will therefore be a desired one. The money manager can recover from a bad performance by using leverage in the future. If he has a good performance early in the evaluation period, he can be more conservative.

4 Implications

Our analysis has direct, practical implications for regulation, performance auditing and agency contracting. In this sense it relates to the growing literature on agency in money management (c.f. Chevalier and Ellison (1997) and Carpenter (2000) and Goetzmann, Ingersoll and Ross (2001)) . In settings in which the Sharpe ratio is used explicitly or implicitly for benchmarking, the use of options, or dynamic replication of derivative payoffs, should be may need to be constrained. Otherwise managers may take actions that may not coincide with their investors' interests. Further, it may pay those allocating assets to compare the distribution of high Sharpe ratio managers with those that can be obtained via an optimal manipulation strategy. In settings for which the use of options is unconstrained, asymmetric performance contracts similar to those used in the hedge fund industry appear to mitigate certain moral hazard problems raised by the use of Sharpe ratios.

The analysis presented here also has applications for the use of Sharpe ratios in asset pricing. Low (1999) finds that large a class of U.S. equities have asymmetric exposure to the index. Glosten and Jagannathan (1994) liken this structural relationship to a derivative-based strategy. In effect, some assets in the U.S. market, primarily small cap stocks, behave as if they are short a put. Our analysis shows that in this case, the Sharpe and Information ratios are potentially biased measures of the attractiveness of an investment.

There may be corporate finance implications for these results as well. To the extent that a corporate manager is evaluated against an explicit benchmark, our strategy shows that he or she has an incentive to choose a capital structure that mimics the payoff of the maximal Sharpe ratio. In the corporate setting, this would mean simultaneously issuing out-of-the-money call warrants and put warrants in a particular proportion. The former is common for certain types of firms. The latter is rare, but not unknown. Our paper provides at least one explanation for the existence of put warrants. In fact, even in settings where the corporate manager is evaluated not on stock returns but on the risk-scaled deviations of corporate earnings against a contemporaneous benchmark, our analysis suggests managers will smooth out large positive income realizations, while recognizing large negative hits.

The results have implications for dynamic portfolio management. Brown, Harlow and Starks (1996) show that mutual fund managers increase variance after a poor showing in the first half of the year. While Jeffrey Busse (1999) disputes this evidence using daily data, our results in this paper suggest that this dynamic behavior is consistent with maximizing the Sharpe ratio. If our conjecture about hedge fund compensation is correct, we would expect to find less dynamic gaming in an industry with asymmetric contracts. Brown, Goetzmann and Park (2000) find some evidence that hedge fund managers increase volatility when they underperform other funds, but not when they underperform a fixed benchmark; thus, the existing empirical evidence is mixed.

Our analysis also has a number of subtle implications concerning the timing of reporting and performance measurement. In particular, the longer the reporting horizon, the more freedom the manager has to discard or shift high returns. For example, for a fund with a monthly Sharpe ratio of 0.6, the fund only wants to discard monthly returns more than 1.54 standard deviations above the mean, and only 5% of the fund's return are wasted. However, if the same fund's performance was measured directly from its annual returns, the fund would want to discard annual returns above –1.14 standard deviations above the mean, and over 60% of the fund's mean return would be wasted.

The shape of the optimal Sharpe ratio leads to further conjectures. Expected returns being held constant, high Sharpe ratio strategies are, by definition, strategies that generate regular, modest profits punctuated by occasional crashes. Our evidence suggests that the "peso problem" may be ubiquitous in any investment management industry that rewards high Sharpe ratios.

5 Conclusion

This paper focuses on methods to manipulate portfolio returns to achieve high Sharpe ratios and related measures. It derives the optimal strategy under certain conditions and shows that the payoff structure resembles a portfolio that is short different fractions of out-of-the-money puts and calls, such that the fund distribution is left skewed. This result poses problems in the measurement and monitoring of investment funds and perhaps corporations in general because it distorts manager incentives. Some distortion may be mitigated by restricting the use of derivatives in the portfolio. In unconstrained settings, however, it may be wiser change incentives to asymmetrically reward managers based upon upside performance, however giving a 20 percent call on the fund to the hedge fund manager further distorts reward-to-risk based performance measurement. This compensation structure has evolved in the hedge fund universe, where portfolio composition is not monitored, but Sharpe ratios and Information ratios are widely used.

A Maximal-Sharpe-Ratio Portfolio in a Complete Market

Consider a portfolio with excess return, x_i in state i. The probability of state i is p_i. The Sharpe ratio of this portfolio is

$$S = \frac{\sum p_i x_i}{\left[\sum p_i x_i^2 - \left(\sum p_i x_i\right)^2\right]^{1/2}} . \tag{36}$$

This is invariant to scaling so with no loss of generality we can fix the expected excess payoff at any nonnegative value $\sum p_i x_i = \bar{x} \geq 0$. Then maximizing the Sharpe ratio of excess returns is equivalent to minimizing the mean squared payoff subject to an expected payoff of $\bar{x}$ with a cost of 0.

Form the Lagrangian[19]

$$\mathcal{L} = \frac{1}{2} \sum p_i x_i^2 + \lambda \left(\bar{x} - \sum p_i x_i\right) + \gamma \left(\sum \hat{p}_i x_i\right) . \tag{37}$$

The first-order conditions for a minimum are

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial x_i} = p_i x_i^* - \lambda p_i + \gamma \hat{p}_i \\ 0 &= \frac{\partial \mathcal{L}}{\partial \lambda} = \bar{x} - \sum p_i x_i^* \\ 0 &= \frac{\partial \mathcal{L}}{\partial \gamma} = \sum \hat{p}_i x_i^* . \end{aligned} \tag{38}$$

The second-order condition for an interior minimum is also met.

Solving the first equation in (38) gives the maximal-Sharpe-ratio return in state i as

$$x_i^* = \lambda - \gamma \hat{p}_i / p_i . \tag{39}$$

[19]The zero-net-wealth budget constraint is expressed here using the risk-neutral probabilities in place of the state prices. Since the state price is $e^{-rT}\hat{p}_i$, a portfolio with a risk-neutral expected excess return of zero has a zero cost.

