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THE OPTIMUM QUANTITY OF MONEY:  
THEORY AND EVIDENCE

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### **ABSTRACT**

In this paper we propose a simple and general model for computing the Ramsey optimal inflation tax, which includes several models from the previous literature as special cases. We show that it cannot be claimed that the Friedman rule is always optimal (or always non-optimal) on theoretical grounds. The Friedman rule is optimal or not, depending on conditions related to the shape of various relevant functions. One contribution of this paper is to relate these conditions to *measurable* variables such as the interest rate or the consumption elasticity of money demand. We find that it tends to be optimal to tax money when there are economies of scale in the demand for money (the scale elasticity is smaller than one) and/or when money is required for the payment of consumption or wage taxes. We find that it tends to be optimal to tax money more heavily when the interest elasticity of money demand is small. We present empirical evidence on the parameters that determine the optimal inflation tax. Calibrating the model to a variety of empirical studies yields an optimal nominal interest rate of less than 1% per year, although that finding is sensitive to the calibration.

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## **I. Introduction**

There has been a long debate among theorists about the optimum quantity of money. We emphasize three stages of that debate. In the first stage, Milton Friedman (1969) argued that, because the social production cost of money is basically zero, the government should provide money at zero cost to its citizens. In a world where money earns a zero nominal return, Friedman's optimum quantity of money corresponds to management of the money supply so that the nominal market return on a risk free bond is zero. With a zero market return, consumers pay no (opportunity) cost by holding money; the cost to consumers equals the production cost. However, Ned Phelps (1973) opened up a second stage of the debate by pointing out that governments are normally in need of revenue and must use distortionary taxes of some kind to obtain that revenue. The inflation tax should be chosen, *a la* Ramsey (1927), so that the marginal deadweight loss of inflation is equated to the marginal deadweight loss from other taxes. Phelps argued that the Ramsey tax policy means a positive nominal interest rate and a correspondingly positive marginal deadweight loss from inflation. In the third stage of the debate, Lucas and Stokey (1983), Kimbrough (1986), Faig (1988), Woodford (1990), Correia and Teles (1995), Chari, Christiano, and Kehoe (1996), and others pointed out that the Ramsey-style reasoning does not necessarily imply a positive tax on money. However, the intuition that leads to this result is not unanimously accepted: while Kimbrough (1986) and Chari, Christiano and Kehoe (1996) say that the key is that money is an intermediate good (if money is an intermediate good, then it should not be taxed), others claim that it is the fact that the marginal cost of producing it is zero (Correia and Teles (1995)), others argue that it depends on whether the shopping time technology exhibits constant, diminishing, or increasing returns to scale (Woodford (1990)), others argue that a key feature is whether the non-inflation tax is a consumption tax or an income tax (Guidotti and Végh (1993)), others point out that the key is whether shopping time is negligible when interest rates are zero (Guidotti and Végh (1993)), and finally others say that it depends on complicated relationships between the interest rate and the scale elasticities of the money demand function, and the consumption elasticity with respect to the tax rate (Faig (1988)).

It is hard to compare the various results in the literature because the assumptions regarding the underlying technologies and available taxes are different. For this reason, we feel that it is

necessary to have a framework which contains all the aforementioned models as particular cases. And the first contribution of this paper is to provide such a framework, which we use to clarify the assumptions that led our predecessors to reach their conclusions. In fact, we show that *it is not possible* to say that the Friedman rule is always optimal or never optimal purely on theoretical grounds. This is true for models of money in the utility function and models of money as means to reduce transaction costs (“shopping time” models).

We show that all those who claimed that the Friedman rule is “always” optimal base their claims on particular (often implicit) assumptions about the utility functions or shopping time technologies. For example, Kimbrough’s (1986) finding that the Friedman rule is always optimal depends crucially on the assumption that the shopping time function is homogeneous of degree one. Chari, Christiano and Kehoe (1996) and Correia and Teles (1995) also claim that the Friedman rule is always optimal as long as the shopping time function is homogenous of degree  $k \geq 1$  and  $k$  any number respectively. These claims rely heavily on the assumption that money is not used to pay taxes and on the assumption that there a no (or at least limited) economies of scale in the holding of money.

The finding that theory alone does not determine the desirability of the Friedman rule suggests that empirical analysis is needed to determine the optimal inflation rate. The second contribution of this paper is to relate the optimal inflation tax to empirically measurable coefficients such as the interest or the scale elasticity of money demand. We provide theoretical formulas and numerical analysis which allow us to compute the optimal inflation tax. For example, we show that among the important determinants of the optimal inflation tax are the interest and the net scale elasticities of money demand as well as the relation between the scale elasticity of money demand and the interest rate. The final contribution of this paper is to use empirical measures of the key elasticities to provide an estimate of the optimal inflation tax in the United States. Following Phelps (1973), the conventional view is that the optimal inflation tax is larger when the interest elasticity of money demand is smaller. This view has been recently challenged by Chari, Christiano and Kehoe (1996) who claim that the optimal inflation tax is zero regardless of the interest elasticity. Our

propositions 9 and 10 and our numerical simulations suggest that the conventional view is correct and that the result in Chari, Christiano and Kehoe depends crucially on their special assumptions.

The rest of the paper is organized as follows. In part II we analyze the theory of optimal taxation of money. Section 1 derives the conditions under which the Friedman rule is optimal when money enters the utility function of the representative consumer. Section 2 shows that when consumption taxes exist, the Feenstra equivalence result between models of “money in the utility function” and “money in the budget constraint” (or “shopping time models”) does not hold, unless one assumes that the tax rate enters the utility function. Section 3 derives the conditions under which it is optimal to set  $R=0$  when money is a means to economize on shopping time. Section 4 relates the findings of sections 1 and 3. Section 5 shows theoretically how to compute the optimal inflation tax and relates it to variables which can be measured empirically. In Part III of the paper we discuss some empirical evidence on the various determinants of the fundamental parameters. Part IV concludes.

## **II. The Theory of the Optimum Quantity of Money.**

The standard methodology in the literature of optimal inflation tax is to set up a dynamic (and perhaps stochastic) general equilibrium model (with discount rates and stochastic factors, and other notation) without capital. Governments begin by choosing a (possibly state contingent) tax policy for all dates. Consumers make (possibly state contingent) decisions regarding consumption, labor supply, and money holdings at each date taking current and future prices and tax policies as given. The Ramsey problem is to choose a tax policy at each date so that, taking into account consumers' reactions to those policies, the government budget constraint balances and consumer utility is maximized. In this dynamic setting, the Ramsey problem is an optimal control problem, and authors in the literature have been especially interested in “stationary” versions of that control problem. “Stationary” means: (i) government consumption and intraperiod production possibilities do not vary over time, (ii) the date  $t$  flow of utility is a time invariant function of date  $t$  variables, and

(iii) any initial real government debt (that cannot be explicitly repudiated) takes the form of consols.<sup>1</sup> The stationary version of the control problem is equivalent to the static Ramsey tax problem posed in this paper. Since we are not the first authors to point out this equivalence,<sup>2</sup> we economize on notation and begin immediately with the static problem. By doing so, however, we need to keep in mind that our analysis ignores the time inconsistency and reputation issues, highlighted by Barro and Gordon (1983) and Judd (1989), as well as incentives present in nonstationary environments to vary tax collections over time, highlighted by Barro (1979), Lucas and Stokey (1983), and Judd (1989).

## 1. Models of Money in the Utility Function (MIUF)

### (i) Households:

Consider a static version of Sidrauski's (1967) monetary model. The consumer's utility function is defined over consumption,  $c$ , real money,  $m$ , and an untaxed good which we identify with leisure,  $l$ :<sup>3</sup>

$$u(c, m, l) \tag{1}$$

We assume  $u_c > 0$ ,  $u_l > 0$ ,  $u_m \geq 0$ , and that  $u(\cdot)$  satisfies the usual concavity properties. Expenditures are made on consumption, consumption taxes which are levied at the flat rate  $\tau$ , and the cost of holding money. Let  $i$  be the nominal interest rate net of any interest that might be paid on money. The cost

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<sup>1</sup>The definition of stationarity employed by Correia and Teles (1995), Faig (1988), Guidotti and Végh (1994), Woodford (1990), and others is more restrictive: there can be no initial government debt, either nominal or real. Correia and Teles (1995), Faig (1988), and Guidotti and Végh (1994) also assume that the initial money stock is zero. Note that the definition of stationary employed here and in the literature does not require government *policy variables* to be constant over time, although the optimal policy does turn out to generate tax and interest rates which are constant over time.

<sup>2</sup> See Woodford (1990), Faig (1988), and others.

<sup>3</sup> In general, the good which we call leisure may represent all kinds of non-taxed activities, including underground economies. For this to be an interesting problem of optimal taxation, a non-taxable good needs to be introduced because, otherwise, taxes are not distortionary.

of holding money for the period is  $R \cdot m$ , where the tax rate on money is  $R \equiv i/(1+i)$ .<sup>4</sup> Normalizing the price of leisure to 1, the budget constraint for the consumer is:

$$Rm + (1+\tau)c \leq T-l \quad (2)$$

where  $T$  is the time endowment. Consumers maximize (1) subject to (2). The solution to their optimization problem yields three Marshallian demands functions:  $m(1+\tau, R)$ ,  $c(1+\tau, R)$  and  $l(1+\tau, R)$ .

(ii) *Government:*

The government's optimal policy - the "Ramsey optimal policy" - maximizes the utility of the representative consumer subject to the government budget constraint and consumers' decision rules. That is, the government chooses  $\tau$  and  $R$  so as to maximize

$$V(1+\tau, R) \quad s.t. \quad \tau c(1+\tau, R) + R m(1+\tau, R) \geq g \quad (3)$$

where  $V(1+\tau, R) = u(c(1+\tau, R), m(1+\tau, R), l(1+\tau, R))$  is the consumer's indirect utility function. We assume that  $g$ , the sum of government consumption and the interest on the initial real government debt, is positive.

The solution  $(\tau^*, R^*)$  to the static Ramsey problem (3) corresponds with the optimal tax rates that come out of the fully dynamic (stationary) Ramsey problem, but it says nothing about the price level and stock of nominal money that solve the fully dynamic problem. If there are no initial nominal assets or liabilities of the government - as often assumed in the literature - then any overall level of nominal money and prices are optimal. If the government is a net nominal debtor, the optimal price level is infinite. In either case, the solution to the fully dynamic Ramsey problem is as if the government repudiated any initial nominal government debts.

The absence of any initial nominal liabilities means that the Ramsey optimal government enjoys extraordinary revenues in the first period from issuing the stock of money. This re-issuing of

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<sup>4</sup> $R=i$  to a very good approximation when the period length is very short.

the initial stock of money is equivalent to a flow of real revenue equal to the real interest rate  $r$  times real money balances. We can think of the revenue associated with the ability to print money, therefore, as having two components: the real interest on the initial stock of real money ( $r \cdot m$ ) and the flow of resources derived from the printing of money at rate  $\pi$  ( $\pi \cdot m$ ). As discussed by Auernheimer (1974), this is why the nominal interest rate (rather than the inflation rate) determines the tax rate on money in the government budget constraint. If the government chooses the Friedman Rule (that is,  $R=0$ ), the revenue derived from the reissuing of the initial stock of money ( $r \cdot m$ ) is exactly enough for the government to finance the reduction of the money stock required by that rule ( $-\pi \cdot m$ ) so that  $r + \pi = 0$ . Alternatively, if the Friedman Rule is implemented by paying interest on money, the resources needed to pay interest on money are exactly the resources generated by the initial reissuing of the stock of money.

Throughout the paper we assume that the government operates in the upward-sloping part of the Laffer curves:

$$G_R > 0 \quad , \quad G_q > 0 \quad (4)$$

where  $G_R = \partial G / \partial R$ ,  $G_q = \partial G / \partial q$ ,  $q \equiv 1 + \tau$ , and where  $G$  is government revenue (the left hand side of the budget constraint in (3), in other words,  $G(1 + \tau, R) \equiv \tau \cdot c(1 + \tau, R) + R \cdot m(1 + \tau, R)$ ).<sup>5</sup> In particular, the conditions are assumed to hold at the Friedman Rule and at the Ramsey optimal policy.

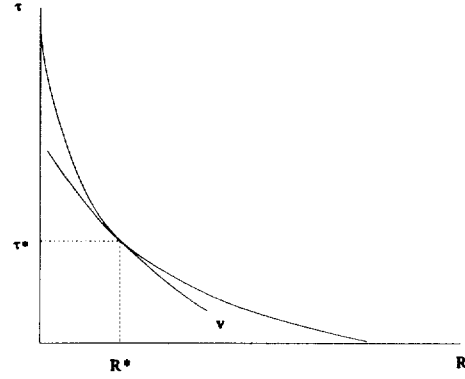
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<sup>5</sup> If either  $G_R$  or  $G_q$  is zero or negative, cutting the corresponding tax would increase welfare without decreasing revenue.



(iii) *Interior Solution.*

Figure 1 illustrates the optimization problem faced by the government. The budget constraint imposes a trade-off between  $\tau$  and  $R$ , a trade-off which we illustrate in the  $[\tau, R]$  plane as a convex curve. The maximization of the consumer's indirect utility function (an indifference curve of which is illustrated in the figure as a less convex curve marked  $v$ ) subject to this constraint delivers, according to the diagram, an interior solution  $\tau^*$  and  $R^*$ .<sup>6</sup> In principle, therefore, there is no reason for  $R^*$  to



**Figure 1**

be zero. If we want to show that the Friedman rule is optimal and the solution interior, then we need to find that the tangency point occurs at exactly  $R^*=0$ .

If the solution is interior, the first order condition equates the consumer's marginal rate of substitution to the government's ability to reallocate tax revenue:

$$V_R / V_q = G_R / G_q \quad (5)$$

For the rest of the paper we use the notation  $c_R = \partial c / \partial R$ ,  $m_R = \partial m / \partial R$ ,  $c_q = \partial c / \partial (1 + \tau)$ ,  $m_q = \partial m / \partial (1 + \tau)$ . With this, we can state the following proposition.

