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CONSUMPTION AND PORTFOLIO DECISIONS WHEN EXPECTED RETURNS ARE TIME VARYING

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ABSTRACT

This paper proposes and implements a new approach to a classic unsolved problem in financial economics: the optimal consumption and portfolio choice problem of a long-lived investor facing time-varying investment opportunities. The investor is assumed to be infinitely-lived, to have recursive Epstein-Zin-Weil utility, and to choose in discrete time between a riskless asset with a constant return, and a risky asset with constant return variance whose expected log return follows and AR(1) process. The paper approximates the choice problem by log-linearizing the budget constraint and Euler equations, and derives an analytical solution to the approximate problem. When the model is calibrated to US stock market data it implies that intertemporal hedging motives greatly increase, and may even double, the average demand for stocks by investors whose risk-aversion coefficients exceed one.

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1 Introduction

The choice of an optimal portfolio of assets is a classic problem in financial economics. In a single-period setting the problem is well understood and analytical solutions for optimal portfolio weights are available in important special cases. When mean-variance analysis is appropriate, for example, optimal portfolio weights are known functions of the first and second moments of asset returns.

In a multi-period setting the problem is far less tractable. Explicit solutions for portfolio weights are available in the special cases where investment opportunities are constant or the investor has log utility and hence acts myopically; but these cases are tractable precisely because they reduce to the familiar single-period problem. Merton (1969, 1971) and Samuelson (1969), followed more recently by Cox and Huang (1989), have shown that in general shifting investment opportunities can have important effects on portfolio choice. They have characterized some properties of optimal portfolios, but their work does not deliver analytical solutions for portfolio weights as functions of state variables. Recently a few papers (Barberis 1996, Brennan, Schwartz, and Lagnado 1995) have used discrete-state numerical methods to find solutions for portfolio weights; but their results, while illuminating, depend on the particular parameter values they assume.

This gap in the literature is particularly unfortunate because there is now considerable evidence that the conditions under which the multi-period portfolio choice problem reduces to the single-period problem do not hold. Many authors have found evidence that expected asset returns vary through time so that investment opportunities are not constant; the evidence for predictable variation in the equity premium, the excess return on stock over Treasury bills, is particularly strong (see Campbell 1987, Campbell and Shiller 1988a,b, Fama and French 1988, 1989, Hodrick 1992, or the textbook treatment in Campbell, Lo, and MacKinlay 1997, Chapter 7). Also, the large literature on the equity premium puzzle finds that average excess stock returns are too high to be consistent with a representative-investor model in which the investor has log utility (see Campbell 1996b, Cecchetti, Lam, and Mark 1994, Cochrane and Hansen 1992, Hansen and Jagannathan 1991, Kocherlakota 1996, Mehra and Prescott 1985, or the textbook treatment in Campbell, Lo, and MacKinlay 1997, Chapter 8).

In this paper we consider the canonical version of the multi-period portfolio choice problem in which an infinitely-lived investor has Epstein-Zin-Weil utility (a generalization of power utility due to Epstein and Zin (1989) and Weil (1989)) and must finance all consumption from the costlessly tradable assets in the portfolio. We work in discrete time and assume that there are two assets: a riskless asset with a constant return, and a risky asset whose expected return, the single state variable for the problem, follows a mean-reverting AR(1) process. The assumption that the riskless return is constant simplifies our analysis and enables us to study the effects of time-variation in the equity premium.

Our strategy is to find an approximation to the portfolio choice problem that can be solved using the method of undetermined coefficients. We approximate the Euler equations of the problem using second-order Taylor expansions, and we replace the investor's intertemporal budget constraint with an approximate constraint that is linear in log consumption and quadratic in the portfolio weight on the risky asset. This enables us to find approximate analytical solutions for consumption and the portfolio weight. The portfolio weight solution is linear in the state variable, while the solution for the log consumption-wealth ratio is quadratic in the state variable.

Our paper builds on the work of Campbell (1993). Campbell considers the simpler problem where only one asset is available for investment and so the agent need only choose consumption. He shows that this problem becomes tractable if one replaces the intertemporal budget constraint by a loglinear approximate constraint. He uses the solution in a representative-agent model to characterize the equilibrium prices of other assets that are in zero net supply, in the spirit of Merton's (1973) intertemporal CAPM. Campbell (1996b) estimates the parameters of the model from US asset market data, while Campbell and Koo (1997) evaluate the accuracy of the approximate analytical solution by comparing it with a discrete-state numerical solution.

This paper is also related to recent work by Kim and Omberg (1996). Kim and Omberg work in continuous time and study the choice between a riskless asset with a constant return and a risky asset whose expected return follows a continuous-time AR(1) (Ornstein-Uhlenbeck) process. They assume that a finite-lived investor has HARA utility of terminal wealth, so their problem does not involve a choice of consumption over time. Kim and Omberg find, without the use of any approximations, that the optimal portfolio weight is linear and the value function is quadratic in the state variable.

The organization of the paper is as follows. Section 2 states the problem we would like to solve, while Section 3 explains our approximate solution method. Section 4 calibrates the model to postwar quarterly and long-run annual US stock market data, and Section 5 concludes.

2 Setup of the problem

2.1 Assumptions

We consider a partial-equilibrium problem in which:

(A1) Wealth consists of two tradable assets. Asset 1 is risky, with one-period log (continuously compounded) return given by $r_{1,t+1}$; asset f is riskless, with constant log return given by $\overline{r}_f$. Therefore, the one-period return on wealth from time t to time $t+1$ is

$$R_{p,t+1} = \alpha_t \left(R_{1,t+1} - \overline{R}_f \right) + \overline{R}_f, \tag{1}$$

where $R_{1,t+1} = \exp\{r_{1,t+1}\}$, $\overline{R}_f = \exp\{\overline{r}_f\}$, and the portfolio weight α_t is the proportion of total wealth invested in the risky asset at time t.

(A2) The expected excess log return on the risky asset is state-dependent. There is one state variable x_t, such that

$$\mathrm{E}_t r_{1,t+1} - \overline{r}_f = x_t. \tag{2}$$

The state variable follows an AR(1):

$$x_{t+1} = \mu + \phi (x_t - \mu) + \eta_{t+1}, \tag{3}$$

where η_{t+1} is a conditionally homoskedastic gaussian martingale difference sequence, i.e., $\eta_{t+1} \sim N\left(0, \sigma_\eta^2\right)$.

(A3) The unexpected log return on the risky asset, denoted by u_{t+1}, is also conditionally homoskedastic and correlated with innovations in the state variable,

$$\mathrm{Var}_t\left(u_{t+1}\right) = \sigma_u^2, \tag{4}$$

$$\mathrm{Cov}_t\left(u_{t+1}, \eta_{t+1}\right) = \sigma_{u\eta}. \tag{5}$$

(A4) The investor's preferences are described by the recursive utility proposed by Epstein and Zin (1989) and Weil (1989):

$$U\left(C_t, \mathrm{E}_t U_{t+1}\right) = \left\{(1-\delta) C_t^{\frac{1-\gamma}{\theta}} + \delta\left(\mathrm{E}_t U_{t+1}^{1-\gamma}\right)^{\frac{1}{\theta}}\right\}^{\frac{\theta}{1-\gamma}}, \tag{6}$$

where $\delta < 1$ is the discount factor, $\gamma > 0$ is the coefficient of relative risk aversion, $\psi > 0$ is the elasticity of intertemporal substitution and the parameter θ is defined as $\theta = (1-\gamma) \,/\, (1-\psi^{-1})$. It is easy to see that (6) reduces to the standard time-separable, power utility function with relative risk aversion γ when $\psi = \gamma^{-1}$; in this case $\theta = 1$ and the nonlinear recursion (6) becomes linear.

(A5) The investor is infinitely-lived.

Assumptions (A1) and (A2) on the number of risky assets and state variables are simplifying assumptions, which we adopt for expositional purposes. The approach of this paper can be applied to a more general setting with multiple risky assets and state variables, at the cost of greater complexity in the analytical solutions to the problem. Assumption (A3) is also a simplification that can be relaxed in order to study the effects of conditional heteroskedasticity on portfolio choice. Assumption (A4) on preferences allows us to separate the effects on optimal consumption and portfolio decisions of the investor's attitude towards risk from the investor's attitude towards consumption smoothing over time. Finally, assumption (A5) allows us to ignore the effects of a finite horizon on portfolio choice, but this assumption too can be relaxed in future work.

2.2 Euler Equations

The individual chooses consumption and portfolio policies that maximize (6) subject to the budget constraint

$$W_{t+1} = R_{p,t+1}\left(W_t - C_t\right), \tag{7}$$

where W_t is total wealth at the beginning of time t and $R_{p,t+1}$ is the return on wealth (1).

Epstein and Zin (1989, 1991) have shown that with this form for the budget constraint, the optimal portfolio and consumption policies must satisfy the following Euler equation for any asset i:

$$1 = \mathrm{E}_t\left[\left\{\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}\right\}^{\theta} R_{p,t+1}^{-(1-\theta)} R_{i,t+1}\right]. \tag{8}$$

When $i = p$, (8) reduces to:

$$1 = \mathrm{E}_t \left[\left\{ \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} R_{p,t+1} \right\}^{\theta} \right]. \tag{9}$$

3 Approximate Solution Method

Our proposed solution method builds on the log-linear approximations to the Euler equation and the intertemporal budget constraint proposed by Campbell (1993, 1996b). By combining the approximations to these equations we can characterize the properties of α_t, the optimal allocation to the risky asset. We then guess a form for the optimal consumption and portfolio policies, we show that policies of this form satisfy the approximate Euler equation and budget constraint, and finally we show that the parameters of the policies can be identified from the primitive parameters of the model.

3.1 Log Euler Equations

The first step in our proposed solution method is to log-linearize the Euler equation (9) to obtain

$$0 = \theta \log \delta - \frac{\theta}{\psi} \mathrm{E}_t \Delta c_{t+1} + \theta \mathrm{E}_t r_{p,t+1} + \frac{1}{2} \mathrm{Var}_t \left(\frac{\theta}{\psi} \Delta c_{t+1} - \theta r_{p,t+1} \right),$$

where lowercase letters denote variables in logs and Δ is the first-difference operator. This expression holds exactly if consumption growth and the return on wealth have a joint conditional lognormal distribution. In our model the return on wealth is conditionally lognormal, because the portfolio weight is known in advance and so the return on wealth inherits the assumed lognormality of the return on the risky asset. Consumption growth, however, is endogenous in our model and so we cannot assume at the outset that it is conditionally lognormally distributed. In fact, our approximate solution implies that consumption growth is not conditionally lognormal unless the elasticity of intertemporal substitution, ψ, is one or the expected return on the risky asset is constant. When these conditions do not hold we must derive the log Euler equation using both a second-order Taylor approximation around the conditional mean of $\{r_{p,t+1}, \Delta c_{t+1}\}$ and the approximation $\log(1+x) \approx x$ for small x.

Reordering terms we obtain the well-known equilibrium linear relationship between expected log consumption growth and the expected log return on wealth:

$$\mathrm{E}_t \Delta c_{t+1} = \psi \log \delta + v_{p,t} + \psi \mathrm{E}_t r_{p,t+1}, \tag{10}$$

where the term $v_{p,t}$ is a time-varying intercept proportional to the conditional variance of log consumption growth in relation to log portfolio returns:

$$v_{p,t} = \frac{1}{2} \left(\frac{\theta}{\psi} \right) \mathrm{Var}_t \left(\Delta c_{t+1} - \psi r_{p,t+1} \right). \tag{11}$$

In a similar fashion we can loglinearize the Euler equation for a general asset, (8). If we subtract the resulting loglinear Euler equation for the riskless asset from the loglinear Euler equation for the risky asset we find:

$$\mathrm{E}_t r_{1,t+1} - \overline{r}_f + \frac{1}{2}\sigma_{1,1,t} = \frac{\theta}{\psi}\sigma_{1,c,t} + (1-\theta)\,\sigma_{1,p,t} \tag{12}$$

where $\sigma_{x,z,t} = \mathrm{Cov}_t\left(x_{t+1} - \mathrm{E}_t x_{t+1}, z_{t+1} - \mathrm{E}_t z_{t+1}\right)$. Under assumption (A3) the conditional variance of the risky asset return, $\sigma_{1,1,t} = \sigma_u^2$, but we avoid making this substitution until we use (A3) to solve the model in section 3.4. Equation (12) is the starting-point for our analysis of optimal portfolio choice.

3.2 Log-linear budget constraint

Following Campbell (1993, 1996b), we also log-linearize the budget constraint (7) around the mean consumption-wealth ratio, and we obtain:

$$\Delta w_{t+1} \approx r_{p,t+1} + \left(1 - \frac{1}{\rho}\right)(c_t - w_t) + k, \tag{13}$$

where $k = \log(\rho) + (1-\rho)\log(1-\rho)/\rho$, and $\rho = 1 - \exp\{\mathrm{E}(c_t - w_t)\}$ is a log-linearization parameter. Note that ρ is endogenous in that it depends on the average log consumption-wealth ratio which is unknown until the model has been solved.

Campbell (1993) and Campbell and Koo (1997) have shown that the approximation (13) is exact when the consumption-wealth ratio is constant over time, and becomes less accurate as the variability of the ratio increases. In our model the consumption-wealth ratio is constant when the elasticity of intertemporal substitution is one or the expected risky asset return is constant; when these conditions do not hold, the consumption-wealth ratio varies and we can only solve for it by using the approximation (13). Hence to check the accuracy of the approximation we would need to compare our solution with a numerical solution of the model.

The log-linearization (13) takes the return on the wealth portfolio as given, and does not relate it to the returns on individual assets. We can push the approach further by using an approximation to the log return on wealth:

$$r_{p,t+1} = \alpha_t\left(r_{1,t+1} - \overline{r}_f\right) + \overline{r}_f + \frac{1}{2}\alpha_t\left(1 - \alpha_t\right)\sigma_{1,1,t}. \tag{14}$$

This approximation holds exactly in a continuous-time model with an infinitesimally small trading interval, where Ito's Lemma can be applied to equation (1).

Combining (13) and (14) we get

$$\Delta w_{t+1} \approx \alpha_t \left(r_{1,t+1} - \overline{r}_f\right) + \overline{r}_f + \left(1 - \frac{1}{\rho}\right)\left(c_t - w_t\right) + k + \frac{1}{2}\alpha_t \left(1 - \alpha_t\right)\sigma_{1,1,t}, \tag{15}$$

which is linear in log returns and log consumption, and quadratic in the portfolio weight α_t.

3.3 Characterizing the optimal portfolio rule

The next step in our solution method is to characterize the optimal portfolio rule by relating the current optimal portfolio choice to future optimal portfolio choices. This will then allow us to guess a functional form for the optimal portfolio policy and to identify its parameters.

Our strategy is to characterize the covariance terms $\sigma_{1,c,t}$ and $\sigma_{1,p,t}$ that appear in the loglinear portfolio-choice Euler equation (12). We first note that, using the trivial equality

$$\Delta c_{t+1} = \left(c_{t+1} - w_{t+1}\right) - \left(c_t - w_t\right) + \Delta w_{t+1} \tag{16}$$

and the budget constraint (15), we can write $\sigma_{1,c,t}$ as

$$\begin{aligned}\sigma_{1,c,t} &= \mathrm{Cov}_t\left(r_{1,t+1}, \Delta c_{t+1}\right) \\ &= \mathrm{Cov}_t\left(r_{1,t+1}, c_{t+1} - w_{t+1}\right) + \alpha_t \mathrm{Var}_t\left(r_{1,t+1}\right) \\ &= \sigma_{1,c-w,t} + \alpha_t \sigma_{1,1,t},\end{aligned}$$

where to obtain the second equality we use the fact that $\mathrm{Cov}_t(x_{t+1}, z_t) = 0$.

Similarly, equation (14) implies that

$$\begin{aligned}\sigma_{1,p,t} &= \mathrm{Cov}_t\left(r_{1,t+1}, \alpha_t r_{1,t+1} + \left(1 - \alpha_t\right)\overline{r}_f + \frac{1}{2}\alpha_t\left(1 - \alpha_t\right)\sigma_{1,1,t}\right) \\ &= \alpha_t \sigma_{1,1,t}.\end{aligned}$$

These expressions can be substituted into (12) to get

$$\mathrm{E}_t r_{1,t+1} - r_{f,t+1} + \frac{1}{2}\sigma_{1,1,t} = \frac{\theta}{\psi}\left(\sigma_{1,c-w,t} + \alpha_t \sigma_{1,1,t}\right) + \left(1 - \theta\right)\alpha_t \sigma_{1,1,t},$$

which can be rearranged, using the fact that $\theta = (1-\gamma) \,/\, (1 - \psi^{-1})$, to get

$$\alpha_t^* = \frac{1}{\gamma}\frac{\mathrm{E}_t r_{1,t+1} - \overline{r}_f + \frac{1}{2}\sigma_{1,1,t}}{\sigma_{1,1,t}} - \frac{1-\gamma}{\gamma\left(\psi - 1\right)}\frac{\sigma_{1,c-w,t}}{\sigma_{1,1,t}}. \tag{17}$$

This equation was first derived by Restoy (1992). It has two terms, each one capturing a different aspect of asset demand. The first term captures that part of asset demand induced exclusively by the current risk premium, hence the adjective "myopic" often used to describe it in the finance literature. The myopic component of asset demand is directly proportional to the asset risk premium and inversely proportional to the individual's relative risk aversion. Our expression for myopic asset demand is the same as the standard solution in those few particular cases in which an exact, closed-form solution for the optimal portfolio policy is known, i.e., when returns are unpredictable, so $\sigma_{1,c-w,t} = 0$, or preferences belong to the log-utility class, so $\gamma = 1$. In fact, (17) generalizes those results, since it says that the only restriction on preferences we need to obtain optimality of the myopic portfolio policy is a coefficient of relative risk aversion equal to unity.[2] Log utility is the special case in which, besides $\gamma = 1$, we also have $\psi = 1$.

The second term is the "intertemporal hedging demand" of Merton (1969, 1971, 1973). It reflects the strategic behavior of the investor who wishes to hedge against future adverse changes in investment opportunities, as summarized by the consumption-wealth ratio.

Although equation (17) gives us meaningful information about the nature of the investor's demand for the risky asset, it is not a complete solution of the model, because the current optimal portfolio allocation in (17) is a function of future portfolio and consumption decisions, which are endogenous to the problem.

The dependence of today's portfolio allocation on future portfolio and consumption choices operates through the conditional covariance $\sigma_{1,c-w,t}$. To see this, note that the the log consumption-wealth ratio can be written, up to a constant, as the discounted present value (with discount factor ρ) of the difference between expected future log returns on wealth and consumption growth rates (Campbell, 1993):

$$c_{t+1} - w_{t+1} = \mathrm{E}_{t+1} \sum_{j=1}^{\infty} \rho^j \left(r_{p,t+1+j} - \Delta c_{t+1+j} \right) + \frac{\rho k}{1-\rho}.$$

This equation follows from combining the log-linear budget constraint (13) with (16), solving forward the resulting difference equation, and taking expectations. If we substitute the expression for expected consumption growth (10) into this equation we obtain:

$$c_{t+1} - w_{t+1} = (1-\psi)\, \mathrm{E}_{t+1} \sum_{j=1}^{\infty} \rho^j r_{p,t+1+j} - \mathrm{E}_{t+1} \sum_{j=1}^{\infty} \rho^j v_{p,t+j} + \frac{\rho}{1-\rho} \left(k - \psi \log \delta \right). \quad (18)$$

[2] See Giovannini and Weil (1989) for a clear statement of this result.

Hence, $\sigma_{1,c-w,t}$ becomes

$$\begin{aligned}\sigma_{1,c-w,t} &= \mathrm{Cov}_t\left(r_{1,t+1} - \mathrm{E}_t r_{1,t+1}, (1-\psi)\left(\mathrm{E}_{t+1} - \mathrm{E}_t\right)\sum_{j=1}^{\infty}\rho^j r_{p,t+1+j}\right) \\ &\quad -\mathrm{Cov}_t\left(r_{1,t+1} - \mathrm{E}_t r_{1,t+1}, \left(\mathrm{E}_{t+1} - \mathrm{E}_t\right)\sum_{j=1}^{\infty}\rho^j v_{p,t+j}\right) \\ &= (1-\psi)\,\sigma_{1,h1,t} - \frac{1}{2}\left(\frac{\theta}{\psi}\right)\sigma_{1,h2,t}, \end{aligned} \tag{19}$$

which depends on the individual's future portfolio and consumption decisions. The second equality in (19) simplifies the notation in the first equality.

Equation (19) gives us a further decomposition of the hedging component of risky asset demand, since we can now write (17) as:

$$\alpha_t^* = \frac{1}{\gamma}\frac{\mathrm{E}_t r_{1,t+1} - \overline{r}_f + \frac{1}{2}\sigma_{1,1,t}}{\sigma_{1,1,t}} - \frac{1-\gamma}{\gamma}\frac{\sigma_{1,h1,t}}{\sigma_{1,1,t}} + \frac{1}{2\gamma}\left(\frac{1-\gamma}{\psi-1}\right)^2\frac{\sigma_{1,h2,t}}{\sigma_{1,1,t}}. \tag{20}$$

This equation says that the hedging component of the total risky asset demand has two terms. The first term arises from the covariance between unexpected stock returns and changes in expectations of future investment opportunities, $\sigma_{1,h1,t}$, and it is influenced only by the agent's attitude towards risk. Depending on the magnitude of the agent's relative risk aversion, a positive covariance $\sigma_{1,h1,t}$ may increase or decrease her portfolio holdings of the risky asset, relative to her position when she acts myopically. The second term arises from the covariance between unexpected stock returns and changes in expectations of the conditional variance of log consumption growth less ψ times the log wealth return, $\sigma_{1,h2,t}$, and it is influenced by the agent's attitude towards both risk and intertemporal substitution. The influence of preference parameters is non-linear in both terms. Of course, in our setting $\sigma_{1,h1,t}$ and $\sigma_{1,h2,t}$ are both endogenous so (20) still falls short of a complete solution to the model.

3.4 Guessing the optimal consumption and portfolio policies

The final step in solving the dynamic optimization problem is to guess a functional form for the optimal consumption and portfolio policies and to identify the parameters of these policies. We guess that the optimal portfolio weight on the risky asset is linear in the state variable, and that the optimal log consumption-wealth ratio is quadratic in the state variable. Hence, we guess that:

$$\begin{aligned} &(\text{ i }) \qquad \alpha^*_{t+j} = a_0 + a_1 x_{t+j} \\ &(\text{ ii }) \qquad c_{t+j} - w_{t+j} = b_0 + b_1 x_{t+j} + b_2 x^2_{t+j} \ , \end{aligned}$$

where $\{a_0, a_1, b_0, b_1, b_2\}$ are fixed parameters to be determined.

Under assumptions (A1)-(A5) we can show that guesses (i)-(ii) are indeed a solution to the intertemporal optimization problem of the recursive-utility-maximizing investor, and we can solve for the unknown parameters $\{a_0, a_1, b_0, b_1, b_2\}$. Details are provided in Appendices A and B; here we give a brief intuitive explanation of the solution.