Multiply (39) by p_i and $\hat{p}_i$ and sum over states. Recognizing that $\sum p_i = \sum \hat{p}_i = 1$ gives

$$
\begin{aligned}
\bar{x} &= \sum p_i x_i^* = \lambda \sum p_i - \gamma \sum \hat{p}_i = \lambda - \gamma \\
0 &= \sum \hat{p}_i x_i^* = \lambda \sum \hat{p}_i - \gamma \sum \hat{p}_i^2/p_i = \lambda - \gamma \sum \hat{p}_i^2/p_i \, .
\end{aligned} \tag{40}
$$

These equations can be solved to determine the multipliers values[20]

$$
\gamma = \frac{\bar{x}}{\sum \hat{p}_i^2/p_i - 1} \qquad \lambda = \bar{x} + \gamma \, . \tag{41}
$$

So the maximal-Sharpe-ratio portfolio is

$$
x_i^* = \bar{x} + \gamma(1 - \hat{p}_i/p_i) \, . \tag{42}
$$

The variance of the "optimized" portfolio is

$$
\begin{aligned}
V &= \sum p_i (x_i^* - \bar{x})^2 = \sum p_i \gamma^2 \left(\hat{p}_i/p_i - 1\right)^2 \\
&= \gamma^2 \left[\sum \hat{p}_i^2/p_i - 2\sum \hat{p}_i + \sum p_i\right] \\
&= \gamma^2 \left[\sum \hat{p}_i^2/p_i - 1\right] = \gamma \bar{x} \, .
\end{aligned} \tag{43}
$$

Therefore, the square of the maximal Sharpe ratio is

$$
S_*^2 = \frac{\bar{x}^2}{V} = \frac{\bar{x}}{\gamma} = \sum \hat{p}_i^2/p_i - 1 \, . \tag{44}
$$

This final sum can also be expressed as $\sum p_i (\hat{p}_i/p_i)^2$ so the square of the maximal Sharpe ratio is one less than the expectation of the square of the realized probability likelihood ratio.

For a continuous state space indexed by ξ, similar analysis yields a maximal squared Sharpe ratio of

$$
S_*^2 = \int \hat{p}^2(\xi)/p(\xi) \mathrm{d}\xi - 1 = \mathbb{E}\left[\hat{p}^2(\xi)/p^2(\xi)\right] - 1 \tag{45}
$$

[20] From (44) below, γ can also be written as $\gamma = \bar{x}/S_*^2$.

for a portfolio with an excess return of

$$x^*(\xi) = \bar{x} + y[1 - \hat{p}(\xi)/p(\xi)] \tag{46}$$

where $y = \bar{x}/S_*^2$ and $\hat{p}(\xi)$ is the risk-neutral probability density for the state space.

If the state space is indexed by a lognormal return on a basis with a continuously-compounded expected rate of return μ and logarithmic variance of σ^2,

$$p(\xi) = \frac{1}{\xi\sigma\sqrt{T}}\phi\left(\frac{\ell\text{n}\xi - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) \tag{47}$$

where $\phi(\cdot)$ is the standard normal density function. The risk-neutral probability density is the same with μ replaced by r, the continuously-compounded interest rate.

To determine the maximal Sharpe ratio portfolio, we need the likelihood ratio

$$\begin{aligned}\frac{\hat{p}(\xi)}{p(\xi)} &= \exp\left[-\frac{\left[\ell\text{n}\xi - \left(r - \frac{1}{2}\sigma^2\right)T\right]^2}{2\sigma^2T} + \frac{\left[\ell\text{n}\xi - \left(\mu - \frac{1}{2}\sigma^2\right)T\right]^2}{2\sigma^2T}\right] \\ &= \xi^{-(\mu-r)/\sigma^2}\exp\left[\frac{\mu-r}{2}T\left(\frac{\mu+r}{\sigma^2}-1\right)\right] .\end{aligned} \tag{48}$$

From (46) and (48), the maximal-Sharpe-ratio portfolio is

$$x^*(\xi) = \bar{x} + y\left(1 - \exp\left[\frac{\mu-r}{2}\left(\frac{\mu+r}{\sigma^2}-1\right)T\right]\xi^{-(\mu-r)/\sigma^2}\right) , \tag{49}$$

and the square of the maximal Sharpe ratio is

$$S_*^2 = \exp\left[(\mu-r)T\left(\frac{\mu+r}{\sigma^2}-1\right)\right]\mathbb{E}\left[\xi^{-2(\mu-r)/\sigma^2}\right] - 1 . \tag{50}$$

Because ξ is lognormally distributed with logarithmic mean $\mathbb{E}[\ell n\xi] = (\mu - \frac{1}{2}\sigma^2)T$ and variance $\text{Var}[\ell n\xi] = \sigma^2 T$, ξ^θ is also lognormally distributed with logarithmic mean $\theta(\mu - \frac{1}{2}\sigma^2)T$ and variance $\theta^2\sigma^2 T$. So the maximal Sharpe ratio is

$$\begin{aligned} S_*^2 &= \exp\left[(\mu - r)T\left(\frac{\mu + r}{\sigma^2} - 1\right)\right] \\ &\quad \cdot \exp\left[-2\frac{\mu - r}{\sigma^2}\left(\mu - \frac{1}{2}\sigma^2\right)T + \frac{1}{2}\left(\frac{4(\mu - r)^2}{\sigma^4}\right)\sigma^2 T\right] - 1 \\ &= \exp\left[(\mu - r)T\left(\frac{\mu + r}{\sigma^2} - 1\right) - 2\frac{\mu - r}{\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)T\right] - 1 \\ &= \exp\left[\frac{(\mu - r)^2}{\sigma^2}T\right] - 1 \,. \end{aligned} \tag{51}$$