**Proposition 1:**

*If the solution to the government optimization problem is interior, then the first order condition to the problem is:*

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<sup>6</sup> The shapes of the budget constraint and the indifferent curves in Figure 1 are drawn this way for expositional simplicity. Any assumptions we make in this regard are made algebraically in the text. Although it is the case in the figure, there is no reason why the Ramsey maximization problem should be concave. Although it can be shown that this is not always true, the literature usually proceeds as if the Ramsey problem were concave.

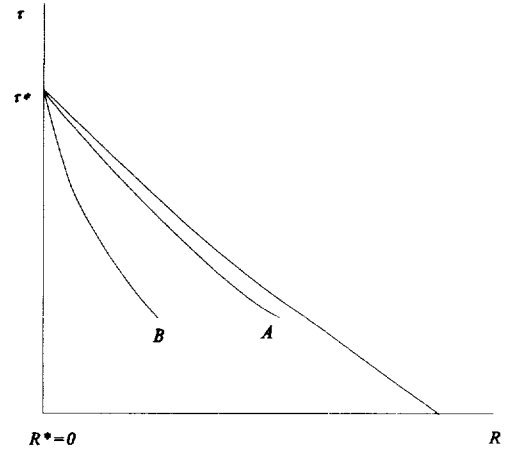
$$\frac{m}{c} = \frac{\tau c_R + R m_R + m}{c + \tau c_q + R m_q} \quad (6)$$

Proof: Apply Roy's identity for the consumer problem to write the left hand side of (6) as the ratio  $m/c$ . Take derivatives of the government budget constraint in (3) to get the right hand side of (6).

The intuition for this result is the following: if the left hand side of equation (6) is greater than the right hand side then, relative to the consumption tax, the inflation tax is too painful compared to the revenue it raises. Under these circumstances, a revenue neutral substitution of consumption taxes for inflation taxes would improve welfare. If the left hand side is smaller, the opposite is true.

*(iv) Non-Interior Solution.*

The Friedman rule of taxation,  $R=0$ , can be optimal if the marginal rate of substitution is equal to the marginal rate of transformation (the slope of the government budget constraint) at  $R=0$ . In this case, the solution satisfies the first order conditions for an interior optimum (6). This is the case depicted by the indifference curve (A) in Figure 2. It can also be optimal, however, if the *marginal rate of substitution at  $R=0$*  is larger than the *marginal rate of transformation at  $R=0$* . This is the case, for example, of indifference curve (B) in Figure 2.



**Figure 2**

In other words, a necessary condition for the taxation (non-taxation) of money to be optimal is:

$$\left. \frac{V_R}{V_q} \right|_{R=0} = \left. \frac{m}{c} \right|_{R=0} < (\geq) \left. \frac{m + \tau c_R}{c + \tau c_q} \right|_{R=0} = \left. \frac{G_R}{G_q} \right|_{R=0} \quad (7)$$

Intuitively, money is not taxed when, relative to the consumption tax, the inflation tax is painful compared to the revenue it raises. The right hand side computes the relative effect of the two prices on government revenue, which includes the amounts purchased by the consumer ( $m$  and  $c$ ), but also the relative distortions of consumption demand  $\tau c_R$  and  $\tau c_q$ . Notice that the relative distortions of money demand are not relevant because we are considering a policy that does not tax money (so the terms multiplying  $R$  vanish from the numerator and denominator of (6)).

We assume that there is *satiation* of money so that  $m$  remains finite at  $R=0$ .<sup>7</sup> If there is satiation (that is, if money remains bounded at  $R=0$ ), the necessary condition (7) for the Friedman rule to be optimal can be manipulated to yield a very simple inequality. This can be expressed in the following proposition:

**Proposition 2:**

*If the Ramsey problem is concave, then its optimum taxes (doesn't tax) money as*

$$\left. \frac{m}{c} \right|_{R=0} > (\leq) \left. \frac{c_R}{c_q} \right|_{R=0} \quad (8)$$

Proof: Multiply the left hand side of (7) by  $c + \tau c_q$  (this term is positive if the Laffer condition holds) and the right hand side by  $m$  ( $m$  is finite at  $R=0$  by assumption). Rearrange and divide both sides by  $c_R$ . Since  $c_R < 0$ , the signs of the inequalities change.

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<sup>7</sup>Even though some authors in the literature assume non-satiation, the satiation assumption is not crucial. What is crucial is that *seigniorage* (ie,  $Rm$ ) fall as  $R$  falls towards zero. If not, it is trivially the case that government policy can be improved by reducing any positive  $R$ . See CCK (1996, example 3.22) for one such example.

For what it is worth, most interesting microeconomic models of money demand - including inventory models and cash-in-advance models - predict satiation (see Mulligan and Sala-i-Martin (1997).)

### 1.A. Particular Case 1: Phelps (1973) and the assumption of zero cross-price elasticities.

If the consumption demand is independent of the interest rate ( $c_R=0$ ) and the demand for money is independent of the consumption tax rate ( $m_q=0$ ), then the right hand side of (8) vanishes so that the left hand side is always larger. It follows that in this case money is taxed by the Ramsey optimal policy. The optimal inflation tax is given by the interior solution (6). When  $c_R=0$  and  $m_q=0$ , the first order condition (6) simplifies to:

$$\frac{\tau}{1+\tau} \cdot \eta_{cq} = \epsilon_R, \quad (9)$$

where  $\eta_{cq}=(\partial c/\partial q)(1+\tau)/c$  is the elasticity of consumption demand with respect to its own price and  $\epsilon_R=(\partial m/\partial R)(R/m)$  is the interest rate elasticity of money demand. Hence, when the cross-price elasticities are zero, the optimal tax system is given by the Phelps rule: the tax rate on money is optimally set at  $R^*>0$  and the inflation tax is large relative to the consumption tax when the interest-elasticity of money demand is small relative to the price elasticity of consumption demand.

### 1.B. Particular Case 2: Money and Consumption are Substitutes.

If consumption and money were *substitutes* in the sense that  $c_R>0$  at  $R=0$ , then the right hand side of (8) is negative (remember that the denominator is negative). Since the left hand side is always positive, proposition 2 says that money is taxed by the Ramsey optimal policy. The interior optimal inflation tax rate in this case is given by the complicated formula in (6), which allows for the cross-price elasticities to be non-zero. It has been argued in the literature that the case of consumption and money being substitutes is not of great empirical relevance.<sup>8</sup> However, if the underground economy is important, then it is possible to think of  $c_R \geq 0$  as plausible as a larger inflation tax forces people away from illegal into market activities if the latter tend to use money less intensively.

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<sup>8</sup> For example, it can be shown that when  $c_R \geq 0$ , the scale elasticity of the implied money demand function is negative, which could be viewed as unrealistic.

### 1.C. Particular Case 3: Money and Consumption are Complements.

Woodford (1990) argues that the particular cases analyzed in subsections 1.A and 1.B are rather uninteresting special cases in the context of the demand for money: the reason why money may enter in the utility function is that money helps us “enjoy” real consumption goods. This suggests that money and consumption would tend to be complements so that the cross-price elasticity should be negative. If we use  $c_R < 0$  in condition (8) above, then the ratio in the right hand side of (8) is positive. Hence, it is no longer obvious that money is taxed by the Ramsey policy: *money may or may not be optimally taxed depending on the size of  $m/c$  relative to the size of  $c_R/c_q$* . If the distortion of consumption tax revenues by inflation at  $R=0$  (that is if  $c_R$ ) were sufficiently small, then it would still be optimal to tax money. Put in another way, only when the change in nominal interest rates triggers a large reduction in consumption it is optimal NOT to tax money. The key question is therefore: how much does inflation affect consumption tax revenues? Note that this is an empirical question that cannot be resolved on theoretical grounds alone.

### 1.D. Particular Case 4: Cash-Good and Credit-Good model (Lucas and Stokey (1987))

As argued by Woodford (1990), the Lucas and Stokey (1987) model of cash-good/credit-good can be thought as a micro-foundation of a MIUF which delivers a negative cross-price elasticity. In the standard cash-good, credit-good model, the utility function is given by  $v(c_1, c_2, l)$ , where  $c_1$  is the cash good,  $c_2$  is the credit good and  $l$  is leisure. If  $c_1$  needs to be purchased with cash, we have  $c_1 = m$ . To transform this into a MIUF problem, define  $c = c_1 + c_2$  as total spending. Substitute  $c_1$  for  $m$  and  $c_2$  for  $c - m$  in the utility function to get  $u(c, m, l) = v(m, c - m, l)$ . This utility function implies that  $m$  and  $c$  are *complements*, even if the original utility function  $v(\cdot)$  is separable. Hence, the analysis in Section 1.C applies for the Lucas-Stokey model.<sup>9</sup>

### 1.E. Particular Case 5: Chari, Christiano, and Kehoe (1996).

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<sup>9</sup>Although we are not aware of an example in the optimal inflation tax literature, another plausible specification of the cash-in-advance constraint for the Lucas-Stokey model is  $c + (\text{taxes}) \leq m$ . In this case, the Lucas-Stokey model is not a special case of the MIUF approach.

Chari, Christiano and Kehoe (1996) (CCK from now on) assume that the utility function (1) is homothetic in money and consumption and weakly separable in leisure and show that, under those conditions, the Friedman rule is optimal. In other words, they assume that  $u(\cdot)$  can be written as

$$u(c, m, l) = \mu(w(c, m), l) \quad (10)$$

where  $w(\cdot)$  is homothetic. Although these homotheticity/separability restrictions may not appear to be very restrictive, they turn out to restrict  $u(\cdot)$  enough to make condition (8) be satisfied with equality. In other words, if  $u(\cdot)$  can be written as (10), then  $c_R/c_q \big|_{R=0} = m/c \big|_{R=0}$  (see Appendix 1 for a proof of this result).

**Proposition 3.**

*If the Ramsey problem is concave and the household utility function satisfies the homotheticity and separability conditions (10) imposed by Chari, Christiano and Kehoe (1996), then money is not taxed by the Ramsey optimal policy.*

Proof: Take  $c_R/c_q \big|_{R=0} = m/c \big|_{R=0}$  and plug it in (8). Condition (8) is satisfied with equality so Proposition (1) says that money is not taxed by the Ramsey optimal policy.

Proposition (3) establishes sufficient conditions for the Friedman rule to be optimal: it is sufficient to assume that the utility function is homothetic and separable to find that the Friedman rule is optimal. In a way, the utility function assumed in CCK imposes just enough complementarity between consumption and money so as to satisfy condition (8) with strict equality. It is not clear, however, how general or realistic these restrictions on the utility function are, given that the CCK model is not related to micro-founded theories of money demand. One way to evaluate the usefulness of the CCK result is to compare the empirical predictions of a model that satisfies their assumptions with data.

One thing we can say, however, is that the fact that the optimum  $R=0$  is reached in this case by setting the two sides of condition (8) are equal suggests that this optimum is a “knife-edge” case

which turns out to be fragile. In terms of Figure 2, the optimality of the Friedman rule discovered by CCK corresponds to an indifference curve like A: the indifference curve is tangent to the government budget constraint exactly at the corner  $R=0$ . Any slight tilting of either the budget constraint or the indifference curve will break the result that the optimum occurs exactly at  $R=0$  so the Friedman rule will not be optimal.<sup>10</sup>

#### 1.F. Particular Case 6: Braun (1994).

Braun (1994) considers a cash-credit goods model in which the utility function is not homothetic. Since the cash-in-advance constraint takes the form  $c_2 \leq m$ , his model is a special case of MIUF model in which  $c$  and  $m$  are aggregated in a globally nonhomothetic way in the utility function. Braun assumes that money (ie, the cash good) is relatively inferior (his utility function is  $m^a/a + (c-m)^b/b + v(l)$  with  $a < b$ ), implying that

$$\left. \frac{c_R}{c_q} \right|_{R=0} < \left. \frac{m}{c} \right|_{R=0}$$

which, by Proposition (2), implies that money is taxed in the Ramsey optimal tax program.

#### 1.G. Irrelevance of having an Income Tax rather than a Consumption Tax in Models of MIUF.

In this section we made the assumption that the real tax was a consumption tax. Some authors in the literature have worked under the alternative assumption of an income or wage tax. In models of MIUF, it turns out that the two are identical when it comes to identifying conditions under which the Friedman rule is optimal so our formulation is general. To see this, consider the budget constraint of household who faces a consumption tax:  $(1+\tau) \cdot c + R \cdot m = T - l$ . Let  $t$  be the tax rate on

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<sup>10</sup> For example, in Mulligan and Sala-i-Martin (1996b) we demonstrate that if the utility function is homothetic and separable as in (10), and if *some* money is held by foreigners (so that their activities do not enter the government's utility function but their money is still taxable through  $R > 0$ ), then the Friedman rule is NOT optimal. The fragility of the CCK result derives from the fact that the introduction of foreign money makes the government constraint in Figure 2 a bit steeper. This is enough to place the optimum away from the corner, given the knife-edge nature of the optimum to begin with.

labor income and define  $1+\tau=1/(1-t)$ . Divide both sides of the budget constraint by  $1+\tau$  and rewrite it as  $c + \hat{R} \cdot m = (T - l) \cdot (1-t)$ , where  $\hat{R}=R \cdot (1-t)$  is the net tax rate on money. Note that the Friedman rule is optimal in the presence of a consumption tax ( $R=0$ ) if and only if it is optimal in the presence of a wage tax ( $\hat{R} = 0$ ).<sup>11</sup>

## 2. The non-validity of the Feenstra (1986) equivalence result when there are taxes.

We have shown some results for models with money in the utility function. It has been argued (Feenstra (1986)) that models of *money in the utility function* (MIUF) and *shopping time models* (STM) are identical if one appropriately redefines consumption expenditures. If so, we should go no further in the investigation of the conditions for the optimality of using the inflation tax because all the results derived in this section would carry over to the next section.<sup>12</sup> Unfortunately, Feenstra's equivalence result does not go through when there are consumption taxes unless one is ready to assume that taxes enter the utility function directly.