The linear portfolio rule (i) has the simplest form consistent with time-variation in the investor's portfolio decisions. This portfolio rule implies that the expected return on the portfolio is quadratic in the state variable x_t, because an increase in x_t increases the expected portfolio return both directly by increasing the expected return on existing risky-asset holdings and indirectly by increasing the investor's allocation to the risky asset. Since the log consumption-wealth ratio is linearly related to the expected portfolio return (see equation (18)), it is natural to guess that the log consumption-wealth ratio is quadratic in the state variable x_t.

Of course, variances and covariances of consumption growth and asset returns also affect the optimal consumption and portfolio decisions. But the homoskedastic linear AR(1) process for x_t implies that all relevant variances and covariances are either linear or quadratic in the current state variable, and thus second-moment effects do not change the linear-quadratic form of the solution. Appendix A states seven lemmas that express important expectations, variances, and covariances as linear or quadratic functions of the state variable.

We now state two propositions that enable us to solve for the unknown coefficients of the model. The propositions are proved in Appendix B, using the lemmas from Appendix A.

Proposition 1 *The parameters defining the linear portfolio policy (i) satisfy the following two-equation system:*

$$a_0 = \left(\frac{1}{2\gamma}\right) - b_1\left(\frac{1-\gamma}{\gamma(\psi-1)}\frac{\sigma_{\eta u}}{\sigma_u^2}\right) - b_2\left(\frac{1-\gamma}{\gamma(\psi-1)}\frac{\sigma_{\eta u}}{\sigma_u^2}2\mu(1-\phi)\right),$$

$$a_1 = \left(\frac{1}{\gamma\sigma_u^2}\right) - b_2\left(\frac{1-\gamma}{\gamma(\psi-1)}\frac{\sigma_{\eta u}}{\sigma_u^2}2\phi\right).$$

Proof. See Appendix B.

By inspection of the system above we see that, were the terms involving $\{b_1, b_2\}$ zero, we would obtain the following optimal portfolio rule:

$$\alpha_t^* = \frac{1}{2\gamma} + \frac{1}{\gamma\sigma_u^2} x_t = \alpha_{t,myopic}^* \tag{21}$$

That is, we would obtain the myopic component of (20) for the optimal portfolio rule. Therefore, those terms involving $\{b_1, b_2\}$ give the hedging component of the demand for the risky asset. The sign and magnitude of the hedging component depend on the sign and magnitude of $\{b_1, b_2\}$, which govern the sensitivity of consumption to changes in the state variable, as well as on $\sigma_{\eta u}/\sigma_u^2$ and $(1-\gamma)/\gamma(\psi-1)$. It is interesting to note that this last factor equals one when the investor has power utility, so in this case preference parameters influence hedging demand only through $\{b_1, b_2\}$. When the expected excess gross return is zero, so $x_t = -\sigma_u^2/2$, we have $\alpha_{t,myopic}^* = 0$, and the demand for the risky asset is entirely hedging demand. We discuss the effects of the parameters on portfolio selection in more detail in our calibration exercise in Section 4.

Lemmas 6 and 7, in Appendix A, and Proposition 1 imply a non-linear equation system for the unknown parameters governing the optimal consumption rule.

Proposition 2 *The parameters defining the consumption policy (ii), $\{b_0,\ b_1,\ b_2\}$, are given by the solution to the following recursive non-linear* ***Fundamental Equation System****:*

$$0 = \Lambda_{10} + \Lambda_{11} b_0 + \Lambda_{12} b_1 + \Lambda_{13} b_1^2 + \Lambda_{14} b_2 + \Lambda_{15} b_2^2 + \Lambda_{16} b_1 b_2,$$

$$0 = \Lambda_{20} + \Lambda_{21} b_1 + \Lambda_{22} b_2 + \Lambda_{23} b_2^2 + \Lambda_{24} b_1 b_2,$$

$$0 = \Lambda_{30} + \Lambda_{31} b_2 + \Lambda_{32} b_2^2,$$

where $\{\Lambda_{i,j}; i = 1, 2, 3, j = 1, ..., 6\}$ are known constants given in Appendix B.

Proof. See Appendix B.

The Fundamental Equation System can be solved recursively. The last equation of the system is a quadratic equation whose only unknown is b_2. It has two possible roots. We show in the next section that one of these roots implies a portfolio weight on the risky asset that is monotonically increasing in x_t; we use this root in our analysis. Once we have solved for b_2, the second equation in the system becomes a linear equation in b_1. Finally, given $\{b_1,\ b_2\}$, the first equation of the system is also linear in b_0. Using the known values of $\{b_0,\ b_1,\ b_2\}$ in Proposition 1, we can find $\{a_0, a_1\}$ and complete the solution of the problem.

3.5 Properties of the solution

Propositions 1 and 2 completely identify the parameters of the optimal policies. They also imply some general properties of the solution that we state below and prove in Appendix C.

Property 1 *When $\gamma \geq 1$, the roots of the third equation of the Fundamental Equation System, defining b_2, are always real and have opposite sign. Moreover, $b_2 \to 0$ as $\gamma \to \infty$. If we select the positive root of the discriminant in the third equation of the Fundamental Equation System, b_2 is negative if $\psi > 1$ and positive if $\psi < 1$, while if we select the negative root, b_2 is negative if $\psi < 1$ and positive if $\psi > 1$.*

When $\gamma < 1$, the roots of the equation are real if the following condition holds:

$$\gamma\left(\phi^2 - \frac{1}{\rho}\right)^2 \sigma_u^2 + (1-\gamma)\left(\phi^2 - \frac{1}{\rho}\right) 4\phi\sigma_{\eta u} - (1-\gamma)\, 4\phi^2\sigma_\eta^2 \geq 0. \tag{22}$$

Provided this condition holds, the roots always have the same sign. They are positive if $\psi < 1$ and negative if $\psi > 1$.

When $\gamma = 0$, the discriminant of the equation is zero, so b_2 is always real and equals $(\psi - 1)/2\phi\sigma_{\eta u}$.

Property 1 tells us when our guessed solution of the approximate model is valid. Real solutions for the parameters can be substituted into the objective function to obtain the value function for the approximate infinite-horizon maximization problem as a quadratic function of the state variable. Thus Property 1 tells us when the value function is finite, that is, it tells us when our approximate problem is well defined. It is the equivalent in our approximate, discrete-time model with time-varying expected returns of the conditions given in Merton (1971) and Svensson (1989) for finite value functions in continuous-time models with constant expected returns.

Property 2 *The slope of the optimal portfolio rule—the coefficient a_1—is always positive when we select the positive root of the discriminant in the third equation of the Fundamental Equation System. Also, $\lim_{\gamma\to\infty} a_1 = 0$ and $\lim_{\gamma\to 0} a_1 = +\infty$, regardless of which root we choose.*

Property 2 generalizes a known comparative-statics result for an investor with power utility facing constant expected returns in a continuous-time model. In that setting the allocation to the risky asset is constant over time and it increases with the expected excess return on the risky asset. In static models with general utility functions, however, it is possible for the allocation to the risky asset to decline with the expected excess return on the risky asset, because the income effect of an increase in the risk premium can overcome the substitution effect. Property 2 shows that this does not happen in our dynamic model when we pick the positive root of the discriminant in the quadratic equation for b_2. With that root the coefficient a_1 is always positive and increases from zero when γ is infinitely large to infinitely large values as γ approaches zero. All the results in our calibration exercise are based on this choice for the root.

Property 3 *The optimal portfolio rule does not depend on ψ for given ρ.*

This property shows that the main preference parameter determining portfolio choice is the coefficient of relative risk aversion γ and not the elasticity of intertemporal substitution ψ. Conditioning on ρ, ψ has no effect on portfolio choice. However, ρ itself is a function of ψ—recall that $\rho = 1-\exp\{E[c_t - w_t]\}$—so the optimal portfolio rule depends on ψ indirectly through ρ. Our calibration results in the next section show that this indirect effect is rather small.

Property 4 $\partial a_1/\partial\mu = 0$ *and* $\partial b_2/\partial\mu = 0$. *That is, neither the slope of the portfolio policy nor the curvature of the consumption policy depend on μ for given ρ.*

Property 4 shows that some aspects of the optimal policy—the sensitivity of the risky asset allocation to the state variable and the quadratic sensitivity of consumption to the state variable—are independent of the average level of the excess return on the risky asset. Of course, other aspects—the average allocation to the risky asset, the average consumption-wealth ratio, and the linear sensitivity of consumption to the state variable—do depend on the average risk premium.

3.6 Some special cases

Having characterized the general solution to the problem, we now consider some special cases. When a closed-form solution has been already obtained using alternative methods, we show that our solution method delivers exactly the same solution.

3.6.1 Log Utility

In this case $\gamma = \psi = 1$. Taking limits as γ approaches one, Lemma 6 in Appendix A implies that $v_0 = v_1 = v_2 = 0$, so $v_{p,t} = 0$. Also, from Lemmas 8 and 9, the covariance terms in (19) are both zero, so $\sigma_{1,c-w,t} = 0$. Substituting $v_{p,t} = 0$ and $\psi = 1$ in (18), we obtain

$$c_{t+1} - w_{t+1} = \frac{\rho}{1-\rho}\left(k - \log \delta\right).$$

After substituting for the value of k, this result simplifies to

$$c_{t+1} - w_{t+1} = \log\left(1 - \rho\right). \tag{23}$$

Moreover, we know from standard arguments that $\rho = \delta$.

We can also obtain this result by solving the Fundamental Equation System in the limit as ψ approaches 1. In this case, the system delivers $b_1 = b_2 = 0$ and b_0 equal to the expression above. Following Giovannini and Weil (1989), we call this optimally constant propensity to consume out of wealth a *myopic consumption policy.*

Substituting log-utility values of γ, $b_1/\left(1-\psi\right)$, and $b_2/\left(1-\psi\right)$ into Proposition 1, we see that

$$\begin{aligned} a_0 &= \frac{1}{2}, \\ a_1 &= \frac{1}{\sigma_u^2}, \end{aligned}$$

so the optimal portfolio policy is the myopic portfolio policy (21) with $\gamma = 1$. The intercept term in (21) is a Jensen's inequality effect. We can see this by rewriting the optimal portfolio rule as

$$\alpha^*_{t,myopic} = \frac{x_t + (\sigma_u^2/2)}{\sigma_u^2}$$

and noting that the expected excess gross return is equal to

$$\mathrm{E}_t\left[\frac{R_{1,t+1}}{R_f}\right] = \exp\left\{x_t + \frac{\sigma_u^2}{2}\right\}.$$

Hence, when $x_t = -\sigma_u^2/2$, which is the value of the expected log excess return consistent with a zero gross excess return, we have $\alpha_t^* = 0$. Also, if the expected return on the risky asset is constant, then α_t^* is constant. In general, when the log excess return is time-varying, the portfolio weight on the risky asset is proportional to the expected excess return next period, and inversely proportional to the variance of the unexpected return on the risky asset. It is positive whenever there is a positive expected excess gross return—$x_t > -\sigma_u^2/2$.

Finally, we note that the optimal portfolio policy for the log-utility investor maximizes the conditional expectation of the log return on wealth, (14). All other policies result in lower expected log portfolio returns.

3.6.2 Unit Elasticity of Intertemporal Substitution

When ψ is equal to one, we have shown above that the Fundamental Equation System implies $b_1 = b_2 = 0$, so $c_{t+1} - w_{t+1} = b_0$ and the myopic consumption policy (23) obtains. Therefore, it is optimal for the individual to consume each period a fixed fraction of her wealth.

However, the agent's optimal portfolio policy is not myopic. This is because, from equation (45) in the proof of Property 3 in Appendix C, we have that $b_2/(\psi - 1)$ and $b_1/(\psi - 1)$ are non-zero constants independent of ψ for given ρ. Therefore, when $\psi = 1$ the terms in $\{b_1, b_2\}$ in the system defining the optimal portfolio policy in Proposition 1 do not vanish, and a non-myopic portfolio policy obtains.

This can also be understood in another way. Setting $\psi = 1$ and noting from (45) that $b_2^2/(\psi - 1) \to 0$ and $b_1 b_2/(\psi - 1) \to 0$ as $\psi \to 1$, Lemma 6 in Appendix A implies $v_0 = v_1 = v_2 = 0$, so the intercept $v_{p,t}$ in the log Euler equation (10) is constant and equal to zero. The covariance terms in Lemmas 8 and 9 are zero too, so from (19) $\sigma_{1,c-w,t} = 0$. However $\sigma_{1,c-w,t} = 0$ does not imply a myopic portfolio policy in this case, because from (17) in order to have a myopic portfolio policy we need

$$\frac{1-\gamma}{\gamma(\psi - 1)} \frac{\sigma_{1,c-w,t}}{\sigma_{1,1,t}} = 0,$$

and it turns out that $\sigma_{1,c-w,t}/(\psi - 1)$does not approach zero as $\psi \to 1$. Giovannini and Weil (1989) emphasize this result.

3.6.3 Unit Relative Risk Aversion

When γ is equal to one, Lemma 6 in Appendix A implies that $v_0 = v_1 = v_2 = 0$ so the intercept term $v_{p,t}$ in the Euler Equation (10) is zero. We also see this by noticing that $\gamma = 1$ implies $\theta = 0$. From Proposition 1 we obtain the same myopic portfolio rule as with log utility. This rule is again the rule that maximizes the conditional expectation of the log portfolio return. However, the consumption-wealth ratio is no longer constant unless $\psi = 1$, as we can see from Lemma 7 in Appendix A. Therefore, unit relative risk aversion implies a myopic optimal portfolio policy and a non-myopic optimal consumption policy. Giovannini and Weil (1989) emphasize this result.

3.6.4 Constant Expected Returns

With constant expected returns, $x_t = \mu \quad \forall t$, so $\sigma_\eta^2 = \sigma_{\eta u} = 0$ and $\phi = 0$. From Lemma 6 in Appendix A we obtain the same portfolio policy as in the log utility case, except that $\gamma \neq 1$:

$$\begin{aligned} a_0 &= \frac{1}{2\gamma}, \\ a_1 &= \frac{1}{\gamma\sigma_u^2}. \end{aligned}$$

This is the well-known result for the optimal portfolio rule when returns are i.i.d. Since x_t is constant, from **(ii)** and Lemma 6 we see that both $v_{p,t}$ and $c_t - w_t$ are trivially constant.

3.6.5 Zero Covariance between Expected and Unexpected Returns

Another interesting case arises when the covariance between unexpected returns on the risky asset and the innovations in the state variable is zero, i.e. $\sigma_{\eta u} = 0$. In this case Proposition 1 delivers again a myopic optimal portfolio rule, as in the previous example. We can also see this result by noting that the covariance terms in Lemmas 8 and 9 in Appendix A are both zero, and so (19) implies that $\sigma_{1,c-w,t} = 0$. However, from Lemmas 6 and 7, $v_{p,t}$ and $c_t - w_t$ are not constant, so $\{b_0, b_1, b_2\}$ are obtained as the solution to the Fundamental Equation System, where the Λ_{ij} parameters are modified tc take into account that $\sigma_{\eta u} = 0$. Similarly $\{v_0, v_1, v_2\}$ are obtained from Lemma 6 after substituting in the optimal values of $\{a_0, a_1, b_0, b_1, b_2\}$.

3.6.6 Non-Persistence in Stock Returns

When $\phi = 0$ expected returns over the next period are time-varying, but expected returns in the more distant future are constant. In this case, Proposition 1 implies the same portfolio slope coefficient a_1 as in the constant-expected-returns case. However, a_0 and the optimal consumption policy are not the same as in that case.

4 Calibration exercise

4.1 The data

An important advantage of our approach is that we can calibrate our model using real data on asset returns. To illustrate this, we use US financial data at a quarterly and annual frequency. In our calibration exercise, the risky asset is the US stock market, and the riskfree asset is a short-term debt instrument. The sample period for quarterly data is 1947.1 - 1995.4. To measure stock returns and dividends we use quarterly returns, dividends and prices on the CRSP value-weighted market portfolio inclusive of the NYSE, AMEX, and NASDAQ markets. The short-term nominal interest rate is the 3-month Treasury bill yield from the Riskfree File on the CRSP Bond tape. To compute the real log riskfree rate, the beginning-of-quarter nominal log yield is deflated by the end-of-quarter log rate of change in the Consumer Price Index from the Ibbotson files on the CRSP tape. Log excess returns are computed as the end-of-quarter nominal log stock return minus the beginning-of-quarter log yield on the riskfree asset. The annual data set is an updated version of the Campbell-Shiller data set on the US stock market, standard in the asset pricing literature. It contains data from 1872 to 1993 on total returns and dividends on the S&P Index, a riskfree rate based on rolling 6-month commercial paper, and the Producer Price Index. Since data on consumption are available only from 1890, we use the sample 1890-1993 in our calibration exercise. A full description of this annual data set can be found in the Data Appendix to Campbell (1996a).

The state variable is taken to be the log dividend-price ratio—for quarterly data it is measured as the log of the total dividend on the market portfolio over the last four quarters divided by the end-of-period stock price. Campbell and Shiller (1988a), Fama and French (1988), Hodrick (1992) and others have found this variable to be a good predictor of stock returns. We estimate the following restricted VAR(1) model:

$$\begin{pmatrix} r_{1,t+1} - \overline{r}_f \\ d_{t+1} - p_{t+1} \end{pmatrix} = \begin{pmatrix} \theta_0 \\ \beta_0 \end{pmatrix} + \begin{pmatrix} \theta_1 \\ \beta_1 \end{pmatrix} (d_t - p_t) + \begin{pmatrix} \epsilon_{1,t+1} \\ \epsilon_{2,t+1} \end{pmatrix}, \tag{24}$$

where $(\epsilon_{1,t+1}, \epsilon_{2,t+1}) \sim N(0, \Omega)$, and report the maximum likelihood estimation results in Table 1. Since (24) is equivalent to a multivariate regression model with the same explanatory variables in all equations, ML estimation is identical to OLS regression equation by equation. The standard errors for the slopes, intercepts and the residual variance-covariance matrix are based on Proposition 11.2 in Hamilton (1994); using these standard errors, which assume that the variables in the model are stationary, the slopes and the elements.of the variance-covariance matrix all appear to be statistically different from zero.

Since $\widehat{\beta}_1$ is high in both data sets, a unit-root test is in order. The Dickey-Fuller unit-root test for the slope of the second equation does not reject the null $H_0 : \beta_1 = 1$ at a 10% significance level when applied to the quarterly data set, while it rejects it at the 2.5% level when applied to the annual data set. There are theoretical reasons to believe that the log dividend-price ratio should be stationary (since it will be stationary if real interest rates, excess stock returns, and real dividend growth rates are stationary), but the failure to reject the unit-root null hypothesis suggests that there is at least a near-unit root in this variable in postwar quarterly data.

Elliott and Stock (1992) have shown that the test $H_0 : \theta_1 = 0$ is not independent of the test $H_0 : \beta_1 = 1$ when $\text{Corr}(\epsilon_{1,t+1}, \epsilon_{2,t+1}) \neq 0$. In that case, they have shown that the t-ratio that tests $H_0 : \theta_1 = 0$ is asymptotically normally distributed when $\beta_1 < 1$, while it has an asymptotic non-standard distribution (see equation (2.5) in Elliott and Stock, 1992) when $\beta_1 = 1$ or $\beta_1 = 1 - c/T$ —where c is an arbitrarily small positive constant and T is the sample size—that is, when the log dividend-price ratio follows a unit-root or near-unit-root process.

In our data sets, $\text{Corr}(\epsilon_{1,t+1}, \epsilon_{2,t+1})$ is estimated to be about -0.7, so the presence of a unit root can affect the distribution of the test statistic for $H_0 : \theta_1 = 0$. Since we have rejected a unit root in the log dividend-yield process in the annual data set, the t-statistic that tests $H_0 : \theta_1 = 0$ is asymptotically normal in that data set and its value of 2.803 enables us to reject that θ_1 is zero. In the quarterly data set we cannot reject that the log dividend-yield process follows a unit root process, which implies that the t-statistic for testing $H_0 : \theta_1 = 0$ has a non-standard distribution. The value of the t-statistic is 2.130. This value may not be large enough to reject $H_0 : \theta_1 = 0$ at the 5% significance level under the alternative non-standard asymptotic distribution, because this distribution has critical values larger than those of the standard normal when $\text{Corr}(\epsilon_{1,t+1}, \epsilon_{2,t+1})$ is negative[3]. We do not pursue this issue further here, and proceed to calculate standard errors under the assumption that the estimated system is stationary, but we note that the standard errors for the postwar data should be treated with some caution.

The parameters in (3), (4), and (5) that define the stochastic structure of our

[3]When $\text{Corr}(\epsilon_{1,t+1}, \epsilon_{2,t+1}) = -1$ and $\beta_1 = 1$, the negative of the t-ratio has a Dickey-Fuller distribution. When $0 \leq |\text{Corr}(\epsilon_{1,t+1}, \epsilon_{2,t+1})| < 1$ and $\beta_1 = 1$, the t-ratio is a linear combination of the Dickey-Fuller distribution—with coefficient equal to $\text{Corr}(\epsilon_{1,t+1}, \epsilon_{2,t+1})$—and the standard Gaussian distribution, so the critical values are closer to those of the Dickey-Fuller distribution the closer is $\text{Corr}(\epsilon_{1,t+1}, \epsilon_{2,t+1})$ to -1.

model can be recovered from the VAR system (24) as follows:

$$\mu = \theta_0 + \theta_1 \frac{\beta_0}{1-\beta_1} \tag{25}$$

$$\phi = \beta_1 \tag{26}$$

$$\sigma_\eta^2 = \theta_1^2 \mathrm{Var}\left(\epsilon_{2,t+1}\right) \tag{27}$$

$$\sigma_u^2 = \mathrm{Var}\left(\epsilon_{1,t+1}\right) \tag{28}$$

$$\sigma_{\eta u} = \theta_1 \mathrm{Cov}\left(\epsilon_{1,t+1}, \epsilon_{2,t+1}\right) \tag{29}$$

Table 1 reports these implied parameters in the quarterly and annual data, along with standard errors computed using the delta method. All of the derived parameters are statistically different from zero. The table also reports the mean log real riskfree rate of interest.

The implied annualized unconditional expected log excess return μ is 5.37% (four times 1.343) in the quarterly post-war period while it is 4.17% in the annual data for the period 1890-1993. On the other hand, the mean log real riskfree rate in the post-war quarterly period is a meager .28% (in annual terms) as compared to the 1.99% average for the annual 1890-1993 period. The higher equity premium in the post-war period reflects the lower short-term real interest rates in the post-WWII era relative to the previous period. These numbers are similar to those reported in Campbell (1996a) and Siegel (1994).[4]

4.2 The benchmark case

Using the parameter estimates in Table 1, we compute the individual's optimal portfolio allocation and consumption-wealth ratio for a range of values of relative risk aversion and elasticity of intertemporal substitution. We set δ, the time discount parameter under time-additive utility, to .94 in annual terms. This is equivalent to a 6.2% log time discount rate in annual terms. We call the optimal portfolio and consumption policies based on these parameters the "benchmark case".

[4]Campbell's Table 2 reports a higher post-war quarterly log real rate, because it is based on quarterly averages of real log yields on a portfolio of 30-day bills, while ours uses the log yield on the bill with maturity closest to 90 days. Campbell's (1996a) risk-free rate is based on the Treasury bill rates in the SBBI File of the CRSP-Indices tape, while our 90-day riskfree rate is taken from Fama's Riskfree File in the CRSP-Bond tape. The former is based on the yield of a portfolio of Treasury bills with maturity closest to 30 days, and then continuously compounded to 90-day rates, while the latter is based on the yield of the single issue with lowest default risk and maturity closest to 90 days. See CRSP manuals (1996) for more detailed information about the data.