B Maximal-Sharpe-Ratio Portfolio with Puts and Calls

Let z be normally distributed $\mathcal{N}(\bar{z}, v^2)$. Let $Z \equiv e^z$, then Z is lognormally distributed with

$$\begin{aligned} \mathbb{E}[Z] &= \mathbb{E}[e^z] = \exp\left(\bar{z} + \frac{1}{2}v^2\right) \\ \text{Prob}\{Z > K\} &= \Phi\left(\frac{\bar{z} - \ell\text{n}K}{v}\right) \\ \int_K^\infty Z\text{d}F(Z) &= \exp\left(\bar{z} + \frac{1}{2}v^2\right)\Phi\left(\frac{\bar{z} - \ell\text{n}K + v^2}{v}\right) \end{aligned} \tag{52}$$

where $\Phi(\cdot)$ is the standard cumulative normal function. All this is standard for the Black-Scholes model where Z represents the stock price at maturity S_T and so $\bar{z} = \ell n S_0 + (\mu - \frac{1}{2}\sigma^2)T$ and $v^2 = \sigma^2 T$. The two additional results we need are the upper and lower noncentral truncated moments of Z.

$$\int_K^\infty Z^\gamma \text{d}F(Z) = \exp\left(\gamma\bar{z} + \frac{1}{2}\gamma^2 v^2\right)\Phi\left(\frac{\bar{z} - \ell\text{n}K + \gamma v^2}{v}\right) \tag{53}$$

$$\int_0^K Z^\gamma \text{d}F(Z) = \exp\left(\gamma\bar{z} + \frac{1}{2}\gamma^2 v^2\right)\Phi\left(-\frac{\bar{z} - \ell\text{n}K + \gamma v^2}{v}\right) \,. \tag{54}$$

Note that the three expressions in equation (52) are all special cases of (53). The first line is $\gamma = 1$, $K = 0$. The second and third lines are $\gamma = 0$ and $\gamma = 1$.

The proof is straightforward. Let $W \equiv Z^\gamma$, then $w \equiv \ell n W$ is normally distributed $\mathcal{N}(\gamma\bar{z}, \gamma^2 v^2)$. And applying the third line in (52) we get

$$\int_K^\infty Z^\gamma \mathrm{d}F(Z) = \int_{K^\gamma}^\infty W \mathrm{d}F_W(W) = \exp\left(\gamma\bar{z} + \frac{1}{2}\gamma^2 v^2\right) \Phi\left(\frac{\gamma\bar{z} - \gamma \ell \mathrm{n} K + \gamma^2 v^2}{\gamma v}\right) \quad (55)$$

which reduces to the first line in (53). Equation (54) follows by complementarity.

C Maximal-Sharpe-Ratio Portfolio with a History

Let $\bar{x}_h$ and Q_h denote the historical average excess return and average squared excess return over the recording period so far. Let x_i be the excess return in state i each period in the future and α be the fraction of the total period which has passed. Then the Sharpe ratio for the entire recording period is

$$S = \frac{\alpha\bar{x}_h + (1-\alpha)\sum p_i x_i}{\sqrt{\alpha Q_h + (1-\alpha)\sum p_i x_i^2 - [\alpha\bar{x}_h + (1-\alpha)\sum p_i x_i]^2}} \,. \quad (56)$$

The variance and hence the average or expected squared excess returns can be expressed as a function of the Sharpe ratio and the average or expected returns so the entire-period Sharpe ratio is

$$\begin{aligned} S &= \frac{\alpha\bar{x}_h + (1-\alpha)\bar{x}_f}{\sqrt{\alpha\bar{x}_h^2(1+S_h^{-2}) + (1-\alpha)\bar{x}_f^2(1+S_f^2) - [\alpha\bar{x}_h + (1-\alpha)\bar{x}_f]^2}} \\ \text{where} \quad \bar{x}_f &\equiv \sum p_i x_i \qquad S_f \equiv \frac{\bar{x}_f}{\sqrt{\sum p_i x_i^2 - \bar{x}_f^2}} \qquad S_h \equiv \frac{\bar{x}_h}{\sqrt{Q_h^2 - \bar{x}_h^2}} \,. \end{aligned} \quad (57)$$

It is clear by inspection of (57) that the entire-period Sharpe ratio is monotonic in the future Sharpe ratio. Therefore, maximizing it requires maximizing the future Sharpe ratio by holding the maximal-Sharpe-ratio portfolio as previously derived, $S_f = S_*$. The only remaining question is what leverage should be used.

If $\bar{x}_h \leq 0$, then the entire-period Sharpe ratio is monotonically increasing in $\bar{x}_f$ so extreme leverage ($\bar{x}_f \to \infty$) should be employed, and

$$S \to \left[\left(1 + S_*^{-2}\right) / (1 - \alpha) - 1\right]^{-1/2} . \tag{58}$$

If $\bar{x}_h > 0$, then there is a finite optimal leverage choice for $\bar{x}_f$. Re-express (57) as

$$1 + S^{-2} = \frac{\alpha \bar{x}_h^2 (1 + S_h^{-2}) + (1 - \alpha) \bar{x}_f^2 (1 + S_f^{-2})}{[\alpha \bar{x}_h + (1 - \alpha) \bar{x}_f]^2} \tag{59}$$

and differentiate with respect $\bar{x}_f$, giving the first order condition

$$\begin{aligned} 0 =& \frac{\partial (1 + S^{-2})}{\partial \bar{x}_f} \\ =& \frac{2(1 - \alpha) \bar{x}_f (1 + S_f^{-2})}{[\alpha \bar{x}_h + (1 - \alpha) \bar{x}_f]^2} - 2(1 - \alpha) \frac{\alpha \bar{x}_h^2 (1 + S_h^{-2}) + (1 - \alpha) \bar{x}_f^2 (1 + S_f^{-2})}{[\alpha \bar{x}_h + (1 - \alpha) \bar{x}_f]^3} . \end{aligned} \tag{60}$$

Solving the first order conditions gives the optimal leverage as

$$\bar{x}_f^* = \bar{x}_h \frac{1 + S_h^{-2}}{1 + S_*^{-2}} . \tag{61}$$

Substituting $\bar{x}_f^*$ into (59) lets us express the entire-period Sharpe ratio as

$$S^2 = \frac{\alpha S_h^2 (1 + S_*^2) + (1 - \alpha)(1 + S_h^2) S_*^2}{1 + (1 - \alpha) S_h^2 + \alpha S_*^2} . \tag{62}$$

