NBER WORKING PAPERS SERIES

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Working Paper No. 3954

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
January 1992

This paper is part of NBER's research program in Asset Pricing. Any opinions expressed are those of the authors and not those of the National Bureau of Economic Research.

NBER Working Paper #3954
January 1992

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ABSTRACT

This paper examines the effect of the labor-leisure choice on portfolio and consumption decisions over an individual's life cycle. The model incorporates the fact that individuals may have considerable flexibility in varying their work effort (including their choice of when to retire). Given this flexibility, the individual simultaneously determines optimal levels of current consumption, labor effort, and an optimal financial investment strategy at each point in his life cycle. We show that labor and investment choices are intimately related. The ability to vary labor supply ex post induces the individual to assume greater risks in his investment portfolio ex ante. The model explains why the young (enjoying greater labor flexibility over their working lives) may take greater investment risks than the old. It also offers an explanation as to why consumption spending is relatively "smooth" despite volatility in asset prices. Finally, the paper provides a compact method for valuing the risky cash flows associated with future wage income.

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1. Introduction

It is a commonplace observation that many individuals have considerable flexibility in their labor decisions — how many hours to work, whether to take on a second job, or when to retire. Such choices involve the traditional tradeoff between monetary income and leisure. (Indeed, we can broaden this tradeoff to include the choice between better or worse paying jobs of varying personal disutilities). The main question we address is this: How does the presence of labor flexibility affect consumption, saving, and portfolio investment decisions over the life cycle?

To answer this question, we employ a life-cycle model that allows continuous consumption decisions and trading in risky financial assets.[1] In our framework, the individual simultaneously determines optimal levels of current consumption, labor effort and leisure, and an optimal investment portfolio at each point in the life cycle. It surely comes as no surprise that realized investment returns influence the individual's consumption and labor decisions. For instance, disastrous portfolio performance will induce

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[1]Among finance theorists, see Mayers (1972), Williams (1978) and Svensson (1988) for studies of the effect of human capital on portfolio choice. Williams (1979) analyzes a life-cycle model in which the individual can increase his human capital by choosing his level of education. He provides an approximate life-cycle solution in which portfolio choice, education, labor supply, and consumption are all endogenously determined. Labor economists have either treated the portfolio mix as given or ignored it altogether in their models of labor market behavior. See Killingsworth (1983) for an extensive survey of life-cycle labor supply models.

the individual to increase his labor supply (so as to earn additional monetary income) and to reduce the amount of leisure (as well as his consumption). Perhaps, a more subtle finding is that the ability to vary labor supply ex post tends to induce the individual to assume greater risks in his investment portfolio ex ante. An individual who has flexibility in choosing how much or how long to work later in life will prefer to invest substantially more of his money in risky assets than if he has no such flexibility. Viewed in this way, labor supply flexibility creates a kind of insurance against adverse investment outcomes. Thus, our framework explains why the young (with greater labor flexibility over their working lifetimes) may take significantly greater investment risks than the old. The analysis also shows how to value the risky cash flows associated with future wage income.

The paper is organized as follows. The next section sets out the basic model and provides a preview of the analysis to come. Section 3 examines the life cycle model in the special case of nonstochastic wage income. Section 4 generalizes the model to include stochastic wage income (which can be perfectly hedged). Section 5 provides a brief summary and concluding remarks.

2. A Life Cycle Model.

In this section we use the lifetime consumption and portfolio choice model of Samuelson (1969) and Merton (1969, 1971) to explore some of the implications of the interaction between portfolio choice and labor supply over the life cycle.[2] We can summarize our model as follows. An individual lives

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[2]For an analysis of these effects in a two period model, see Bodie and Samuelson (1989).

from time 0 to T. He begins with financial wealth F(0). At any time t, his current wealth F(t) is determined by his past saving and investment decisions. In addition, he possesses wealth in the form of his human capital. This human capital H(t) embodies the present value of his future labor income. Human capital is in part stochastic (since future wages are uncertain) and in part under his own control. (The individual will adjust his future labor supply in light of changes in his level of wealth and changes in wages.) At each point in time, the individual may invest his financial wealth in one of two assets: a risk-free asset offering a certain return of r or a risky asset offering an instantaneous expected return of α.

At each point in time t, the individual determines his rate of consumption, C(t), of a single good and the proportion of his financial wealth, $\hat{x}(t)$, to invest in the risky asset. In addition, he determines the quantity of labor that he will supply (for instance, the number of hours worked) and the amount of leisure to consume. We consider two labor supply settings: i) Under flexible labor supply, the individual can vary his labor and leisure continuously. At any point in time, he determines his leisure L(t) and his labor h(t) subject to L(t) + h(t) = 1. (Here, setting the right side equal to one is simply a convenient normalization.) Given the prevailing wage w(t), the individual earns labor income equal to w(t)(1 − L(t)). ii) Under fixed labor supply, the individual determines his choice of labor and leisure once and for all. Once set, the labor decision cannot be subsequently changed.

Thus, the individual consumes a fixed amount of leisure L over the duration of the life cycle.

Whatever the labor setting (flexible or fixed), the individual sets his decision variables — $C(t)$, $\hat{x}(t)$, and $L(t)$ (or L) -- optimally, conditional on his information to date: his current financial wealth and the future dynamics of the asset returns and his uncertain wage.

The individual's objective is to maximize his discounted lifetime expected utility given by:

$$E_t[\int_0^T e^{-\delta s}u(C(s),L(s))ds], \tag{1}$$

where u denotes the individual's utility function and δ is his rate of time preference. Here, E_t denotes the expectation operator, conditional on all relevant information at time t. The utility function u is assumed to be strictly concave in its arguments. Note that the lifetime expected utility function is intertemporally additive and independent.[3]

The instantaneous rate of return on the riskless asset is r. The price of the risky asset (in a frictionless market with continuous trading) follows the Ito process:

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[3]The analysis can be generalized to include non-additive and other path-dependent effects on preferences. See Bergman (1985), Constantinides (1990), and Svensson (1989). Moreover, this formulation can be generalized to include an uncertain terminal date (death) and the inclusion of a bequest function $B(F(T),T)$. Merton (1990a, section 6.4) shows that these more realistic preference functions do not affect the basic structure of the optimal portfolio demand functions.

$$dP = \alpha P dt + \sigma P dz, \quad (2)$$

where α is the instantaneous expected return per unit time and σ^2 is the instantaneous conditional variance. The wage paid to labor also follows an Ito process:

$$dw = gwdt + \sigma^* w dz^*, \quad (3)$$

(The instantaneous means and variances of these processes are assumed to be constant unless otherwise noted.) In much of the analysis to follow, we place further restrictions on the stochastic behavior of the wage (in order to obtain manageable analytic results). We consider two main cases: i) a nonstochastic wage ($\sigma^* = 0$) and ii) a stochastic wage that is perfectly correlated with the risky asset ($\sigma^* dz^* = k\sigma dz$, where k is a non-zero coefficient). This completes the description of the model.