We consider relative risk aversion coefficients $\gamma = \{.75,\ 1,\ 1.50,\ 2,\ 4,\ 10,\ 20,\ 40\}$, and elasticity of intertemporal substitution coefficients $\psi = \{1/.75,\ 1,\ 1/1.50,\ 1/2,\ 1/4,\ 1/10,\ 1/20,\ 1/40\}$. The literature on the equity premium puzzle has shown that high levels of risk aversion are needed to reconcile aggregate consumption data with asset market data in the standard power-utility framework; here we are able to compare the portfolio allocations and consumption rules implied by low and high risk aversion coefficients. We consider low elasticities of intertemporal substitution, both because we want to include the power-utility cases in which the elasticity of intertemporal substitution is the reciprocal of risk aversion, and because low elasticities of intertemporal substitution seem to be required to explain the insensitivity of consumption growth to real interest rates in postwar US data (Hall 1988, Campbell and Mankiw 1989).

The calibration of the model also requires us to set a value for ρ, the log-linearization parameter in the budget constraint. However, this value is exogenous only when $\psi = 1$. In this case the optimal consumption policy is myopic and $\rho = \delta$. In all other cases, ρ is endogenous because it is a monotonic function of the endogenous optimal expected log consumption-wealth ratio — $\rho = 1 - \exp\{\mathrm{E}(c_t - w_t)\}$. For given values of ψ and γ, we use a recursion to compute ρ jointly with $\{a_0, a_1, b_0, b_1, b_2\}$. We first set $\rho = \delta$ and then find the optimal values of $\{a_0, a_1, b_0, b_1, b_2\}$ given this value of ρ. For these optimal values we then compute $E(c_t - w_t)$ and a new value of ρ, for which a new set of optimal policies is computed. We proceed with this recursion until the absolute value of the difference between two consecutive values of ρ is less than 10^{-4}.

Tables 2 to 9 and Figures 1 to 6 report the results of this exercise. To make it easier to interpret our results, we have normalized the parameters defining the optimal portfolio and consumption policies **(i)** and **(ii)**, so that the intercepts of the optimal policy functions are the optimal allocation to stocks and the optimal consumption-wealth ratio when the expected gross excess return, $\mathrm{E}_t\left[R_{1,t+1}/R_f\right]$, is zero. The expected gross excess return is zero when the expected log excess return x_t is equal to $-\sigma_u^2/2$. Therefore, the parameters reported in the tables are a_0^*, a_1, b_0^*, b_1^* and b_2 in

$$\alpha_t = a_0^* + a_1\left(x_t + \frac{\sigma_u^2}{2}\right) \tag{30}$$

and

$$c_t - w_t = b_0^* + b_1^*\left(x_t + \frac{\sigma_u^2}{2}\right) + b_2\left(x_t + \frac{\sigma_u^2}{2}\right)^2, \tag{31}$$

where $a_0^* = a_0 - a_1(\sigma_u^2/2)$, $b_0^* = b_0 - b_1(\sigma_u^2/2) + b_2(\sigma_u^4/4)$, $b_1^* = b_1 - b_2\sigma_u^2$, and a_1 and b_2 are not starred because they coincide with the original parameters.

Each table contains two panels. Panel A is based on the postwar quarterly return process estimated in Panel A of Table 1. Panel B is based on the long-run annual return process estimated in Panel B of Table 1. The main diagonal of each panel corresponds to standard power-utility preferences, since the elasticity of intertemporal substitution is the reciprocal of risk aversion along the main diagonal.

The parameters given in the tables summarize the optimal decisions of a recursive-utility individual who observes the true process for returns. Since we do not observe the true process but must estimate it, we also report standard errors for these parameters, calculated using the delta method. The delta method requires the computation of derivatives of the parameters of interest (for example, a_1) with respect to $\{\theta_0, \theta_1, \beta_0, \beta_1, \Omega\}$. Since no analytical formulae are available, we use two-sided numerical derivatives based on a proportional perturbation parameter equal to 10^{-4}. These standard errors are reported in the lower part of each panel.

4.3 The optimal portfolio rule

Tables 2 to 5 and Figure 1 summarize the optimal portfolio decision. Table 2 reports a_0^*, the optimal allocation to stocks when the expected gross excess return is zero, while Table 3 reports a_1, the slope of the optimal portfolio policy α_t^*. Table 4 reports the mean total demand for stocks as a fraction of wealth, while Table 5 reports the mean hedging demand for stocks as a fraction of the mean total demand for stocks.

Figure 1, which is divided into four panels, illustrates the portfolio rule α_t^* using the parameter values estimated from the annual data set. Figure 1a fixes ψ at $1/0.75$ and plots α_t^* for a wide range of γ values; Figure 1b repeats this exercise fixing ψ at $1/4$. Figures 1c and 1d, on the other hand, fix γ at 0.75 and 4, respectively, and plot α_t^* for a wide range of ψ values. In all these figures we consider values of x_t in the interval $(\mu - 2\sigma_x^2, \mu + 2\sigma_x^2)$, and the horizontal axis is the log of the expected gross excess return, i.e., $\log \mathrm{E}_t\left[R_{1,t+1}/\overline{R}_f\right] = x_t + \sigma_u^2/2$. All figures are based on parameter estimates from the annual data set. The right vertical line intersects the horizontal axis at $\log \mathrm{E}\left[R_{1,t+1}/\overline{R}_f\right] = \mu + \sigma_x^2/2 + \sigma_u^2/2$.

The most striking lesson from the tables, and from Figure 1, is that relative risk aversion is far more important than the elasticity of intertemporal substitution in determining the optimal portfolio allocation to stocks. The variation in parameters across rows of the tables, as γ changes, is far greater than the variation across columns, as ψ changes. Similarly, the α_t^* lines in Figures 1c and 1d are all close together, whereas those in Figures 1a and 1b vary widely in both slope and intercept. This result can be understood by recalling Property 3 of our solution, i.e., that α_t^* depends on ψ only through the dependence of ρ on ψ. Our calibration results show that this indirect effect through ρ is rather small.

Table 2 shows that the intercept of α_t^*, a_0^*, is negative when $\gamma < 1$, while it is positive when $\gamma > 1$. It is zero when $\gamma = 1$, as we already know from the analysis of the special case with unit relative risk aversion. These results hold regardless of the value of ψ.[5] To understand this behavior, recall that a_0^* is the optimal allocation to stocks when the excess gross return is zero. Since (21) implies that the myopic demand for stocks is zero at this level of the expected excess gross return, a_0^* is completely determined by hedging demand. Thus with our return-generating process the investor has positive hedging demand when she is more risk averse than a logarithmic investor, and negative hedging demand when she is less risk-averse than a logarithmic investor.[6]

This result can be explained in intuitive terms as follows. We have estimated a return-generating process which has a negative sign for $\sigma_{\eta u}$, the covariance between unexpected stock returns and revisions in expected future stock returns. This implies that stocks tend to have high returns when expected future returns fall, that is, when the investment opportunity set deteriorates. There are offsetting considerations that determine an investor's attitudes towards such assets. On the one hand an investor with low risk aversion ($\gamma < 1$) will want to hold assets that deliver wealth when wealth is most productive, that is, when investment opportunities are good. The negative covariance $\sigma_{\eta u}$ will give this investor a negative hedging demand. On the other hand an investor with high risk aversion ($\gamma > 1$) will want to hold assets that deliver wealth in unfavorable states of the world, that is, when investment opportunities are poor. The negative covariance $\sigma_{\eta u}$ will give this investor a positive hedging demand.[7] Interestingly, the hedging demand is not monotonic in risk aversion because an extremely risk-averse investor will limit her exposure to the risky asset in all states of the world; thus the coefficient a_0^* first rises and then falls with risk aversion.

The sign of hedging demand is sensitive to the average level of excess stock returns, μ. We have estimated μ to be positive and quite large; hence the investor normally has a long position in stocks and a decrease in the expected stock return represents a deterioration in investment opportunities. If μ were negative, however,

[5]The lower part of panel A shows that a_0^* is significantly different from zero in the quarterly model. In the annual model we cannot reject the hypothesis that $a_0^* = 0$ in most cases.

[6]Kim and Omberg (1996) describe the case of negative hedging demand as positive "speculative" demand for the risky asset, since the individual is "speculating" rather than "hedging" against changes in the opportunity set.

[7]Campbell (1993, 1996b) makes a similar point in the context of a model with an exogenous portfolio return process. He points out that for a given level of risk aversion greater than one, the equilibrium equity premium will be lower when stocks have a negative covariance $\sigma_{\eta u}$. Kim and Omberg (1996) derive a related result in their model with HARA preferences over terminal wealth; they show that the sign of a_0^* is the same as the sign of $(1-\gamma)\,\sigma_{\eta u}$.

the investor would normally have a short position in stocks. In this case a decrease in the expected stock return would represent an improvement in investment opportunities and the sign of hedging demand would reverse. An investor with $\gamma < 1$ would have a positive hedging demand for stocks (equivalently, a negative hedging demand for the risky portfolio with a short position in stocks) while an investor with $\gamma > 1$ would have a negative hedging demand for stocks (a positive hedging demand for the short position).

Table 3 shows that the coefficient a_1, the slope of the α_t^* function, is positive for all levels of γ and ψ as implied by Property 2 of our solution. Like the intercept a_0^*, the slope a_1 varies substantially across γ for a given level of ψ, but changes very little across ψ for a given level of γ. As γ increases, a_1 rapidly approaches zero, indicating that the optimal portfolio rule is very responsive to changes in expected excess returns when the individual is close to risk neutral but is almost flat when the individual is highly risk averse. This finding is also implied by Property 2 of our solution. The standard errors in the lower part of each panel show that a_1 is much more precisely estimated than the intercept of the optimal portfolio rule. One reason for this greater precision is that, as we showed in Property 4, the slope of the portfolio rule is not sensitive to the mean excess stock return μ whereas the intercept does depend on μ.

Table 4 reports the mean optimal allocation to stocks as a percentage of total wealth. The mean allocation is positive at all levels of γ and ψ. Both the quarterly and annual models imply that, on average, a recursive-utility individual with low or moderate levels of risk aversion will short the riskless asset in order to hold more than 100% of her wealth in the risky asset. Large levels of relative risk aversion are needed to keep mean stock demand below 100%; this is a manifestation of the equity premium puzzle in our model with exogenous asset returns and endogenous portfolios.

Table 5 shows that on average hedging demand is a very important part of total stock demand for investors whose relative risk aversion coefficients are not close to one. Mean hedging demand is calculated using (21), by setting $x_t = \mu$ and subtracting from the total risky-asset allocation the total allocation when $\gamma = 1$ divided by the level of relative risk aversion:

$$\begin{aligned}\alpha_{t,hedging}^*(\mu;\gamma,\psi) &= \alpha_t^*(\mu;\gamma,\psi) - \alpha_{t,myopic}^*(\mu;\gamma,\psi) \\ &= \alpha_t^*(\mu;\gamma,\psi) - \frac{1}{\gamma}\alpha_t^*(\mu;1,\psi)\ .\end{aligned}$$

Mean hedging demand is negative, and often large, for investors with risk aversion coefficients of 0.75; for the illustrated risk aversion coefficients above one it is positive and accounts for at least 20% of stock demand and often above 50%. Thus intertemporal hedging motives can easily double the equity demand of risk-averse investors.

This makes it harder to explain the equity premium puzzle with moderate levels of risk aversion, a point emphasized by Campbell (1996b).

The results in Tables 4 and 5 can be related to the argument of Siegel (1994) that long-run investors should not try to time the stock market, but should buy and hold large equity positions because these positions involve little risk at long horizons. Siegel's estimates of long-run stock market risk are low because of the mean-reversion in stock returns that we have captured with our VAR system. In our model rational investors should time the market (the coefficient a_1 is always positive), but as risk-aversion increases the propensity to time the market declines (a_1 approaches zero) while intertemporal hedging demand remains an important component of total equity demand. Thus our model might provide an approximate rationalization for the investment strategy advocated by Siegel.

4.4 The optimal consumption-wealth ratio

Tables 6 to 9 and Figure 2 summarize the optimal consumption policy, following a structure similar to that used to present the optimal portfolio policy. Table 6 presents $\exp\{b_0^*\}$, and Tables 7 and 8 report b_1^* and b_2. We report $\exp\{b_0^*\}$ rather than b_0^* because $\exp\{b_0^*\}$ equals the optimal consumption wealth-ratio C_t/W_t when the expected excess gross return, $\mathrm{E}_t\left[R_{1,t+1}/\overline{R}_f\right]$, is zero. Finally, Table 9 reports the exponentiated mean of the optimal log consumption-wealth ratio. Figure 2 is similar to Figure 1, but it plots $C_t/W_t = \exp\{c_t - w_t\}$ instead of α_t^*.

The tables and figure reveal an important difference between the optimal consumption rule and the optimal portfolio rule: The optimal consumption-wealth ratio is very sensitive to both the level of the elasticity of intertemporal substitution and the level of risk aversion, while the optimal portfolio rule moves noticeably only with the level of risk aversion. Consider for example the exponentiated intercept of the optimal consumption rule shown in Table 6. This corresponds to the optimal consumption-wealth ratio when $\mathrm{E}_t\left[R_{1,t+1}/\overline{R}_f\right] = 1$. The variation as one moves along a row of Table 6 is just as great as the variation as one moves up or down a column; this is not true in Tables 2 to 4. More specifically, Table 6 displays interesting patterns. At low levels of risk aversion γ, the optimal consumption-wealth ratio decreases as the elasticity of substitution ψ rises (a movement along a row from right to left). At high levels of risk aversion, on the other hand, the optimal consumption-wealth ratio increases with ψ. Similarly, at low levels of the elasticity of substitution ψ—specifically, $\psi < 1$—the optimal ratio rises with risk aversion γ, while at high levels of ψ, it declines with γ. The optimal ratio is independent of γ when $\psi = 1$, as we already know from our analysis of this special case in Section 3.6.2. These patterns are repeated in Table 9, which reports the exponentiated mean optimal log consumption-wealth ratio, and they are illustrated in Figures 2a through 2d, where the vertical sorting of the C_t/W_t curves is reversed as we move from $\psi = 1/0.75$ in Figure 2a to $\psi = 1/4$ in Figure 2b, and from $\gamma = 0.75$ in Figure 2c to $\gamma = 4$ in Figure 2d.

Tables 7 and 8 summarize the behavior of b_1^* and b_2, the parameters that determine the slope and curvature of the optimal log consumption-wealth ratio. These parameters both decline with ψ; they are positive for $\psi < 1$, at the right of the table, while they are negative for $\psi > 1$, at the left. They are zero when $\psi = 1$, as we already know from our analysis of the case with unit elasticity of intertemporal substitution in Section 3.6.2.

The sensitivity of the optimal log consumption-wealth ratio to the state variable is given by

$$\frac{\partial\left(c_t - w_t\right)}{\partial\left(x_t + \sigma_u^2/2\right)} = b_1^* + 2b_2\left(x_t + \sigma_u^2/2\right) = b_1^* + 2b_2 \log \mathrm{E}_t\left[R_{1,t+1}/\overline{R}_f\right] .$$

Figure 2 shows that this sensitivity is modest for most parameter values; the curves for the optimal consumption policies tend to be rather flat. This is particularly true for high values of risk aversion γ. One can reach the same conclusion by comparing the consumption-wealth ratios given in Tables 6 and 9. The ratio in Table 6 is the optimal consumption-wealth ratio when the expected excess gross return on stock is zero (an unusually low value), while the ratio in Table 9 is a measure of the "average" optimal consumption-wealth ratio. These ratios are generally close to one another.

The standard errors in the tables follow the same pattern as those in the tables describing the optimal portfolio choice. That is, the intercepts are estimated with less precision than the parameters determining the slope and curvature of the optimal policy, and the quarterly data set generally offers more precise estimates than the annual data set.

To interpret these results, it is useful to recall the log-linear equation for the log consumption-wealth ratio, (18), which we reproduce here for convenience with a one-period shift in time index from $t+1$ to t:

$$c_t - w_t = (1-\psi)\,\mathrm{E}_t \sum_{j=1}^{\infty} \rho^j r_{p,t+j} - \mathrm{E}_t \sum_{j=1}^{\infty} \rho^j v_{p,t-1+j} + \frac{\rho}{1-\rho}\left(k - \psi \log \delta\right).$$

Using this equation, we can distinguish three different effects that operate on the optimal consumption-wealth ratio. The first is the *intertemporal substitution effect*, which works through expected future portfolio returns $\mathrm{E}_t r_{p,t+j}$ and is controlled by the elasticity of intertemporal substitution ψ. This effect is present even when there is no uncertainty and all asset returns are riskless. The second is the *precautionary savings effect*, the willingness to save because returns are uncertain. This works through expected future variances of consumption growth in relation to portfolio returns $\mathrm{E}_t v_{p,t-1+j}$, where equation (11) gives $v_{p,t}$ as $v_{p,t} = (\theta/2\psi)\mathrm{Var}_t(\Delta c_{t+1} - \psi r_{p,t+1})$. This effect is controlled by both risk aversion γ and the elasticity of intertemporal substitution ψ. These two effects interact with a third effect, the *portfolio composition effect*, which arises because in our setup the individual can control either the expected portfolio return $\mathrm{E}_t r_{p,t+1}$ or the variability of return $\mathrm{Var}_t r_{p,t+1}$ through her investment decision. The literature on growth has traditionally considered the intertemporal substitution effect in a framework in which the rate of return on wealth is given to the individual. Similarly, the literature on precautionary savings has traditionally considered an uncertain income stream over which the individual has no control. In our model, however, the expected return on wealth and the degree of return uncertainty are endogenous, even though the individual cannot control both simultaneously. She faces a trade-off, familiar in the asset pricing literature, between risk and return.

Equation (18) also illustrates the fact that the consumption-wealth ratio is determined by long-run considerations. The terms that appear on the right-hand side of the equation are expected discounted values of all future expected returns and variances, not current expected returns and variances. This explains why the current value of the state variable has only a small effect on the optimal consumption-wealth ratio, as shown in Figure 2.

We can now interpret the detailed patterns in Tables 6 and 9. Consider first the right-hand column of Table 9. This gives the exponentiated mean optimal log consumption-wealth ratio for an individual who is extremely reluctant to substitute consumption intertemporally ($\psi = 1/40$, close to zero). Such an individual wishes to maintain a constant expected consumption growth rate regardless of current investment opportunities. She can do this by consuming the long-run average return on her portfolio, with a precautionary-savings adjustment for risk. But both the risk and the average return are endogenous here. If the investor is highly risk-averse, as she is at the bottom of the column ($\gamma = 40$), then she holds almost all her wealth in the riskless asset and earns a low return with little risk; if she is close to risk-neutral, as she is at the top of the column ($\gamma = 0.75$), she borrows at the riskless interest rate to earn a high but risky leveraged return. This explains why the mean consumption-wealth ratio is so much higher at the top of the column than at the bottom.

To clarify this interpretation, Table 10 reports the unconditional mean log portfolio return, $\mathrm{E}[r_{p,t+1}]$.[8] The mean log returns in the right-hand column of Table 10 are close to the optimal consumption-wealth ratios given in the right-hand column of Table 9. They are particularly close at high levels of risk aversion, shown at the bottom of the tables; at the top of the tables the two variables diverge because the mean log return reaches a maximum when the coefficient of relative risk aversion $\gamma = 1$, and starts to fall when risk aversion declines from this level, whereas the optimal consumption-wealth ratio keeps on rising as γ falls below one. We have already emphasized the fact that the investor with unit risk aversion maximizes the conditional expectation of the log portfolio return; hence this investor must also have the highest unconditional expected log portfolio return. The increase in the average consumption-wealth ratio as γ falls below one is caused by the precautionary savings effect, which turns negative when ψ and γ are on the same side of unity. We explore this effect in more detail in the next section.

Now consider what happens as the individual becomes more willing to substitute intertemporally, that is, as ψ increases and we move to the left in Table 9. If we hold fixed the variance terms in (18), the derivative of $c_t - w_t$ with respect to ψ is $-[\rho/(1-\rho)](\mathrm{E}_t[(1-\rho)/\rho]\sum_{j=1}^{\infty}\rho^j r_{p,t+j} + \log\delta)$, which is negative if the long-run expected portfolio return exceeds the rate of time preference and positive otherwise. Ignoring precautionary savings effects, an individual who is willing to substitute intertemporally will have higher saving and lower current consumption than an individual who is unwilling to substitute intertemporally if the time-preference-adjusted rate of return on saving is positive, but will have lower saving and higher current consumption if the adjusted return on saving is negative.

Table 9 illustrates this pattern very well. Investors with low risk aversion γ at the top of the table choose portfolios with high average returns, so a higher elasticity of intertemporal substitution ψ corresponds to a lower average consumption-wealth ratio. Highly risk-averse investors at the bottom of the table choose safe portfolios with low average returns, so for these investors a higher ψ corresponds to a higher average consumption-wealth ratio.

Our discussion so far has concentrated on the average level of consumption in

[8] We can compute the long-term or unconditional expected log return on wealth by taking unconditional expectations in Lemma 4, i.e., by calculating $\mathrm{E}[\mathrm{E}_t(r_{p,t+1})]$, which gives:

$$\begin{aligned}\mathrm{E}\left[r_{p,t+1}\right] &= \overline{r}_f + p_0 + p_1\,\mathrm{E}\,x_t + p_2\,\mathrm{E}\,x_t^2 \\ &= \overline{r}_f + p_0 + p_1\mu + p_2\left(\sigma_x^2 + \mu^2\right)\end{aligned}$$

where p_0, p_1, and p_2 are functions of a_0 and a_1 defined in Lemma 4. We can rewrite p_0, p_1, and p_2 as functions of the normalized parameters a_0^* and a_1 by noticing that $a_0^* = a_0 - a_1\left(\sigma_u^2/2\right)$.

relation to wealth. We now give some intuition about the sensitivity of the optimal ratio to the state variable x_t. Although we have noted that the slope of the optimal consumption policy is always small in absolute value relative to the intercept, it is negative when $\psi > 1$, and positive when $\psi < 1$. Moreover, it increases in absolute value as γ decreases. The intertemporal substitution effect and the portfolio composition effect explain this pattern. As x_t increases so does the expected return on wealth, causing income and substitution effects on consumption. When $\psi > 1$ the substitution effect dominates and the investor will cut consumption to exploit favorable investment opportunities. When $\psi < 1$ the income effect dominates and the investor will increase consumption because a given quantity of wealth can sustain a greater flow of consumption. The effect of risk aversion appears because the state variable x_t increases only the expected return on the risky asset, not the expected return on the riskless asset. An investor with a low risk aversion coefficient is more heavily invested in the risky asset and thus her portfolio return is more sensitive to changes in x_t.