Table 1: Annual Perfect Sorting Ability Within 1-Digit SIC-CODE: Zero-Investment Portfolio Returns

Year	SIC 0 Agriculture	SIC 1 Mining	SIC 2 Construction	SIC 3 Manufacturing	SIC 4 Utilities+	SIC 5 Wholesale	SIC 6 Retail	SIC 7 Finance+	SIC 8 Services	SIC 9 Other	equal-weighted portfolio
1981/01-12	0.332	0.499	0.504	0.553	0.444	0.583	0.471	0.613	0.825	0.655	0.5479
1982/01-12	0.537	0.743	0.621	0.723	0.506	0.911	0.507	0.806	0.751	3.180	0.9283
1983/01-12	0.819	0.564	0.549	0.667	0.432	0.672	0.481	0.658	0.606	0.574	0.6022
1984/01-12	0.261	0.445	0.438	0.462	0.446	0.502	0.411	0.507	0.621	0.754	0.4847
1985/01-12	1.337	0.491	0.561	0.587	0.429	0.625	0.573	0.801	0.750	0.509	0.6662
1986/01-12	0.589	0.595	0.516	0.547	0.429	0.579	0.419	0.581	0.548	1.048	0.5852
1987/01-12	0.899	0.606	0.519	0.593	0.397	0.581	0.381	0.659	0.654	0.184	0.5474
1988/01-12	0.342	0.510	0.469	0.567	0.416	0.639	0.336	0.655	0.608	nan	0.5046
1989/01-12	0.233	0.642	0.568	0.586	0.433	0.594	0.488	0.689	0.963	0.596	0.5793
1990/01-12	0.422	0.392	0.506	0.578	0.338	0.518	0.423	0.642	0.531	0.351	0.4702
1991/01-12	0.566	0.684	0.773	0.791	0.491	0.926	0.713	0.933	1.221	nan	0.7885
1992/01-12	0.599	0.516	0.496	0.681	0.335	0.653	0.494	0.787	0.695	nan	0.5840
1993/01-12	0.506	0.516	0.530	0.720	0.418	0.652	0.375	0.746	0.788	nan	0.5836
1994/01-12	0.304	0.383	0.494	0.581	0.338	0.532	0.285	0.681	0.575	nan	0.4637
1995/01-12	0.961	0.584	0.672	0.838	0.511	0.726	0.435	0.930	0.909	nan	0.7296
1996/01-12	0.818	0.873	0.548	0.740	0.499	0.692	0.373	0.923	0.684	nan	0.6834
1997/01-12	0.600	0.683	0.647	0.690	0.557	0.708	0.560	0.845	0.642	nan	0.6594
1998/01-12	1.199	0.716	0.598	0.781	0.597	0.869	0.413	1.135	0.838	nan	0.7939
1999/01-12	0.810	0.792	0.712	1.546	1.222	0.763	0.412	1.952	0.884	nan	1.0103
AvgNumPerPfio	9.8	110.2	410.5	611.8	313.2	251.2	643.8	209.2	68.5	2.2	263.0
Mean	0.64	0.59	0.56	0.70	0.49	0.67	0.45	0.82	0.74	0.87	0.64
Std.Dev.	0.31	0.13	0.09	0.23	0.19	0.12	0.10	0.31	0.17	0.90	0.15
Sharpe Ratio	**2.05**	**4.44**	**6.49**	**3.05**	**2.54**	**5.37**	**4.68**	**2.61**	**4.33**	**0.97**	**4.26**

Table 2: Maximal Sharpe Ratio for Various Parameter Values

	$T=1$ Year				$T=1$ Month			
	σ	$\mu-r$			σ	$\mu-r$		
		5%	10%	15%		5%	10%	15%
Maximal Sharpe	15%	0.343	0.748	1.311	15%	0.096	0.194	0.295
	20%	0.254	0.533	0.869	20%	0.072	0.145	0.219
S_*	25%	0.202	0.417	0.658	25%	0.058	0.116	0.175
Basis Sharpe	15%	0.323	0.631	0.923	15%	0.096	0.192	0.287
	20%	0.241	0.471	0.690	20%	0.072	0.144	0.215
S_ξ	25%	0.192	0.375	0.548	25%	0.058	0.115	0.172
Improvement	15%	6.0%	18.6%	42.0%	15%	0.5%	1.4%	2.8%
	20%	5.2%	13.1%	26.0%	20%	0.4%	1.0%	1.9%
$S_*/S_\xi - 1$	25%	5.2%	11.2%	20.0%	25%	0.4%	0.9%	1.5%
$\mu_{\text{app}} - \mu$	15%	31.0	197.4	703.2	15%	2.4	14.1	42.4
in basis pts.	20%	26.7	139.1	430.3	20%	2.1	10.3	28.7
eq. (22)	25%	26.7	118.1	329.3	25%	2.1	8.9	22.9

Table 3: Skewness and Kurtosis of Maximal Sharpe Ratio Portfolio

$T=1$ year					$T=1$ month				
σ	Skewness		Kurtosis		σ	Skewness		Kurtosis	
	Basis	MSR	Basis	MSR		Basis	MSR	Basis	MSR
15%	0.456	−2.663	3.372	17.801	15%	0.130	−0.590	3.030	3.625
20%	0.614	−1.750	3.678	8.898	20%	0.174	−0.438	3.054	3.344
25%	0.778	−1.322	4.096	6.260	25%	0.217	−0.349	3.084	3.217

Figure 1: The Sharpe Ratio Maximizing Portfolio.