In a typical STM, households maximize a utility function which depends on  $c$  and  $l$  only,  $U(c, l)$ , subject to a constraint that includes the shopping time  $Rm + (1+\tau)c + v([1+\tau]c, m) = 1-l$ , where  $v(\cdot)$  is the shopping time function.<sup>13</sup> Typically this function is increasing in  $c$  (consumption takes time) and decreasing in  $m$  (money economizes on the shopping time.) Define  $(1+\tau)e$  as expenditures necessary to consume  $c$ :

$$e(\tau, c, m) = (1+\tau)c + v([1+\tau]c, m) \quad (12)$$

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<sup>11</sup>If the MIUF specification is the reduced form of a cash-credit goods model, our equivalence proof requires that money is used to pay neither income taxes (as assumed by Braun (1994) and others) nor consumption taxes. In order for the equivalence to obtain in the fully dynamic Ramsey problem, initial real government debt must, as often assumed in the literature, be zero.

<sup>12</sup> This argument, for example, is used by Guidotti and Végh (1993) as a reason for restricting their attention to STM.

<sup>13</sup> In the next section we discuss this case in more detail.



with  $e_e > 0$ ,  $e_\tau > 0$  and  $e_m < 0$ . Implicitly (12) defines  $c$  as a function of  $e$ ,  $\tau$  and  $m$ ,  $c(e, \tau, m)$  with  $c_e > 0$ ,  $c_\tau > 0$ , and  $c_m < 0$ . If we plug this in the utility function and the budget constraint we get that the original STM problem can be written as a MIUF problem:

$$\hat{U} = u(e, \tau, l, m) \quad s.t. \quad (1 + \tau)e + Rm = T - l \quad (13)$$

with  $u_e > 0$ ,  $u_\tau > 0$ ,  $u_l > 0$ , and  $u_m > 0$ . The key difference between this and the problem derived in the previous section is that the utility function in (13) depends on taxes directly. Note that if we set  $\tau=0$  (as Feenstra does), then the STM problem is identical to our MIUF problem. If  $\tau > 0$ , however, the equivalence holds only if we assume that taxes enter the utility function. Since we do not know the properties of this model when it comes to optimal taxation issues, we cannot claim generality for the results derived in the previous section. We need to derive explicit conditions for the case when money economizes on shopping time.

### 3. Shopping Time Models (STM)

#### (i) Households:

Consider now a static version of a “transactions” monetary model where, in the spirit of the inventory models of demand for money, money reduces the transaction costs or “shopping time” of purchasing consumption goods rather than entering the consumer’s utility function. Utility is therefore defined only over consumption and leisure.

$$U(c, l) \quad (14)$$

We assume that the usual properties apply to this utility function.

#### (ii) Consumption Tax, Wage Tax and the Consumer’s Budget Constraint

In Section 1 we show that, in models of MIUF, a Ramsey problem with a consumption tax is equivalent to a Ramsey problem with a wage tax. This equivalence does not hold for all versions of

the STM. If shopping time receives the same treatment under the consumption and wage taxes and neither consumption nor wage taxes are paid with money, then the equivalence does hold and the proof is as presented in Section 1. However, one peculiarity of the previous literature is the assymmetric treatment of the amount of taxes paid with money. In particular, models with *consumption taxes* include the implicit assumption that *all taxes are paid with money*.<sup>14</sup> Thus, if  $v(\cdot)$  is the time spent “transacting” or “shopping” during a period, and  $\tau c$  is consumption taxes, then  $v(\cdot)$  would depend on  $(1+\tau)c$ . In contrast, when researchers consider a *wage tax*, it is implicitly assumed that the *wage taxes are not paid with money* so the shopping time function  $v(\cdot)$  is assumed to depend only on consumption and money balances.<sup>15</sup> Results from the previous literature thereby convolve the potential differences between a consumption and a wage tax with the issue of whether tax payments require money. Because there is an equivalence in some cases between consumption and wage tax models, we abstract from the former issue by analyzing consumption taxes only and focus on the latter issue by writing the shopping time function as  $v([1+\lambda\tau]c, m)$ . In this shopping time technology we assume that *all* consumption expenditure must be paid with money but we parameterize the *fraction of taxes that must be paid with money* with the term  $\lambda$ .<sup>16</sup> The fraction  $\lambda$  turns out to be an important determinant of the optimality of the Friedman rule.

The individual’s budget constraint is, therefore:

$$Rm + (1+\tau)c \leq T - l - v([1+\lambda\tau]c, m) \quad (15)$$

The left-hand side is expenditures on money, consumption, and consumption taxes (levied at a flat rate  $\tau$ ). The right-hand side is time spent working, which is the total time endowment ( $T$ ) net of leisure ( $l$ ) and shopping time for the period  $v([1+\lambda\tau]c, m)$ . Note that, when  $\lambda=1$ , all taxes are paid

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<sup>14</sup> See for example Kimbrough (1986) and Guidotti and Végh (1993).

<sup>15</sup> See for example, Faig (1988), Guidotti and Végh (1993), Correia and Teles (1995), and Chari, Christiano and Kehoe (1996).

<sup>16</sup> Our discussion implicitly assumes that the velocity of money used to pay taxes is the same as the velocity of money used to purchase consumption goods. If not, the parameter  $\lambda$  might also be interpreted as a measure of the difference between the velocities. For example, we would have  $\lambda < 1$  if all taxes were paid with money but the velocity of tax money were larger.

with money as has been assumed in the literature with consumption taxes. When  $\lambda=0$ , no taxes are paid with money as has been assumed in the literature the wage taxes. To see this, just set  $\lambda=0$  in (15) and divide both sides by  $(1+\tau)$ . Let “ $t$ ” be the income tax rate and define  $l-t=I/(1+\tau)$  and  $\hat{R} = (1 - t) \cdot R$  to get:  $\hat{R} \cdot m + c \leq [T - l - v(c, m)](1 - t)$  which is exactly the budget constraint found in papers that assume an income tax.<sup>17</sup>

To condense our notation, let us use the notation  $(1+\lambda\tau) \cdot c \equiv x$  to denote the first argument of the shopping time function  $v$ . Our assumptions about the shape of the  $v(\cdot)$  function are

$$\frac{\partial v}{\partial x} \equiv v_x \geq 0, \quad \frac{\partial v}{\partial m} \equiv v_m \leq 0, \quad \frac{\partial^2 v}{\partial m^2} \equiv v_{mm} > 0, \quad \frac{\partial^2 v}{\partial m \partial x} \equiv v_{mx} < 0, \quad v_m = 0 \text{ at finite } m \quad (16)$$

The first two derivatives reflect the assumption that consumption takes time and that money saves on this time. The next two (second order) derivatives restrict the concavity properties of  $v(\cdot)$ : The assumption  $v_{mm} > 0$  makes the *consumer* problem concave. The assumption  $v_{mx} < 0$  is not strictly necessary, but we make it because it implies a positive scale elasticity of money demand, which we think is realistic. The last assumption says that there is some finite stock of real money for which additional money cannot reduce shopping time. That is, the satiation point is finite.

The first order conditions for this consumer problem are:

$$\frac{U_c}{U_l} \equiv D(c, l) = 1 + \tau + (1+\lambda\tau)v_x, \quad -v_m = R \quad (17)$$

The second order conditions are  $\Delta \equiv -(D_c - D \cdot D_l) + (1+\lambda\tau)^2 \{H(x, m)/v_{mm}\} > 0$ , where  $H(x, m)$  is the Hessian of the shopping time function  $v(\cdot)$  and is given by  $H = v_{xx} \cdot v_{mm} - v_{xm}^2$ .

The solution to this consumer problem yields the optimum demands as functions of the prices  $\tau$  and  $R$ , which satisfy the first order conditions and the budget constraint. Let's call these Marshallian demand functions  $c(\tau, R)$ ,  $l(\tau, R)$ , and  $m(\tau, R)$ .

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<sup>17</sup> Note that the tax rate on money  $\hat{R}$ , is different from  $R$  because it is the tax rate *net* of income taxes. However,  $\hat{R}$  goes to zero as  $R$  goes to zero. So the conditions for the optimality of the Friedman rule ( $R=0$ ), will also make  $\hat{R} = 0$  optimal.

(iii) *Government:*

The Ramsey optimal policy maximizes the utility of the representative consumer subject to the government budget constraint and consumers' decision rules. The problem is of the same form as the problem posed in (3). As in Section 1, we assume that the government operates on the upward-sloping parts of its Laffer curves so condition (4) holds.

The consumer's first order condition  $v_m(x,m) = -R$  implies that the demand for money can be expressed as a(n implicit) function of  $x$  and  $R$ ,  $\theta(x,R)$ . From now on, we refer to the elasticity of this money demand function with respect to  $x$  as the “scale elasticity” and denote it  $\beta(x,m)$ . We refer to the absolute value of the elasticity with respect to  $R$  as the “interest elasticity” and denote it  $\epsilon_R$ . Using the implicit function theorem on  $\theta(x,R)$ , we get that the scale and interest elasticities of money demand are given by:

$$\beta(x,m) = - \frac{x}{m} \cdot \frac{v_{xm}}{v_{mm}} , \quad \epsilon_R = \frac{R}{m} \cdot \frac{1}{v_{mm}} \quad (18)$$

If Ramsey optimal  $R$  is 0, then it must be the case that at  $R=0$ , the marginal rate of substitution is greater than or equal to the marginal rate of transformation, as depicted in Figure 2. In other words, if  $R=0$  has to be an optimum, then it must be the case that the slope of the indifference curve at the corner  $R=0$  is at least as large as the slope of the budget constraint. Using this condition, we derive the following key proposition:

**Proposition 4:**

*If the Ramsey problem is concave, then its optimum taxes (doesn't tax) money as*

$$\frac{\tau}{\Delta} \left( [\beta - 1] m (1 + \lambda v_x) - \lambda m x \frac{H}{v_{mm}} \right) \Big|_{R=0} < ( \geq ) \lambda v_x c m \Big|_{R=0} . \quad (19)$$

Proof: See Appendix 2.

It is interesting to note that this condition relates the optimality of the inflation tax to the size of the scale elasticity of money demand,  $\beta$ . For example, if we are willing to assume that the

fraction of taxes paid in cash is zero,  $\lambda=0$ , or that the Hessian of the shopping time function is zero when  $R=0$ ,  $H=0$ , then the second term in (19) vanishes. It follows that the desirability of the inflation tax hinges on whether the scale elasticity of money demand is  $\beta$  is less than one.

The key point, however, is that the size of  $\beta$  at  $R=0$  will in general be related to  $H$  and the other variables in (19), and that all these variables will be driven by the specific assumptions on the functional form of the shopping time function,  $v(\cdot)$ . In other words, once one makes specific assumptions about  $v(\cdot)$ , then the values at  $R=0$  of  $H$ ,  $\beta$ ,  $v_x$ , and the other relevant parameters in (19) are fully determined and, as a result, so is the desirability of the Friedman rule. We will next show how various authors in the literature have arrived at particular cases of (19) by imposing restrictions on the cost function  $v(\cdot)$ .

### 3.A. Particular Case 1. Kimbrough (1986).

Kimbrough (1986) is usually credited with the notion that the Friedman rule “*should always be followed even when distortionary taxes are levied to raise revenue, in sharp contrast to the earlier public finance based literature on the inflation tax by Phelps (1973). The crucial feature of the model employed here that accounts for this difference in results is the explicit recognition of money’s role as an intermediate good that helps to effect the conversion of scarce resources into final consumption goods.*” (Kimbrough (1986), p 283.) According to Kimbrough, the intuition for his result is that, once one has argued that money is an intermediate good, then the Diamond and Mirrlees (1971) theorem suggests that (under some conditions) intermediate goods should not be taxed.

We can analyze Kimbrough’s results and intuition by considering his assumed functional form for the shopping time technology:

$$v((1+\tau) \cdot c, m) = (1+\tau) \cdot c \cdot L\left(\frac{m}{(1+\tau) \cdot c}\right) \quad (20)$$

with  $L' \leq 0$ ,  $L'' > 0$ , and  $v_x|_{R=0}=0$ . In Appendix 3 we show that this specification implies  $H|_{R=0}=0$  and  $\beta|_{R=0}=1$ . This leads to the following

**Proposition 5.**

*If the government problem is concave and  $v(\cdot)$  takes the form in Kimbrough (1986), then Ramsey optimal policy does not tax money.*

Proof: The assumptions  $H|_{R=0}=0$  and  $\beta|_{R=0}=1$  make the left hand side of condition (19) vanish while the assumption  $v_x|_{R=0}=0$  makes the right hand side vanish. It follows that condition (19) holds with equality so our Proposition (4) suggests that, for the Kimbrough assumptions, the Friedman Rule is optimal.

Our Propositions (4) and (5) show that Kimbrough's finding is only partly the result of modeling money as an intermediate input: if the parameters of the model were such that the left hand side of (19) were negative, it would be optimal to tax money, even if money is an intermediate input. Our results also suggest that, for the particular functional forms assumed by Kimbrough, his assumption of a consumption tax (that is,  $\lambda=1$ ) is irrelevant: his results would hold for any value of  $\lambda$ . It is also worth noting that with  $\lambda>0$ , Kimbrough's assumption  $v_x|_{R=0}=0$  is crucial for his results to hold: if  $v_x|_{R=0}>0$ , then the left hand side of condition (19) would be positive and the left hand side would be negative so our proposition would call for the taxation of money.<sup>18</sup>

### **3.B. Particular Case 2. Homogeneous Shopping Time Technology and Taxes NOT Paid with Money** (Chari, Christiano and Kehoe (1996) and Correia and Teles (1995).)

Chari, Christiano and Kehoe (1996) and Correia and Teles (1995) (from now on CT) use a homogeneous of degree  $k$  shopping time function and an income tax (which is equivalent to  $\lambda=0$  in our notation, or taxes not paid with money) to prove that the Friedman rule is optimal.