4.5 The optimal growth rates of consumption and wealth

So far we have characterized the level of consumption in relation to wealth, and have seen that this is determined primarily by long-run considerations. In order to get a clearer view of precautionary savings effects, we now turn to a short-run analysis of expected portfolio returns and consumption growth over a single period. Lemmas 4 and 5 in Appendix A show that the expected single-period portfolio return $\mathrm{E}_t r_{p,t+1}$ and the expected single-period consumption growth rate $\mathrm{E}_t \Delta c_{t+1}$ are quadratic functions of x_t. Figures 3a and 4a plot the $\mathrm{E}_t r_{p,t+1}$ function, and Figures 3b and 4b plot the $\mathrm{E}_t \Delta c_{t+1}$ function, using the annual return-generating process and the preference parameters $\{\psi = 1/0.75, \gamma = 4\}$ and $\{\psi = 1/4, \gamma = 4\}$, respectively. The figures show that both functions are convex.

We have already noted that the conditional expected log portfolio return $\mathrm{E}_t r_{p,t+1}$ is maximized by an investor with unit risk aversion who sets $\alpha_t = (1/2) + (x_t/\sigma_u^2)$. In this case $\min\{\mathrm{E}_t r_{p,t+1}\} = \overline{r}_f$. Hence, whenever $\gamma \neq 1$, the minimum expected log portfolio return must be lower than $\overline{r}_f$. Figures 3a and 4a illustrate this possibility. Similarly, the minimum expected gross portfolio return must be less than the simple gross riskless interest rate $\overline{R}_f$ whenever $\gamma \neq 1$. The explanation is that an investor whose relative risk aversion is greater than one has a positive intertemporal hedging demand for the risky asset when its gross expected excess return is zero (a state indicated by a vertical line in Figures 3a and 4a). By continuity, this investor also has a positive demand for the risky asset when its gross expected excess return is slightly negative, and this will imply a negative gross expected excess return on the portfolio. Conversely, an investor whose relative risk aversion is less than one has a negative hedging demand and will short the risky asset even when its gross expected excess return is slightly positive, again implying a negative gross expected excess return on the portfolio.

The $E_t \Delta c_{t+1}$ curve has a similar shape to the $E_t r_{p,t+1}$ curve. Expected consumption growth rates are negative around the point $x_t = -c_1/2c_2$, where c_1 and c_2 are parameters defined in Lemma 5 in Appendix A; this is the value of x_t at which the curve reaches its minimum. The convexity of the curve varies with both ψ and γ. For the particular cases plotted in Figures 3b and 4b, the convexity decreases with ψ, for a given value of γ ($\gamma = 4$). This is generally true, because as ψ decreases the individual cares more about smoothing consumption and expected consumption growth becomes less sensitive to the state variable.

A precautionary savings effect is also driving the behavior of the curve. To understand this effect it is useful to analyze the Euler equation (10), and particularly its intercept $v_{p,t}$ as given in (11), which are reproduced here for convenience:

$$E_t \Delta c_{t+1} = v_{p,t} + \psi \left(E_t r_{p,t+1} + \log \delta \right),$$

$$v_{p,t} = \frac{1}{2} \left(\frac{1-\gamma}{\psi - 1} \right) \mathrm{Var}_t \left(\Delta c_{t+1} - \psi r_{p,t+1} \right) .$$

The intercept $v_{p,t}$ shows how the conditional variance of optimal consumption growth relative to wealth growth affects expected consumption growth. When the investment opportunity set is constant, this intercept is constant, but in our model with time-varying investment opportunities it moves with the state variable x_t. Proposition 1 shows that $v_{p,t}$ is a quadratic function of the state variable, with coefficients that are non-linear functions of the underlying parameters of the model. Figures 3c and 4c plot $E_t \Delta c_{t+1}$ and $\psi \left(E_t r_{p,t+1} + \log \delta \right)$ against $\left(E_t r_{p,t+1} + \log \delta \right)$. Thus $v_{p,t}$ is the distance between the two curves in the figures. Because both $E_t \Delta c_{t+1}$ and $\left(E_t r_{p,t+1} + \log \delta \right)$ are quadratic functions of x_t, a plot of one against the other gives a hyperbola whose vetex is $(\min E_t r_{p,t+1}, \min E_t \Delta c_{t+1})$. Nevertheless, the arms of the hyperbola appear to be very close, rendering the relationship very close to linear.

Figures 3c and 4c show that precautionary savings, as measured by $v_{p,t}$, can have either sign. From (11), we have that $\mathrm{sign}(v_{p,t}) = \mathrm{sign}((1-\gamma)/(\psi - 1))$, so $v_{p,t}$ is positive whenever γ and ψ are on opposite sides of unity, but negative whenever they are on the same side of unity. When preferences are isoelastic, $\psi = \gamma^{-1}$, and $v_{p,t} > 0$. But as we move away from isoelastic preferences, precautionary savings may be negative. It is important to notice that the sign of $v_{p,t}$ does not depend on having a state-dependent investment opportunity set—we have already shown in section 3.6.4 that $v_{p,t} \neq 0$, but constant, when expected returns are constant as long as $\gamma \neq 1$ and $\psi \neq 1$. Therefore, the sign of precautionary savings must be related to the preference structure rather than to the stochastic structure of the model. From standard results on the power-utility model we know that precautionary savings are positive because these preferences exhibit prudence, i.e. a positive third derivative of the utility function. We have not characterized the general value function under recursive-utility preferences, but the sign of $v_{p,t}$ suggests that the third derivative of the value function may be negative when γ and ψ are simultaneously above or below one.

Figures 3c and 4c also show that the magnitude of precautionary savings (in absolute value) increases with $\mathrm{E}_t r_{p,t+1}$—that is, as x_t moves away from $-c_1/2c_2$. However, for the range of values of x_t considered here, the magnitude of $v_{p,t}$ does not increase dramatically—the maximum value of $v_{p,t}$ is slightly below 2 percentage points—which is consistent with our earlier result that short-term effects do not have a large impact on the optimal consumption-wealth ratio.

As a final way to explore the implications of our model for short-run consumption behavior, in Table 11 we report the unconditional standard deviation of consumption innovations for each set of preferences we have considered. The table shows that investors with low risk aversion have extremely volatile consumption growth, for their consumption inherits the volatility of their portfolios. Investors with unit elasticity of substitution in consumption have constant consumption-wealth ratios and so their consumption volatility equals their portfolio volatility. Investors with low elasticity of intertemporal substitution have somewhat less volatile consumption, because they react to mean-reversion in stock returns by cutting their consumption-wealth ratios when the stock market rises. A 1% innovation in wealth causes these investors to increase consumption by less than 1%; they know that a 1% increase in consumption could not be sustained, even with 1% greater wealth, because the increase in wealth is accompanied by a decrease in expected portfolio returns. Investors with high elasticity of intertemporal substitution respond to the decrease in expected returns by cutting saving, so their consumption is more volatile than their portfolio returns. Similar results are reported by Campbell (1996a) for a model with an exogenous portfolio return process.

4.6 Portfolio allocation and consumption over time

Our results can also be summarized by plotting the optimal equity allocations and consumption-wealth ratios over time. Figure 5 does this for our quarterly model, while

Figure 6 does it for our annual model. We use preference parameters $\{\psi = 1/4, \gamma = 0.75\}$ in Figures 5a and 6a, and $\{\psi = 1/4, \gamma = 4\}$, corresponding to power utility with moderate risk aversion, in Figures 5b and 6b. The left hand part of each figure shows the optimal equity allocation, while the right hand part shows the optimal consumption-wealth ratio. The solid lines are the point estimates, while the dotted and dashed lines are 95% asymptotic confidence intervals based on asymptotic standard errors (conditional on x_t) calculated using the delta method. The horizontal lines in the equity-allocation plots represent 0% and 100% holdings.

The figures show that stock holdings are highly volatile while the optimal ratio of consumption to wealth ratio is more stable, but spikes up in periods where expected returns and optimal stock holdings are unusually high. The investor with lower risk aversion, in Figures 5a and 6a, holds on average a much larger proportion of her wealth in stocks and her consumption-wealth ratio is also larger on average. But both investors are keen stock-market investors. In our model investors do not face restrictions on short sales, so we allow the optimal allocation to stocks to be either larger than 100% or negative. Figure 5 shows that in the postwar quarterly model both investors usually want to short the riskless asset and invest more than 100% of their wealth in the market, except in periods of unusually low dividend yields such as the early 1970's and the 1990's.

Barberis (1995) has obtained similar results for a Bayesian investor who maximizes power utility defined over terminal wealth and uses the log dividend-price ratio as a state variable; with a 10-year investment horizon and access to historical data over the period 1927-1993, Barberis' investor, who is not allowed to short assets, is mostly 100% invested in stocks. Brennan, Schwartz, and Lagnado (1995) have studied a similar problem with power utility of terminal wealth, three state variables, three assets, and weekly portfolio rebalancing. They also do not allow short sales, and their optimal strategy for the period 1972-1992 frequently switches between 100% cash and 100% stocks. Their optimal strategy is more volatile than Barberis' or ours because they allow for a larger number of state variables. Both Barberis and Brennan, Schwartz, and Lagnado also include long-term bonds in their analysis, but bonds do not play a major role in the optimal portfolio.

5 Conclusion

One of the major objectives of modern financial economics has been to put investment advice on a scientific basis. This task has been accomplished for investors who have short horizons or constant investment opportunities. Unfortunately most investors have long horizons, and there is considerable evidence that they face time-varying expected returns on risky assets. Financial economists have not been able to give such investors precise quantitative advice about their portfolio strategies.

Our goal has been to remedy this situation. Because the intertemporal consumption and portfolio choice problem is highly intractable when expected returns are time varying, we have resorted to an approximation. We have replaced the Euler equations and budget constraint of the exact problem with approximate equations that are much easier to solve, and we have explored in detail the analytical solution of the approximate problem. We have shown, for example, that in our model investors always increase their risky asset allocations when the expected return on the risky asset increases, and that portfolio allocations depend on the elasticity of intertemporal substitution only indirectly through the effect of this elasticity on the average level of consumption relative to wealth.

Our approximate solution is exact when the elasticity of intertemporal substitution is one and the time interval between consumption and portfolio decisions is infinitesimally small. As a next step in this research agenda, we plan to check the accuracy of the analytical approximate solution in other cases by comparing it with a discrete-state numerical solution for a wide range of parameter values. We hope that the approximate solution will prove to be workably accurate, but we also believe that it gives economic insight into the nature of the problem, and should provide useful starting values for more precise numerical solutions in particular cases.

We have used our model to assess the quantitative importance of intertemporal hedging demand for risky assets by long-lived investors. After calibrating the model to US stock market data, we find that intertemporal hedging motives can easily double the average total demand for stocks by investors whose coefficients of relative risk aversion exceed one. This implies that static models of portfolio choice can be seriously misleading, and should be used only with great caution.

The approach of this paper can be applied to many related problems. For simplicity we have considered only a single risky asset and a single state variable, but it is straightforward to consider multiple risky assets and state variables. We can explore horizon effects, in the manner of Barberis (1995), by assuming that the investor has a finite rather than infinite horizon. We can allow the riskless interest rate to vary over time, and can consider investor choices among indexed or nominal bonds of different maturities. We may be able to allow for time-variation in the volatility of risky asset returns, and even for the presence of exogenous labor income in the investor's budget constraint. We believe that in all these cases, there is much understanding to be gained by taking an analytical rather than a purely numerical approach to the problem.

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A Appendix A: Some Useful Lemmas

In this appendix we state as lemmas and prove nine useful results. We use some of them to prove, in Appendix B, the main propositions of the paper.

Lemma 1 *The conditional expectation of future values of the state variable is a linear function of its current value, while the conditional expectation of future values of the squared state variable is a quadratic function of the current state variable:*

$$E_{t+1}x_{t+j} = \mu + \phi^j (x_t - \mu) + \phi^{j-1}\eta_{t+1} ,$$

$$E_{t+1}x_{t+j}^2 = \mu^2 \left(1 - \phi^j\right)^2 + \frac{1 - \phi^{2(j-1)}}{1 - \phi^2}\sigma_\eta^2 + 2\mu\phi^j \left(1 - \phi^j\right) x_t + \phi^{2j}x_t^2$$
$$+\phi^{2(j-1)}\eta_{t+1}^2 + 2\phi^{1+2(j-1)} (x_t - \mu)\, \eta_{t+1} + 2\mu\phi^{j-1}\eta_{t+1} .$$

Proof. By simple forward recursion of x_t and x_t^2 in (3) we have

$$x_{t+j} - \mu = \phi^j (x_t - \mu) + \sum_{l=0}^{j-1} \phi^l \eta_{t+j-l}$$

and

$$(x_{t+j} - \mu)^2 = \phi^{2j} (x_t - \mu)^2 + \sum_{l=0}^{j-1} \phi^{2l}\eta_{t+j-l}^2 + 2\phi \sum_{l=0}^{j-1} \phi^{2l} (x_{t+j-1-l} - \mu)\, \eta_{t+j-l} . \tag{32}$$

The results stated in the lemma follow from the expressions above, after taking conditional expectations at time $(t + 1)$, E_{t+1}, and the martingale assumption (A2), which implies $E_{t+l-1}\eta_{t+l} = 0$, $E_{t+1}\eta_{t+l} = 0$, and $E_{t+1}\left[x_{t+l-1-l}\eta_{t+l}\right] = 0$, $\forall l > 1$.□

Lemma 2 *The innovation in next period's squared state variable is linear in the current state variable:*

$$x_{t+1}^2 - E_t x_{t+1}^2 = \left(\eta_{t+1}^2 - \sigma_\eta^2\right) + \left(2\mu\,(1 - \phi) + 2\phi x_t\right) \eta_{t+1}.$$

Proof. The proof for this lemma is similar to that for Lemma 1. From (32), find x_{t+1}^2 by setting $j = 1$:

$$x_{t+1}^2 = \mu^2 (1 - \phi)^2 + 2\mu\phi (1 - \phi)\, x_t + \phi^2 x_t^2 + \eta_{t+1}^2 + \left[2\phi\,(x_t - \mu) + 2\mu\right] \eta_{t+1} .$$

Lemma 2 then follows by applying the conditional expectations operator E_t to this expression, under the martingale assumption (A2).□

Lemma 3 *The unexpected return on the risky asset and the conditional variance of the risky asset are given by*

$$r_{1,t+1} - E_t r_{1,t+1} = u_{t+1} ,$$

$$\sigma_{1,1,t} = \sigma_u^2 .$$

Proof. This result follows trivially from (A2) and (A3). It is stated here as a Lemma for completeness.□

Lemma 4 *The expected portfolio return next period is quadratic in the current state variable, and the unexpected portfolio return is linear in the current state variable:*

$$E_t r_{p,t+1} = \overline{r}_f + p_0 + p_1 x_t + p_2 x_t^2 \, ,$$

$$r_{p,t+1} - E_t r_{p,t+1} = (a_0 + a_1 x_t)\, u_{t+1} \, ,$$

where

$$p_0 = a_0 (1 - a_0) \frac{\sigma_u^2}{2} \, ,$$

$$p_1 = a_0 + a_1 (1 - 2a_0) \frac{\sigma_u^2}{2} \, ,$$

$$p_2 = a_1 - a_1^2 \frac{\sigma_u^2}{2} \, .$$

Proof. From (14) and our guess **(i)** on the optimal portfolio rule, we have that

$$\begin{aligned} \mathrm{E}_t\, r_{p,t+1} &= \alpha_t\, \mathrm{E}_t \left[r_{1,t+1} - \overline{r}_f \right] + \overline{r}_f + \frac{\sigma_u^2}{2} \alpha_t (1 - \alpha_t) \\ &= (a_0 + a_1 x_t)\, x_t + \overline{r}_f + \frac{\sigma_u^2}{2} \left((a_0 + a_1 x_t) - (a_0 + a_1 x_t)^2 \right) \, , \end{aligned}$$

where the last line follows from (2). Reordering terms we get a quadratic expression in x_t whose coefficients are those given in the statement of the proposition.

The expression for $r_{p,t+1} - \mathrm{E}_t\, r_{p,t+1}$ also follows from (14) and guess **(i)**, as well as (A1) — constant r_f — and (A2)—(2),

$$\begin{aligned} r_{p,t+1} - \mathrm{E}_t\, r_{p,t+1} &= \alpha_t \left((r_{1,t+1} - \overline{r}_f) - \mathrm{E}_t \left[r_{1,t+1} - \overline{r}_f \right] \right) \\ &= (a_0 + a_1 x_t)\, u_{t+1} \, . \end{aligned} \tag{33}$$

□

Lemma 5 *Expected optimal consumption growth over the next period is quadratic in the current state variable, and unexpected consumption growth is linear in the current state variable:*

$$\begin{aligned} E_t \Delta c_{t+1} &= E_t r_{p,t+1} + E_t \left[c_{t+1} - w_{t+1} \right] - \frac{1}{\rho} (c_t - w_t) + k \\ &= c_0 + c_1 x_t + c_2 x_t^2 \, , \end{aligned}$$

$$\begin{aligned} \Delta c_{t+1} - E_t \Delta c_{t+1} = (a_0 + a_1 x_t)\, u_{t+1} + b_1 \eta_{t+1} + b_2 \left(2\mu (1 - \phi) + 2\phi x_t \right) \eta_{t+1} \\ + b_2 \left(\eta_{t+1}^2 - \sigma_\eta^2 \right) \, , \end{aligned}$$

where

$$c_0 = \overline{r}_f + a_0(1-a_0)\frac{\sigma_u^2}{2} + k + b_0\left(1-\frac{1}{\rho}\right) + b_1(\mu(1-\phi)) + b_2\left(\mu^2(1-\phi)^2 + \sigma_\eta^2\right) ,$$

$$c_1 = a_0 + a_1(1-2a_0)\frac{\sigma_u^2}{2} + b_1\left(\phi - \frac{1}{\rho}\right) + b_2(2\mu\phi(1-\phi)) ,$$

$$c_2 = a_1 - a_1^2\frac{\sigma_u^2}{2} + b_2\left(\phi^2 - \frac{1}{\rho}\right) .$$

Proof. From (16) and (13), we can write

$$\Delta c_{t+1} = r_{p,t+1} + (c_{t+1} - w_{t+1}) - \frac{1}{\rho}(c_t - w_t) + k . \tag{34}$$

Therefore,

$$\begin{aligned}
\mathrm{E}_t\,\Delta c_{t+1} &= \mathrm{E}_t\, r_{p,t+1} + \mathrm{E}_t(c_{t+1} - w_{t+1}) - \frac{1}{\rho}(c_t - w_t) + k \\
&= a_0(1-a_0)\frac{\sigma_u^2}{2} + \left(a_0 + a_1\frac{\sigma_u^2}{2} - a_0 a_1 \sigma_u^2\right)x_t + a_1\left(1 - a_1\frac{\sigma_u^2}{2}\right)x_t^2 + \overline{r}_f \\
&\quad + b_0\left(1-\frac{1}{\rho}\right) + b_1\left(\mathrm{E}_t\, x_{t+1} - \frac{1}{\rho}x_t\right) + b_2\left(\mathrm{E}_t\, x_{t+1}^2 - \frac{1}{\rho}x_t^2\right) + k \\
&= a_0 x_t + a_1 x_t^2 + \overline{r}_f + b_0\left(1-\frac{1}{\rho}\right) + b_1\left[\mu(1-\phi) + \left(\phi - \frac{1}{\rho}\right)x_t\right] \\
&\quad + b_2\left[\mu^2(1-\phi)^2 + \sigma_\eta^2 + 2\mu\phi(1-\phi)x_t + \left(\phi^2 - \frac{1}{\rho}\right)x_t^2\right] + k ,
\end{aligned}$$

where the second equality follows from Lemma 4 and our guess **(ii)** on the optimal consumption policy, and the last line follows from (3) and Lemmas 1 and 2. Re-ordering terms we get the expression for $\mathrm{E}_t\,\Delta c_{t+1}$ in Lemma 5 as well as $\{c_0, c_1, c_2\}$.

The expression for unexpected consumption growth follows from (34), the expression for the unexpected portfolio return in Lemma 4, and from the expression for the unexpected log consumption-wealth ratio that obtains from guess (ii), (3) and Lemma 2:

$$\begin{aligned}
c_{t+1} - w_{t+1} - \mathrm{E}_t(c_{t+1} - w_{t+1}) &= b_1(x_{t+1} - \mathrm{E}_t\, x_{t+1}) + b_2\left(x_{t+1}^2 - \mathrm{E}_t\, x_{t+1}^2\right) \\
&= b_1\eta_{t+1} + b_2\left[\left(\eta_{t+1}^2 - \sigma_\eta^2\right) + (2\mu(1-\phi) + 2\phi x_t)\eta_{t+1}\right] . \qquad (35)
\end{aligned}$$

□

Lemma 6 *The time-varying intercept in the Euler equation for portfolio returns (11) is a quadratic function of the state variable:*

$$v_{p,t} = v_0 + v_1 x_t + v_2 x_t^2 ,$$

where

$$v_0 = a_0^2\left[(1-\gamma)(\psi-1)\frac{1}{2}\sigma_u^2\right] + b_1^2\left[\left(\frac{1-\gamma}{\psi-1}\right)\frac{1}{2}\sigma_\eta^2\right] + b_2^2\left[\left(\frac{1-\gamma}{\psi-1}\right)\left(\sigma_\eta^2 + 2\mu^2(1-\phi)^2\right)\sigma_\eta^2\right]$$

$$-a_0 b_1 \left[(1-\gamma)\sigma_{\eta u}\right] - a_0 b_2 \left[(1-\gamma) 2\mu (1-\phi)\sigma_{\eta u}\right] + b_1 b_2 \left[\left(\frac{1-\gamma}{\psi-1}\right) 2\mu (1-\phi)\sigma_\eta^2\right] ,$$

$$v_1 = b_2^2 \left[\left(\frac{1-\gamma}{\psi-1}\right) 4\mu\phi (1-\phi)\sigma_\eta^2\right] + a_0 a_1 \left[(1-\gamma)(\psi-1)\sigma_u^2\right] - a_0 b_2 \left[(1-\gamma) 2\phi\sigma_{\eta u}\right]$$

$$-a_1 b_1 \left[(1-\gamma)\sigma_{\eta u}\right] - a_1 b_2 \left[(1-\gamma) 2\mu (1-\phi)\sigma_{\eta u}\right] + b_1 b_2 \left[\left(\frac{1-\gamma}{\psi-1}\right) 2\phi\sigma_\eta^2\right] ,$$

$$v_2 = a_1^2 \left[(1-\gamma)(\psi-1)\frac{1}{2}\sigma_u^2\right] + b_2^2 \left[\left(\frac{1-\gamma}{\psi-1}\right) 2\phi^2\sigma_\eta^2\right] - a_1 b_2 \left[(1-\gamma) 2\phi\sigma_{\eta u}\right] .$$

Proof. From (11), (13) and (16), we have

$$\begin{aligned} v_{p,t} &= \frac{1}{2}\left(\frac{\theta}{\psi}\right) \mathrm{Var}_t \left(\Delta c_{t+1} - \psi r_{p,t+1}\right) \\ &= \frac{1}{2}\left(\frac{\theta}{\psi}\right) \mathrm{E}_t \left[\left(\Delta c_{t+1} - \mathrm{E}_t \Delta c_{t+1}\right) - \psi \left(r_{p,t+1} - \mathrm{E}_t r_{p,t+1}\right)\right]^2 \\ &= \frac{1}{2}\left(\frac{\theta}{\psi}\right) \mathrm{E}_t \left[(1-\psi)\left(r_{p,t+1} - \mathrm{E}_t r_{p,t+1}\right) + \left(c_{t+1} - w_{t+1}\right) - \mathrm{E}_t \left(c_{t+1} - w_{t+1}\right)\right]^2 . \end{aligned}$$