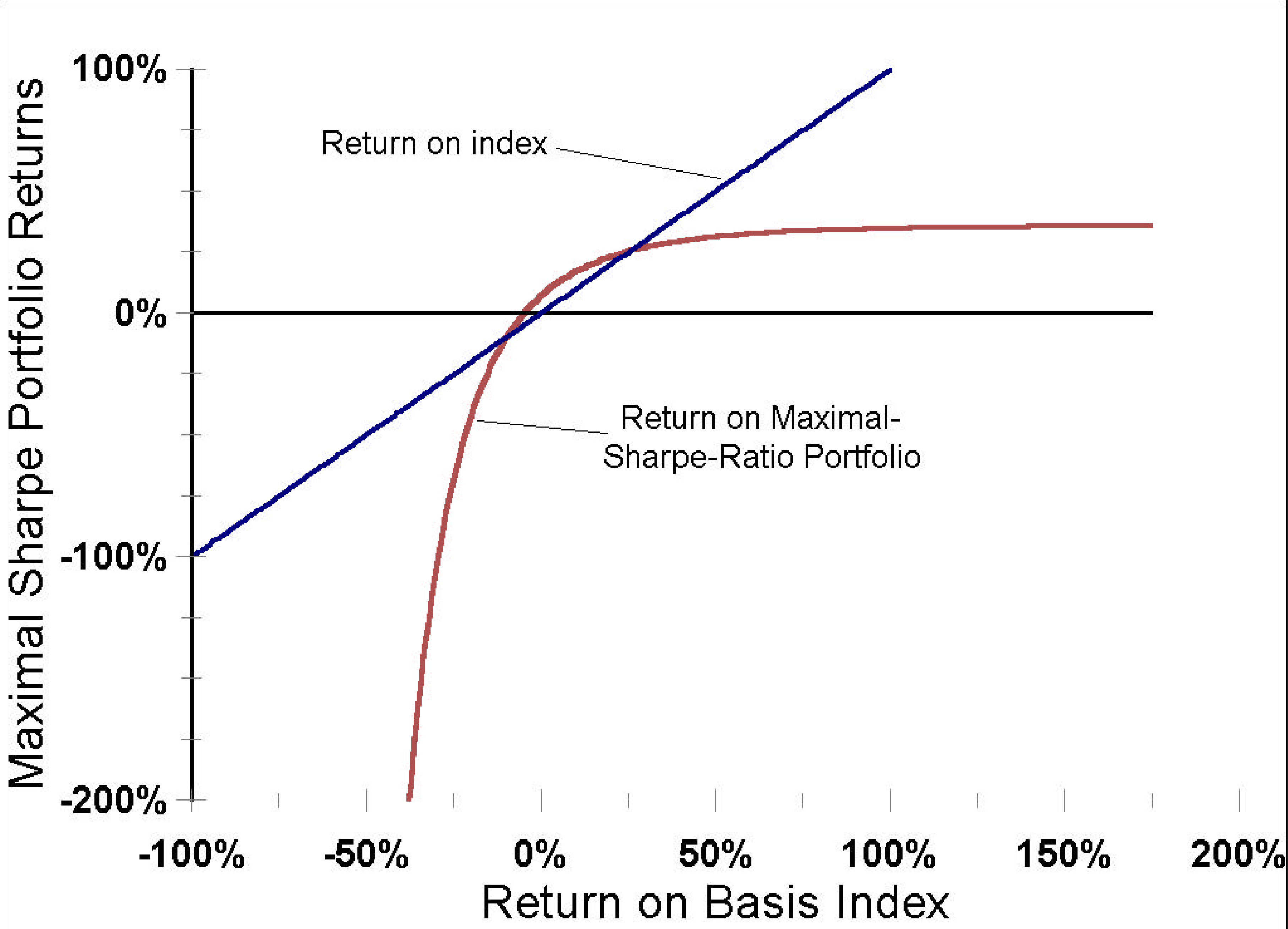

100%
0%
-100%
-200%
Maximal Sharpe Portfolio Returns
Return on index
Return on Maximal-
Sharpe-Ratio Portfolio
-100%
-50%
0%
50%
100%
150%
200%
Return on Basis Index

Figure 1

Figure 2: The Distribution of the Sharpe Ratio Maximizing Portfolio.