A Preview of Basic Results

The continuous-time consumption and portfolio model has been applied to a wide range of financial problems. The present analysis employs this framework with two important additions. First, we explicitly examine the optimal tradeoff between a pair of goods, a single consumption good and leisure. This extension is easily handled by treating the two as a "composite" good. At any point in time, the optimal consumption of each good is determined as it would be in any static decision. Let y(t) denote the individual's total expenditure on C and L at time t. At each time t, the individual maximizes:

$$u(C(t), L(t)) \quad (4)$$

subject to: $\quad C(t) + w(t)L(t) = y(t).$

Here, the consumption good serves as the numeraire, and the current wage represents the price (or opportunity cost) of leisure. The individual determines the respective optimal quantities of the goods C* and L* for any y(t). Given these optimal quantities, we can construct the individual's indirect utility function $v(y(t), w(t)) \equiv u(C^*(t), L^*(t))$ and apply the main results from the continuous-time model for a single good with state-dependent utility function v. From this, his optimal level of total consumption spending y(t) and his investment strategy $\hat{x}(t)$ are then determined.

The second addition is that our analysis explicitly examines the role of human capital in determining the individual's optimal investment and consumption decisions. Human capital (properly valued) influences these decisions in the same way that financial wealth does. In particular, the characteristics of the individual's investment in the risky asset are best understood if viewed in terms of his total (financial and human capital) wealth. This observation is elementary but nonetheless important especially early in the life cycle when the great majority of an individual's total wealth is human capital in the form of future wage income. To limit attention to the individual's financial wealth alone leads to a systematic underestimate of the investment resources at the individual's disposal.

The individual's human capital is essentially the same as a financial asset, except that it is not traded. How should this non-traded asset be valued? The crucial insight is that if the risks of an individual's human capital (representing capitalized future wage income) can be hedged using financial securities, then it is valued by the individual as if it were a traded asset. And this is so even if the standard moral hazard problem renders the direct sale of human capital infeasible. To do the valuation, the risk characteristics of these future cash flows must be determined. For instance, if the individual's future wage income is nonstochastic (or if its risk is nonsystematic), the value of his human capital is found by discounting these future cash flows at the risk-free rate of return r. In section 4, we analyze the more general case of stochastic wage income.[4]

With this discussion as background, we can now describe the individual's life-cycle decisions as a step-by-step process. Consider the flexible labor case.

Step 1. At each given time t, estimate the value of the individual's human capital by valuing future wage income as if it were a traded asset. (The procedures for doing this are outlined in section 4.) Also determine the risk characteristics of these future cash flows.

Step 2. Compute the individual's total wealth — the sum of his financial wealth and his human capital (found in step 1).

Step 3. Determine the individual's optimal level of total spending, y(t), on the two goods, consumption and leisure. (This is a standard application of the continuous-time consumption model.) The optimal rate of total spending depends on the individual's current total wealth (found in step

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[4]Finance theorists, Merton (1971), Mayers (1972), Fischer (1975), Williams (1978, 1979), and Breeden (1979), have recognized the potential importance of human capital in affecting life-cycle wealth. Losque (1978) derives valuation formulas under restrictive conditions. Svensson (1988) provides valuation expressions in a more general class of cases.

2) as well as his remaining years of life. With y(t) determined, then solve the static optimization problem in equation 4 to find C(t) and L(t).

Step 4. Determine the optimal gross amount and proportion of the individual's total wealth (found in step 2) to invest in the risky asset.

Step 5. Using the risk characteristics of the wage flows (found in step 1), estimate the dollar-value equivalent exposure to the risky asset of the capitalized value of these flows. We call this the individual's implicit exposure to risk embodied in his uncertain future wage income. Adjust the gross amount in step 4 by subtracting this implicit investment in the risky asset to determine the optimal "explicit" investment out of current financial wealth to make in the risky asset.

As a simple illustration, consider an individual in the middle of his working life who enjoys complete labor flexibility. The value of his accumulated financial wealth is $300,000. To estimate his potential human capital, suppose that he will consume no leisure over his remaining lifetime but instead he will supply the maximum amount of labor: $L(t) = 0$ and $h(t) = 1$ for all $t \leq T$. Compute the value of his human capital under this assumption. Suppose that the result of such a calculation is a human capital value of $500,000. Furthermore, suppose that the risk characteristics of this human capital make it equivalent to holding $400,000 in the riskless asset and $100,000 in the risky asset. (His future wages have relatively modest risk.) Taking his financial capital and human capital together, the individual's total wealth is $800,000. According to step 3, his rate of spending on current consumption (on the consumption good and on leisure) should be based on this total value. In addition, the amount of leisure he "purchases" (by reducing his labor supply and earning a proportionally lower income) depends on the current wage.

Now consider the individual's investment behavior. Suppose that in step 4, he has determined his optimal investment proportions out of total wealth to be 60% in the riskless asset and 40% in the risky asset — that is, $480,000 and $320,000 gross investments in the respective assets. Since his human capital is already equivalent to having a $100,000 investment in the risky asset, the optimal explicit investment in this asset is $320,000 - $100,000 = $220,000. The remaining $80,000 of his financial wealth is invested in the riskless asset. To the outside observer who views only his financial assets, it appears that the individual is taking a relatively risky portfolio position. He is investing 220,000/300,000 or 73% of his current financial wealth in the risky asset. The more accurate picture is that the individual has a total effective investment of $320,000 in the risky asset (representing a desired 40% of his total wealth). The investment implications of this example are summarized in Table 1 (Case 1).