If  $v(\cdot)$  is homogeneous of degree  $k$ , it can be written as:

$$v([1+\lambda\tau]c, m) = ([1+\lambda\tau]c)^k \cdot L\left(\frac{m}{([1+\lambda\tau]c)}\right) \quad (21)$$

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<sup>18</sup> This was first pointed out by Guidotti and Végh (1993).

with  $L' \leq 0$  and  $L'' > 0$ . An important implicit assumption of this formulation is that it implies that the scale elasticity approaches one as  $R$  approaches zero. That is, if  $v(\cdot)$  is homogenous of any degree, then  $\beta(x, m)|_{R=0} = 1$  (see Appendix 3 for a proof of this result). Under these assumptions, the following proposition holds:

**Proposition 6.**

*If the Ramsey problem is concave,  $v(\cdot)$  is homogeneous and taxes are NOT paid with money (so  $\lambda=0$ ), then the Ramsey optimal policy does not tax money.*

Proof: plug  $\lambda=0$  and  $\beta=1$  in condition (19) and get that the two sides vanish. When the two sides of (19) are equal, the necessary condition for  $R=0$  is satisfied so the Ramsey policy does not tax money.

Using this specification, CCK (1996) show that "if  $v(\cdot)$  is homogenous of degree  $k$  and  $k \geq 1$ , the Friedman rule is optimal" (see Proposition 5 in their paper). CT (1995), on the other hand argue that "the optimality of the Friedman rule is a general result (for all degrees of homogeneity) in models where money reduces transaction costs and alternative taxes are distortionary...[as long as] the cost of producing money is insignificant" (see first paragraph of their conclusions.)

Our finding indicates that this result is not general: the result in CCK and CT follows immediately from the particular cost function assumed, (which implies a money demand function with a scale elasticity  $\beta=1$  at  $R=0$ ), and the assumption that money is NOT used to pay taxes so  $\lambda=0$ . In other words, if you restrict the shopping time function  $v(\cdot)$  to be homogeneous, and you assume  $\lambda=0$ , then you are *assuming* that the two sides of condition (19) are always zero. Under these assumptions, our Proposition (4) also calls for the non-taxation of money. But note that, far from being general, this results depends crucially on the assumed functional form of  $v(\cdot)$  and the assumed value of  $\lambda=0$ .

As it was the case for the CCK model of MIUF discussed in Section 1, the fact that the optimum  $R=0$  is reached in this case by setting the two sides of condition (19) equal to zero suggests

that this optimum is a “knife-edge” case which may easily break down as soon as some of the particular assumptions of the model are violated. In terms of Figure 2, the optimality of the Friedman rule discovered by CCK and CT corresponds to an indifference curve like A: the indifference curve is tangent to the government budget constraint exactly at the corner  $R=0$ . Any slight tilting of either the budget constraint or the indifference curve will break the interior result at exactly  $R=0$  so the Friedman rule will not be optimal.<sup>19</sup>

Another interesting implication of this analysis is that, if  $\lambda=0$ , then Friedman Rule satisfies the first order conditions of the Ramsey problem regardless of the degree of homogeneity,  $k$ . This confirms the finding in CT, who generalize the result in CCK. However, one needs to be careful with the second order conditions of the Ramsey problem because this problem might not be concave. For example, we have generated examples with  $k<0$  where the Ramsey problem is not concave so the usual first order conditions derived by CCK and CT do not characterize a maximum. In this case, proposition 4 does not apply, and the Friedman Rule is not optimal.<sup>20</sup>

### 3.C. Particular Case 3. Homogeneous Shopping Time Technology and Taxes Paid with Money

If we keep the assumption that the cost function is homogeneous of degree  $k$ , but we now allow for a fraction of the real taxes to be paid with money (so  $\lambda>0$ ), then it is no longer true that the two sides of (19) vanish. In fact, using the properties of the shopping time function at  $R=0$  derived in Appendix 3, it is easy to see that, when  $k\geq 1$  and  $\lambda>0$ , the left hand side of (19) is unambiguously negative and the right hand side is unambiguously positive so it is optimal to tax money, unless one makes the additional assumption that  $v|_{R=0} = v_x|_{R=0} = 0$ . In fact, if the Laffer conditions hold, this result also applies to the case  $k\leq 1$ . We state this result in the following proposition.

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<sup>19</sup> As it was also the case for the model of MIUF, the results in CCK-CT disappear in the sense of the Friedman rule being suboptimal as soon as one introduces foreign money. This small change makes the government constraint a bit steeper, and this is enough to overturn the CCK-CT result.

<sup>20</sup> Both CCK (1996) and CT (1995) ignore the issue of concavity of the Ramsey problem and focus their attention to the first order condition of the problem.



**Proposition 7.**

*If the Ramsey problem is concave,  $v(\cdot)$  is homogeneous, the Laffer conditions hold,  $v_x \neq 0$  at  $R=0$ , and some taxes ARE paid with money (so  $\lambda > 0$ ), then the Ramsey optimal policy DOES tax money.*

Proof: For  $k \geq 1$ , substitute  $H(\cdot)|_{R=0} = v_{mm}k(k-1)v/x^2$  and  $v_x|_{R=0} = vk/x$ , and  $\beta|_{R=0} = 1$  in condition (19) to get  $-\tau(k-1)/\Delta < (\geq) c$ . When  $k \geq 1$ , the left hand side is unambiguously negative so Proposition 4 calls for  $R^* > 0$ . For  $k < 1$ , note that the Laffer conditions imply  $\tau/(\Delta \cdot c) < 1$ . Hence,  $\tau(k-1)/(\Delta \cdot c) < 1$ . Use this and the properties of the homogeneous  $v(\cdot)$  function at  $R=0$  to get that the left hand side of (19) is less than the right hand side so  $R^* > 0$ .

This proposition suggests that the main result in CCK is fragile: since they do not impose any restrictions on  $v_x$  at  $R=0$ ,  $v_x$  is not necessarily zero.<sup>21</sup> Proposition 7, then, suggests that the CCK result depends crucially on the assumption that NO money is used to pay taxes (so  $\lambda=0$ ) in the sense that, as soon as some money is used to pay taxes ( $\lambda > 0$ ), the Friedman rule is not optimal even if the rest of their technological assumptions are satisfied.

**3.D. Particular Case 4: Faig (1988)**

Faig (1988) arrives at the conclusion that money should be taxed or not taxed depending on the relative size of various elasticities and, in particular, the size of the scale elasticity,  $\beta$ . His results can be derived with the following shopping time function:

$$v([1 + \lambda\tau] \cdot c, m) = ([1 + \lambda\tau] \cdot c)^\beta \cdot L\left(\frac{m}{([1 + \lambda\tau] \cdot c)^\beta}\right), \quad (22)$$

with  $\beta$  being a constant, and  $L' \leq 0$ , and  $L'' > 0$ . We can now use our Proposition (4) along with the

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<sup>21</sup> In Section 4 we argue that the assumption  $v_x=0$  at  $R=0$  may not be too realistic if, for example, there is the possibility of money being stolen or lost and some “shopping time” needs to be spent to avoid these outcomes even at  $R=0$ .

Faig cost function to analyze his results. In Appendix 3 we show that, under this specification,  $v_{xm}|_{R=0} = -(\beta/x)m \cdot v_{mm}$  and  $v_{xx}|_{R=0} = (\beta/x)^2 m \cdot v_{mm}$ . These two equalities, in turn, imply  $H|_{R=0} = 0$ .<sup>22</sup> If we apply this result to our condition (19) we get:

**Proposition 8:**

*If  $v_x|_{R=0} = 0$ , the government problem is concave and the cost function  $v(\cdot)$  is (20), then the Ramsey optimal policy taxes (doesn't tax) money as  $\beta < 1$  ( $\beta \geq 1$ ) at  $R=0$ .*

Proof: Plug  $H|_{R=0} = 0$  to condition (19) so its left hand side becomes  $(\beta-1)m$  and apply proposition (4).

Faig's result says that the inflation tax is more likely to be desirable, the smaller the scale elasticity of money demand,  $\beta$ .

Faig uses his condition and some "realistic" values for the various parameters to analyze whether an inflation tax is optimal in the United States. Following Baumol and Tobin, he assumes that the interest and scale elasticities are  $1/2$ . He concludes that, even with this small scale elasticity, the optimal inflation tax is zero. The problem with this analysis is that Faig's own specification implies that the interest elasticity of money demand  $\epsilon_R$  goes to zero (not to  $1/2$  as Faig assumes) as  $R$  goes to zero.<sup>23</sup> And condition (19) says that if  $\beta = 1/2 < 1$  the Ramsey optimal policy does tax money.

**3.E. Particular Case 5. Guidotti and Végh (1993) when NO Taxes are Paid with Money.**

In the first part of their paper, Guidotti and Végh (1993) consider a wage tax that is not paid

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<sup>22</sup> Faig further assumes  $\lambda=0$ . It is easy to see that his conclusion could have been reached with any value of  $\lambda$ , given that in condition (19)  $\lambda$  multiplies  $H$  and  $H$  is equal to zero with the Faig technology.

<sup>23</sup>Strictly speaking, Faig's condition (condition (7) in his paper) is: "tax money if  $\epsilon_R|_{R=0} \geq (1-\beta) \cdot ([1+\tau]c_\tau/c) \cdot (g/(1+\tau)c)$ ", rather than "tax money if  $\beta < 1$ ". Note, however, that under his specification,  $\epsilon_R|_{R=0} = -(R/m)(1/v_{mm})$ . Since  $v_{mm} > 0$  and  $m > 0$ , it follows that  $\epsilon_R|_{R=0} = 0$  as  $R=0$ . Since  $c_\tau < 0$ , it follows that, at  $R=0$ , Faig's condition (7) becomes  $\beta < 1$ .

with money. In our notation, this is equivalent to setting  $\lambda=0$ . Condition (10) in their paper reads that a necessary condition for the Friedman rule to be optimal is

$$\Delta\tau(\beta-1)m \cdot v_{mm}^2 = 0.$$

With this equality in hand, they argue that since  $\beta$  is not equal to one in general, it must be optimal to tax money. However, our analysis proves that this condition is sufficient but not necessary. In other words, with  $\lambda=0$ ,  $\beta>1$  makes the left hand side of (19) strictly positive so the condition for taxing money fails. Their incorrect conclusion, therefore, results from ignoring the possibility of a corner solution at  $R=0$  when  $\beta>1$ .

### 3.F. Particular Case 6: Guidotti and Végh (1993) when ALL Taxes are Paid with Money.

When  $\lambda=1$ , Guidotti and Végh derive a condition analogous to our (19) (it is condition (21) in their paper) except that theirs is an equality. Because they assume that the solution is interior, they get the correct answer whenever this is the case. For example, when  $\beta=1$  and  $H=0$ , they correctly show that the key to the optimality of Friedman's rule is  $v_x=0$  at  $R=0$ . However, their analysis is incorrect when the solution is a corner, which is the case when  $\beta>1$ .

## 4. When Friedman is Not Optimal: What is the Optimal Inflation Tax?

In previous sections, we analyzed the desirability of having a zero or non-zero nominal interest rate. We now characterize the size of the optimal interest rate whenever we find that it is not zero. For this purpose, we have the following Propositions 9 and 10. Before stating the proposition, we define  $\epsilon_{cq}$  as the (absolute value of the) Hicksian elasticity of consumption with respect to its own price,  $q \equiv 1+\tau$ .

### Proposition 9.

*If  $\lambda=0$  and the government problem is concave, then a revenue neutral increase (decrease) in  $R$  and decrease (increases) in  $\tau$  is optimal when:*

$$\epsilon_R < (>) \epsilon_{cq} \cdot (1 - \beta) \cdot \frac{g - Rm(1 - \beta)}{(1 + \tau)c} \quad (23)$$

Proof: See Appendix 4.

This condition says that raising the tax rate on money is optimal whenever the interest elasticity of money is small relative to the tax elasticity of consumption.<sup>24</sup> This result is reminiscent of the Phelps (1973) elasticity-based approach to optimal taxation: the optimal inflation tax will tend to be large when the interest elasticity of money demand is small. It also says that the optimal inflation rate is increasing in the size of the scale elasticity of money demand,  $\beta$ .<sup>25</sup>

The intuition for this result is the following: A consumption tax distorts the consumption-leisure margin. An inflation tax also distorts this margin, in addition to wasting resources according to the shopping time function  $v(x,m)$ . However, we (and those in the literature before us) assume that a consumption tax must be "flat" - it must produce an average tax rate that is independent of the quantity consumed - while inflation *might* be a regressive tax on consumption - it may produce an average tax rate that declines with the quantity consumed. To see this, consider again the consumer's budget constraint:

$$(1+\tau)c + Rm + v[(1+\lambda\tau)c, m] = 1 - l$$

The "average consumption tax rate" - defined as resources lost per unit consumed - is:

$$ATR \equiv \tau + Rm/c + v/c$$

When the scale elasticity of money demand ( $\beta$ ) is unity and  $v(x,m)$  exhibits constant returns to scale, ATR is independent of  $c$  and the entire tax system effectively taxes consumption at a flat rate. Of the two taxes in the model, the consumption tax is the flat tax that does not waste resources, so it is optimal to refrain from taxing money. However,  $\beta < 1$  and a function  $v$  that exhibits diminishing returns to scale means that the inflation tax acts as a *regressive* tax on consumption - the ATR falls with  $c$  when  $R > 0$ . All else equal, a regressive tax is more desirable than a flat tax from the Ramsey

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<sup>24</sup> Faig derives a very similar condition.