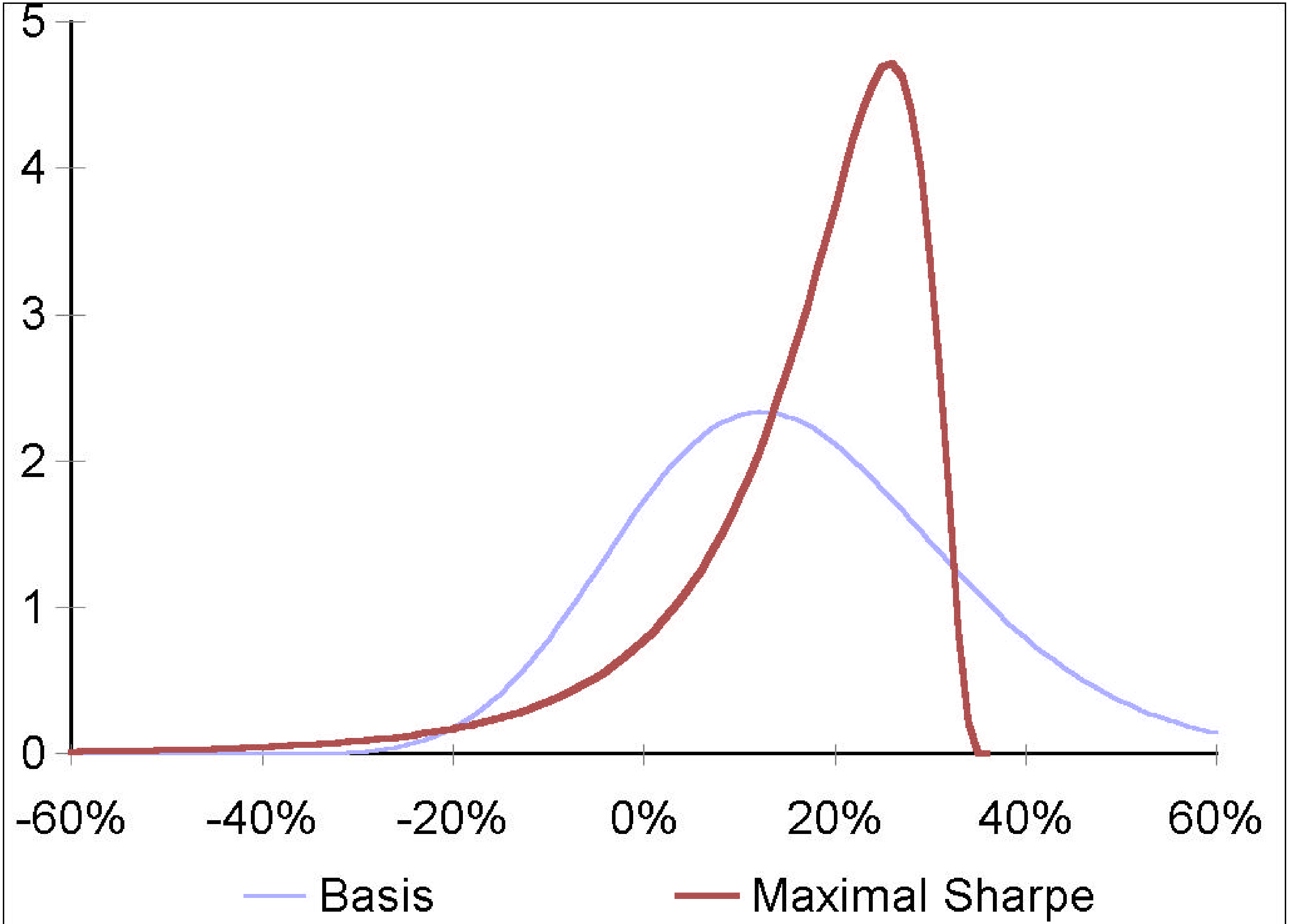


Figure 2

Figure 3: Improvement in the Sharpe Ratio using One Call Option.

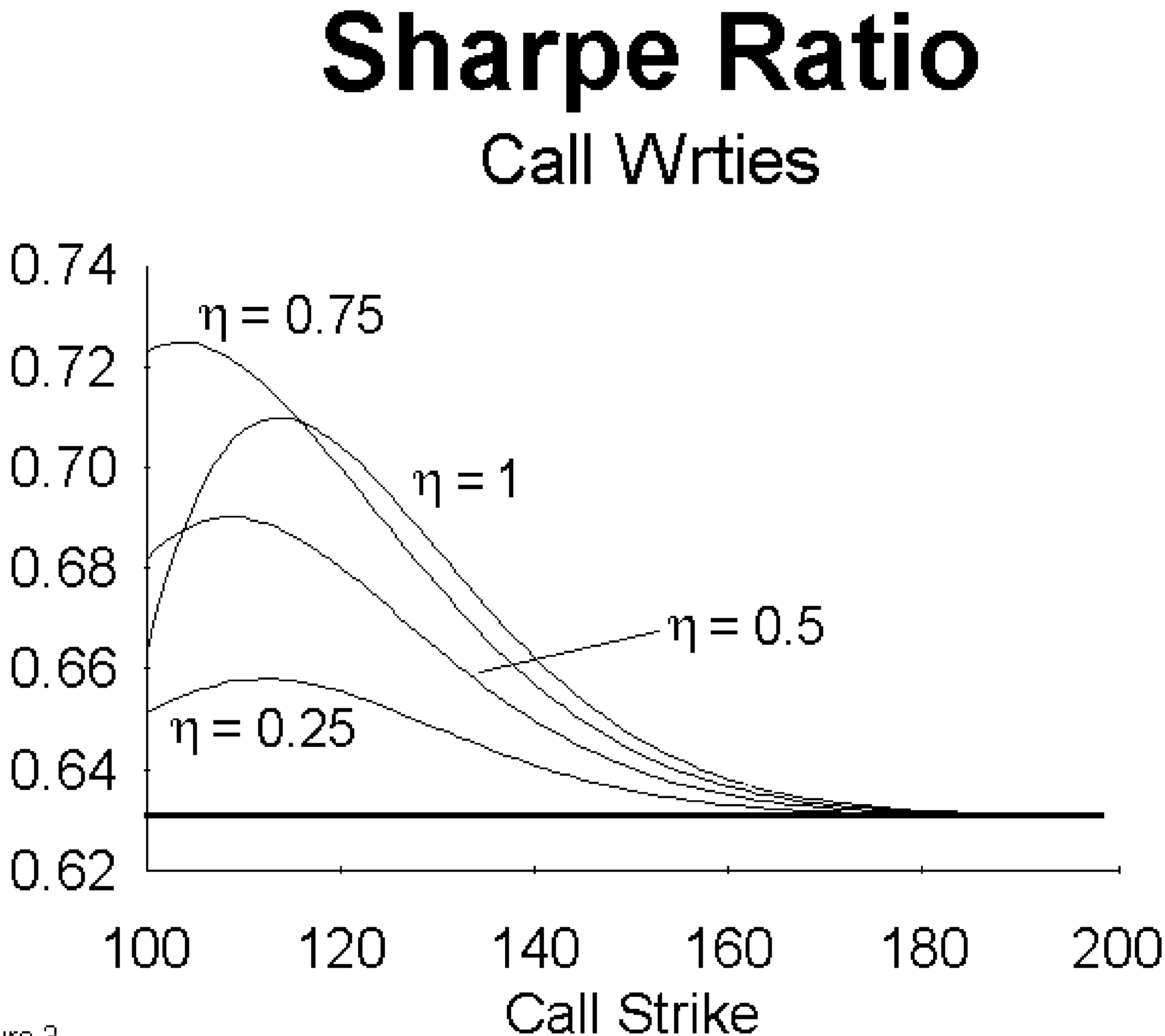

Sharpe Ratio
Call Wrties
0.74
0.72
0.70
0.68
0.66
0.64
0.62
η = 0.75
η = 1
η = 0.5
η = 0.25
100
120
140
160
180
200
Call Strike

Figure 3

Figure 4: Payoff on the maximal-Sharpe-ratio portfolio with options.