Now consider a second individual identical to the first except that his labor supply is fixed over his working life. Suppose that his working hours are $h = .6$ and his leisure hours are $L = .4$ which implies that the value of his human capital is (.6)($500,000) = $300,000. Table 1 (case 2) shows the investment implications for this individual. Note that he invests $180,000 of his financial wealth (or 60%) in the risky asset -- a smaller proportion than the first individual. Why is this the case? The explanation stems from a "wealth" effect. With flexible labor, the first individual has, in effect, a

Table 1

Case 1. Flexible Labor
Maximum human capital
is $500,000

Implicit or Explicit Investment	Total Wealth	Human Capital	Financial Wealth
Total	$800,000	$500,000	$300,000
Riskless Asset	$480,000	$400,000	$80,000
Risky Asset	$320,000	$100,000	$220,000

Case 2. Fixed Labor
Human capital is $300,000
(h = .6, L = .4)

Implicit or Explicit Investment	Total Wealth	Human Capital	Financial Wealth
Total	$600,000	$300,000	$300,000
Riskless Asset	$360,000	$240,000	$120,000
Risky Asset	$240,000	$60,000	$180,000

Case 3. Flexible Labor
Risky future wages

Implicit or Explicit Investment	Total Wealth	Human Capital	Financial Wealth
Total	$800,000	$500,000	$300,000
Riskless Asset	$480,000	$250,000	$230,000
Risky Asset	$320,000	$250,000	$70,000

greater store of human capital upon which to draw. Since this human capital is nearly riskless, the individual must "rebalance" his total wealth holdings by investing a larger proportion of his financial wealth in the risky asset. By contrast, the second individual (by precommitting himself to a fixed amount of leisure over his working life) has a smaller amount of human capital from which to invest and, therefore, requires much less rebalancing. Note that the wealth effect can be viewed quite generally. The individual's amount of human capital depends not only on his degree of labor flexibility but also on the length of his remaining working life. An individual near retirement has little human capital to draw upon and so will undertake little in the way of rebalancing. His explicit investment proportions out of his current financial wealth will be very close to his optimal proportions out of his total wealth (as determined in step 4).

Finally, consider a third individual identical to the first except that his future wages are relatively more risky. Suppose that his human capital is equivalent to an implicit holding of $250,000 in the riskless asset and $250,000 in the risky asset. In this instance (Table 1, Case 3), the individual makes an explicit investment of only $70,000 in the risky asset to ensure his desired overall investment proportion (40% of his total wealth). The reason for his relatively conservative financial investment is a "substitution" effect. Since the risk profile of his wage is already equivalent to having 50% of his human capital invested in the risky asset, he need invest relatively little of his financial wealth (23%) in this asset in order to achieve his desired overall proportion (40%). Here, the pure substitution effect associated with the risky human capital outweighs the pure wealth effect.[5]

To sum up, these simple examples suggest the ways in which an individual's human capital (both its riskiness and its flexibility) can greatly influence his investment behavior. In the next sections, we turn to the formal model and analysis.

3. A Nonstochastic Wage

We begin with the case in which the individual's wage is nonstochastic ($\sigma^* = 0$ in (3)). For reasons already discussed, it is convenient to frame the analysis in terms of total wealth, $W(t)$. The dynamics of total wealth are determined by the initial condition, $W(0) = F(0) + H(0)$, where $H(0)$ is still to be evaluated and by the dynamic budget equation:

$$dW = [(x(\alpha - r) + r)W - C - wL]dt + \sigma x W dz. \tag{5}$$

Here, x denotes the fraction of <u>total</u> wealth invested in the risky asset. (Recall that $\hat{x}$ is the proportion of <u>financial</u> wealth invested in the asset.)

<u>Flexible Labor Supply</u>. The individual's optimization problem is to maximize the expected utility in (1) subject to (5). To apply stochastic dynamic programming (as for example in Merton 1990a), we first define the derived utility function

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[5]The individual's explicit investment proportion in the risky asset will equal his desired overall proportion if and only if his human capital is equivalent to holding this desired proportion. In this knife-edge case, the substitution effect exactly matches the wealth effect.

$$J(W,w,t) = \max E_t[\int_t^T e^{-\delta s} u(C(s),L(s))ds], \tag{6}$$

conditional on the current value of total wealth W(t) and the wage w(t). The maximization in (6) is with respect to C(t), L(t), and x(t). By the Bellman optimality conditions (Merton, 1990a, eqs. 4.17a and 6.66), J must satisfy

$$0 = \max_{(C,W)} \{u(C,L)e^{-\delta t} + J_W[(x(\alpha-r) + r)W - C - wL] + J_t + J_w gw + .5x^2W^2\sigma^2 J_{WW}\}, \tag{7}$$

where subscripts denote partial derivatives with respect to the designated arguments.[6] The resulting first order conditions are:

$$u_C(C*(t),L*(t))e^{-\delta t} - J_W = 0, \tag{8}$$

$$u_L(C*(t),L*(t))e^{-\delta t} - wJ_W = 0, \tag{9}$$

and

$$J_W(\alpha-r) + x*\sigma^2 WJ_{WW} = 0. \tag{10}$$

Dividing (9) by (8), we find

$$u_L/u_C = w, \tag{11}$$

which is simply the familiar optimality condition for the static problem in (4). After rearranging (10), we have that

$$x*W = -[J_W/J_{WW}][(\alpha-r)/\sigma^2]. \tag{12}$$

Note that $-J_W/J_{WW}$ is the reciprocal of the coefficient of absolute risk aversion of the individual's derived utility function.

The solution is completed by interpreting and valuing the individual's total wealth W(t). Total wealth is simply the sum of his financial wealth and

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[6]For analytical convenience, we have not included the Kuhn-Tucker multipliers associated with the constraints $C \geq 0$, $L \geq 0$, and $L \leq 1$. Although explicit inclusion of these constraints will in general influence the optimal policies, it would not materially change the basic behavioral results derived in the unconstrained analysis.

the value of his human capital. Since future wages are nonstochastic, the value of future wage cash flows is determined by discounting them at the risk-free rate of interest r. Thus, the individual's remaining human capital at time t is:

$$H(t) = w(t)(1 - e^{-r(T-t)})/(r-g). \qquad (13)$$

Finally, using this formula establishes the initial total wealth condition, $W(0) = F(0) + H(0)$.

Remark 1. It should be understood that H(0) is the present value of the individual's future wage income under the hypothesis, L = 0. That is, it is as if the individual realized 100% of his potential labor income. In effect, one can think of the individual as transforming his maximum potential human capital into financial wealth and basing his portfolio investment and expenditures on this. We should emphasize that the individual is not literally "selling" his human capital (since there is no market for this intangible asset). Nonetheless, he can achieve the economic equivalent of such a sale by dynamic hedging in traded securities. This is accomplished by selling short the riskless asset and investing the resulting positive cash balance in the risky asset. (Of course, the short sale must be designed so that his future liabilities from the sale are exactly matched and offset by his future wage cash flows.)

Fixed Labor Supply. In this case, the individual is unable to change his labor supply so that L(t) = L throughout his lifetime. The analysis is similar to (and indeed simpler than) the one for flexible labor. Begin by defining $I(W,t,L) = E_t[\int_t^T e^{-\delta s}u(C(s),L)ds]$ in analogy to equation (6) above.