<sup>25</sup> Strictly speaking, the right hand side of (23) is increasing in  $\beta$  if  $(\tau c)/(Rm) < 1 - 2\beta < 1$ . In other words, if the revenue from consumption taxation is smaller than the revenue from money creation.

point of view.<sup>26</sup> So the Ramsey tradeoff with  $\beta < 1$  is whether to utilize a regressive tax and waste the resources  $v$  or to forego the regressive inflation tax and economize on  $v$ . The value of a regressive tax relative to economizing on  $v$  depends on:

- (i) how much consumption taxes affect the consumption-leisure choice (ie, the magnitude of  $\epsilon_{cq}$ )
- (ii) how much inflation distorts the money holding decision (ie, the magnitude of  $\epsilon_r$ )
- (iii) the degree of regressivity (ie, the magnitude of  $1-\beta$ )

Our expression (23) shows that higher  $\epsilon_{cq}$  and  $(1-\beta)$  tend to increase the optimal inflation tax while a higher  $\epsilon_r$  tends to decrease it. The reason is that a high  $\epsilon_{cq}$  increases the value of the regressive taxation of consumption while higher  $(1-\beta)$  increases the degree of regressivity. Higher  $\epsilon_r$  increases the distortion of the money-holding decision which increases the waste of resources  $v$ .

Proposition 9 assumes that  $\lambda=0$ . Unfortunately, we are not aware of an expression as simple as (23) for the case for  $\lambda>0$ . In order to characterize the optimal policy in this case, we have Proposition 10, its corollary and some numerical examples.

**Proposition 10.**

*If the shopping time function  $v(x,m)$  is homogeneous of degree one and the Laffer conditions hold, then the Ramsey optimal inflation tax depends only on the monetary parameters (the fraction of taxes paid with money,  $\lambda$ , and the shape of the shopping time function  $v(\cdot)$ ).*

Proof: See Appendix 5.

This proposition will be useful in the computation and interpretation of our numerical simulations. It is interesting to note, however, that this proposition says that, if there are constant returns to scale, then the optimal inflation tax rate is *independent* of the size of government

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<sup>26</sup> The reason is that the deadweight loss per dollar of revenue is lower with a regressive tax system: the deadweight loss depends on the *marginal* tax rate while the revenue depends on the *average* tax rate. In a regressive tax system the *average* tax rate is always larger than *marginal* tax rate.

spending. This suggests that the fact that the cross-country correlation between the size of government spending as a fraction of GDP and the inflation rate is nearly zero is not evidence against the hypothesis that governments choose inflation according to the optimal theory of taxation.

**Corollary.** If the shopping time function takes the form:

$$v(x, m) \equiv x L(x/m) + \gamma x, \text{ where } L(z) \equiv A \frac{(z - \bar{z})^2}{z} \text{ defined on } z \geq \bar{z} > 0$$

The optimal inflation tax rate is:

$$R^* = 2A\bar{z}^2 \lambda \frac{2A\bar{z}\sqrt{1 + (4A\bar{z} - \gamma)\lambda^2\gamma} - (1 - \lambda\gamma)(2A\bar{z} - \gamma) + 2A\bar{z}\gamma}{[1 + \lambda(2A\bar{z} - \gamma)]^2} \quad (24)$$

Proof: Substitute functional forms into part (iii) of the proof of the Proposition and solve the quadratic equation.

### III. The Optimum Quantity of Money: Evidence

Our analysis shows that the optimum quantity of money depends on the interest elasticity of money demand, the importance of scale economies in the holding of money, the degree of distortion of the consumption-leisure choice by taxes on consumption, the fraction of taxes paid with money, and money demand behavior at low interest rates. This section of our paper reviews available evidence on this factors and computes the optimal inflation tax. Our numerical examples utilize the STM, although a comparison of STM and the MIUF sections of our paper reveals that many of the same empirical questions arise in both settings. So our review of the evidence is just as relevant for calibrating the MIUF model as for calibrating the STM.<sup>27</sup>

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<sup>27</sup>See Braun (1994) for one study that calibrates a MIUF model to compute the Ramsey optimal inflation tax.

### Evidence on the size of $\beta$

The results in the theory section highlight the importance of knowing empirically whether the scale elasticity is less than one ( $\beta < 1$ ) and how this scale elasticity might vary with the interest rate. It is important to stress that the relevant scale elasticity is the “scale elasticity net of wages”. This is an important point because it is the case in many data sets that consumption is correlated with wages. Higher wages in the STM tend to lead people to use more money as the value of time induces them to spend less time shopping. This may tend to bias empirical estimates of the scale elasticity upwards. It is clear, however, that in the STM model (and in the MIUF model where money is a substitute for leisure), the right estimate of  $\beta$  is net of wages.

In cross-sectional regressions of household demand deposit holdings on household income, Mankiw (1992) finds an income elasticity of roughly 0.5. Using a very similar data set and empirical specification, we find an income elasticity of nearly 1.<sup>28, 29</sup> Using aggregate time-series data, Friedman (1959) and Friedman and Schwartz (1982) find income elasticities greater than one while Meltzer (1963b) and Lucas (1988) find income elasticities of one. However, all five estimates are upward biased estimates of  $\beta$ . The reason was just outlined: the value of time increases the demand for money in the STM model but is an omitted variable which is highly positively correlated with income in the five studies. Karni (1974) and others have included measures of the value of time in empirical money demand equations estimated with aggregate time series data. Although multicollinearity of the measures of scale and the value of time is a problem in these studies, Karni (1974) and others suggest that  $\beta < 1$ .

Meltzer (1963a) and Mulligan (1997b) estimate scale elasticities of money demand across

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<sup>28</sup>We use the 1989 Survey of Consumer Finances (SCF) while Mankiw used the 1983 Survey. We believe that Mankiw measures the amount that a household has in its largest checking account. We find in the 1989 Survey that richer households are much more likely to hold multiple checking accounts and that the estimated income elasticity of demand deposits is larger when the amounts in all checking accounts are summed rather than measuring only a single account. See Mulligan and Sala-i-Martin (1996a) for detailed information on our SCF sample and measurements.

<sup>29</sup> Both these studies measure only demand deposits rather than the sum of currency and demand deposits, so it may be that the income elasticity of the sum is smaller than the demand deposit elasticity.

firms.<sup>30</sup> Meltzer estimates  $\beta=1$  and Mulligan estimates  $\beta=0.75$  with more recent data. Because labor markets function to some degree, it is likely that the variability of the value of time across firms is very much smaller than the variable of the scale of operations, so Meltzer and Mulligan have estimates that are not substantially biased because of an omitted value of time variable.

Propositions 4, 6, 7 and their corresponding examples show that it matters for the optimal inflation tax question how  $\beta$  varies with the interest rate. In particular, (assuming the Ramsey problem is concave) the Friedman Rule is optimal when the shopping time function is homogeneous, money is bounded, and  $\lambda=0$  because that functional form implies that  $\beta$  approaches 1 as  $R$  approaches 0.<sup>31</sup> We know of no direct evidence on the relationship between  $\beta$  and  $R$ , although the empirical findings of Mulligan and Sala-i-Martin (1996a) suggest that the scale elasticity of the household demand for money does approach one for interest rates that are only slightly below current rates (see below). Given the lack of empirical evidence on this issue we assume that  $\beta$  approaches 1 as  $R$  approaches 0 as implied by a homogeneous shopping time function. Our assumption in this regard has a quantitatively important effect on the Ramsey optimal inflation tax rate.<sup>32</sup>

#### Evidence on substitution of market for nonmarket activity

$\epsilon_{cq}$  is the Hicksian own price elasticity of consumption. It measures how compensated changes of the relative price of consumption and leisure motivate substitution of consumption for leisure. As Faig (1988) suggests, one might approximate  $\epsilon_{cq}$  with a compensated labor supply elasticity. Contrary to Faig's suggestions, we believe that the existing labor supply literature is

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<sup>30</sup>Meltzer studies a cross-section of industries while Mulligan studies a micro cross-section of firms.

<sup>31</sup>The Ramsey problem must be concave in the neighborhood of the Friedman Rule when the shopping time function is homogenous of degree  $k \geq 0$ , money is bounded,  $\lambda=0$ , and the Laffer conditions hold. We have found examples of nonconcave Ramsey problems when the shopping time function is homogenous of degree  $k < 0$ , money is bounded,  $\lambda=0$ , and the Laffer conditions hold.

<sup>32</sup>The scale elasticity does not approach one in Braun's (1994) cash-credit goods model, which is an important reason why he tends to find fairly large optimal inflation taxes.



consistent with a *large* compensated labor supply elasticity. One fruitful way to estimate such a compensated elasticity is to look at the labor force participation decisions of women. Mincer (1962) and Killingsworth (1983) do so and find elasticities ranging of 1 or larger.<sup>33</sup> Because there can be no wealth effect of a higher wage on the labor supply of someone who does not participate, we are comfortable in interpreting the female participation estimates as estimates of compensated labor supply elasticities. With some restrictions on intertemporal preferences, one can also estimate the static uncompensated labor supply elasticity from temporal wage and labor supply changes because one expects the substitution effect of anticipated temporary wage changes to dominate the wealth effect. Mulligan provides a number of intertemporal labor supply elasticity estimates, many of which are as large as 1 or 2. When we calibrate the utility function in our numerical examples, we do so in a way that implies  $\epsilon_{cq}$  is fairly near one, but less than 2.

#### Evidence on the interest elasticity of money demand when interest rates are small

In the STM, assumptions about the shopping time function  $v(x,m)$  imply that the interest elasticity approaches zero as  $R$  approaches zero. Moreover, our theoretical analysis shows that the Ramsey optimal inflation tax depends on exactly how the interest and scale elasticities change as  $R$  approaches zero. We are aware of only one empirical study of this question, Mulligan and Sala-i-Martin (1996a), so their modeling and empirical results play a central role in our calibration of the STM.

Three issues arise in calibrating the interest elasticity in the STM. First, what is the level of interest elasticity at a given interest rate? Second, how does the interest elasticity vary with the interest rate? Third, how might the different behavior of households and firms be combined into that of a single representative agent? On the first point, Mulligan and Sala-i-Martin (1996a) estimate the interest elasticity of household money demand for a variety of interest rates show that it is as small (in absolute value) as 0.1 for moderately small interest rates. Mulligan (1997b) finds interest

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<sup>33</sup>Although the theory suggests that both taxes and pretax wages should effect labor supply through their influence on the price of leisure, several empirical studies find very small effects of taxes on female labor supply. See Mroz (1987) for a survey.

elasticities between -1.2 and -0.7 for firms in the 1970's and 1980's.

In the models that we have studied, the interest elasticity of money demand is at least as large in magnitude as the wage elasticity.<sup>34</sup> The wage elasticity therefore puts a lower bound on the magnitude of the interest elasticity. Although it is often difficult to empirically separate the wage elasticity from the scale elasticity, Mulligan (1997b) is able to do so in a cross-section of firms by looking at wages and money holdings by firms in different regions. He estimates a wage elasticity of 0.75. Karni (1974) estimates using aggregate time series data are also consistent with a wage elasticity of 0.75.<sup>35</sup>

On the second point, Mulligan and Sala-i-Martin (1996a) provide a model and some evidence for households. The basic idea of the model is that agents can pay a fixed cost to “adopt a financial technology” (which can be as simple as opening a mutual fund account) which makes their money holdings more interest sensitive. The incentives to pay the fixed cost are increasing in the interest rate and the scale of activity.

On the third point, the magnitude of the interest elasticity of money demand appears to be larger for firms than for households. Because firms tend to operate on a much larger scale than do households, the difference between households and firms is consistent with the assumptions we have made about the function  $v(x,m)$  or with the microfoundations for that function provided by Mulligan and Sala-i-Martin (1996a) and others. We therefore apply the Mulligan and Sala-i-Martin model to both households and firms, but assume that the average firm operates on a scale which is 10,000 times larger than the scale of an average household. Although we recognize that a fuller analysis would explicitly model the heterogeneity of agents using money - and Faig (1988) is one step in this direction - we obtain the money demand of a single representative agent by summing the money

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<sup>34</sup> In the STM  $R$  is DEFINED to be the price of money relative to leisure so  $\epsilon_R$  is exactly the interest elasticity and exactly the wage elasticity. The STM model makes the special assumption that transactions take time but not goods. If transactions took both time and goods, then  $R$  would be the ratio of the interest rate to a “transactions price index” which would be a weighted average of the price of time  $w$  and the price of goods used in transactions. It follows that the wage elasticity would be less than  $\epsilon_R$  because  $w$  would only be a fraction of the “price of transactions”.

<sup>35</sup> The wage elasticity is also relevant for comparing shopping time models with other models. Braun's (1994) cash-credit goods model, for example, implies that the wage elasticity is zero.

demand of a representative household and a representative firm. In order to determine the relative importance of the representative household and firm in the sum, we use Mulligan's (1997a) review of two data sets that provide some information on the relative importance of households and firms in the ownership of M1. His review concludes that, at current interest rates, firms own about twice as many demand deposits as households. If firms hold currency equal to two days retail sales and each adult holds \$100 in currency, then a lot of currency is unaccounted for but the calculation suggests that households may hold more currency than do firms.<sup>36</sup> Based on these calculations, we assume that firms hold 50% more M1 than do households at current interest rates.

The Mulligan and Sala-i-Martin model, the Allais-Baumol-Tobin inventory model, cash-in-advance models, and other micromodels of the demand for money imply that money demand is bounded. In many CIA models with  $R=0$ , for example, there is no reason to hold more money than the present value of all future consumption. In the Mulligan and Sala-i-Martin model, the demand for money is limited by the amount of liquid assets that an agent owns. Given that we assume that the interest elasticity varies with the interest rate as in the Mulligan and Sala-i-Martin model, we also assume that satiation occurs when all "liquid assets" have been converted into money. In 1995 in the U.S.  $(\text{consumption} + \text{taxes})/(\text{liquid assets}) \approx 1$ , so we assume in our preferred numerical examples that the satiation point for  $x/m$  is unity.<sup>37</sup> This calibration of the satiation point together with our restrictions on the interest elasticity deliver realistic predictions for velocity and other variables in our numerical examples.

#### Shopping time that does not economize on foregone interest

Another crucial parameter in the theory is the effect of consumption on shopping time under

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<sup>36</sup>The household survey data of Avery et al (1986) suggest that a typical households does hold about \$100 in currency.