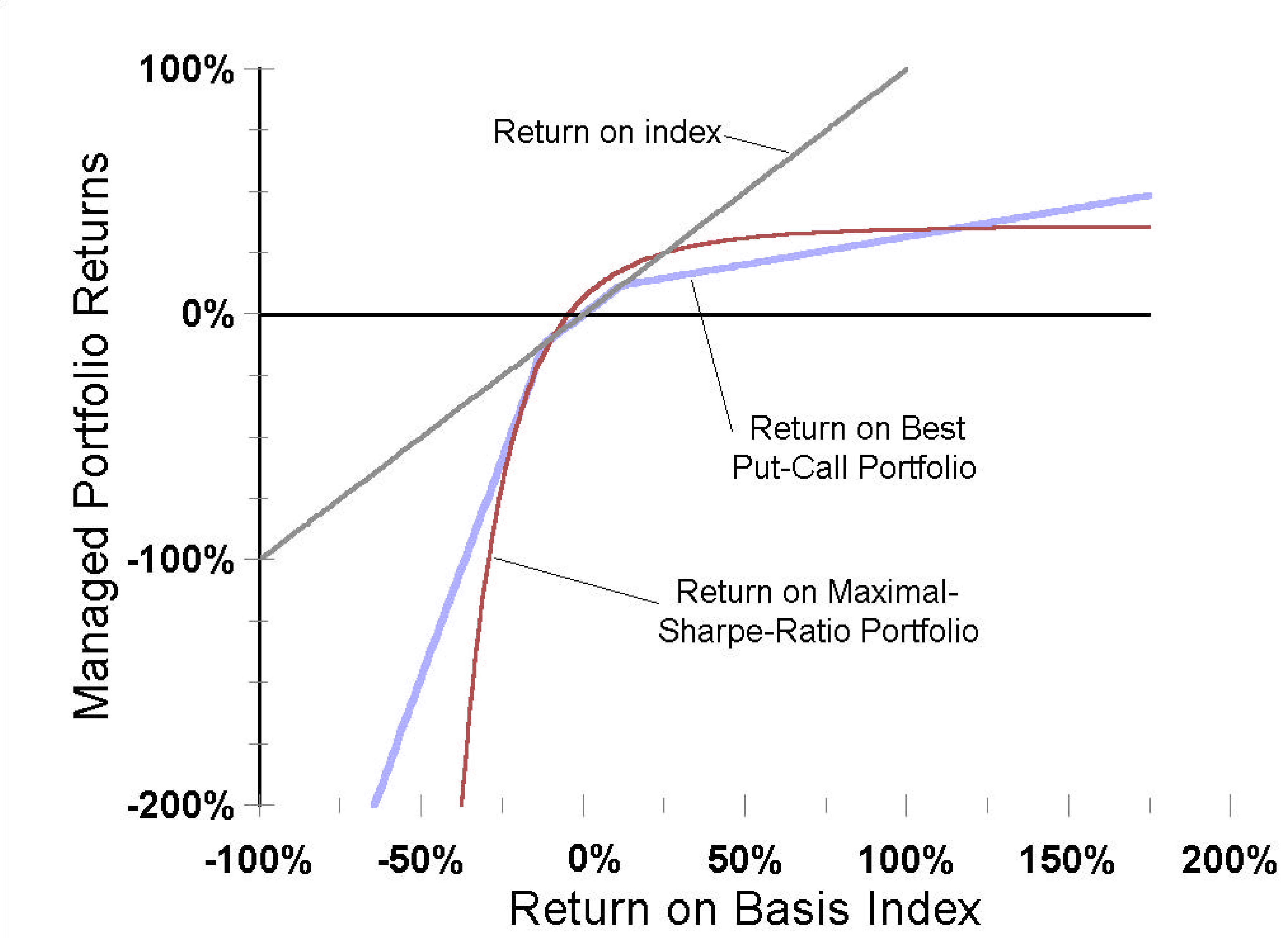
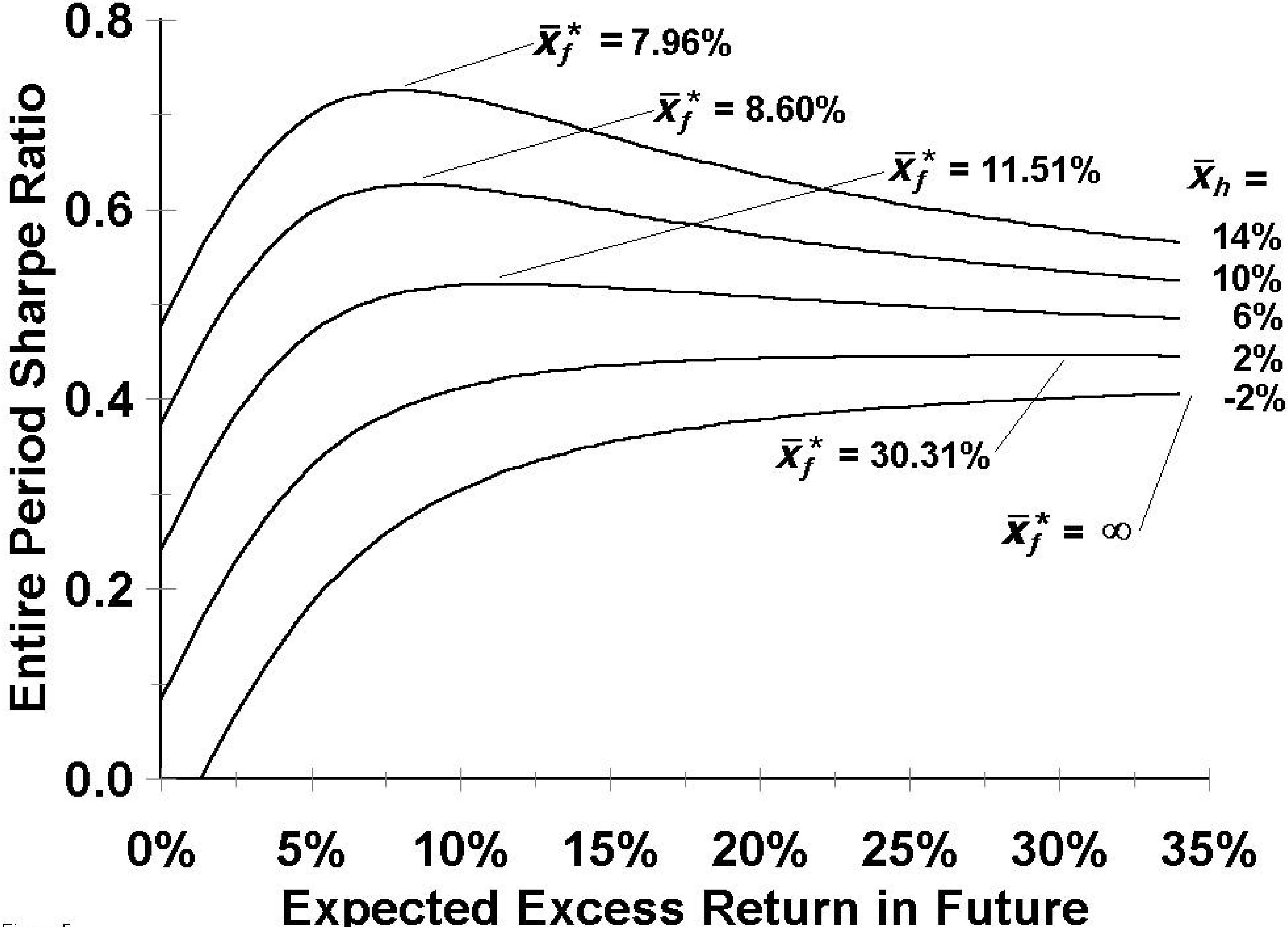