Then the first order conditions are:

$$u_C(C'(t),L')e^{-\delta t} - I_W = 0, \tag{8'}$$

and

$$I_W(\alpha - r) + x'\sigma^2 W I_{WW} = 0. \tag{10'}$$

The individual chooses his optimal (fixed) consumption of leisure L' such that

$$L' = \text{argmax } I(W(0),0,L). \tag{9'}$$

Finally, the individual's optimal portfolio investment is given by

$$x'W = -[I_W/I_{WW}][(\alpha - r)/\sigma^2]. \tag{12'}$$

<u>Remark 2</u>. There are two key differences between the flexible and fixed labor cases. First, comparing (9) and (9'), we see that leisure is determined as a flow variable in the former case and as a stock variable in the latter case. In the flexibility case, the individual continuously varies his consumption of leisure in response to changes in his total wealth and in the wage rate. Obviously, the individual's welfare suffers when he has no such flexibility. (I is strictly smaller than J.) Second, the value of the individual's "fungible" human capital (i.e. the amount available to support consumption) differs between the two cases. In contrast to (13), the value of fungible human capital when leisure is fixed at L' is:

$$H(t) = [(1 - L')w(t)](1 - e^{-r(T-t)})/(r-g). \tag{13'}$$

In the flexible labor case, the individual incorporates 100% of his remaining potential human capital into his current wealth. He then proceeds to purchase the consumption good and to "buy back" leisure. In the fixed labor case, he can call on only (1-L') of his potential human capital for the consumption good since he is committed to consuming L' in leisure. (Of course, he possesses an implicit asset called "capitalized" leisure.) Though equations (12) and (12') appear similar, the respective values of total wealth (W) differ due to the different values of the individual's fungible human capital.

Applications

The following simple examples provide concrete illustrations of the points above.

Example 1. Let $u(C,L) = \log(C) + \Gamma\log(L)$. It is easy to check that $J_W/J_{WW} = I_W/I_{WW} = -W$. Using (12) and (12'), it follows that

$$\hat{x}^* = x^*W/F = [(\alpha-r)/\sigma^2][1 + H/F] \tag{14}$$

and

$$\hat{x}' = x'W/F = [(\alpha-r)/\sigma^2][1 + (1 - L')(H/F)] \tag{14'}$$

A comparison of (14) and (14') shows that the proportion invested in the risky asset is unambiguously greater with a flexible labor supply than with a fixed labor supply. Note that as the individual grows older and H(t) declines, the difference between $\hat{x}^*$ and $\hat{x}'$ becomes smaller. This implies that flexibility in labor supply is more important in the portfolio decisions of the young than of the old.

Example 2. Let $u(C,L) = C^aL^b/a$. Here, a and b are restricted to be of the same sign and must satisfy $a + b \leq 1$. It follows that $J_W/J_{WW} = -W/(1-a-b)$ and $I_W/I_{WW} = -W/(1-a)$. In turn,

$$\hat{x}^* = [(\alpha-r)/((1-a-b)\sigma^2)][1 + H/F] \tag{15}$$

and

$$\hat{x}' = [(\alpha-r)/((1-a))\sigma^2)][1 + (1 - L')(H/F)] \tag{15'}$$

For flexible labor supply, the derivation of J_W/J_{WW} follows from the static optimality condition $u_L/u_C = w$, which reduces to $C/L = (a/b)w$. After substituting this condition into $u(C,L)$, the indirect utility function takes the form $v = dy^{a+b}$, where y denotes the total spending on the goods. It is then straightforward to compute J_W/J_{WW}.

Remark 3. Together, the log and power (isoelastic) functions comprise the class of utility functions having constant relative risk aversion (CRRA). The parameters a and b satisfy $a = 1 - RRA_C$ and $b = 1 - RRA_L$, where RRA_C and RRA_L are the coefficients of relative risk aversion for consumption and leisure. In the log case, we have $RRA_C = RRA_L = 1$, so that $a = b = 0$.) For the CRRA class, the individual's optimal investment is proportional to total wealth. For log utility, the "wealth effect" — the gap between H and $(1-L')H$ — is the sole difference between $\hat{x}*$ and $\hat{x}'$. For general isoelastic utility, $\hat{x}*$ and $\hat{x}'$ differ not only because of the wealth effect, but also because of the difference between (a+b) and (a), the "coefficient" effect. If the utility function is less risk averse than the log function (a and b are positive), then the wealth and coefficient effects work in the same direction. An individual who enjoys labor flexibility unambiguously invests a greater portion of his wealth in the risky asset than his counterpart whose labor supply is fixed. If the utility function is more risk averse than the log function (a and b are negative), this second effect works against the wealth effect. Comparing (15) and (15'), we see that for sufficiently small values of H (occurring, for instance, late in life), the coefficient effect may outweigh the wealth effect, causing $\hat{x}'$ to be greater than $\hat{x}*$.

The wealth effect suggests that in "normal" circumstances, the individual with flexible labor will invest a greater proportion of his financial wealth in the risky asset than his counterpart whose labor is fixed. However, it is worth making the obvious observation: An individual with flexible labor must actually want to exercise this option ex post if $\hat{x}^*$ is to exceed $\hat{x}'$. The next example illustrates the point.

Example 3. Let $u(C,L) = V(C + h(L))$, where V and h are increasing and concave functions. It is easy to verify that the individual's optimal choice of leisure ex post does not vary over time — $L^*(t) = L'$, for all t regardless of current wealth $W(t)$ or the wage $w(t)$. For this class of utility functions, labor flexibility serves no insurance function and provides no advantage. The solutions in the flexible and fixed labor supply cases are identical in all respects and, therefore, $\hat{x}^* = \hat{x}'$.

Example 4. Let $u(C,L) = V(g(C) + L)$, where V and g are concave functions. This is the converse of Example 3. Assuming an interior optimum, the individual adjusts his leisure to offset fluctuations in consumption that might result from fluctuations in investment income. That is, $C^*(t) = C'$, for all t regardless of the level of current wealth. The investor in effect adjusts L to provide himself perfect insurance against wealth fluctuations. If the function V is linear (so that the individual is risk neutral with respect to fluctuations in leisure), the investor will place the maximum amount in the risky asset. Under fixed labor supply, he will limit investment in the risky asset due to risk aversion (since g is concave).