<sup>37</sup>Liquid assets (currency, bank deposits, Eurodollars, savings bonds, short-term treasuries, banker's acceptances, and commercial paper) were \$5.7 trillion, consumption expenditures were \$5.0 trillion, and general government purchases were \$1.4 trillion (U.S. Council of Economic Advisors, 1997). Because  $x$  is a flow and  $m$  is a stock, our calibration of the satiation ratio  $x/m$  pins down the period length. We calibrate this ratio to annual data so the period length in our numerical examples is one year.

the Friedman Rule ( $v_x|_{R=0}$ ). If the only purpose of “shopping time” or “shoe-leather costs” were to economize on foregone interest, it seems obvious that shopping time would be zero at  $R=0$  regardless of the level of consumption. However, there may be other reasons for shopping time. As one example, note that some of the features that make an asset “money” - such as the ease of rapid transfer between parties - also make money an object of theft. Some “shopping time” is therefore spent avoiding theft rather than economizing on interest. This shopping time would still exist under the Friedman Rule and presumably would vary with the amount of consumption, so  $v_x$  would not be zero as  $R$  approaches zero. We are unaware of direct estimates of this and other forms of shopping time that do not economize on foregone interest, so we assume that the value of this extra shopping time somewhere between one and ten percent of consumption expenditures.

#### Computation of the Optimal Inflation Tax

What is the optimal inflation tax? We use the empirical results reviewed above to calibrate a numerical version of our model and check some of the numerical results with the closed form solution (24) for the optimal inflation tax that is available in some special cases. In our numerical example, we assume two functional forms for the shopping time function  $v(x,m)$ :

$$v(x,m) \equiv x^k L(x/m) + \gamma x \quad \text{or} \quad v(x,m) \equiv x^k L(x/m) + \gamma \sqrt{x} \quad > 0$$

where  $L(z) \equiv A \frac{(z - \bar{z})^2}{z}$  defined on  $z \geq \bar{z}$ . Both functional forms imply that the interest elasticity of money demand approaches zero as  $R$  approaches zero in a way that conforms closely to the empirical results (discussed above) of Mulligan and Sala-i-Martin (1996a). For  $k=1$ , the first functional form is homogeneous of degree one and satisfies the assumptions made by Correia & Teles (1995), Chari, Christiano, and Kehoe (1996), Faig (1988), and Guidotti and Végh (1993). For  $k=1/2$ , the second function form is homogeneous of degree  $1/2$  and satisfies the assumptions made by Correia & Teles (1995), and Faig (1988). Our only departure in both cases is that we allow some taxes to affect the scale variable  $x$  in the shopping time function. The functional form assumptions above do not coincide with those of Kimbrough (1986) except in the special case  $\gamma=0$  because

Kimbrough assumes  $v = 0$  at the satiation point whereas we assume in the numerical examples that  $v = \gamma x$  or  $v = \gamma x^{1/2}$  at the satiation point.

Proposition 10 shows that the optimal inflation tax depends only on the monetary parameters in the special case  $k=1$ . This special case is included among our numerical examples, but for those examples to which the proposition does not apply, we assume a CES utility function:

$$u(c, l) \equiv \frac{\sigma}{\sigma - 1} \left[ c^{(\sigma - 1)/\sigma} + \alpha l^{(\sigma - 1)/\sigma} \right]$$

The result from our numerical examples are shown in Table 1. A parameterization -  $\gamma=0.01$ ,  $k=.4$ ,  $A=0.0035$ ,  $\bar{z}=1$ ,  $\sigma=1.5$ ,  $\alpha=1$  - that matches pretty closely the empirical evidence reviewed above is displayed in the first column. The optimal inflation tax is very small in this example, about 0.01% per year. Since U.S. interest rates are not equal to 0.01% per year, the realism of the example must be evaluated by the values taken on by the model variables when a suboptimal policy, such as  $R=5\%$  per year, is followed. At  $R=5\%/yr$ , the net scale elasticity  $\beta$  is 0.73, the interest elasticity is 0.45, shopping time as a fraction of GNP is 0.02, annual consumption and GNP velocities are roughly 3, government spending is 21% of GNP, and the consumption tax rate is 26%.

The second term in the shopping time function is linear in  $x$  in the examples reported in the first column of the table. We have no evidence to support this assumption, so we consider in the second column the square root case. The square root case turns out to be pretty similar to the linear case. To help readers evaluate the importance of the extra or second term in the shopping time function, the fraction of shopping time attributable to that second term at  $R = 5\%/year$  is reported for each example.

Since closed form solutions for the optimal inflation tax are reported in (24) for  $k=1$ , we compare these special cases with our preferred parameterization in Table 1. Increasing  $k$  from 0.4 to 1 has an important effect on the net scale elasticity - as expected - but does not substantially affect the optimal inflation tax or values of the variables at  $R = 5\%/year$ . The fourth column shows that changing the importance of the second shopping time term,  $\gamma$ , does affect the optimal inflation tax although it does not affect the money demand function.

The fifth column shows that changing the shopping time parameters in order to lower the interest elasticity of money demand has a substantial affect on the optimal inflation tax. With  $\gamma=0.1$ ,  $A=1/9$ , and  $\bar{z} = 3$ , the optimal inflation tax rate is 21%/year. The parameter change substantially lowers the interest elasticity at  $R=5\%$ /year.

The final column improves on the fifth column by lowering  $k$  so that there are realistic scale economies. With  $k=-5$ , the net scale elasticity of money demand at  $R=5\%$ /year is 0.77. The optimal inflation tax rate in this example is 61%/year. Columns 2, 3, 5, and 6 show that scale economies increase the optimal inflation tax rate, but the effect is of negligible magnitude when  $\gamma$  is small and/or  $\epsilon_R$  is large.

All of the numerical examples reported in Table 1 set  $\lambda=1$ . Equation (24) as well as our experience with these and other numerical examples suggests that  $R^*$  is very nearly linear in  $\lambda$ . Thus to compute  $R^*$  for a numerical example that is the same as one in the Table with the exception of  $\lambda$ , just multiply  $\lambda$  by the  $R^*$  reported in the Table.<sup>38</sup>

#### The Relation Between the Optimal Inflation tax and the Interest Rate Elasticity.

Following Phelps's (1973) seminal paper, the conventional wisdom has been that it is optimal to tax money more heavily when the interest elasticity of money demand is low and vice versa. CCK argue that the conventional wisdom is incorrect since a model that satisfies their assumptions delivers a zero inflation tax for all interest rate elasticities. Since we have shown that the CCK result on the optimality of the Friedman rule is fragile and not general (it holds under their very specific assumptions and it collapses, for example, when some taxes are paid with money), we can also say that their challenge of the conventional view is fragile. Our Proposition 9 shows that the optimal inflation tax will be larger in economies with small interest rate elasticities. Proposition 9 is restrictive in the sense that assumes that no taxes are paid with money, but our numerical examples (some of which are reported in Table 1) confirm that the conventional view is correct in general, and it is even correct under the technological assumptions of homogeneity made explicit in

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<sup>38</sup>Remember from Proposition 6 that, when  $v$  is homogeneous and the Ramsey problem is concave,  $R^* = 0$  when  $\lambda=0$ .

CCK, as long as we allow a positive fraction of the taxes paid with money ( $\lambda > 0$ ). As an example, compare the parameterizations in Columns 4 and 5 in Table 1. The value of  $\epsilon_r$  at 5% interest rates falls from 0.467 in Column 4 to 0.024 in Column 5. Associated with this decline in interest rate elasticity, the optimum interest rate increases from 0.078 to 21. Repeated experimentation has shown that this negative association tends to be quite robust.

#### IV. Conclusions

In this paper we analyzed the conditions under which it is optimal for the government to use an inflation tax in models of money in the utility function as well as in shopping time models. We show the following:

(1) In the model of MIUF, it is not always true that money is taxed in the Ramsey optimal tax program. The taxation of money is optimal ( $R > 0$ ) if money and consumption are substitutes (in the sense that  $c_r > 0$ ) or if the cross-price elasticity is zero ( $c_r = 0$ , which is the case emphasized by Phelps). Since this is a condition that may not be satisfied empirically, it is interesting to see what happens in the case when money and consumption are complements in the sense that  $c_r < 0$ . In this case, we found that the Friedman rule is optimal ONLY when the absolute size of  $c_r$  is larger than  $(m/c)c_q$  at  $R=0$ . One can find restrictions on the utility function that make  $c_r \geq (m/c)c_q$  or  $c_r < (m/c)c_q$ , which suggests that the optimality of an inflation tax cannot be resolved only on theoretical grounds.

(2) We show that, in the presence of taxation, Feenstra's result which says that models of money in the utility function are equivalent to models in which money economizes on shopping time does not hold.

(3) We provide a shopping time model which embodies previous models as particular cases. This allows us to compare the various results found in the literature. The main findings here are:

(A) We show that the Kimbrough (1986) claim that the Friedman Rule is optimal follows from his particular shopping time function and that it does not hold in general.

(B) Correia and Teles (1995) and Chari, Christiano and Kehoe (1996) derive the result that Friedman's rule is optimal if the shopping time technology is homogenous of any degree. We show that this result is fragile. It depends on their assumption that there cannot be scale

elasticities of money ( $\beta=1$ ) at  $R=0$ , that the Ramsey problem is concave and that NO taxes are paid with money. As soon as some taxes are paid with money, we find that with their technology, it is not optimal to follow the Friedman rule, unless other restrictions on the shopping time technology are imposed.

(C) We find that Guidotti and Végh's (1993) strong results in favor of taxing money come from neglecting the possibility of corner solutions.

(4) Hence, our finding is that it cannot be claimed that the Friedman rule is optimal (or optimal) purely on theoretical grounds. It depends on the various elasticities of the model, as suggested by Faig (1988).

(5) We relate the key conditions for the optimal taxation of money to *empirically observable* magnitudes such as the (net) scale elasticity of money demand and the interest elasticity of money demand, both evaluated as  $R$  tends to zero.

(6) After deriving the conditions for taxing money, we derive the size of the optimal inflation tax and we relate it to magnitudes that can be estimated empirically. Two key money demand elasticities are the interest elasticity and the consumption or scale elasticity NET OF WAGES. Our results accord with the conventional wisdom which says that economies with smaller interest rate elasticities will tend to have larger optimal inflation rates and vice versa. This result is reminiscent of Phelps (1973).

(7) We then use our own estimates of the relevant elasticities as well as other measures found in the literature to estimate whether the conditions for the optimality of the Friedman rule hold in the United States, and if they don't, to estimate the optimal inflation tax. Our finding is that it is NOT optimal to set  $R=0$ , but that the optimal inflation tax is small - probably smaller than 1 percent per year. Our conclusion in this regard depends on our estimates of the interest elasticity and other monetary parameters.

One of the main lessons of the present paper is that the optimality of the Friedman rule ( $R=0$ ) depends crucially on the assumptions of the utility function in models of MIUF and the shopping time function in the STM. The strong dependence of policy implications on functional form



assumptions shifts the emphasis of the debate to two new areas. First, can micro-founded models of money demand suggest useful restrictions on the shape of the shopping time or the utility functions? For example, it can be shown that some well known inventory models of money demand are inconsistent with homogeneous of degree  $k \geq 1$  shopping time functions assumed by Kimbrough (1986) and Chari, Christiano and Kehoe (1996).<sup>39</sup> Results derived under the assumption these homogeneity assumptions should therefore be taken with caution. Second, our work has begun to show how results from the empirical money demand can be used to place restrictions on shopping time and utility function. Again, let's take the  $v(\cdot)$  homogeneous of degree  $k \geq 1$  function as an example. This shopping time function implies that both the net and gross scale elasticities of money demand are larger than one, an implication which appears to be inconsistent with the findings of Meltzer (1963), Lucas (1988) or Braun and Christiano (1994) and others. Other relevant empirical results include the magnitude of the interest elasticity and how it varies with the interest rate, the way in which taxes affect the demand for money, and the amount of shopping time that is spent for purposes other than economizing on foregone interest.

Finally, we should point out several limitations of our analysis. First, government spending is given in all the formulations although, in a political equilibrium, the amount of public spending might depend on the kinds of taxes being used. Second, we evaluate the tradeoff between the inflation tax and "perfectly flat" consumption or wage taxes - taxes which have a single rate and no deductions or exemption. The inflation tax may be more or less efficient when compared with "real world" taxes such as an income tax with loopholes, progressivity, or tax rates that depend on the rate of inflation.<sup>40</sup> Third, we focus on the taxation of monetary transactions and ignore many distortions associated with inflation. The taxation of money can in theory be separated from the rate of growth of prices (eg., by paying interest on money), but we recognize that the two seem to be

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<sup>39</sup> For example the Allais-Baumol-Tobin model without integer constraints and with withdrawal cost  $\gamma = \gamma_0 + \gamma_1 2m$ , where  $\gamma_0$  is a fixed cost component and  $\gamma_1 2m$  is a component proportional to the amount withdrawn implies a shopping time technology  $v(\cdot) = \gamma_0 c / 2m + \gamma_1 c$ . If  $\gamma_1 > 0$ , then this function is not homogenous. If  $\gamma_1 = 0$ , then  $v(\cdot)$  is homogenous but of degree 0, not degree  $k \geq 1$  as assumed by CCK.

<sup>40</sup> See Feldstein (1996) for one such analysis.

linked in practice. Fourth, we follow the previous literature and assume that money is a complement with taxable activity. In fact, an important fraction of the stock of money is held by the illegal sector (underground economy) or foreign citizens. Fifth, we follow the previous literature in assuming that the process of government spending and borrowing does not require money. Finally, the previous literature has also assumed that the Ramsey problem is concave, an assumption which we make explicit in each of our propositions. However, we have generated nonconcave examples that are both consistent with the assumptions of Correia and Teles (1995) and with well known inventory models of the demand for money. In some cases, first order conditions of the Ramsey problem are satisfied at the Friedman Rule, but the Friedman Rule is a local minimum of that problem.