If we substitute in the bracketed expression above (33) and (35) for $(r_{p,t+1} - \mathrm{E}_t r_{p,t+1})$ and $(c_{t+1} - w_{t+1}) - \mathrm{E}_t (c_{t+1} - w_{t+1})$ and compute E_t under assumptions (A2) and (A3), we find that $v_{p,t}$ is a quadratic function of x_t, with the coefficients given in the statement of the lemma. □

Lemma 7 *The parameters defining the optimal consumption rule (ii) satisfy the following three-equation system:*

$$v_0 = k - \psi \log\delta + (1-\psi)\overline{r}_f + (1-\psi) a_0 (1-a_0)\frac{\sigma_u^2}{2} + b_0\left(1 - \frac{1}{\rho}\right) + b_1 \left(\mu (1-\phi)\right) + b_2 \left(\mu^2 (1-\phi)^2 + \sigma_\eta^2\right),$$

$$v_1 = a_0 (1-\psi) + (1-\psi) a_1 (1-2a_0)\frac{\sigma_u^2}{2} + b_1\left(\phi - \frac{1}{\rho}\right) + b_2 \left(2\mu\phi (1-\phi)\right),$$

$$v_2 = a_1 (1-\psi) - (1-\psi) a_1^2 \frac{\sigma_u^2}{2} + b_2 \left(\phi^2 - \frac{1}{\rho}\right).$$

Proof. This follows from the log-linearized Euler equation for the optimal portfolio given in (10), and Lemmas 4, 5 and 6. From (10) and Lemmas 4 and 6,

$$\begin{aligned} \mathrm{E}_t \Delta c_{t+1} &= \psi \log\delta + v_{p,t} + \psi \mathrm{E}_t r_{p,t+1} \\ &= \psi \log\delta + \psi \overline{r}_f + v_0 + \psi a_0 (1-a_0)\frac{\sigma_u^2}{2} \\ &\quad + \left(v_1 + \psi\left(a_0 + a_1\frac{\sigma_u^2}{2} - a_0 a_1 \sigma_u^2\right)\right) x_t + \left(v_2 + \psi a_1 \left(1 - a_1 \frac{\sigma_u^2}{2}\right)\right) x_t^2 , \end{aligned} \quad (36)$$

which is a quadratic function of the state variable. But from Lemma 5 we have that $\mathrm{E}_t \Delta c_{t+1}$ is also quadratic in x_t:

$$\mathrm{E}_t \Delta c_{t+1} = c_0 + c_1 x_t + c_2 x_t^2 , \quad (37)$$

where c_0, c_1, and c_2 are given in Lemma 5. Equating coefficients on the right hand side of (36) and (37), the lemma follows immediately.□

Lemma 8 *The covariance between unexpected stock returns and changes in expected portfolio returns is linear in the state variable:*

$$Cov_t\left(r_{1,t+1} - E_t r_{1,t+1}, (1-\psi)(E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j r_{p,t+1+j}\right)$$
$$= (1-\psi)\left((p_1 + p_2 2\mu)\frac{\rho}{1-\rho\phi} - p_2\mu\frac{\rho\phi}{1-\rho\phi^2}\right)\sigma_{\eta u} + (1-\psi)\left(p_2\frac{\rho\phi}{1-\rho\phi^2}\sigma_{\eta u}\right)x_t\ ,$$

where $\{p_1, p_2\}$ *are given in Lemma 4.*

Proof. Lemma 4 implies:

$$\begin{aligned}
(\mathrm{E}_{t+1} - \mathrm{E}_t)\sum_{j=1}^{\infty}\rho^j r_{p,t+j+1} &= p_1\sum_{j=1}^{\infty}\rho^j(\mathrm{E}_{t+1} - \mathrm{E}_t)x_{t+j} + p_2\sum_{j=1}^{\infty}\rho^j(\mathrm{E}_{t+1} - \mathrm{E}_t)x_{t+j}^2 \qquad (38)\\
&= p_1\eta_{t+1}\sum_{j=1}^{\infty}\rho^j\phi^{j-1} + \frac{p_2\sigma_\eta^2}{1-\phi^2}\sum_{j=1}^{\infty}\rho^j\left(\phi^{2j} - \phi^{2(j-1)}\right) + p_2\eta_{t+1}^2\sum_{j=1}^{\infty}\rho^j\phi^{2(j-1)}\\
&\quad + p_2\eta_{t+1}(x_t - \mu)\phi^{-1}\sum_{j=1}^{\infty}\rho^j\phi^{2j} + p_2\eta_{t+1}2\mu\phi^{-1}\sum_{j=1}^{\infty}\rho^j\phi^j\\
&= \left[p_1\frac{\rho}{1-\rho\phi} + p_2(x_t - \mu)\frac{\rho\phi}{1-\rho\phi^2} + p_2 2\mu\frac{\rho}{1-\rho\phi}\right]\eta_{t+1}\\
&\quad + p_2\frac{\rho}{1-\rho\phi^2}\eta_{t+1}^2 - p_2\frac{\rho}{1-\rho\phi^2}\sigma_\eta^2\ ,
\end{aligned}$$

where the second equality follows from Lemma 2 and the third one follows after computing the infinite summations in the second one and reordering terms.

The result stated in the lemma follows immediately from assumptions (A2) and (A3) about the distribution of (u_{t+1}, η_{t+1}), the expression above and the properties of the covariance operator.□

Lemma 9 *The covariance between unexpected stock returns and changes in the expected value of the intercept in the Euler equation (11) is linear in the state variable:*

$$Cov_t\left(r_{1,t+1} - E_t r_{1,t+1}, (E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j v_{p,t+j}\right)$$
$$= \left((v_1 + v_2 2\mu)\frac{\rho}{1-\rho\phi} - v_2\mu\frac{\rho\phi}{1-\rho\phi^2}\right)\sigma_{\eta u} + \left(v_2\frac{\rho\phi}{1-\rho\phi^2}\sigma_{\eta u}\right)x_t\ ,$$

where $\{v_0, v_1, v_2\}$ *are given in Lemma 6.*

Proof. Lemma 6 implies:

$$(\mathrm{E}_{t+1}-\mathrm{E}_t)\sum_{j=1}^{\infty}\rho^j v_{p,t+j}=v_1\sum_{j=1}^{\infty}\rho^j\left(\mathrm{E}_{t+1}-\mathrm{E}_t\right)x_{t+j}+v_2\sum_{j=1}^{\infty}\rho^j\left(\mathrm{E}_{t+1}-\mathrm{E}_t\right)x_{t+j}^2 .$$

But the right hand side of the equality above is identical to (38), except that we have v_1 and v_2 instead of p_1 and p_2. Therefore, we must have that:

$$\begin{aligned}(\mathrm{E}_{t+1}-\mathrm{E}_t)\sum_{j=1}^{\infty}\rho^j v_{p,t+j} &= \left[v_1\frac{\rho}{1-\rho\phi}+v_2\left(x_t-\mu\right)\frac{\rho\phi}{1-\rho\phi^2}+v_2 2\mu\frac{\rho}{1-\rho\phi}\right]\eta_{t+1}\\ &\quad +v_2\frac{\rho}{1-\rho\phi^2}\eta_{t+1}^2-v_2\frac{\rho}{1-\rho\phi^2}\sigma_\eta^2 ,\end{aligned}$$

from which the lemma follows, under the distributional assumptions (A2) and (A3).□

B Appendix B: Proofs of Propositions

B.1 Proof of Proposition 1

From (A2), $\mathrm{E}_t\, r_{1,t+1}-\overline{r}_f=x_t$ and from (A3), $\sigma_{1,1,t}=\sigma_u^2$. Also, from guess **(ii)**,

$$\begin{aligned}\sigma_{1,c-w,t} &= \mathrm{Cov}_t\left(r_{1,t+1},c_{t+1}-w_{t+1}\right)\\ &= \mathrm{Cov}_t\left[r_{1,t+1}-\mathrm{E}_t\, r_{1,t+1}, b_1\left(x_{t+1}-\mathrm{E}_t\, x_{t+1}\right)+b_2\left(x_{t+1}^2-\mathrm{E}_t\, x_{t+1}^2\right)\right]\\ &= \mathrm{Cov}_t\left[u_{t+1}, b_1\eta_{t+1}+b_2\left(\eta_{t+1}^2-\sigma_\eta^2\right)+b_2\left(2\mu\left(1-\phi\right)+2\phi x_t\right)\eta_{t+1}\right]\\ &= b_1\sigma_{\eta u}+b_2\left[2\mu\left(1-\phi\right)+2\phi x_t\right]\sigma_{\eta u} ,\end{aligned}$$

where the second line follows from substituting guess **(ii)** for $c_{t+1}-w_{t+1}$, the third line follows from (A2) and Lemmas 2 and 3, and the last line follows from (A3) and the assumption of joint normality of u_{t+1} and η_{t+1}.

Using these results we can re-write (17) as

$$\alpha_t^*=\frac{1}{\gamma}\frac{x_t}{\sigma_u^2}+\frac{1}{2\gamma}-\frac{1-\gamma}{\gamma\left(\psi-1\right)}\frac{\sigma_{\eta u}}{\sigma_u^2}\left\{b_1+b_2\left[2\mu\left(1-\phi\right)+2\phi x_t\right]\right\} , \tag{39}$$

which is linear in x_t. But our guess **(i)** on the optimal portfolio policy is that α_t^* is linear in the state variable,

$$\alpha_t^*=a_0+a_1x_t .$$

Grouping terms in (39) we obtain a_0 and a_1 as stated in Proposition 1.□

B.2 Proof of Proposition 2

The proof for this proposition follows from Lemmas 6 and 7 and Proposition 1. Lemma 6 defines a non-linear equation system for $\{v_0,v_1,v_2\}$, $\{a_0,a_1\}$ and $\{b_0,b_1,b_2\}$:

$$\begin{aligned}v_0 &= V_{11}a_0^2+V_{12}b_1^2+V_{13}b_2^2+V_{14}a_0b_1+V_{15}a_0b_2+V_{16}b_1b_2\\ v_1 &= V_{21}b_2^2+V_{22}a_0a_1+V_{23}a_0b_2+V_{24}a_1b_1+V_{25}a_1b_2+V_{26}b_1b_2\\ v_3 &= V_{31}a_1^2+V_{32}b_2^2+V_{33}a_1b_2 ,\end{aligned}$$

where the coefficients V_{ij} are functions of the primitive parameters of the model (both those defining the preference structure and those defining the stochastic structure of the model) and are immediately identifiable from the statement of the system in Lemma 6. For example, $V_{11} = (1-\gamma)(\psi-1)\sigma_u^2/2$, and so on.

Similarly, Lemma 7 defines a second system for $\{v_0, v_1, v_2\}$, $\{a_0, a_1\}$ and $\{b_0, b_1, b_2\}$:

$$
\begin{aligned}
v_0 &= B_{10} + B_{11}b_0 + B_{12}b_1 + B_{13}b_2 + B_{14}a_0 + +B_{15}a_0^2 \\
v_1 &= B_{21}a_0 + B_{22}b_1 + B_{23}b_2 + B_{24}a_1 + B_{25}a_0a_1 \\
v_2 &= B_{31}a_1 + B_{32}b_2 + B_{33}a_1^2 \ ,
\end{aligned}
$$

where the coefficients B_{ij} are functions of the primitive parameters of the model. For example, $B_{10} = k - \psi \log \delta + (1-\psi)\overline{r}_f$, and so on.

Finally, Proposition 1 defines another system for $\{a_0, a_1\}$ and $\{b_0, b_1, b_2\}$:

$$
\begin{aligned}
a_0 &= A_{10} + A_{11}b_1 + A_{12}b_2 \\
a_1 &= A_{20} + A_{21}b_2 \ ,
\end{aligned}
$$

where, again the coefficients A_{ij} are also functions of the primitive parameters of the model and are immediately identifiable from the statement of the system in the proposition. For example, $A_{10} = 1/2\gamma$, and so on.

By equating the right-hand sides of the first and second system we obtain another system whose unknowns are $\{a_0, a_1\}$ and $\{b_0, b_1, b_2\}$. But the third system defines $\{a_0, a_1\}$ as linear combinations of $\{b_1, b_2\}$. Substituting this system into the one obtained by combining the first and second systems, we obtain the Fundamental Equation System given in the statement of the proposition, that depends only on $\{b_0, b_1, b_2\}$, the parameters of the optimal log consumption-wealth ratio. The coefficients Λ_{ij} relate to the coefficients A_{ij}, B_{ij} and V_{ij} as follows:

$$
\begin{aligned}
0 &= -B_{10} + (V_{11} - B_{15})A_{10}^2 - B_{14}A_{10} + (-B_{11})b_0 \\
&\quad + [2(V_{11} - B_{15})A_{10}A_{12} - B_{12} + V_{14}A_{10} - B_{14}A_{11}]b_1 + \left[(V_{11} - B_{15})A_{11}^2 + V_{12} + V_{14}A_{11}\right]b_1^2 \\
&\quad + [2(V_{11} - B_{15})A_{10}A_{12} - B_{13} + V_{15}A_{10} - B_{14}A_{12}]b_2 + \left[(V_{11} - B_{15})A_{12}^2 + V_{13} + V_{15}A_{12}\right]b_2^2 \\
&\quad + [2(V_{11} - B_{15})A_{11}A_{12} + V_{14}A_{12} + V_{15}A_{11} + V_{16}]b_1b_2, \\
0 &= -B_{21}A_{10} + (V_{22} - B_{25})A_{10}A_{20} - B_{24}A_{20} + [-B_{21}A_{11} - B_{22} + (V_{22} - B_{25})A_{11}A_{20} + V_{24}A_{20}]b_1 \\
&\quad + [-B_{21}A_{12} - B_{23} + (V_{22} - B_{25})(A_{10}A_{21} + A_{12}A_{20}) + V_{23}A_{10} + V_{25}A_{20} - B_{24}A_{21}]b_2 \\
&\quad + [V_{21} + (V_{22} - B_{25})A_{12}A_{21} + V_{23}A_{12} + V_{25}A_{21}]b_2^2 \\
&\quad + [(V_{22} - B_{25})A_{11}A_{21} + V_{23}A_{11} + V_{24}A_{21} + V_{26}]b_1b_2, \\
0 &= (V_{31} - B_{33})A_{20}^2 - B_{31}A_{20} + [2(V_{31} - B_{33})A_{20}A_{21} - B_{31}A_{21} - B_{32} + V_{33}A_{20}]b_2 \\
&\quad + \left[(V_{31} - B_{33})A_{21}^2 + V_{32} + V_{33}A_{21}\right]b_2^2.
\end{aligned}
$$

□

C Appendix C: Proofs of Properties

C.1 Proof of Property 1

The third equation of the Fundamental Equation System characterizes the behavior of b_2. This equation is:

$$0 = \Lambda_{30} + \Lambda_{31} b_2 + \Lambda_{33} b_2^2 .$$

Substituting Λ_{ij}'s for their values we get:

$$0 = \frac{(\psi - 1)}{2\gamma\sigma_u^2} - \left[\frac{(1-\gamma)\, 2\phi\sigma_{\eta u}}{\gamma\sigma_u^2} + \left(\phi^2 - \frac{1}{\rho}\right)\right] b_2 + \left[\frac{2(1-\gamma)\,\phi^2\left[\sigma_{\eta u}^2 + \gamma\left(\sigma_u^2\sigma_\eta^2 - \sigma_{\eta u}^2\right)\right]}{(\psi - 1)\,\gamma\sigma_u^2}\right] b_2^2 , \quad (40)$$

This equation has two roots, that we denote $\{b_{21}, b_{22}\}$. A sufficient (but not necessary) condition for these roots to be real is that

$$\Lambda_{32}\Lambda_{30} \leq 0 ,$$

i.e.,

$$\frac{(1-\gamma)\,\phi^2\left[\sigma_{\eta u}^2 + \gamma\left(\sigma_u^2\sigma_\eta^2 - \sigma_{\eta u}^2\right)\right]}{\gamma^2\sigma_u^4} \leq 0$$

which is always true when $\gamma > 1$, since $(1-\gamma) < 0$

$$\sigma_u^2\sigma_\eta^2 - \sigma_{\eta u}^2 = \sigma_u^2\sigma_\eta^2 \left(1 - \operatorname{Corr}(u, \eta)\right) \geq 0 , \quad (41)$$

and all other terms in the expression for $\Lambda_{32}\Lambda_{30}$ are positive.

If $\gamma < 1$, $\Lambda_{32}\Lambda_{30} > 0$, so the roots are real if

$$\Lambda_{31}^2 - 4\Lambda_{32}\Lambda_{30} \geq 0 ,$$

which gives (22) in text after simple algebraic manipulations. The case $\gamma = 1$ is straightforward, since the third equation of the Fundamental Equation System becomes linear, and delivers:

$$b_2 = \frac{(\psi - 1)}{2\sigma_u^2\left(\phi^2 - \frac{1}{\rho}\right)} .$$

When $\gamma = 0$, it is easy to check that we can write the third equation of the Fundamental Equation System as

$$0 = (\psi - 1)^2 - (\psi - 1)\, 4\phi\sigma_{\eta u} b_2 + 4\phi^2\sigma_{\eta u}^2 b_2^2 ,$$

which is a quadratic equation whose discriminant is zero and delivers

$$b_2 = \frac{(\psi - 1)}{2\phi\sigma_{\eta u}} . \quad (42)$$

Finally, the case $\gamma \to \infty$ implies $b_2 = 0$, since the numerator of the solution to the quadratic equation is $O(\gamma)$, while the numerator is $O(\gamma^2)$.

To analyze the sign of the roots, rewrite the equation as

$$
\begin{aligned}
0 &= \frac{(\psi-1)^2}{4(1-\gamma)\phi^2\left[\sigma_{\eta u}^2+\gamma\left(\sigma_u^2\sigma_\eta^2-\sigma_{\eta u}^2\right)\right]} \\
&\quad -\left[\frac{(\psi-1)\sigma_{\eta u}}{\phi\left[\sigma_{\eta u}^2+\gamma\left(\sigma_u^2\sigma_\eta^2-\sigma_{\eta u}^2\right)\right]}+\frac{(\psi-1)\gamma\sigma_u^2\left(\phi^2-\frac{1}{\rho}\right)}{2(1-\gamma)\phi^2\left[\sigma_{\eta u}^2+\gamma\left(\sigma_u^2\sigma_\eta^2-\sigma_{\eta u}^2\right)\right]}\right]b_2+b_2^2
\end{aligned}
\tag{43}
$$

or

$$0=\widetilde{\Lambda}_{30}+\widetilde{\Lambda}_{31}b_2+b_2^2\,.$$

From standard theory on quadratic equations, the product of the roots is given by $\widetilde{\Lambda}_{30}$, which is negative when $\gamma > 1$ and positive when $\gamma < 1$:

$$b_{21}\cdot b_{22}=\widetilde{\Lambda}_{30}\begin{cases}<0 & \text{if } \gamma>1\\ \\ >0 & \text{if } \gamma<1\end{cases}.$$

Therefore, when $\gamma > 1$, the roots are real and have opposite sign and, when $\gamma < 1$, the roots have the same sign—provided they are real.

Similarly, from standard theory on quadratic equations,

$$b_{21}+b_{22}=-\widetilde{\Lambda}_{31}\,,$$

which is always positive if $\{\psi < 1, \gamma < 1, \phi\sigma_{\eta u} < 0\}$ or $\{\psi > 1, \gamma > 1, \phi\sigma_{\eta u} > 0\}$, and always negative if $\{\psi > 1, \gamma < 1, \phi\sigma_{\eta u} < 0\}$ or $\{\psi < 1, \gamma > 1, \phi\sigma_{\eta u} > 0\}$, since $(\phi^2-\rho^{-1}) < 0$, because $0 < \rho < 1$, $|\phi| < 1$ and, from (41), the term in brackets in the denominator is positive.

Hence, when $\gamma < 1$ and $\phi\sigma_{\eta u} < 0$, both roots are positive if $\psi < 1$ and negative if $\psi > 1$. When $\gamma < 1$ and $\phi\sigma_{\eta u} > 0$, the same results still obtain, provided the condition for real roots (22) holds —this condition implies $\gamma\sigma_u^2(\phi^2-\rho^{-1})^2 > -(1-\gamma)2\phi\sigma_{\eta u}(\phi^2-\rho^{-1})$, which is sufficient to obtain the results for this case.

When $\gamma > 1$, the roots of the equation alternate in sign. If $\phi\sigma_{\eta u} > 0$, we can write the expression for the roots of the equation as

$$b_2=(\psi-1)\,\frac{-(A+B)\pm\sqrt{(A+B)^2+C}}{-D}\,,$$

where A, B, C, D are positive constants —provided that $\gamma > 1$ and $\phi\sigma_{\eta u} > 0$— so choosing the positive root of the discriminant delivers $b_2 < 0$ if $\psi > 1$ and $b_2 > 0$ if $\psi < 1$, and the opposite obtains if we choose the negative root. If $\phi\sigma_{\eta u} < 0$, we can write the expression for the roots of the equation as

$$b_2=(\psi-1)\,\frac{(-A+B)\pm\sqrt{(-A+B)^2+C}}{-D}\,,$$

so the same results obtain.□

C.2 Proof of Property 2

To prove Property 2 we need to consider two cases, the case in which $\phi\sigma_{\eta u} < 0$ and the case in which $\phi\sigma_{\eta u} > 0$.

C.2.1 Case $\phi\sigma_{\eta u} < 0$

From the last part of the proof of Property 1, we have that $b_2/(\psi - 1) < 0$ when we select the value of b_2 associated with the positive root of the discriminant of the third equation of the Fundamental Equation System.

Plugging this result into the second equation of Proposition 1 we obtain immediately that $a_1 > 0$ when $\gamma > 1$, and $\phi\sigma_{\eta u} < 0$, since all the terms in the equation are positive. Also, when $\gamma = 1$, the second term in the equation is zero, so $a_1 = 1/\sigma_u^2 > 0$. However, when $\gamma < 1$, while the first term is positive, the second is negative, so we need to prove whether the sum of both terms is positive. Solving for the positive root in (40) and plugging the result in the second equation in Proposition 1, we find:

$$a_1 = \frac{4\gamma\phi^2\sigma_\eta^2\sigma_u^2 - 2\gamma\phi\sigma_{\eta u}\sigma_u^2\left(\phi^2 - \frac{1}{\rho}\right)}{4\gamma\phi^2\sigma_u^2\left[\sigma_{\eta u}^2 + \gamma\left(\sigma_\eta^2\sigma_u^2 - \sigma_{\eta u}^2\right)\right]} + \frac{\sqrt{\left[2\gamma\phi\sigma_{\eta u}\sigma_u^2\left(\phi^2 - \frac{1}{\rho}\right)\right]^2 + 16\gamma(1-\gamma)\phi^3\sigma_{\eta u}^3\sigma_\eta^2\left(\phi^2 - \frac{1}{\rho}\right) - 16\gamma(1-\gamma)\phi^4\sigma_{\eta u}^2\sigma_u^2\sigma_\eta^2}}{4\gamma\phi^2\sigma_u^2\left[\sigma_{\eta u}^2 + \gamma\left(\sigma_\eta^2\sigma_u^2 - \sigma_{\eta u}^2\right)\right]}, \tag{44}$$

when $\phi\sigma_{\eta u} < 0$. Since the denominator is always positive, the sign of the slope depends on the sign of the numerator. A straightforward analysis of the numerator shows that a couple of sufficient conditions for it to be positive are

$$\sigma_\eta^2 \frac{2\phi^2}{\left(\frac{1}{\rho} - \phi^2\right)} \geq -\phi\sigma_{\eta u},$$

and

$$\sigma_\eta^2 \frac{\phi^2}{\left(\frac{1}{\rho} - \phi^2\right)} < -\phi\sigma_{\eta u}.$$

But if the first sufficient condition is violated, the second one is immediately verified, so $a_1 > 0$.