A Numerical Example

The following numerical example illustrates exact life-cycle investment and saving dynamics. At age 31, an individual has $100,000 of financial wealth and has the opportunity to earn maximum labor income of $60,000 per year (if he were to consume no leisure). We abstract from his retirement decision and simply specify that he works until age 71.[7] That is, his planning problem ends at age 71, at which time he plans to have zero financial wealth. (One may think of him as working until he dies and having no bequest motive.)

Table 2 and Figure 1 summarize the individual's optimal life-cycle behavior in the case of log utility, $u = \log(C) + .5\log(L)$, and for real asset returns: $r = .03$, $\alpha = .09$, and $\sigma^2 = .12$. Labor is flexible in Part a of Table 2; it is fixed in Parts b and c. In Part b, the individual's yearly consumption of leisure is set at $L = .55$, matching the expected consumption of leisure in the flexible labor case. In Part c, leisure is fixed at the ex ante optimal value of leisure, $L' = .33$.

Table 2's lead column shows the individual's optimal investment proportion as a multiple of his current financial wealth.[8] Observe that under

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[7]Our focus here is solely on the advantages of labor flexibility during the individual's working life. Obviously, choosing when to retire is another way of exercising labor supply flexibility.

[8]It is straightforward to confirm that for the given asset returns, the agent optimally invests 50% of his current total wealth (F + H) in the risky asset. Table 2's lead column shows the corresponding proportion of the agent's financial wealth allocated to the risky asset.

Table 2
Life-Cycle Behavior

	Proportion in Risky Asset	Human Capital	Financial Wealth	Consumption	Leisure
	$\hat{x}$	$\bar{H}$	$\bar{F}$	$\bar{C}$	$\bar{L}$
a) Flexible Labor					
Age 31	7.5	1398	100	65.9	.55
Age 41	3.7	1187	188	65.9	.55
Age 51	2.3	902	249	65.9	.55
Age 61	1.6	518	225	65.9	.55
b) Fixed Labor (L = .55)					
Age 31	3.6	630	100	48.2	.55
Age 41	2.5	535	135	48.2	.55
Age 51	1.8	407	154	48.2	.55
Age 61	1.4	234	129	48.2	.55
c) Fixed Labor (L' = .333)					
Age 31	5.1	931	100	68.1	.33
Age 41	3.0	791	156	68.1	.33
Age 51	2.1	601	191	68.1	.33
Age 61	1.5	346	166	68.1	.33

Assumptions: $u = \log(C) + .5\log(L)$
$r = .03$, $\alpha = .09$, and $\sigma^2 = .12$
w = \$60,000 per year
Initial wealth is \$100,000

All variables are in thousands of dollars
except x which is a multiple of financial wealth.

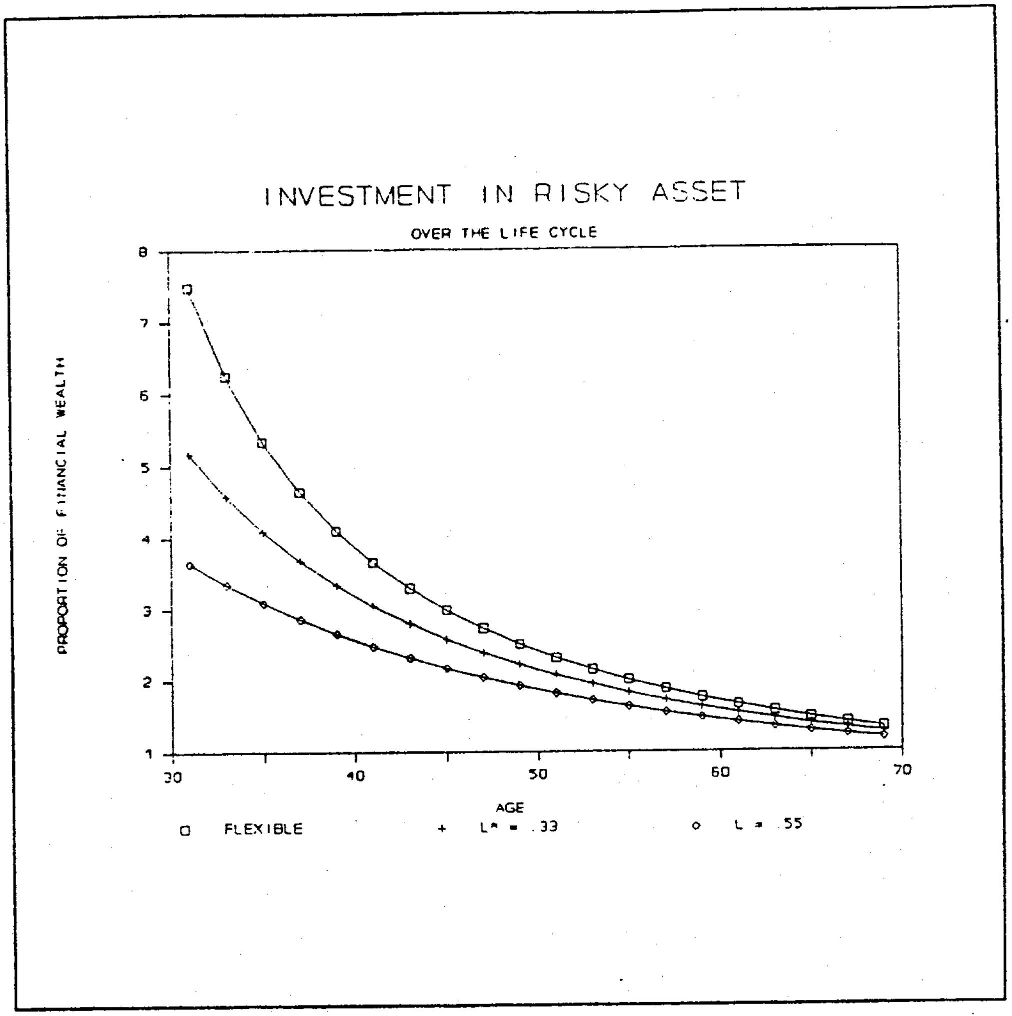

Figure 1. Investment in the Risky Asset as a Multiple of Financial Wealth

ASSUMPTIONS: $U = \log(C) + .5\log(L)$;
$r = .03$, $\alpha = .09$, and $\sigma^2 = .12$.

either labor supply regime, the values of $\hat{x}$ are well in excess of one — that is, the individual borrows at the risk-free rate to finance his investment in the risky asset. The pure wealth effect of the individual's riskless human capital causes a significant rebalancing of his investment portfolio. As the example illustrates, the individual's degree of leverage is greatest early in the life-cycle and when his labor supply is flexible. Casual empiricism suggests that young workers do tend to have highly leveraged portfolios. The major asset held by the young is residential real estate financed in large part with mortgage loans. The model also predicts that households with greater labor flexibility will tend to have riskier investment portfolios.