Table 1: The Optimal Inflation Tax in Six Numerical Examples						
shopping time parameters <sup>a,b</sup>						
form of extra shopping time term	linear	sqrt	linear	linear	linear	linear
marginal shopping time when $R=0$ , $\gamma$	0.01	0.01	0.01	0.1	0.1	0.1
degree of homogeneity, $k$	0.4	0.4	1	1	1	-5
multiplicative factor, $A$	.0035	.0035	.0035	.0035	1/9	.0003
satiation $x/m$ , $\bar{z}$	1	1	1	1	3	3
fraction of taxes paid with money, $\lambda$	1	1	1	1	1	1
optimal tax on money (%/yr), $R^*$	0.012	0.008	0.007	0.078	21	61
values with $R = 5\%/yr^c$						
net scale elasticity, $\beta$	0.731	0.731	1	1	1	0.768
interest elasticity, $\epsilon_R$	0.449	0.449	0.467	0.467	0.024	0.039
shopping time/GNP, $v(x,m)/(1-l)$	0.018	0.022	0.017	0.096	0.090	0.090
extra term's share of shopping time, $1 - x^k L(x/m)/v(x,m)$	0.550	0.644	0.570	0.930	0.998	0.997
consumption velocity, $c/m$	2.497	2.496	3.091	2.990	2.367	2.406
GNP velocity, $(1-l)/m$	3.246	3.261	4.020	4.370	3.432	3.486
government/GNP, $g/(1-l)$	0.213	0.212	0.214	0.221	0.220	0.220
tax rate, $\tau$	0.257	0.257	0.262	0.307	0.299	0.298
<sup>a</sup> Nonmonetary parameters are $\sigma = 1.5$ , $\alpha = 1$ , $g = 0.1$ for all numerical examples. As shown in Proposition 10, nonmonetary parameters are irrelevant for the optimal inflation tax when $k=1$ and the extra shopping time term is linear <sup>b</sup> The shopping time function takes the form $v(x,m) = x^k L(x/m) + \gamma x$ or $x^k L(x/m) + \gamma x^{1/2}$ ; $L(z) \equiv A(z-\bar{z})^2/z$ . <sup>c</sup> Frequency is annual. Values are computed so that consumer choices of $c$ , $l$ , and $m$ maximize utility taking $\tau$ and $R$ as given and $\tau$ balances the government budget constraint at $R = 0.05$						

## Appendix 1. Properties of the CCK Utility Function.

In this appendix we show that the CCK utility function (10) implies  $c_R / c_q \big|_{R=0} = m/c \big|_{R=0}$ . The CCK problem is to choose  $c$ ,  $m$ , and  $l$  so as to maximize  $u(w(c,m), l)$  subject to  $q \cdot c + Rm = T - l$  with  $u_1 > 0$ ,  $u_2 > 0$ , and SOC, and where  $w(\cdot)$  is homothetic. In the above we define  $1 + \tau = q$  ( $q$  is the effective price of consumption, including the consumption tax).

Given the homotheticity properties of  $w(\cdot)$ , this problem is equivalent to:

$$\max \mu(c \cdot \tilde{\phi}(m/c), l) \text{ st } qc + Rm = 1 - l \text{ with } \mu_1 > 0, \mu_2 > 0, \tilde{\phi}' \geq 0 \text{ and SOC.}$$

Define  $D \equiv \mu_1/\mu_2$ ,  $\Delta \equiv D \cdot D_2 - D_1$  ( $\Delta$  is positive by SOC),  $1/\sigma \equiv -c\phi D_1/D$ . Define  $g(\cdot)$  as the function whose invers is  $g^{-1}(x) = \tilde{\phi}(x)/\tilde{\phi}'(x) - x$  (with  $g^{-1} > 0$  and  $g' \geq 0$ ). The first order conditions for the consumer problem involve  $q/R = g^{-1}(m/c)$ . Invert  $g^{-1}(\cdot)$  to get

$$\text{FOC1: } m/c = g(q/R).$$

$$\text{FOC2: } D[c\phi(q/R), 1 - qc(1 + f(q/R))] \cdot \phi(q/R) + q[1 + f(q/R)].$$

$$\text{Define } \phi(q/R) \equiv \tilde{\phi}(q/R) \text{ and } f(q/R) \equiv g(q/R)/(q/R).$$

$$\text{Lemma 1: } g' = - \frac{(\tilde{\phi}')^2}{\tilde{\phi}'' \tilde{\phi}}$$

$$\text{Proof: By definition of } g(\cdot), \quad \frac{1}{g'} = 1 - \frac{\tilde{\phi}'' \cdot \tilde{\phi}}{(\tilde{\phi}')^2} = 1. \text{ Invert } 1/g' \text{ and get lemma 1.}$$

$$\text{Lemma 2: } \lim_{R \rightarrow 0} \tilde{\phi}'(g(q/R)) \cdot [q/R] = \lim_{R \rightarrow 0} \tilde{\phi}(g(q/R)) < \infty$$

$$\text{Proof: } \lim_{x \rightarrow \infty} \frac{x}{\frac{1}{\tilde{\phi}'(g(x))}} = \frac{\infty}{\infty} = \lim_{x \rightarrow \infty} \frac{1}{\frac{-\tilde{\phi}'' g'}{(\tilde{\phi}')^2}} \quad (\text{the second equality uses L'Hopital's Rule}). \text{ Using}$$

$$\text{Lemma 1, this expression can be rewritten as } \lim_{x \rightarrow \infty} \frac{1}{\frac{g}{g \cdot \tilde{\phi}}} = \lim_{x \rightarrow \infty} \tilde{\phi}$$

$$\text{Lemma 3(i) } \lim_{R \rightarrow 0} f(q/R) = 0$$

*Proof:*  $\lim_{R \rightarrow 0} f(q/R) = \lim_{R \rightarrow 0} \frac{R m}{q c} = 0$  . This last equality holds because money is bounded below infinity.

**Lemma 3(ii)**  $\lim_{R \rightarrow 0} \frac{f(q/R)}{R} \in (0, \infty)$  .

*Proof:* using the definition of  $f(\cdot)$  and  $g(\cdot)$ ,  $\lim_{R \rightarrow 0} \frac{f(q/R)}{R} = \lim_{R \rightarrow 0} \frac{m}{q c} \in (0, \infty)$  , since  $m$  is bounded below infinity.

**Lemma 3(iii):**  $\frac{f'}{f} \frac{q}{R} = \frac{g'}{g} \frac{q}{R} - 1$  .

*Proof:* By definition,  $f(x)=g(x)/x$ . Take derivatives of this expression with respect to  $x$ , multiply by  $x$  and divide by  $f(\cdot)$  to get *Lemma 3(iii)*.

**Lemma 3(iv):**  $\lim_{R \rightarrow 0} \frac{f'}{f} \frac{q}{R} = -1$  .

*Proof:* If  $g(\cdot)$  is bounded then the limit of the elasticity  $\lim_{R \rightarrow 0} \left[ \frac{g'}{g} \frac{q}{R} \right]$  is zero under quite general conditions. Use this and 3(iii) to get 3(iv).

**Lemma 3(v):**  $\lim_{R \rightarrow 0} f'(q/R) q/R = 0$  .

*Proof:* From 3(i) and 3(iv), get that  $\lim_{R \rightarrow 0} \left[ \frac{f'}{f} \frac{q}{R} \right] f = -1 \cdot 0 = 0$  .

**Lemma 4:**  $\lim_{R \rightarrow 0} \frac{\Phi'(q/R)}{\Phi(q/R)f(q/R)} = 0$  .

*Proof:* (a):  $\Phi'/f = \Phi' x \frac{x(g'/g)}{x(g'/g) - 1}$ . (b)  $\lim_{x \rightarrow \infty} \frac{\Phi'}{f} = \lim_{x \rightarrow \infty} \Phi'(x) x \cdot \lim_{x \rightarrow \infty} \frac{x(g'/g)}{x(g'/g) - 1}$ . Use *Lemma 2*

says that the first limit is a finite number. Since  $\lim_{R \rightarrow 0} \left[ \frac{g'}{g} \frac{q}{R} \right] = 0$  , the second limit is 0/-1 so the product is zero.

**Proposition:** If the Utility Function Satisfies the properties of homotheticity and separability (10)

imposed by CCK, then  $\left. \frac{\partial c / \partial R}{\partial c / \partial (1+\tau)} \right|_{R=0} = \frac{m}{c} \Big|_{R=0}$ .

*Proof:* (i) Take derivatives of FOC 2 with respect to  $q$ :

$$[D_1 \phi - D_2 q(1+f)] \phi c_q = D \phi' / R(1/\sigma - 1) + (1 + \phi c D_2)[1 + f + f' q / R].$$

(ii) Take derivatives of the same function with respect to  $R$ :

$$[D_1 \phi - D_2 q(1+f)] \phi c_R = -(q/R)[D \phi' (1/\sigma - 1)/R + (1 + \phi c D_2) f' q / R].$$

(iii) Divide (i) by (ii):

$$\frac{c_q}{c_R} = - \frac{R}{q} - \frac{R}{q} \frac{(1 + \phi c D_2) \cdot (1+f)}{\frac{D \phi'}{R} \left( \frac{1}{\sigma} - 1 \right) + (1 + \phi c D_2) \frac{f' q}{R}} = - \frac{R}{q} - \left( \frac{R}{q} \right)^2 \frac{1}{f'} \frac{1}{\frac{\phi'}{f' \phi} \frac{1/\sigma - 1}{1 + \phi c D_2} + \frac{1}{1+f}}$$

Take the limit of this expression as  $R$  goes to zero:

$$\lim_{R \rightarrow 0} \frac{c_R}{c_q} = -0 - \frac{\lim_{R \rightarrow 0} \left( \frac{R}{q} \right)^2 \frac{1}{f'}}{\frac{1/\sigma - 1}{1 + \phi c D_2} \lim_{R \rightarrow 0} \frac{\phi'}{f' \phi} + \lim_{R \rightarrow 0} \frac{1}{1+f}}$$

Note that  $\lim_{R \rightarrow 0} f = \lim_{R \rightarrow 0} \frac{\phi'}{f' \phi} = 0$ , we get that  $\lim_{R \rightarrow 0} \frac{c_R}{c_q} = - \lim_{R \rightarrow 0} \left( \frac{R}{q} \right)^2 \frac{1}{f'}$ . Since  $\lim_{R \rightarrow 0} \frac{f' q}{R} = -f$ , we

can write the limit as:  $\lim_{R \rightarrow 0} \frac{c_R}{c_q} = - \lim_{R \rightarrow 0} \frac{1}{f q / R} = \lim_{R \rightarrow 0} \frac{1}{g} = \lim_{R \rightarrow 0} \frac{c}{m}$ .

## Appendix 2: Proof of Proposition 4.

Consumers Maximize (14) subject to (15) and to the conditions of  $v(\cdot)$  in (16). FOC's are (17). The solution to this program are the demand functions  $c(\tau, R)$ ,  $l(\tau, R)$ ,  $m(\tau, R)$ .

### Lemma 1:

If  $m^d = L(x, R)$ , then

$$(1.I) \quad L_x = -v_{mx}/v_{mm}$$

$$(1.II) \quad L_R = -1/v_{mm}$$

and (1.III)  $L < \infty$ .

*Proof:* From the FOC,  $v_m(x, m) = v_m(x, L(x, R)) = -R$ .

(1.I) Take the total derivative of the above expression for  $v_m$  with respect to  $x$  and get  $v_{xm} + v_{mm} \partial L / \partial x = 0$ . Rearrange to get (1.I). The inequality follows from the assumptions (16):  $v_{mx} < 0$  and  $v_{mm} > 0$ .

(1.II) Take the total derivative of the above expression for  $v_m$  with respect to  $R$  and get  $v_{mR} \partial L / \partial R = -1$ . Rearrange and multiply both sides and get (ii). The inequality follows from the assumption  $v_{mR} > 0$ .

(1.III)  $v_m(x, L) = 0$  at finite  $L$  by assumption (16)

### Lemma 2.

The demand functions  $c(\tau, R)$ ,  $l(\tau, R)$ , and  $m(\tau, R)$  satisfy:

- (2.I)  $l_R = -(D \cdot c_R + m)$  (or  $m = -(D \cdot c_R + l_R)$ )  
 (2.II)  $l_\tau = -(D \cdot c_\tau + [1 + \lambda v_x]c)$  (or  $(1 + \lambda v_x)c = -(D \cdot c_\tau + l_\tau)$ )  
 (2.III)  $m_\tau = -([1 + \lambda \tau]c_\tau + \lambda c) \cdot v_{xm} / v_{mm}$   
 (2.IV)  $m_R = -(1 + \lambda \tau)c_R v_{xm} / v_{mm} - 1 / v_{mm}$ ,  
 where  $D(\cdot)$  is defined in (19) as  $D = U_c(c, l) / U_l(c, l)$ .

**Proof:**

The budget constraint is given by  $(1 + \tau)c(\tau, R) + Rm(\tau, R) = I - l(\tau, R) - v([1 + \lambda \tau] \cdot c(\tau, R), m(\tau, R))$

(2.I) Take derivatives of both sides of the above constraint with respect to  $R$  and use the FOC  $v_m = -R$  to get  $(1 + \tau + (1 + \lambda \tau)v_x) \cdot c_R + m = -l_R$ . Use the definition of  $D$  to get (2.I).

(2.II) Take derivatives of both sides of the above expression with respect to  $\tau$  and use  $v_m = -R$ .

The demand for money depends on the two prices  $R$  and  $\tau$ :  $m(\tau, R) = L([1 + \lambda \tau] \cdot c(\tau, R), R)$ .

(2.III) Take derivatives with respect to  $\tau$  to get  $m_\tau = L_x(\lambda c + [1 + \lambda \tau]c_\tau)$ . Use lemma (1.I) to substitute for  $L_x$  and get (2.III).