C.2.2 Case $\phi\sigma_{\eta u} > 0$

From the last part of the proof of Property 1, we have that $b_2/(\psi - 1) < 0$ when $\gamma < 1$ —provided that the condition for for real roots (22) in (40) holds. Plugging this result into the second equation of Proposition 1 we obtain immediately that $a_1 > 0$ when $\gamma < 1$ and $\phi\sigma_{\eta u} < 0$, since all the terms in the equation are positive. Therefore, the slope of the optimal portfolio policy is always positive when $\gamma < 1$ and $\phi\sigma_{\eta u} > 0$ no matter what root we select for the discriminant of the third equation of the Fundamental Equation System.

When $\gamma > 1$, solving for the negative root of the discriminant in (40) and plugging the result in the second equation of Proposition 1, we find again (44), which is always positive when $\gamma > 1$. If we solve for the positive root of the discriminant in (40) and we plug the result into the second equation of Proposition 1, we find an expression similar to (44), except that the second term is subtracted. A sufficient condition for this expression to be positive is $\phi\,\mathrm{Corr}(\eta_{t+1}, u_{t+1}) > \phi\,\mathrm{Corr}(\eta_{t+1}, u_{t+1})^3$, which is always true because $\mathrm{sign}(\phi\sigma_{\eta u}) = \mathrm{sign}(\phi\,\mathrm{Corr}(\eta_{t+1}, u_{t+1}))$ and $|\,\mathrm{Corr}(\eta_{t+1}, u_{t+1})| \leq 1$.

C.2.3 The limiting behavior of a_1

Regardless of the sign of the covariance and ϕ, $a_1 \to +\infty$ as as $\gamma \to 0$ and $a_1 \to 0$ as $\gamma \to +\infty$. To prove the first result, we use (42), i.e. $b_2 \to (\psi - 1)/2\phi\sigma_{\eta u}$ as $\gamma \to 0$. Substituting this result into the the equation for a_1 in Proposition 1 and taking limits as $\gamma \to 0$ we find :

$$\begin{aligned}
\lim_{\gamma \to 0} a_1 &= \frac{1}{\sigma_u^2}\left(\lim_{\gamma \to 0} \frac{1}{\gamma} - \frac{1-\gamma}{\gamma}\right) \\
&= \frac{1}{\sigma_u^2} \lim_{\gamma \to 0} \frac{1}{\gamma} \\
&= +\infty \,.
\end{aligned}$$

Similarly, from (44) we have that the numerator of a_1 is $O(\gamma)$, while the numerator is $O(\gamma^2)$. Hence, taking limits as $\gamma \to \infty$ we have that $a_1 \to 0$.□

C.3 Proof of Property 3

A straightforward analysis of the solutions to the Fundamental Equation System reveals that we can write b_1 and b_2 as

$$\begin{aligned}
b_1 &= (\psi - 1)\ \mathrm{f}_1(\gamma, \rho) \,, \\
b_2 &= (\psi - 1)\ \mathrm{f}_2(\gamma, \rho) \,,
\end{aligned} \tag{45}$$

where $\mathrm{f}_1(\gamma, \rho)$ and $\mathrm{f}_1(\gamma, \rho)$ are functions that do not depend on ψ. After substitution in the equation system in Proposition 2, we find that the parameters defining the optimal portfolio rule, $\{a_0, a_1\}$ do not depend on ψ for given ρ. However, ρ itself is a function of ψ—recall that $\rho = 1 - \exp\{E[c_t - w_t]\}$—so the optimal portfolio rule depends on ψ indirectly through ρ.□

C.4 Proof of Property 4

To prove this result, notice that the third equation of the Fundamental Equation System, that determines b_2, is found by equating the right hand side of the third equation in Lemmas 6 and 7, and substituting out a_1 using the second equation in Proposition 1. None of these equations depend on μ.□

TABLE 1

Stochastic Process for Returns

Estimated model:

$$\begin{pmatrix} r_{1,t+1} - \overline{r}_f \\ d_{t+1} - p_{t+1} \end{pmatrix} = \begin{pmatrix} \theta_0 \\ \beta_0 \end{pmatrix} + \begin{pmatrix} \theta_1 \\ \beta_1 \end{pmatrix} (d_t - p_t) + \begin{pmatrix} \epsilon_{1,t+1} \\ \epsilon_{2,t+1} \end{pmatrix}$$

$$\begin{pmatrix} \epsilon_{1,t+1} \\ \epsilon_{2,t+1} \end{pmatrix} \sim (0, \Omega), \qquad \Omega = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$$

Derived model:

$$\begin{aligned} r_{1,t+1} - \overline{r}_f &= x_t + u_{t+1} \\ x_{t+1} &= \mu + \phi\,(x_t - \mu) + \eta_{t+1} \end{aligned}$$

$$\begin{pmatrix} u_{1,t+1} \\ \eta_{2,t+1} \end{pmatrix} \sim (0, \Sigma), \qquad \Sigma = \begin{pmatrix} {\sigma^2}_u & \sigma_{u,\eta} \\ \sigma_{u,\eta} & {\sigma^2}_\eta \end{pmatrix}$$

(A) Quarterly Model: 1947.1 - 1995.4

Estimated model:

$$\begin{pmatrix} r_{1,t+1} - \overline{r}_f \\ d_{t+1} - p_{t+1} \end{pmatrix} = \begin{pmatrix} -0.027 \\ (0.020) \\ 0.000 \\ (0.000) \end{pmatrix} + \begin{pmatrix} 4.610 \\ (1.953) \\ 0.954 \\ (0.022) \end{pmatrix} (d_t - p_t) + \begin{pmatrix} \epsilon_{1,t+1} \\ \epsilon_{2,t+1} \end{pmatrix}$$

$$\widehat{\Omega} = \begin{pmatrix} 5.294E-3 & -4.176E-5 \\ (0.540E-3) & (0.520E-5) \\ -4.176E-5 & 6.523E-7 \\ (0.520E-5) & (0.666E-7) \end{pmatrix} \qquad R^2 = \begin{pmatrix} 0.028 \\ 0.910 \end{pmatrix}$$

Derived model:

$$x_{t+1} = \underset{(0.005)}{1.343E-2} + \underset{(0.022)}{0.954} \; (x_t - \mu) + \eta_{t+1}$$

$$\widehat{\Sigma} = \begin{pmatrix} 5.294E-3 & -0.193E-3 \\ (0.540E-3) & (0.085E-3) \\ -0.193E-3 & 1.386E-5 \\ (0.085E-3) & (0.007E-5) \end{pmatrix}$$

$\bar{r}_f = .071E-2 \qquad {\sigma_x}^2/{\sigma_u}^2 = 2.913E-2 \qquad corr(\eta, u) = -0.711$

TABLE 1 (ctd.)

Stochastic Process for Returns

Estimated model:

$$\begin{pmatrix} r_{1,t+1} - \bar{r}_f \\ d_{t+1} - p_{t+1} \end{pmatrix} = \begin{pmatrix} \theta_0 \\ \beta_0 \end{pmatrix} + \begin{pmatrix} \theta_1 \\ \beta_1 \end{pmatrix} (d_t - p_t) + \begin{pmatrix} \epsilon_{1,t+1} \\ \epsilon_{2,t+1} \end{pmatrix}$$

$$\begin{pmatrix} \epsilon_{1,t+1} \\ \epsilon_{2,t+1} \end{pmatrix} \sim (0, \Omega), \qquad \Omega = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$$

Derived model:

$$\begin{aligned} r_{1,t+1} - \bar{r}_f &= x_t + u_{t+1} \\ x_{t+1} &= \mu + \phi (x_t - \mu) + \eta_{t+1} \end{aligned}$$

$$\begin{pmatrix} u_{1,t+1} \\ \eta_{2,t+1} \end{pmatrix} \sim (0, \Sigma), \qquad \Sigma = \begin{pmatrix} \sigma^2{}_u & \sigma_{u,\eta} \\ \sigma_{u,\eta} & \sigma^2{}_\eta \end{pmatrix}$$

(B) Annual Model: 1890 - 1993

Estimated model:

$$\begin{pmatrix} r_{1,t+1} - \bar{r}_f \\ d_{t+1} - p_{t+1} \end{pmatrix} = \begin{pmatrix} 0.613 \\ (0.205) \\ -0.626 \\ (0.192) \end{pmatrix} + \begin{pmatrix} 0.185 \\ (0.066) \\ 0.798 \\ (0.062) \end{pmatrix} (d_t - p_t) + \begin{pmatrix} \epsilon_{1,t+1} \\ \epsilon_{2,t+1} \end{pmatrix}$$

$$\widehat{\Omega} = \begin{pmatrix} 3.190E-2 & -2.101E-2 \\ (0.445E-2) & (0.361E-2) \\ -2.101E-2 & 2.818E-2 \\ (0.361E-2) & (0.393E-2) \end{pmatrix} \qquad R^2 = \begin{pmatrix} 0.070 \\ 0.613 \end{pmatrix}$$

Derived model:

$$x_{t+1} = \underset{(0.013)}{4.165E-2} + \underset{(0.062)}{0.798} (x_t - \mu) + \eta_{t+1}$$

$$\widehat{\Sigma} = \begin{pmatrix} 3.319E-2 & -0.388E-2 \\ (0.445E-2) & (0.155E-2) \\ -0.388E-2 & 0.096E-2 \\ (0.155E-2) & (0.393E-2) \end{pmatrix}$$

$\bar{r}_f = 1.992E-2 \quad \sigma_x{}^2/\sigma_u{}^2 = 7.964E-2 \quad corr(\eta, u) = -0.701$

TABLE 2

Optimal Percentage Allocation to Stocks When the Expected Gross Excess Return Is Zero

(Intercept of the Optimal Portfolio Policy)

$$\alpha_t = a_0^* \times 100$$

(A) Quarterly Model: 1947:1 - 1995:4

R.R.A.	E.I.S.							
	Intercepts (Allocations)							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	-38.00	-28.93	-22.95	-20.63	-17.79	-16.34	-15.88	-15.69
1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
1.50	25.79	22.39	19.55	18.34	16.72	15.85	15.57	15.44
2.00	31.73	28.84	26.28	25.10	23.51	22.62	22.33	22.19
4.00	27.42	27.32	27.23	27.18	27.11	27.06	27.05	27.04
10.0	14.37	15.43	16.61	17.24	18.24	18.96	19.19	19.31
20.0	7.81	8.64	9.59	10.11	10.99	11.56	11.76	11.85
40.0	4.07	4.57	5.16	5.48	6.04	6.39	6.53	6.60
	Standard Errors							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	20.11	10.57	6.32	5.12	4.11	3.41	3.14	3.51
1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
1.50	11.76	8.40	5.97	5.01	4.12	3.68	3.56	3.50
2.00	14.22	11.01	8.43	7.35	29.41	5.26	5.30	5.22
4.00	12.21	10.93	9.68	9.06	8.17	7.66	7.49	7.41
10.0	6.48	6.47	6.46	6.45	6.46	6.46	6.46	6.47
20.0	3.55	3.70	3.87	3.97	4.15	4.28	4.33	4.36
40.0	1.85	1.98	2.13	2.21	2.37	2.48	2.52	2.54

Note: Portfolio allocations and their standard errors are given in percentage points and are based on the parameter values for the return processes presented in Panel A and B of Table 1. Panel A allocations are based on estimates for the period 1947:1-1995:4, while Panel B allocations are based on estimates for the period 1890-1993. **The values in the main diagonal correspond to the power utility case.**

TABLE 2 (ctd.)

Optimal Percentage Allocation to Stocks When the Expected Gross Excess Return Is Zero

(Intercept of the Optimal Portfolio Policy)

$\alpha_t = a_0^* \times 100$

(B) Annual Model: 1890 - 1993

R.R.A.	E.I.S.							
	Intercepts (Allocations)							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	-17.31	-15.35	-13.70	-12.97	-11.98	-11.44	-11.27	-11.18
1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
1.50	12.16	11.44	10.79	10.49	10.05	9.81	9.73	9.69
2.00	15.08	14.55	14.06	13.82	13.47	13.27	13.21	13.18
4.00	13.23	13.46	13.68	13.79	13.98	14.09	14.12	14.14
10.0	7.02	7.46	7.95	8.20	8.61	8.87	8.96	9.00
20.0	3.83	4.15	4.50	4.69	5.01	5.20	5.27	5.31
40.0	2.00	2.19	2.40	2.51	2.70	2.83	2.87	2.89
	Standard Errors							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	16.48	7.77	4.19	4.76	6.77	7.77	8.13	8.30
1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
1.50	6.58	6.95	7.59	7.96	8.53	8.84	8.95	9.01
2.00	12.75	12.96	13.32	13.54	13.91	14.16	14.25	14.29
4.00	19.69	20.44	21.19	21.65	22.07	22.39	22.49	22.54
10.0	13.65	15.31	17.13	18.11	19.68	20.64	20.98	21.15
20.0	8.08	9.39	10.94	11.81	13.26	14.22	14.56	14.73
40.0	4.38	5.20	6.20	6.77	7.76	8.42	8.66	8.78

Note: Portfolio allocations and their standard errors are given in percentage points and are based on the parameter values for the return processes presented in Panel A and B of Table 1. Panel A allocations are based on estimates for the period 1947:1-1995:4, while Panel B allocations are based on estimates for the period 1890-1993. **The values in the main diagonal correspond to the power utility case.**

TABLE 3

Slope of the Optimal Portfolio Policy (a_1)

(A) Quarterly Model: 1947:1 - 1995:4

R.R.A.	E.I.S.							
	Slopes							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	225.23	227.67	229.68	230.58	231.79	232.48	232.70	232.79
1.00	188.87	188.87	188.87	188.87	188.87	188.87	188.87	188.87
1.50	143.15	142.20	141.32	140.92	140.34	140.02	139.92	139.86
2.00	115.38	114.56	113.79	113.41	112.89	112.58	112.48	112.43
4.00	65.01	64.98	64.95	64.94	64.92	64.91	64.90	64.90
10.0	28.14	28.45	28.78	28.95	29.21	29.39	29.45	29.48
20.0	14.46	14.71	14.97	15.10	15.33	15.47	15.51	15.54
40.0	7.33	7.48	7.64	7.73	7.86	7.95	7.98	8.00
	Standard Errors							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	24.13	24.23	24.42	24.52	24.67	24.77	24.81	24.80
1.00	19.28	19.28	19.28	19.28	19.28	19.28	19.28	19.28
1.50	15.91	15.51	15.16	15.03	14.82	14.71	14.68	14.66
2.00	14.10	13.69	13.31	13.13	15.36	12.81	12.73	12.71
4.00	9.66	9.57	9.49	9.46	9.41	9.39	9.39	9.38
10.0	4.81	4.92	5.05	5.13	5.24	5.34	5.37	5.38
20.0	2.60	2.70	2.81	2.88	2.98	3.05	3.07	3.09
40.0	1.35	1.42	1.49	1.53	1.59	1.63	1.65	1.66

Note: The numbers in the upper portion of the panel report the change - in percentage points - in the optimal allocation to stocks when the expected log excess return increases by one percent - on a quarterly basis in Panel A and on an annual basis in Panel B -. The numbers in the lower part are their standard errors. They are all based on the parameter estimates for the return processes presented in Panel A and B of Table 1. Panel A numbers are based on estimates for the period 1947:1-1995:4, while Panel B numbers are based on estimates for the period 1890-1993. **The values in the main diagonal correspond to the power utility case.**

TABLE 3 (ctd.)

Slope of the Optimal Portfolio Policy (a_1)

(B) Annual Model: 1890 - 1993

R.R.A.	E.I.S.							
	Slopes							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	38.80	38.96	39.10	39.17	39.26	39.32	39.34	39.35
1.00	31.34	31.34	31.34	31.34	31.34	31.34	31.34	31.34
1.50	22.74	22.69	22.64	22.62	22.58	22.56	22.56	22.55
2.00	17.87	17.84	17.80	17.79	17.76	17.75	17.74	17.74
4.00	9.65	9.67	9.68	9.69	9.70	9.71	9.71	9.71
10.0	4.06	4.09	4.12	4.13	4.16	4.17	4.18	4.18
20.0	2.07	2.09	2.11	2.12	2.13	2.15	2.15	2.15
40.0	1.04	1.05	1.07	1.07	1.08	1.09	1.09	1.09
	Standard Errors							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	6.00	5.65	5.41	5.31	5.20	5.15	5.14	5.13
1.00	4.37	4.37	4.37	4.37	4.37	4.37	4.37	4.37
1.50	3.31	3.35	3.39	3.42	3.46	3.47	3.48	3.49
2.00	2.86	2.88	2.91	2.93	2.96	2.97	2.98	2.98
4.00	2.01	2.03	2.05	2.06	2.07	2.08	2.08	2.08
10.0	1.05	1.10	1.14	1.16	1.20	1.22	1.23	1.23
20.0	0.58	0.62	0.66	0.68	0.71	0.73	0.74	0.74
40.0	0.30	0.33	0.35	0.37	0.39	0.41	0.41	0.41

Note: The numbers in the upper portion of the panel report the change - in percentage points - in the optimal allocation to stocks when the expected log excess return increases by one percent - on a quarterly basis in Panel A and on an annual basis in Panel B -. The numbers in the lower part are their standard errors. They are all based on the parameter estimates for the return processes presented in Panel A and B of Table 1. Panel A numbers are based on estimates for the period 1947:1-1995:4, while Panel B numbers are based on estimates for the period 1890-1993. **The values in the main diagonal correspond to the power utility case.**

TABLE 4

Mean Optimal Percentage Allocation to Stocks

$$\alpha_t = [a_0^* + a_1(\mu + \sigma_u^2/2)] \times 100$$

(A) Quarterly Model: 1947:1 - 1995:4

Allocations

R.R.A.	E.I.S. 1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	324.04	337.05	346.26	350.02	354.81	357.37	358.17	358.52
1.00	303.60	303.60	303.60	303.60	303.60	303.60	303.60	303.60
1.50	255.91	250.97	246.72	244.86	242.31	240.92	240.48	240.26
2.00	217.19	213.00	209.18	207.41	204.97	203.59	203.15	202.93
4.00	131.92	131.78	131.64	131.57	131.46	131.40	131.38	131.37
10.0	59.61	61.17	62.87	63.77	65.20	66.20	66.53	66.69
20.0	31.06	32.28	33.65	34.38	35.63	36.42	36.69	36.83
40.0	15.85	16.60	17.44	17.90	18.68	19.16	19.36	19.45

Standard Errors

R.R.A.	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	43.12	39.13	38.83	39.05	39.45	39.72	39.82	39.83
1.00	30.99	30.99	30.99	30.99	30.99	30.99	30.99	30.99
1.50	31.95	28.97	26.89	26.18	25.28	24.85	24.73	24.66
2.00	31.45	28.30	25.82	24.80	48.43	23.08	22.84	22.75
4.00	23.82	22.59	21.48	20.97	20.28	19.91	19.79	19.73
10.0	12.30	12.42	12.56	12.66	12.83	13.00	13.05	13.08
20.0	6.71	6.95	7.23	7.38	7.67	7.86	7.92	7.96
40.0	3.50	3.68	3.89	4.00	4.20	4.33	4.39	4.41

Note: Portfolio allocations and their standard errors are given in percentage points and are based on the parameter values for the return processes presented in Panel A and B of Table 1. Panel A allocations are based on estimates for the period 1947:1-1995:4, while Panel B allocations are based on estimates for the period 1890-1993. **The values in the main diagonal correspond to the power utility case.**

TABLE 4 (ctd.)

Mean Optimal Percentage Allocation to Stocks

$$\alpha_t = [a_0^* + a_1(\mu + \sigma_u^2/2)] \times 100$$

(B) Annual Model: 1890 - 1993

Allocations

R.R.A.	E.I.S. 1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	206.18	209.03	211.50	212.61	214.17	215.03	215.30	215.44
1.00	180.54	180.54	180.54	180.54	180.54	180.54	180.54	180.54
1.50	143.13	142.13	141.20	140.76	140.12	139.77	139.66	139.60
2.00	118.03	117.30	116.60	116.26	115.77	115.49	115.39	115.35
4.00	68.83	69.15	69.46	69.61	69.86	70.00	70.05	70.08
10.0	30.41	31.02	31.67	32.01	32.55	32.89	33.01	33.07
20.0	15.74	16.17	16.64	16.89	17.30	17.56	17.65	17.69
40.0	8.01	8.26	8.54	8.69	8.94	9.10	9.15	9.18

Standard Errors

R.R.A.	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	44.97	35.24	29.94	28.45	27.22	26.90	26.85	26.83
1.00	25.16	25.16	25.16	25.16	25.16	25.16	25.16	25.16
1.50	23.60	24.30	25.15	25.62	26.38	26.71	26.85	26.92
2.00	26.88	27.26	27.78	28.10	28.63	28.99	29.11	29.17
4.00	29.77	30.63	31.49	31.98	32.48	32.83	32.95	33.01
10.0	19.12	21.03	23.11	24.22	26.00	27.06	27.44	27.63
20.0	11.12	12.65	14.43	15.42	17.06	18.15	18.53	18.72
40.0	5.99	6.94	8.09	8.75	9.87	10.62	10.89	11.03

Note: Portfolio allocations and their standard errors are given in percentage points and are based on the parameter values for the return processes presented in Panel A and B of Table 1. Panel A allocations are based on estimates for the period 1947:1-1995:4, while Panel B allocations are based on estimates for the period 1890-1993. **The values in the main diagonal correspond to the power utility case.**

TABLE 5

Percentage Mean Hedging Demand Over Mean Total Demand

$$\alpha_{t,hedging}(\mu;\gamma,\psi)/\alpha_t(\mu;\gamma,\psi) = [1 - (\alpha_t(\mu;1,\psi)/\gamma)] \times 100$$

(A) Quarterly Model: 1947:1 - 1995:4

R.R.A.	E.I.S.							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	−24.92	−20.10	−16.91	−15.65	−14.09	−13.27	−13.02	−12.91
1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
1.50	20.91	19.35	17.96	17.34	16.47	15.99	15.83	15.76
2.00	30.11	28.73	27.43	26.81	25.94	25.44	25.27	25.19
4.00	42.47	42.40	42.34	42.31	42.26	42.24	42.23	42.22
10.0	49.07	50.36	51.71	52.39	53.43	54.14	54.37	54.48
20.0	51.12	52.97	54.89	55.85	57.39	58.32	58.63	58.78
40.0	52.12	54.27	56.48	57.59	59.36	60.39	60.79	60.97

(B) Annual Model: 1890 - 1993

R.R.A.	E.I.S.							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	−16.75	−15.16	−13.81	−13.22	−12.40	−11.95	−11.80	−11.73
1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
1.50	15.91	15.32	14.76	14.49	14.11	13.89	13.82	13.78
2.00	23.52	23.04	22.58	22.36	22.03	21.84	21.77	21.74
4.00	34.43	34.73	35.02	35.16	35.39	35.53	35.57	35.59
10.0	40.64	41.80	43.00	43.60	44.53	45.11	45.31	45.40
20.0	42.64	44.17	45.76	46.57	47.83	48.59	48.85	48.98
40.0	43.63	45.36	47.16	48.08	49.51	50.39	50.69	50.83

Note: The numbers in the table are based on the parameter values for the return processes presented in Panel A and B of Table 1. Panel A allocations are based on estimates for the period 1947:1-1995:4, while Panel B allocations are based on estimates for the period 1890-1993. **The values in the main diagonal correspond to the power utility case.**

TABLE 6

Optimal Percentage Consumption-Wealth Ratio When the Expected Gross Excess Return Is Zero (Exponentiated Intercept of the Optimal Consumption Policy)

$$C_t/W_t = \exp\{b_0^*\} \times 100$$

(A) Quarterly Model: 1947:1 - 1995:4

R.R.A.	E.I.S.							
	Consumption-Wealth Ratios							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	0.72	1.53	2.01	2.20	2.45	2.58	2.63	2.65
1.00	0.94	1.53	1.91	2.05	2.24	2.34	2.38	2.39
1.50	1.25	1.53	1.74	1.82	1.92	1.97	1.99	2.00
2.00	1.41	1.53	1.63	1.66	1.71	1.73	1.74	1.74
4.00	1.68	1.53	1.40	1.34	1.26	1.21	1.19	1.18
10.0	1.87	1.53	1.21	1.06	0.83	0.70	0.66	0.64
20.0	1.94	1.53	1.14	0.94	0.65	0.48	0.42	0.39
40.0	1.98	1.53	1.09	0.88	0.55	0.35	0.29	0.26
	Standard Errors							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	0.71	0.00	0.37	0.51	0.70	0.82	0.86	0.85
1.00	0.56	0.00	0.34	0.46	0.49	0.72	0.75	0.77
1.50	0.37	0.00	0.26	0.37	0.51	0.59	0.62	0.63
2.00	0.29	0.00	0.22	0.32	0.30	0.52	0.53	0.54
4.00	0.15	0.00	0.14	0.20	0.29	0.34	0.36	0.37
10.0	0.07	0.00	0.07	0.10	0.15	0.18	0.19	0.19
20.0	0.03	0.00	0.04	0.05	0.08	0.10	0.11	0.11
40.0	0.02	0.00	0.02	0.03	0.04	0.05	0.06	0.06

Note: Consumption-Wealth ratios and their standard errors are given in percentage points and are based on the parameter values for the return processes presented in Panel A and B of Table 1. Panel A allocations are based on estimates for the period 1947:1-1995:4, while Panel B allocations are based on estimates for the period 1890-1993. **The values in the main diagonal correspond to the power utility case.**

TABLE 6 (ctd.)