The other columns of Table 2 show <u>expected</u> values of the key wealth and consumption variables.[9] The second column tracks the individual's human capital — the present value of his future labor earnings — over the life cycle. This component of wealth is non-stochastic. Under flexible labor, human capital embodies the individual's maximum labor income ($60,000 per annum), before income is withdrawn to purchase the consumption good and leisure. In the case of fixed labor, human capital measures the present discounted value of the individual's actual yearly earnings, (1-L)H. As emphasized earlier, this is the key difference between the fixed and flexible cases. In the former, investments in the risky asset are "out of" actual

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[9]Analytic expressions for the expected investment and consumption behavior with a single consumption good can be found in Merton (1969 and 1971). Extending these results to the case of flexible labor supply (i.e. two consumption goods) is straightforward.

human capital; in the latter, they are based on maximum, potential human capital.

Table 2's third column shows the expected value of financial wealth (a stochastic variable) over the life cycle. Note that the (average) growth in financial wealth is much greater under flexible labor supply than under fixed labor supply (for either $L = .55$ or $L' = .33$). This is a direct result of the greater investment in the risky asset when labor is flexible.

The final two columns of Table 2 show the life-cycle pattern of consumption and leisure. For convenience, we have assumed a particular rate of time preference: $\delta = .06$. This knife-edge value ensures that the individual's expected consumption and leisure behavior is constant over the life cycle.[10] In the flexible labor case, the individual's portfolio investments (by which his human capital is transformed into financial wealth) support an expected annual consumption flow of $65,900 and expected leisure of .55. By contrast, when labor and leisure are fixed (at $L = .55$), the expected consumption stream is only $48,200 per year.

It is important to recognize that consumption and leisure entries in Table 2 overstate the <u>utility</u> differences between the fixed and flexible cases. Fortunately, exact analytic expressions for the individual's lifetime expected utility can be readily derived. A natural way to express the welfare cost associated with fixed labor supply is to compute the proportional

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[10] Since the agent invests 50% of his total wealth in the risky asset, the overall expected return on his (total) wealth is (.5)(3%) + (.5)(9%) = 6%. Choosing a matching rate of time preference ensures a level consumption stream on average.

increase in lifetime wealth (as of age 31) necessary to leave the individual as well off under fixed labor as under flexible. For Part b, the proportional increase in wealth is 24%. This is to say that on top of his initial lifetime wealth ($730,000), the individual would need an additional $175,000 to bring him the same level of utility as he enjoys with flexible labor. In Part c, the individual makes an ex ante optimal leisure choice, $L = .33$. Here, one computes the compensating differential to be $133,000 (an increase in lifetime wealth from $1,032,000 to $1,165,000).

It is interesting to observe that the individual's optimal ex ante choice of leisure ($L' = .33$) is considerably smaller than the amount of leisure ($\bar{L} = .55$) expected to be consumed in the case of flexible labor. It is straightforward to carry out the requisite optimization in each case to obtain closed-form expressions for the decision variables of interest. The simplest expressions emerge when F/H (the ratio of initial financial wealth to lifetime human capital) is zero. In this case, the optimal ex ante choice of leisure is simply $L' = \Gamma/(1+\Gamma)$. Leisure is determined once and for all (as if it were a "stock" variable). Here, L' depends only on the individual's utility trade-off, not on any aspects of security returns.[11]

By contrast, leisure is determined optimally as a flow variable in the case of flexible labor supply. In accord with our earlier usage, let y(t) denote the optimal level of spending (on consumption and leisure) supported at

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[11]In the case of logarithmic utility, the agent's expected lifetime utility (under optimal decisions) is proportional to $\text{Log}((1-L)W) + \Gamma\text{Log}(L)$. Thus, L' depends only on Γ, measuring the trade-off between the stocks of wealth and leisure.

time t. It is easy to check that $L(t) = [\Gamma/(1+\Gamma)][y(t)/w(t)]$. In the present example, the expected level of spending, $\bar{y}$, is constant over the life cycle. (Of course, the actual level of total spending varies with investment performance.) Accordingly, $\bar{L} = [\Gamma/(1+\Gamma)][\bar{y}/w)]$ As long as the risky asset's expected return exceeds the risk-free return, the expected level of sustainable spending exceeds the flow of labor earnings: $\bar{y} > w$. Consequently, $\bar{L}$ is greater than L'.

The expected levels of spending on the consumption good are identical in the cases of flexible and fixed labor supply (for $F = 0$). The confirmation is straightforward. Under fixed labor supply, the individual's human capital is $(1-L')H$ implying that expected flow spending on the consumption good is $(1-L')\bar{y} = [1/(1+\Gamma)]\bar{y}$. Under a flexible labor regime, $C(t) = [1/(1+\Gamma)]y(t)$ implying $\bar{C} = [1/(1+\Gamma)]\bar{y}$. In the present example, if initial wealth is zero instead of $100,000, we find $L' = .33$ and $\bar{L} = .51$. In addition, the expected annual rate of consumption is $61,500 with either fixed or flexible labor supply.

<u>Remark 4</u>. Similar dynamic results apply in the broader case of isoelastic utility. Flexible labor supply induces the individual to invest a greater proportion of his financial wealth in the risky asset, $\hat{x}^* > \hat{x}'$. In the life-cycle model, there is also a compounding effect: with flexible labor, larger initial investments in the risky asset lead to more rapid accumulation of wealth on average, leading to still greater gross investments in the risky asset. The difference in investment behavior between the fixed and flexible labor cases is greatest early in the life-cycle when the individual's stock of

human capital is greatest. Moreover, the welfare advantage of labor flexibility is significant for typical numerical examples.[12] All of this applies when the wage does not vary stochastically over time.

4. A Stochastic Wage.

We now turn our attention to the case of an uncertain wage. Throughout this section we adopt the restriction:

$$dw = gwdt + k\sigma wdz. \tag{3'}$$

Note that the stochastic portions of the change in wealth (dW) and the change in wage (dw) are now given by the same Weiner process (dz). That is, changes in the individual's wages are instantaneously perfectly correlated with the risky asset.[13] Since unanticipated wage increases are likely to occur in favorable investment climates, we further assume perfect positive correlation. If, as appears to be the case, wages are less volatile than equity prices, then $0 < k < 1$.