100%
0%
-100%
-200%
Managed Portfolio Returns
Return on index
Return on Best
Put-Call Portfolio
Return on Maximal-
Sharpe-Ratio Portfolio
-100%
-50%
0%
50%
100%
150%
200%
Return on Basis Index

Figure 4

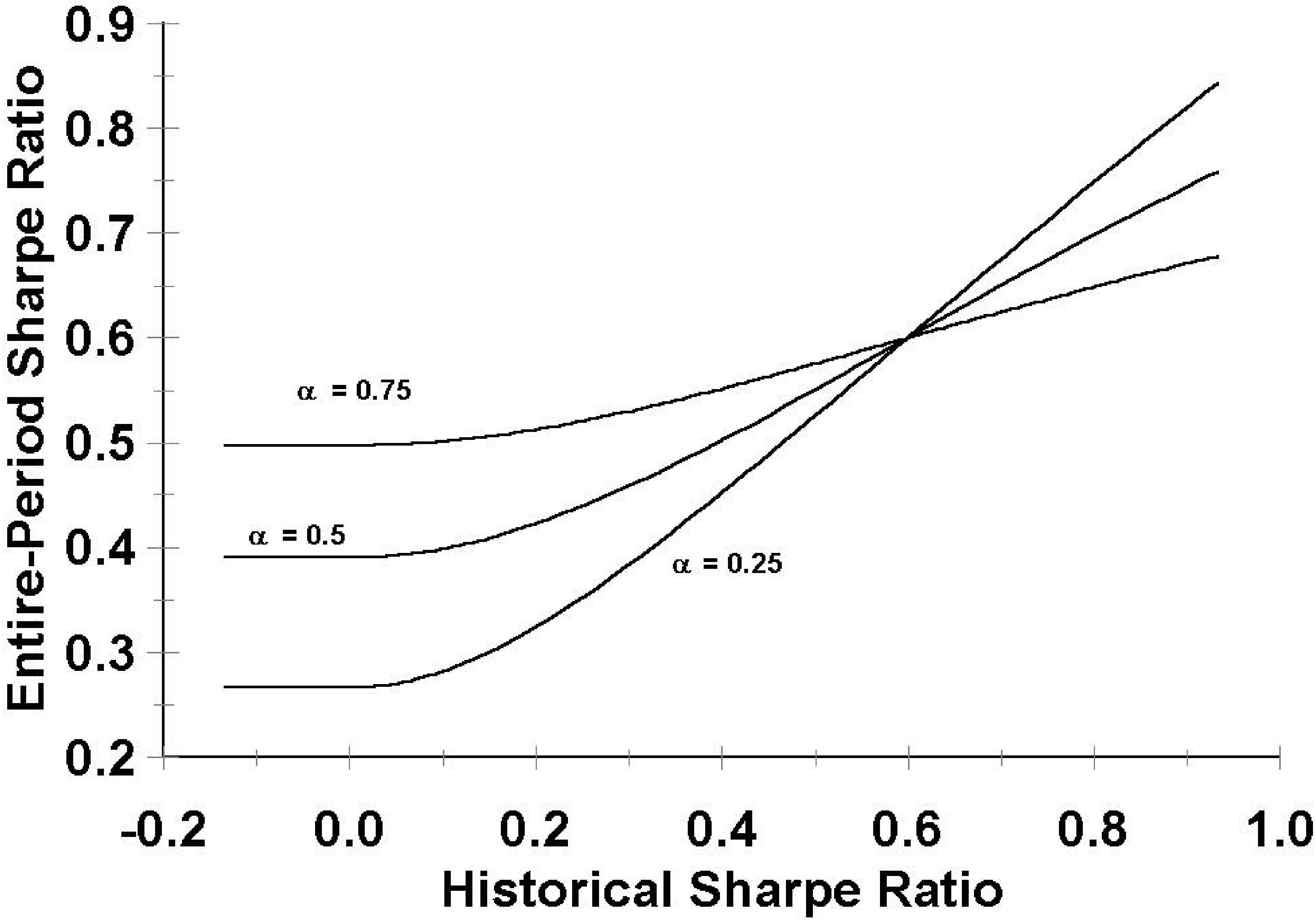
Figure 5: Maximizing the Sharpe Ratio with a History of Returns.



0.8
0.6
0.4
0.2
0.0
Entire Period Sharpe Ratio
$\bar{x}_f^* = 7.96\%$
$\bar{x}_f^* = 8.60\%$
$\bar{x}_f^* = 11.51\%$
$\bar{x}_h =$
14%
10%
6%
2%
-2%
$\bar{x}_f^* = 30.31\%$
$\bar{x}_f^* = \infty$
0%
5%
10%
15%
20%
25%
30%
35%
Expected Excess Return in Future

Figure 5

Figure 6: Maximizing the Sharpe Ratio with a History of Returns.



0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
Entire-Period Sharpe Ratio
α = 0.75
α = 0.5
α = 0.25
-0.2
0.0
0.2
0.4
0.6
0.8
1.0
Historical Sharpe Ratio

Figure 6

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