(2.IV) Take derivatives with respect to  $R$  and get  $m_R = L_x(1 + \lambda \tau)c_R + L_R$ . Use lemmas (1.I) and (1.II) to substitute for  $L_x$  and  $L_R$  respectively to get (2.IV).

**Lemma 3.**

$$(3.I) \quad c_\tau = \frac{-[1 + \lambda v_x](1 + c D_\rho) + \lambda c[1 + \lambda \tau](v_{xm}^2 / v_{mm} - v_{xx})}{\Delta}$$

$$(3.II) \quad l_\tau = \frac{[1 + \lambda v_x](D(1 + c D_\rho) - \Delta c) - D\lambda c[1 + \lambda \tau](v_{xm}^2 / v_{mm} - v_{xx})}{\Delta}$$

$$(3.III) \quad c_R = \frac{[1 + \lambda \tau] v_{xm} / v_{mm} - D_l m}{\Delta} < 0$$

$$(3.IV) \quad l_R = \frac{(D D_R - \Delta)m - [1 + \lambda \tau] D v_{xm} / v_{mm}}{\Delta}.$$

Where  $\Delta$  is defined in (19),  $\Delta \equiv -(D_c - D \cdot D_\rho) + (1 + \lambda \tau)^2 \{H(x, m) / v_{mm}\} > 0$  and  $H \equiv v_{xx} \cdot v_{mm} - v_{xm}^2$  is the Hessian of  $v(\cdot)$ . **Proof:**

$D(c(\tau, R), l(\tau, R)) = 1 + \tau + [1 + \lambda \tau] \cdot v_x([1 + \lambda \tau] \cdot c(\tau, R), m(\tau, R))$ .

(3.I) Take derivatives of both sides of the above expression with respect to  $\tau$ . Use Lemma (2.II) to substitute for  $l_\tau$ , rearrange and simplify using the definition of  $\Delta$  to get (3.I).

(3.II) From lemma (2.II),  $l_\tau = -(D \cdot c_\tau + [1 + \lambda v_x]c)$ . Substitute  $c_\tau$  from lemma (3.I) and get (3.II).

(3.III) Take derivatives of the above expression for  $D(\cdot)$  with respect to  $R$ . Use lemmas (2.I) and (2.IV) to substitute for  $l_R$  and  $m_R$  respectively. Rearrange and use the definition of  $\Delta$  to get (3.III).

(3.IV) Use lemma (2.II)  $l_R = -(D \cdot c_R + m)$ . Substitute  $c_R$  by the expression given in lemma (3.III) and get (3.IV).

**Lemma 4.**

$$\begin{aligned}
c_R l_\tau - c_\tau l_R &= \\
&= \frac{1}{\Delta} \left( [\beta(x,m)-1] m [1+\lambda v_x] - \lambda m x \frac{H}{v_{mm}} \right) = \\
&= m c_\tau - c (1+\lambda v_x) c_R =
\end{aligned}$$

where  $\beta(x,m) \equiv -(v_{xm}/v_{mm}) \cdot (x/m)$  is the scale elasticity of money demand.

Proof:

Substitute  $c_\tau$ ,  $l_\tau$ ,  $c_R$ , and  $l_R$  using lemmas (3.I), (3.II), (3.III), and (3.IV) respectively and rearrange to get:

$$c_R l_\tau - c_\tau l_R = \frac{1}{\Delta} \left( \left[ -\frac{x}{m} \frac{v_{xm}}{v_{mm}} - 1 \right] m (1+\lambda v_x) - \lambda m x \left[ v_{xx} - \frac{v_{xm}^2}{v_{mm}} \right] \right) \text{ Use the definition of } \beta(x,m)$$

to substitute for the expression in the first squared bracket, use the definition of the Hessian,  $H$ , to substitute for the expression in the second squared bracket, and use the definition of  $x=(1+\lambda\tau)c$  to get the first equality in lemma (4).

To get the second inequality in lemma (4), rewrite the above expression as

$$c_R l_\tau - c_\tau l_R = \frac{m}{\Delta} \left( \lambda x \left[ \frac{v_{xm}^2}{v_{mm}} - v_{xx} \right] - (1+\lambda v_x) \right) - \left( \frac{x}{m} \frac{v_{xm}}{v_{mm}} m (1+\lambda v_x) \right)$$

Subtract and add the expression  $c(1+\lambda v_x) \cdot m \cdot D_l$  to the first and second term respectively and note that the first term becomes  $m \cdot c_\tau$  by lemma (3.I). Similarly, lemma (3.III) says that the second term is equal to  $c(1+\lambda v_x) c_R$ , which is the second equality we want to prove.

**Lemma 5.**

If the government problem (3) is concave (and the laffer conditions (5) hold) at  $R=0$ , then tax (don't tax) money as

$$\tau(m \cdot c_\tau - c [1+\lambda \cdot v_x] \cdot c_R) \Big|_{R=0} < (\geq) \lambda \cdot v_x \cdot c \cdot m \Big|_{R=0}$$

Proof:

A necessary condition for the taxing (not taxing) of money to be optimal is:

$$\frac{V_\tau}{V_R} \Big|_{R=0} > (\leq) \frac{g_\tau}{g_R} \Big|_{R=0} \text{ where } V(\tau, R) \text{ is the indirect utility function } V(\tau, R) = U[c(\tau, R), l(\tau, R)] \text{ and}$$

$g(\tau, R) = \tau \cdot c(\tau, R) + R \cdot m(\tau, R)$ . From the FOC of individuals  $V_\tau/V_R = |V_\tau|/|V_R| = -(Dc_\tau + l_\tau)/(Dc_R + l_R)$  where  $D$  is the marginal rate of substitution  $D = U_c/U_l$ . From the government budget constraint,  $g_\tau/g_R = (c + \tau c_\tau + R m_\tau)/(\tau c_R + m + R m_R)$ . Using these equalities at  $R=0$ , we can rewrite the above inequality as  $-(Dc_\tau + l_\tau)(\tau c_R + m) \Big|_{R=0} > (\leq) -(Dc_R + l_R)(\tau c_\tau + c) \Big|_{R=0}$ . Rearrange and use lemma (2.I) to substitute  $Dc_R + l_R = -m$  and lemma (2.II) to substitute  $Dc_\tau + l_\tau = -(1+\lambda v_x)$  to get proposition 3.



**Proposition 4,**

Use Lemma (5) and substitute  $[c_R l_\tau - c_\tau l_R]$  by the expression in lemma (4) to get Proposition 4.

**Appendix 3: Properties of various shopping time functions.****(A) Properties of the Kimbrough (1986) Cost Function:**

Assume  $v(x, m) = xL(m/x)$  with  $L' \leq 0$ ,  $L'' > 0$  and  $L|_{R=0} = L'|_{R=0} = 0$ . Also, he assumes  $\lambda = 1$ .

Take derivatives of  $v(\cdot)$ :

$$v_x = L - (m/x)L', \quad v_m = L', \quad v_{mm} = L''/x, \quad v_{mx} = -(m/x)v_{mm}, \quad v_{xx} = -(m/x^2)L' + (m^2/x^3)L'' - (m/x^2)L'$$

At  $R=0$ ,  $L'=0$  so the derivatives become:

$$v_x = 0, \quad v_m = 0, \quad v_{mm} = L''/x, \quad v_{mx} = -(m/x)v_{mm}, \quad v_{xx} = (m^2/x^2)v_{mm}$$

From these derivatives, it follows that

$$H|_{R=0} = v_{xx}v_{mm} - v_{mx}^2 = (m^2/x^2)v_{mm}v_{mm} - [(m/x)v_{mm}]^2 = 0$$

$$\beta|_{R=0} = [-(v_{xm}/v_{mm})x/m]|_{R=0} = 1.$$

**(B) Properties of the homogenous of degree  $k \geq 0$  Cost Function (Chari, Christiano, Kehoe (1996) and Correia and Teles (1995)):**

Assume  $v(x, m) = x^k L(m/x)$  with  $L' \leq 0$ ,  $L'' > 0$  and  $L'|_{R=0} = 0$ .

$$v_x = (k/x)v - x^{k-2}mL', \quad v_m = x^{k-1}L', \quad v_{mm} = x^{k-2}L'', \quad v_{mx} = (k-1)x^{k-2}L' - (m/x)v_{mm},$$

$$v_{xx} = -(k/x^2)v - (k/x)v_x - (k-2)x^{k-3}mL' - x^{k-2}mL'' - (m/x^2).$$

Evaluate these derivatives at  $R=0$  and get

$$v_m = 0, \quad v_x = (k/x)v|_{R=0}, \quad v_{mx} = -(m/x)v_{mm}, \quad v_{xx} = k(k-1)v/x^2 - (m/x)^2v_{mm}.$$

$$(I) \quad H(\cdot)|_{R=0} = (v_{xx}v_{mm} - [v_{mx}]^2) = [k(k-1)v/x^2 - (m/x)^2v_{mm}]v_{mm} - [-(m/x)v_{mm}]^2 = v_{mm}k(k-1)v/x^2.$$

$$(II) \quad \beta = -(v_{xm}/v_{mm})x/m = 1.$$

**(C) Properties of the Faig (1988) Cost Function:**

Assume  $V(x, m) = x^\beta L(m/x^\beta)$  with  $L' \leq 0$ ,  $L'' > 0$  and  $L|_{R=0} = L'|_{R=0} = 0$ .

Compute the derivatives:

$$v_x = (\beta/x)v - \beta mL'/x, \quad v_m = L', \quad v_{mm} = x^{-\beta}L'', \quad v_{mx} = -(\beta m/x)v_{mm}, \quad v_{xx} = -\beta v/x^2 + \beta v_x/x + \beta mL'/x^2 + \beta^2 m^2 x^{-\beta-2}L''$$

$$\text{At } R=0, L'=0 \text{ and } v=0 \text{ so } v_x=0, \quad v_m=0, \quad v_{mm}=x^{-\beta}L'', \quad v_{mx}=-(\beta m/x)v_{mm}, \quad v_{xx}=-(\beta^2 m^2/x^2)v_{mm}$$

Use these derivatives to get

$$H|_{R=0} = [v_{xx}v_{mm} - v_{mx}^2]|_{R=0} = (-\beta^2 m^2/x^2)v_{mm}v_{mm} - [(\beta m/x)v_{mm}]^2 = 0.$$

**Appendix 4: Proof of Proposition 9.**

(i) If  $\lambda=0$ ,  $V_\tau/V_R = c/m$ . This follows from the FOC of individuals  $V_\tau/V_R = (-Dc_\tau l)/(-Dc_R l_R) = (1 + \lambda v_x)c/m$ . Evaluate this at  $\lambda=0$ .

(ii) If  $\lambda=0$ ,  $c_R l_\tau - c_\tau l_R = (m/\Delta)(\beta-1) = mc_\tau \cdot c \cdot c_R$ . This follows directly from evaluating Lemma 4 at  $\lambda=0$ .

$$(iii) \text{ If } \lambda=0, \quad \frac{g_\tau}{g_R} = \frac{\tau c_\tau + c + RL_x \cdot c_\tau}{\tau c_\tau + m + RL_x c_R + RL_R}$$

$$(iv) \quad V_\tau/V_R > (<) g_\tau/g_R \text{ as } \frac{c}{m} > (<) \frac{c_\tau(\tau + RL_x) + c}{(\tau + RL_x)c_R + m(1 - \epsilon_R)}. \text{ This expression follows from (i) and}$$

$$(iii). \text{ Rearrange this expression to get } \epsilon_R < (>) \frac{\tau + RL_x}{cm} (c \cdot c_R - m \cdot c_\tau). \text{ Use (ii) to rewrite this}$$

expression as  $\epsilon_R < (>) \frac{1-\beta}{c} \frac{\tau c + \beta R m}{\Delta} \frac{1}{c}$ .

(v) With  $\lambda=0$ ,  $-\epsilon_{cq} = (1+\tau)c_\tau/c + (1+\tau)D/\Delta$ . Using Lemma 3.1 at  $\lambda=0$ , we get that  $\Delta c_\tau + c D_\tau = -1$ . It follows that  $\epsilon_{cq} = (1+\tau)/(\Delta c)$ .

Plug this expression for  $\epsilon_{cq}$  in the above inequality to get  $\epsilon_R < (>) (1-\beta) \frac{\tau c + \beta R m}{c} \frac{\epsilon_{cq}}{1+\tau}$ . Now

we can use the government budget constraint to rewrite  $\tau c + \beta R m = g - (1-\beta)Rm$  to get Proposition 9.

#### Appendix 5: Proof of Proposition 10.

(i) Since the Laffer conditions hold, the government budget constraint  $g = G(\tau, R)$  implicitly defines a function  $\tau(R)$ . Substituting this function into the consumer's indirect utility function, the Ramsey problem maximizes  $u(R) \equiv v(\tau(R), R)$ .  $u(R)$  satisfies (where  $H$  is the Hessian of  $v(x, m)$ ):

$$u'(R) = \frac{c m u_l}{g_\tau} \left[ \lambda \left( v_x(x, m)(1 - \epsilon_R) - \beta R \frac{m}{x} \right) + \left( (1 - \beta)(1 + \lambda v_x) + \lambda x \frac{H}{v_{mm}} \right) \frac{\tau + \beta R \frac{m}{c}}{c m} - \epsilon_R \right]$$

(ii) The homogeneity of  $v$  and the consumer's optimal choice of  $m$  implies that the ratio  $x/m$  depends only on  $R$  and the shape of  $v$ .

(iii) The homogeneity of  $v$  implies that  $\beta=1$  and  $H=0$  and that the shopping time function can be written as  $v(x, m) = xL(x/m)$ . We have:

$$u'(R) = \frac{c m u_l}{g_\tau} \left[ \lambda L(x/m) + \lambda \gamma - \frac{\epsilon_R}{1 - \epsilon_R} \left( 1 + \lambda R \frac{m}{x} \right) \right]$$

(iv) With the sign of  $u'(R)$  a function of only the monetary variables  $\lambda$ ,  $(x/m)$ ,  $\epsilon_R$ , and  $R$ , the optimal inflation tax depends only on those variables.

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