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[12]Life-cycle behavior for general isoelastic utility functions closely resembles that displayed for the logarithmic case. The sole difference is an effect related to the degree of risk aversion. For utility functions more risk averse than "log", the difference between the flexible and fixed cases is diminished. For example, for the more risk-averse utility function with $a = -2$ and $b = -1$, the necessary compensating differentials in wealth fall to 3% and 5% when labor is fixed ex ante respectively at L' and L (the expected value under flexible leisure).

[13]If this "complete market" assumption (with respect to opportunities for hedging human capital risk) is relaxed, then the capitalized value of the individual's human capital will be a "personal" one, dependent on the individual's preferences and initial financial wealth. However, the impact on his consumption and investment behavior will be qualitatively similar. For an analysis in this more general case, see Svensson (1988).

The analysis of the individual's optimality conditions proceeds as in the previous section. The dynamic budget equation (5) remains the same as before. The Bellman optimality condition (7) changes to:

$$0 = \max_{(C,W)} \{u(C,L)e^{-\delta t} + J_W[(x(\alpha-r) + r)W - C - wL] + J_t + J_w gw + .5x^2W^2\sigma^2 J_{WW} + J_{Ww}xwk\sigma W + .5J_{WW}k^2w^2\}, \tag{7'}$$

where the two new terms reflect the stochastic behavior of the wage. The resulting optimality conditions pertaining to C*(t) and L*(t) are exactly as before. The optimality conditions for the individual's portfolio investment are:

$$x^*W = -[J_W/J_{WW}][(\alpha-r)/\sigma^2] - [J_{Ww}/J_{WW}]kw, \tag{16}$$

and

$$x'W = -[I_W/I_{WW}][(\alpha-r)/\sigma^2] - [I_{Ww}/I_{WW}]kw, \tag{16'}$$

under flexible and fixed labor, respectively. Since u is strictly concave, so are I and J; therefore, J_{WW} and I_{WW} are both negative. If J_{Ww} is negative (positive), the presence of wage risk implies a reduction (increase) in the demand for the risky asset.

The presence of wage uncertainty involves both wealth and substitution effects. As shown in the previous section, there is a wealth effect — that is, the individual's asset demands depend on his total wealth, financial wealth plus human capital. (We show below exactly how this human capital is valued.) Second, there is a substitution effect. The individual already holds an implicit investment in the risky asset in the form of his future human capital. Consequently, his explicit investment in the risky asset is simply the difference between his desired "total exposure" and this implicit investment. In short, the presence of risky wage income creates a differential demand for the risky asset. Otherwise identical individuals will pursue different investment strategies due to differences in the risk characteristics of their human capital.

Suppose the individual enjoys complete labor flexibility. Let $H(w(t),t)$ denote the value of the individual's human capital at time t, conditional on the current wage. Using contingent claims analysis, we derive the following results:

1. The individual's human capital is economically equivalent to an investment of kH in the risky asset and (1-k)H in the risk-free asset.
2. The total value of the individual's human capital is:

$$H(w(t),t) = [w(t)/\mu][1 - e^{-\mu(T-t)}], \tag{17}$$

where $\mu = r + k(\alpha-r) - g$.

Before proceeding with the formal derivation of these two propositions, we show how they can be used to follow our five-step procedure for determining the individual's optimal portfolio allocation. We begin by using (17) to demonstrate our assertion that when an individual has a risky future wage (with a stochastic part that is instantaneously perfectly correlated with the risky asset), it is as if he has an implicit investment in the asset itself. To do so, we show that the value of human capital can be expressed as a function of the price of the asset. From equations (2) and (3'), we have that $dw/w = (g-k\alpha)dt + kdP/P$. From Ito's lemma, it follows that $w(t) = w(0)e^{\gamma t}[P(t)/P(0)]$, where $\gamma = g - k\alpha + k(1-k)\sigma^2/2$. By substitution for w(t), we can rewrite (17) as:

$$H = [w(0)/\mu][P(t)/P(0)]^k e^{\gamma t}[1 - e^{-\mu(T-t)}]. \tag{17'}$$

Thus, although the value of human wealth is linear in the current wage, it is in general a nonlinear function of the risky asset price.

If future wages are riskless ($k = 0$), the value of human capital reduces to the risk-free case of the previous section. In this case, we observe from (17′) that the value of human capital does not depend on the price of the risky asset. If $k = 1$, each dollar of human capital is proportional to the price of the risky asset. Finally, for $0 < k < 1$, the present value of the individual's human capital is a strictly concave function of the risky asset's price. Note the similarity in form between (17) and (13). For a risky wage with expected zero growth ($g = 0$), the required discount rate is $r + k(\alpha - r)$ reflecting the implicit investments in the risky and riskless assets. For a growing wage ($g > 0$), equation (17) represents the classic present-value formula for an exponentially growing asset. Finally, note that the valuation formulas, (17) or (17′), are independent of the individual's preferences and his endowment.

The final step in the analysis of optimal portfolio behavior uses (17) to derive expressions for $\hat{x}^*(t)$ and $\hat{x}'(t)$. Consider first the case of flexible labor. Let $D^*(t)$ denote the investment demand (in dollars) for the risky asset. From the preceding analysis, we know that

$$D^*(t) = x^*W(t) - kH(w(t),t) = x^*F + (x^* - k)H(w(t),t). \tag{18}$$

In short, the individual's explicit investment in the risky asset equals his desired gross investment (given by (16)) minus his implicit investment. In turn,

$$\hat{x}^*(t) = D^*(t)/F(t) = x^* + (x^*-k)H/F. \tag{19}$$

In the case of fixed labor, the corresponding expression is

$$\hat{x}'(t) = D'(t)/F(t) = x' + (x'-k)(1-L')H/F. \tag{19'}$$

Comparing (19) and (19') leads to two observations:

1. We see that $\hat{x}^* = x^*$ if and only if $k = x^*$; similarly, $\hat{x}' = x'$ if and only if $k = x'$. There is no need for rebalancing if and only if the individual's implicit investment in the risky asset (via his human capital) matches his ideal proportion. (This is clearly a knife-edge case.)

2. For $x^* = x'$ (as in the case of logarithmic utility) and for relatively "safe" wage profiles ($k < x^*$), then $\hat{x}^* > \hat{x}'$. That is, labor flexibility induces the individual to increase his investment in the risky asset. Conversely, if the wage profile is very risky ($k > x^*$), optimal rebalancing leads to the reverse effect: $\hat{x}^* < \hat{x}'$.