Optimal Percentage Consumption-Wealth Ratio When the Expected Gross Excess Return Is Zero

(Exponentiated Intercept of the Optimal Consumption Policy)

$$C_t/W_t = \exp\{b_0^*\} \times 100$$

(B) Annual Model: 1890 - 1993

Consumption-Wealth Ratios

R.R.A.	E.I.S. 1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	4.50	6.00	7.12	7.59	8.22	8.57	8.68	8.73
1.00	4.84	6.00	6.87	7.23	7.70	7.95	8.03	8.07
1.50	5.58	6.00	6.33	6.47	6.65	6.74	6.77	6.79
2.00	5.93	6.00	6.04	6.06	6.07	6.07	6.07	6.07
4.00	6.54	6.00	5.49	5.24	4.89	4.69	4.62	4.59
10.0	6.98	6.00	5.04	4.58	3.89	3.48	3.35	3.28
20.0	7.14	6.00	4.87	4.31	3.47	2.97	2.80	2.72
40.0	7.22	6.00	4.77	4.16	3.24	2.69	2.50	2.41

Standard Errors

R.R.A.	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	7.94	0.00	5.33	7.46	10.25	11.75	12.23	12.46
1.00	5.39	0.00	3.90	5.52	7.67	8.84	9.22	9.41
1.50	2.74	0.00	2.34	3.40	4.89	5.70	5.97	6.10
2.00	1.63	0.00	1.49	2.19	3.19	3.76	3.95	4.04
4.00	0.42	0.00	0.37	0.53	0.76	0.88	0.92	0.94
10.0	0.14	0.00	0.22	0.36	0.65	0.86	0.94	0.98
20.0	0.09	0.00	0.16	0.28	0.52	0.71	0.77	0.81
40.0	0.05	0.00	0.10	0.17	0.33	0.44	0.49	0.51

Note: Consumption-Wealth ratios and their standard errors are given in percentage points and are based on the parameter values for the return processes presented in Panel A and B of Table 1. Panel A allocations are based on estimates for the period 1947:1-1995:4, while Panel B allocations are based on estimates for the period 1890-1993. **The values in the main diagonal correspond to the power utility case.**

TABLE 7

Linear Coefficient of the Optimal Consumption Policy (b_1^*)

(A) Quarterly Model: 1947:1 - 1995:4

R.R.A.	E.I.S.							
	Coefficients							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	-9.89	0.00	5.84	7.83	10.05	11.02	11.29	11.44
1.00	-8.98	0.00	5.80	7.90	10.33	11.50	11.84	12.00
1.50	-6.73	0.00	5.05	7.09	9.66	10.96	11.36	11.55
2.00	-5.52	0.00	4.54	6.50	9.11	10.50	10.94	11.16
4.00	-3.18	0.00	3.16	4.73	7.08	8.48	8.95	9.18
10.0	-1.39	0.00	1.61	2.52	4.00	4.99	5.33	5.51
20.0	-0.72	0.00	0.88	1.40	2.29	2.89	3.10	3.21
40.0	-0.36	0.00	0.46	0.74	1.23	1.56	1.68	1.74
	Standard Errors							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	5.56	0.00	1.73	2.09	2.45	2.84	3.03	2.81
1.00	4.53	0.00	1.74	2.03	2.24	2.60	2.67	2.71
1.50	3.10	0.00	1.55	2.10	2.48	2.73	2.81	2.85
2.00	2.43	0.00	1.45	1.93	8.07	3.03	2.88	2.93
4.00	1.29	0.00	1.12	1.63	2.35	2.78	2.92	2.99
10.0	0.54	0.00	0.64	1.01	1.64	2.10	2.26	2.34
20.0	0.27	0.00	0.37	0.60	1.03	1.34	1.46	1.52
40.0	0.14	0.00	0.20	0.33	0.58	0.76	0.83	0.87

Note: The numbers in the table report the linear coefficient - and its standard error - of the optimal consumption policy when expected log excess returns are expressed in decimal points. They are based on the parameter values for the return processes presented in Panel A and B of Table 1: Panel A numbers are based on estimates for the period 1947:1-1995:4, while Panel B numbers are based on estimates for the period 1890-1993. **The values in the main diagonal correspond to the power utility case.**

TABLE 7 (ctd.)

Linear Coefficient of the Optimal Consumption Policy (b_1^*)

(B) Annual Model: 1890 - 1993

Coefficients

R.R.A.	E.I.S. 1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	-1.06	0.00	0.80	1.13	1.53	1.74	1.80	1.83
1.00	-1.15	0.00	0.90	1.27	1.75	2.00	2.08	2.12
1.50	-0.78	0.00	0.68	0.98	1.40	1.64	1.71	1.75
2.00	-0.65	0.00	0.60	0.88	1.29	1.52	1.59	1.63
4.00	-0.39	0.00	0.40	0.61	0.93	1.12	1.19	1.22
10.0	-0.17	0.00	0.20	0.31	0.49	0.61	0.65	0.67
20.0	-0.09	0.00	0.11	0.17	0.27	0.34	0.36	0.38
40.0	-0.05	0.00	0.06	0.09	0.14	0.18	0.19	0.20

Standard Errors

R.R.A.	E.I.S. 1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	1.11	0.00	0.27	0.52	1.12	1.56	1.72	1.79
1.00	0.54	0.00	0.43	0.77	1.42	1.88	2.04	2.12
1.50	0.46	0.00	0.58	0.92	1.53	1.85	1.98	2.04
2.00	0.65	0.00	0.69	1.05	1.63	1.99	2.12	2.18
4.00	0.68	0.00	0.74	1.12	1.74	2.12	2.25	2.31
10.0	0.39	0.00	0.50	0.75	1.30	1.63	1.75	1.81
20.0	0.22	0.00	0.30	0.49	0.83	1.07	1.15	1.20
40.0	0.12	0.00	0.17	0.27	0.47	0.62	0.67	0.70

Note: The numbers in the table report the linear coefficient - and its standard error - of the optimal consumption policy when expected log excess returns are expressed in decimal points. They are based on the parameter values for the return processes presented in Panel A and B of Table 1: Panel A numbers are based on estimates for the period 1947:1-1995:4, while Panel B numbers are based on estimates for the period 1890-1993. **The values in the main diagonal correspond to the power utility case.**

TABLE 8

Quadratic Coefficient of the Optimal Consumption Policy (b_2)

(A) Quarterly Model: 1947:1 - 1995:4

R.R.A.	E.I.S.							
	Coefficients							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	-383.39	0.00	319.16	459.37	649.61	752.96	785.69	802.37
1.00	-325.48	0.00	277.09	400.65	569.62	663.56	693.47	708.20
1.50	-248.45	0.00	222.05	324.35	467.93	548.92	575.11	588.06
2.00	-201.19	0.00	185.92	273.51	398.90	470.79	494.23	505.85
4.00	-113.97	0.00	113.61	170.28	255.12	305.93	322.85	331.30
10.0	-49.39	0.00	52.79	80.56	123.98	151.40	160.70	165.39
20.0	-25.38	0.00	27.93	42.93	66.95	82.22	87.46	90.11
40.0	-12.87	0.00	14.39	22.20	34.83	42.91	45.76	47.18
	Standard Errors							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	166.38	0.00	122.07	172.85	240.28	284.29	300.63	296.96
1.00	140.11	0.00	110.73	154.77	168.02	251.77	262.78	268.20
1.50	106.63	0.00	89.89	132.33	185.92	217.32	227.49	232.52
2.00	86.46	0.00	77.27	113.11	83.07	198.80	203.14	207.91
4.00	49.36	0.00	50.01	75.36	113.94	137.45	145.36	149.32
10.0	21.61	0.00	24.42	37.82	59.54	74.01	78.97	81.50
20.0	11.15	0.00	13.18	20.62	33.12	41.40	44.30	45.77
40.0	5.66	0.00	6.86	10.79	17.47	21.91	23.55	24.36

Note: The numbers in the table report the quadratic coefficient - and its standard error - of the optimal consumption policy when expected log excess returns are expressed in decimal points. They are based on the parameter values for the return processes presented in Panel A and B of Table 1: Panel A numbers are based on estimates for the period 1947:1-1995:4, while Panel B numbers are based on estimates for the period 1890-1993. **The values in the main diagonal correspond to the power utility case.**

TABLE 8 (ctd.)

Quadratic Coefficient of the Optimal Consumption Policy (b_2)

(B) Annual Model: 1890 - 1993

R.R.A.	E.I.S.							
	Coefficients							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	-15.41	0.00	13.89	20.31	29.34	34.44	36.08	36.90
1.00	-12.82	0.00	11.67	17.12	24.85	29.24	30.67	31.37
1.50	-9.50	0.00	8.99	13.31	19.57	23.22	24.41	25.00
2.00	-7.57	0.00	7.32	10.90	16.15	19.25	20.27	20.78
4.00	-4.16	0.00	4.23	6.37	9.62	11.58	12.24	12.57
10.0	-1.77	0.00	1.88	2.86	4.39	5.34	5.67	5.83
20.0	-0.90	0.00	0.98	1.49	2.31	2.82	3.00	3.09
40.0	-0.46	0.00	0.50	0.76	1.19	1.45	1.54	1.59
	Standard Errors							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	5.55	0.00	4.08	6.99	12.71	17.08	18.63	19.42
1.00	3.42	0.00	3.85	6.34	10.96	14.23	15.41	16.01
1.50	3.01	0.00	3.64	5.77	9.77	11.74	12.58	13.01
2.00	3.11	0.00	3.46	5.35	8.41	10.38	11.07	11.41
4.00	2.63	0.00	2.78	4.15	6.46	7.83	8.29	8.52
10.0	1.41	0.00	1.62	2.51	3.94	4.84	5.16	5.31
20.0	0.77	0.00	0.93	1.47	2.35	2.94	3.15	3.25
40.0	0.41	0.00	0.50	0.80	1.30	1.64	1.76	1.82

Note: The numbers in the table report the quadratic coefficient - and its standard error - of the optimal consumption policy when expected log excess returns are expressed in decimal points. They are based on the parameter values for the return processes presented in Panel A and B of Table 1: Panel A numbers are based on estimates for the period 1947:1-1995:4, while Panel B numbers are based on estimates for the period 1890-1993. **The values in the main diagonal correspond to the power utility case.**

TABLE 9

Percentage Exponentiated Mean Optimal Log Consumption-Wealth Ratio

$$C_t/W_t = \exp\{E[c_t - w_t]\} \times 100$$

$$E[c_t - w_t] = b_0^* + b_1^*(\mu + \sigma_u^2/2) + b_2(\sigma_x^2 + \mu^2 + \mu\sigma_u^2 + \sigma_u^4/4)$$

(A) Quarterly Model: 1947:1 - 1995:4

R.R.A.	E.I.S.							
	Consumption-Wealth Ratios							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	0.52	1.53	2.52	3.02	3.76	4.21	4.36	4.43
1.00	0.71	1.53	2.35	2.75	3.35	3.71	3.83	3.89
1.50	1.01	1.53	2.06	2.33	2.72	2.95	3.03	3.07
2.00	1.18	1.53	1.89	2.07	2.33	2.49	2.54	2.57
4.00	1.52	1.53	1.55	1.55	1.56	1.57	1.57	1.57
10.0	1.79	1.53	1.27	1.14	0.94	0.81	0.77	0.75
20.0	1.90	1.53	1.17	0.98	0.69	0.52	0.46	0.43
40.0	1.96	1.53	1.11	0.89	0.57	0.37	0.30	0.27
	Standard Errors							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	0.55	0.00	0.39	0.56	0.78	0.90	0.94	0.95
1.00	0.44	0.00	0.36	0.51	0.71	0.82	0.85	0.87
1.50	0.31	0.00	0.28	0.41	0.58	0.68	0.71	0.72
2.00	0.25	0.00	0.24	0.35	0.51	0.59	0.62	0.63
4.00	0.14	0.00	0.15	0.22	0.33	0.40	0.42	0.43
10.0	0.07	0.00	0.07	0.10	0.16	0.20	0.21	0.22
20.0	0.03	0.00	0.04	0.06	0.09	0.11	0.11	0.12
40.0	0.02	0.00	0.02	0.03	0.04	0.06	0.06	0.06

Note: Consumption-Wealth ratios and their standard errors are given in percentage points and are based on the parameter values for the return processes presented in Panel A and B of Table 1. Panel A allocations are based on estimates for the period 1947:1-1995:4, while Panel B allocations are based on estimates for the period 1890-1993. **The values in the main diagonal correspond to the power utility case.**

TABLE 9 (ctd.)

Percentage Exponentiated Mean Optimal Log Consumption-Wealth Ratio

$$C_t/W_t = \exp\{E[c_t - w_t]\} \times 100$$

$$E[c_t - w_t] = b_0^* + b_1^*(\mu + \sigma_u^2/2) + b_2(\sigma_x^2 + \mu^2 + \mu\sigma_u^2 + \sigma_u^4/4)$$

(B) Annual Model: 1890 - 1993

R.R.A.	E.I.S.							
	Consumption-Wealth Ratios							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	3.86	6.00	8.10	9.15	10.70	11.63	11.93	12.09
1.00	4.20	6.00	7.75	8.61	9.88	10.62	10.87	11.00
1.50	5.04	6.00	6.94	7.41	8.10	8.51	8.65	8.71
2.00	5.46	6.00	6.53	6.80	7.19	7.43	7.50	7.54
4.00	6.24	6.00	5.76	5.64	5.46	5.36	5.32	5.30
10.0	6.84	6.00	5.16	4.74	4.10	3.72	3.59	3.53
20.0	7.06	6.00	4.93	4.39	3.57	3.08	2.91	2.83
40.0	7.18	6.00	4.80	4.20	3.29	2.74	2.55	2.46
	Standard Errors							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	7.12	0.00	5.91	8.50	11.99	13.89	14.49	14.79
1.00	4.79	0.00	4.13	5.99	8.52	9.91	10.36	10.58
1.50	2.34	0.00	2.25	3.33	4.90	5.81	6.11	6.26
2.00	1.27	0.00	1.28	1.91	2.86	3.42	3.61	3.70
4.00	0.33	0.00	0.34	0.52	0.79	0.96	1.01	1.04
10.0	0.29	0.00	0.39	0.63	1.05	1.34	1.45	1.50
20.0	0.19	0.00	0.27	0.44	0.75	0.97	1.04	1.08
40.0	0.11	0.00	0.16	0.26	0.44	0.57	0.62	0.64

Note: Consumption-Wealth ratios and their standard errors are given in percentage points and are based on the parameter values for the return processes presented in Panel A and B of Table 1. Panel A allocations are based on estimates for the period 1947:1-1995:4, while Panel B allocations are based on estimates for the period 1890-1993. **The values in the main diagonal correspond to the power utility case.**

TABLE 10

Long-Term Expected Log Return on Wealth

$E[r_{p,t+1}] \times 100$

(A) Quarterly Model: 1947:1 - 1995:4

R.R.A.	E.I.S.							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	3.91	3.88	3.86	3.84	3.83	3.82	3.82	3.81
1.00	3.97	3.97	3.97	3.97	3.97	3.97	3.97	3.97
1.50	3.83	3.81	3.79	3.79	3.78	3.77	3.77	3.77
2.00	3.55	3.53	3.51	3.49	3.48	3.47	3.47	3.47
4.00	2.56	2.56	2.56	2.56	2.56	2.56	2.56	2.56
10.0	1.34	1.36	1.39	1.40	1.42	1.44	1.44	1.45
20.0	0.76	0.78	0.80	0.82	0.84	0.85	0.86	0.86
40.0	0.43	0.44	0.46	0.47	0.48	0.49	0.49	0.50

(B) Annual Model: 1890 - 1993

R.R.A.	E.I.S.							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	10.99	10.96	10.93	10.91	10.89	10.87	10.87	10.87
1.00	11.33	11.33	11.33	11.33	11.33	11.33	11.33	11.33
1.50	10.80	10.78	10.77	10.76	10.75	10.74	10.74	10.74
2.00	9.94	9.93	9.91	9.90	9.89	9.88	9.88	9.87
4.00	7.36	7.37	7.39	7.39	7.40	7.41	7.41	7.41
10.0	4.60	4.64	4.67	4.69	4.72	4.74	4.75	4.75
20.0	3.39	3.41	3.44	3.46	3.49	3.50	3.51	3.51
40.0	2.71	2.73	2.75	2.76	2.78	2.79	2.79	2.79

TABLE 11

Volatility of Consumption Growth

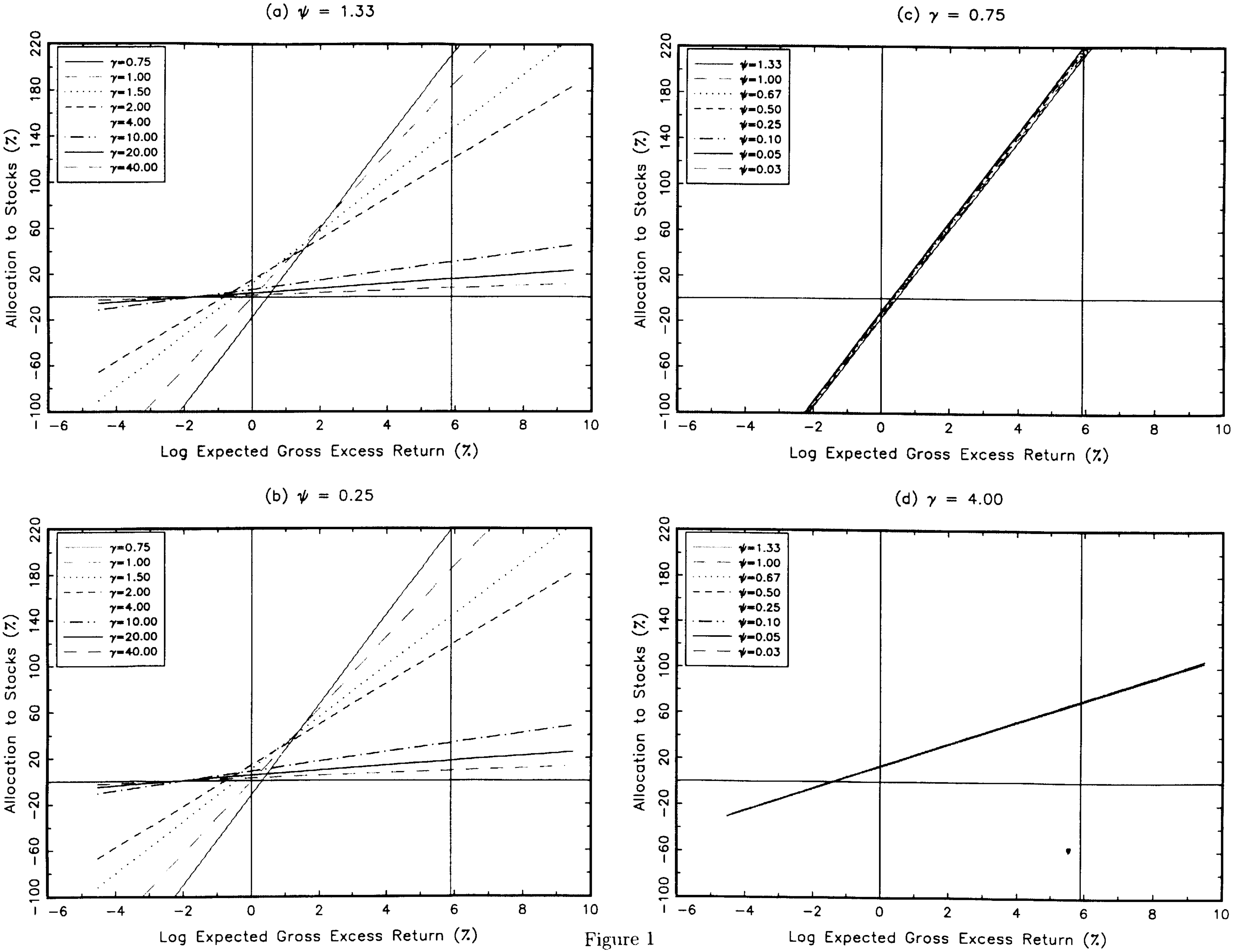
$\sigma\left(\Delta c_{t+1} - E_t\left[\Delta c_{t+1}\right]\right) \times 100$

(A) Quarterly Model: 1947:1 - 1995:4

R.R.A.	E.I.S.							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	37.78	32.04	28.52	27.34	26.08	25.57	25.43	25.36
1.00	33.74	27.93	24.19	22.90	21.53	20.97	20.82	20.75
1.50	27.11	22.34	18.95	17.73	16.42	15.92	15.79	15.73
2.00	22.56	18.65	15.72	14.64	13.50	13.09	13.00	12.96
4.00	13.32	11.25	9.53	8.87	8.22	8.05	8.04	8.04
10.0	5.93	5.14	4.44	4.17	3.93	3.93	3.96	3.98
20.0	3.07	2.70	2.36	2.22	2.12	2.14	2.16	2.18
40.0	1.57	1.38	1.22	1.15	1.10	1.12	1.14	1.15

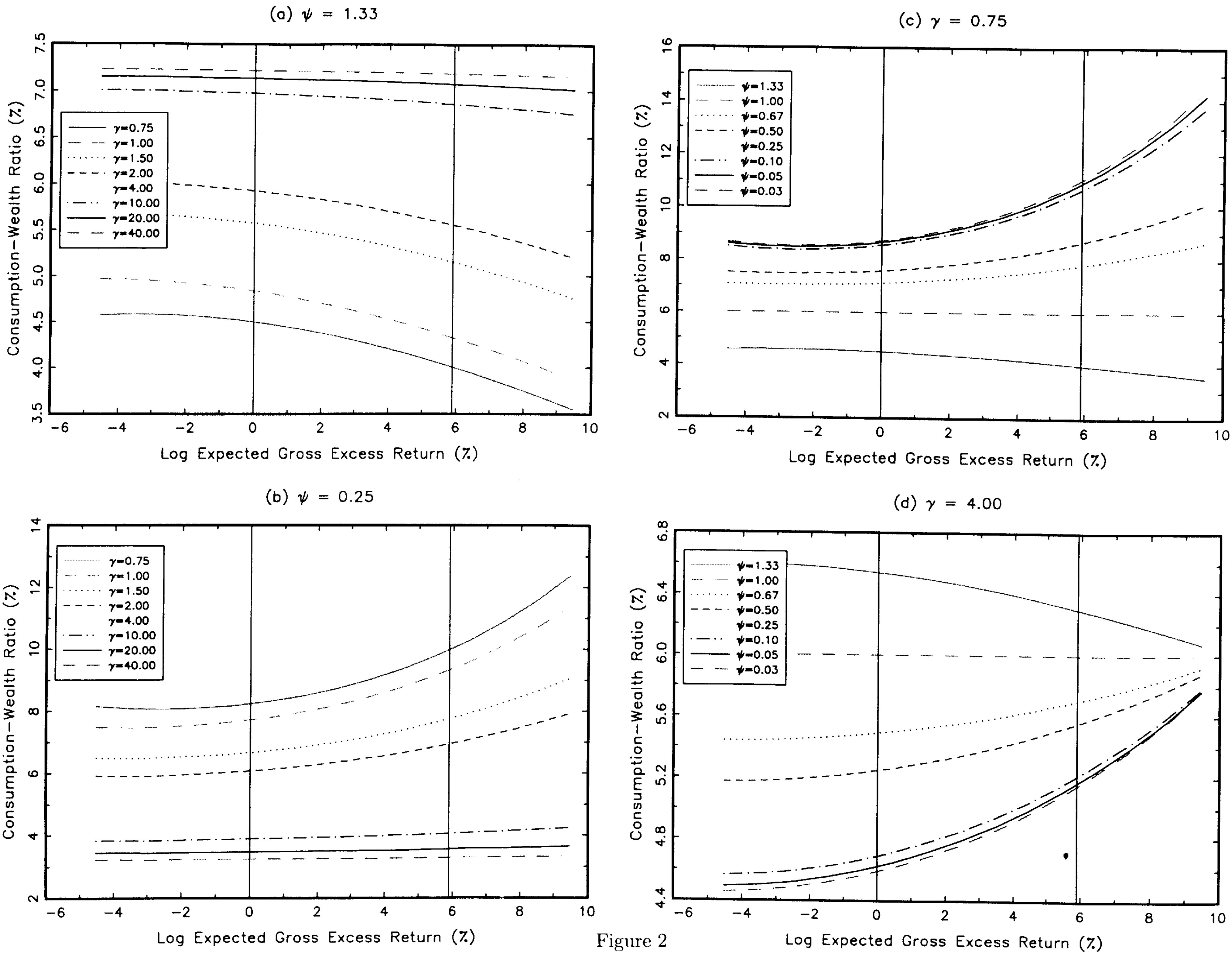
(B) Annual Model: 1890 - 1993

R.R.A.	E.I.S.							
	1/.75	1.00	1/1.5	1/2	1/4	1/10	1/20	1/40
0.75	58.08	51.71	47.07	45.31	43.28	42.37	42.11	41.99
1.00	49.46	43.22	38.66	36.93	34.97	34.12	33.89	33.78
1.50	37.54	32.84	29.13	27.67	25.97	25.24	25.04	24.95
2.00	30.41	26.59	23.50	22.26	20.83	20.24	20.08	20.02
4.00	17.26	15.21	13.44	12.72	11.91	11.61	11.55	11.52
10.0	7.50	6.69	5.96	5.66	5.33	5.24	5.23	5.23
20.0	3.86	3.47	3.10	2.95	2.79	2.75	2.76	2.76
40.0	1.96	1.76	1.58	1.51	1.43	1.42	1.42	1.42



(a) ψ = 1.33
γ=0.75
γ=1.00
γ=1.50
γ=2.00
γ=4.00
γ=10.00
γ=20.00
γ=40.00
Allocation to Stocks (%)
Log Expected Gross Excess Return (%)
(b) ψ = 0.25
γ=0.75
γ=1.00
γ=1.50
γ=2.00
γ=4.00
γ=10.00
γ=20.00
γ=40.00
Allocation to Stocks (%)
Log Expected Gross Excess Return (%)
(c) γ = 0.75
ψ=1.33
ψ=1.00
ψ=0.67
ψ=0.50
ψ=0.25
ψ=0.10
ψ=0.05
ψ=0.03
Allocation to Stocks (%)
Log Expected Gross Excess Return (%)
(d) γ = 4.00
ψ=1.33
ψ=1.00
ψ=0.67
ψ=0.50
ψ=0.25
ψ=0.10
ψ=0.05
ψ=0.03
Allocation to Stocks (%)
Log Expected Gross Excess Return (%)

Figure 1



(a) ψ = 1.33
Consumption-Wealth Ratio (%)
Log Expected Gross Excess Return (%)
3.5
4.0
4.5
5.0
5.5
6.0
6.5
7.0
7.5
-6
-4
-2
0
2
4
6
8
10
γ=0.75
γ=1.00
γ=1.50
γ=2.00
γ=4.00
γ=10.00
γ=20.00
γ=40.00
(b) ψ = 0.25
Consumption-Wealth Ratio (%)
Log Expected Gross Excess Return (%)
2
4
6
8
10
12
14
-6
-4
-2
0
2
4
6
8
10
γ=0.75
γ=1.00
γ=1.50
γ=2.00
γ=4.00
γ=10.00
γ=20.00
γ=40.00
(c) γ = 0.75
Consumption-Wealth Ratio (%)
Log Expected Gross Excess Return (%)
2
4
6
8
10
12
14
16
-6
-4
-2
0
2
4
6
8
10
ψ=1.33
ψ=1.00
ψ=0.67
ψ=0.50
ψ=0.25
ψ=0.10
ψ=0.05
ψ=0.03
(d) γ = 4.00
Consumption-Wealth Ratio (%)
Log Expected Gross Excess Return (%)
4.4
4.8
5.2
5.6
6.0
6.4
6.8
-6
-4
-2
0
2
4
6
8
10
ψ=1.33
ψ=1.00
ψ=0.67
ψ=0.50
ψ=0.25
ψ=0.10
ψ=0.05
ψ=0.03

Figure 2

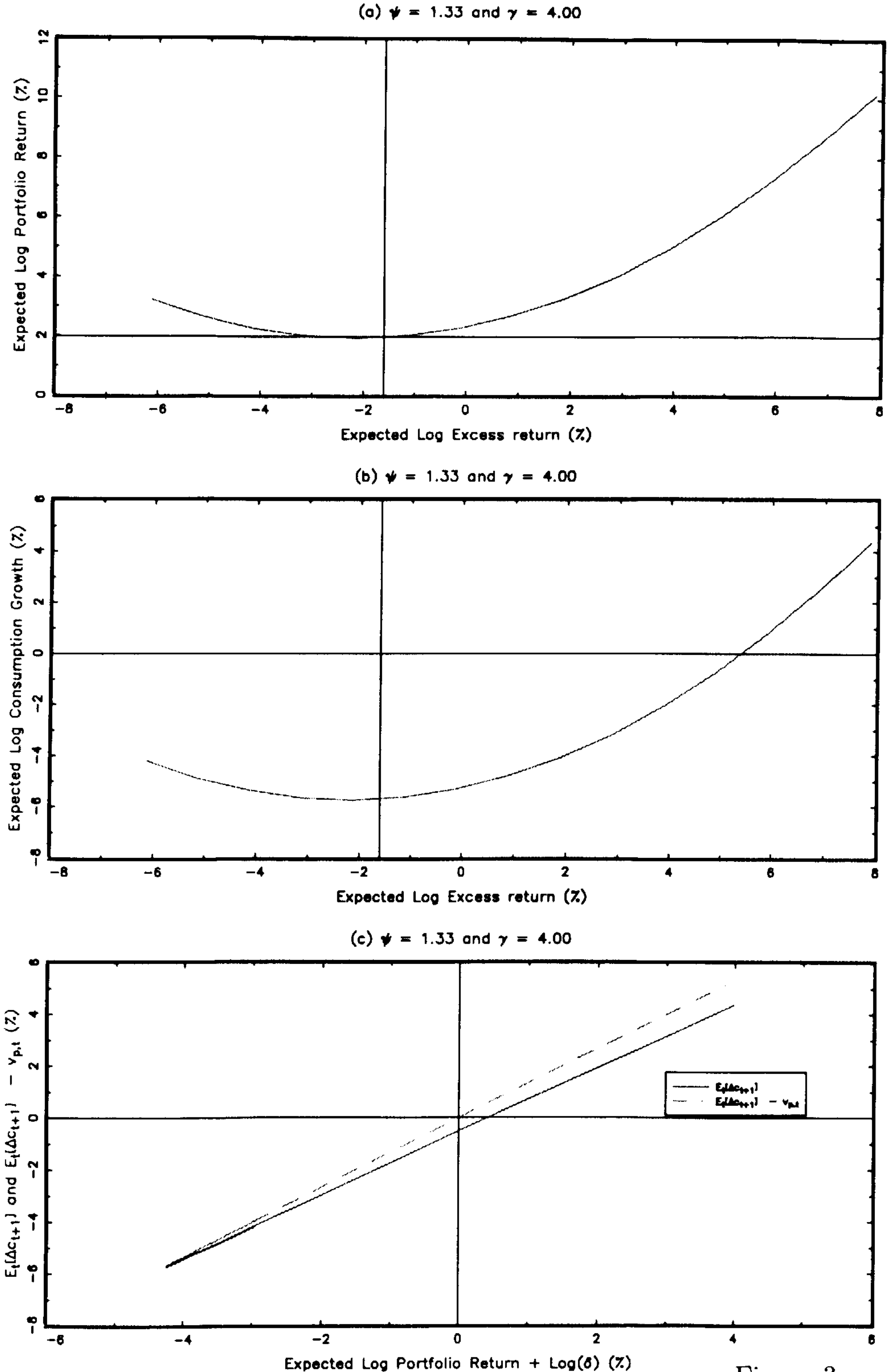


Figure 3

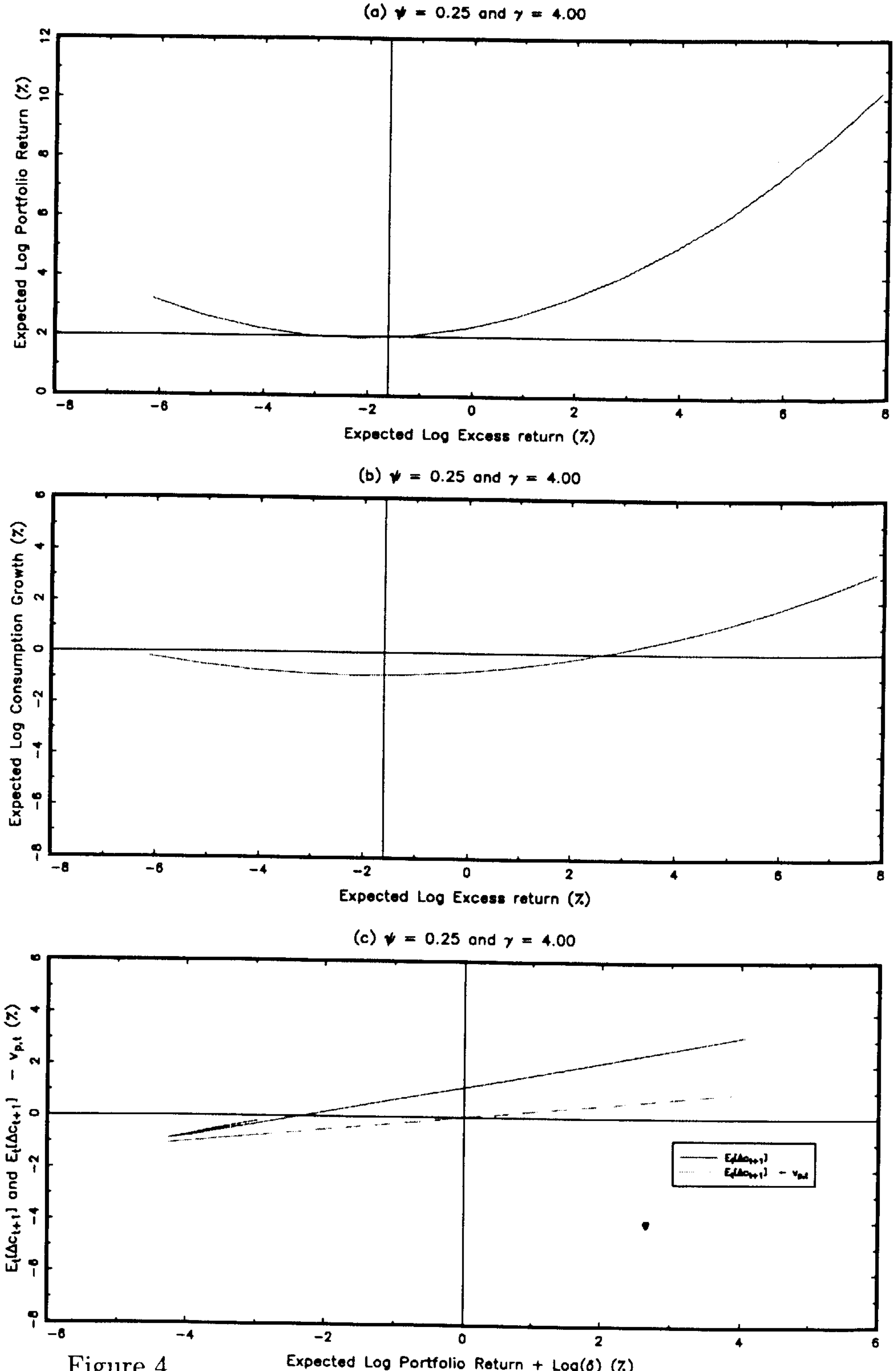
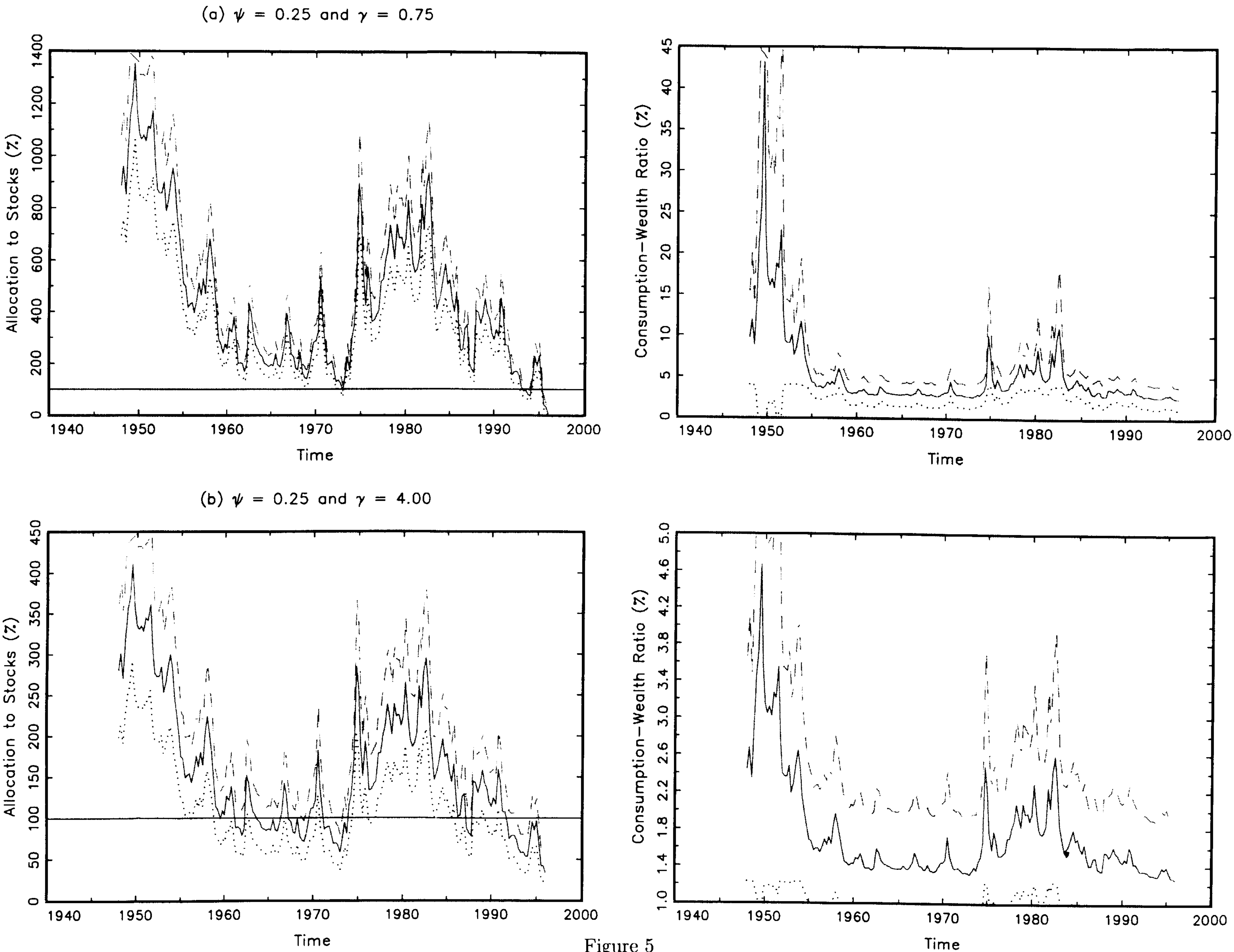
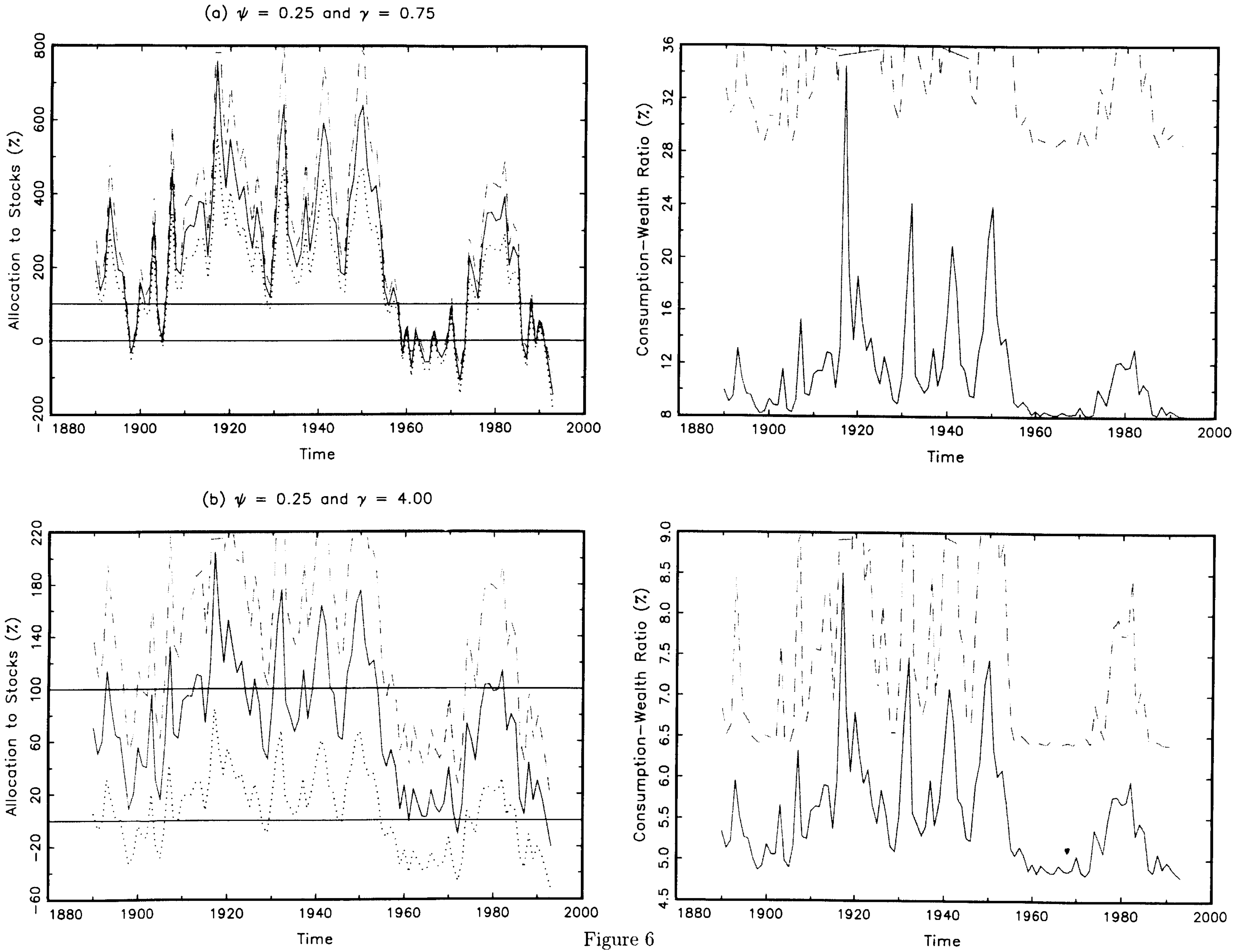


Figure 4



(a) ψ = 0.25 and γ = 0.75
Allocation to Stocks (%)
Consumption-Wealth Ratio (%)
Time
(b) ψ = 0.25 and γ = 4.00
Allocation to Stocks (%)
Consumption-Wealth Ratio (%)
Time

Figure 5



(a) ψ = 0.25 and γ = 0.75
Allocation to Stocks (%)
Consumption–Wealth Ratio (%)
Time
(b) ψ = 0.25 and γ = 4.00
Allocation to Stocks (%)
Consumption–Wealth Ratio (%)
Time

Figure 6