Valuing Human Capital. We have used the example above as a concrete and intuitive illustration of how to incorporate risky human capital in the optimal portfolio choice problem with flexible and fixed labor supply. We now sketch a general method for using contingent claims analysis to derive formulas like (17) to value risky, non-marketed cash flows.[14] The steps in the analysis are as follows:

1. Let $H(P(t),t)$ denote the value of the individual's human capital at time t conditional on the current price of the risky asset $P(t)$ as in (17'). As

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[14]Merton (1990a) provides a general analytic treatment of the valuation method. Merton (1992) applies the method to optimal investment strategies for university endowment funds.

argued earlier, the optimal rules can be derived as if the individual sells off his human capital and invests this value in financial assets. To see this, suppose that he invests $H_P P$ of the total proceeds from the sale in the risky asset and the remainder, $H - H_P P$, in the riskless asset. (Here, H_P denotes $\partial H/\partial P$.) Furthermore, suppose that the individual distributes income from the portfolio at the flow rate w(t). This ensures that the individual's "investment" generates cash flows exactly equal to his (uncertain) future wages. Thus, this investment strategy replicates the cash flows from the individual's future wage income.

2. We claim that H(P(t),t) is given by the function F(P,t) that satisfies the fundamental partial differential equation:

$$.5P^2\sigma^2F_{PP} + rPF_P - rF + F_t + w(t) = 0. \tag{20}$$

The proof follows along the lines of Merton (1992, section 4). Write down the dynamic equations for dH (based on the replicating strategy in step 1). Write down the analogous equation for dF using Ito's lemma and the fact that F satisfies (20). The result is an ordinary differential equation in (dH - dF) that has the unique solution H = F for all t, if H is set equal to F(P(0),0) at t = 0. This confirms the identity of H and F.

3. Solve the partial differential equation (20) to arrive at H(P(t),t). Appropriate substitutions from (2) and (3) lead to the expression for H(w(t),t) in (17).

Remark 5. The general approach using contingent claims analysis is useful to place our example in perspective. The virtue of such a "bare bones" example is that it leads to a simple, intuitive result. (Facing a stochastic wage with $\sigma* = k\sigma$ is equivalent to holding a portion k of human capital in the risky asset.) The more important point is that contingent claims analysis is equally applicable in more general models — for instance, when the riskiness of the wage varies over the life cycle, i.e. k varies with time.[15]

5. Summary and Conclusion.

The model developed in this paper suggests that labor supply flexibility can play an important role in household asset allocation. Our analysis produces a number of propositions and hypotheses amenable to empirical testing. A short list of these include:

1. Accounting for human capital is crucial to explaining investment, labor, and consumption behavior of rational economic agents.

2. Ignoring human capital constitutes an "omitted variable" problem. This problem is most severe with respect to individuals who are early in their working lives and who enjoy a relatively high degree of labor flexibility.

3. Under "normal" circumstances, an individual will tend to exhibit more conservative investment behavior as he nears retirement. This hypothesis is supported by two effects. First, as the individual moves through his working

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[15]Under certain restrictions, Svensson (1988) extends the analysis to the important case where uncertain human capital can only be imperfectly hedged by the available financial assets.

life, he expends human capital leading to less and less rebalancing of his financial investments. (The main asset of young workers is their future earning power. By retirement, financial wealth has grown, but the human capital asset is fully depleted.) Second, it is reasonable to hypothesize that for most individuals, the degree of labor flexibility diminishes over the life cycle.[16] For this reason, the effective human capital on which the individual can draw also declines. Hypothesis 3 accords well with the "conventional wisdom" offered by investment counselors. Guides to personal investing usually advise movements from riskier assets like stocks to more conservative fixed-income securities as individuals near retirement. This conventional wisdom is well accepted but often for the wrong reasons.[17] Our analysis suggests a rational basis for this behavior.

4. At any given age in the life-cycle, greater labor flexibility will induce greater risk taking in an individual's financial investments (other things equal). Testing this hypothesis requires identifying measures of labor flexibility. Occupational category provides one such measure. ("Flexible labor" occupations are those that offer opportunities for working extra hours, taking extra jobs, or delaying retirement.) Family status is a second potential measure. (Households with multiple potential workers probably

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[16]Our model also specifies the important exceptions to "normal" circumstances. For instance, an individual would exhibit greater risk taking with age, i) if his wage was very risky (so that the substitution effect outweighs the wealth effect), or ii) if wages become systematically less risky over the life-cycle.

[17]See Samuelson (1989) for a discussion and critique of the conventional wisdom.

enjoy greater labor flexibility than single workers.) Given a suitable measure of flexibility, one would expect to find a positive correlation between holdings of risky assets and labor flexibility in a cross-section of households. The link between occupational status and investment risk taking remains to be tested empirically.

5. Labor flexibility induces "stabilizing" labor supply behavior by individuals. From a macroeconomic point of view, one would expect to see a positive labor supply response to financial "shocks" that adversely affect household financial wealth (holding prevailing wages fixed). Put another way, many individuals can and will vary their leisure consumption in response to financial shocks. By contrast, if an individual's labor supply were absolutely fixed, he would be forced to make all adjustments via his current and future consumption of goods and services. In short, labor supply flexibility supplies one reason why consumption spending on goods and services may be relatively stable in the face of financial shocks. Many researchers have found consumption relatively smooth compared to income and asset prices.[18] In addition to other explanations for this phenomenon, our model suggests that at least some of the apparent smoothness may be due to an omitted variable — the consumption of leisure. If flexibility is important, much of the adjustment will be made via leisure/labor behavior, leaving the consumption of goods and services relatively smooth.

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[18] For discussions and explanations for consumption smoothing, see Hall (1988), Black (1990), Constantinides (1990), and Grossman and Laroque (1990)

6. <u>At any given age in the life cycle, the riskier is an individual's human capital, the lower will be his financial investment in risky assets</u>. One way to test the empirical relevance of the "substitution" hypothesis is to take a cross section of job categories and occupations. For each occupation, one would first estimate the correlation between average wages and market movements. With these correlation estimates, one would test whether, other things equal, risk taking investment behavior is associated with occupations characterized by low levels of wage risks.[19]

In conclusion, it is well to note that the main virtue of the present model — its simplicity — is also an important limitation. We have modeled labor flexibility in a quite primitive fashion. Obviously, the opportunity to vary continuously one's labor supply without cost is a far cry from the workings of actual labor markets. A more realistic model would allow limited flexibility in varying labor and leisure. One current research objective is to analyze the retirement problem as an optimal stopping problem and to evaluate the accompanying portfolio effects.

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[19]As an application of propositions 4 and 6 that is "close to home," consider a financial economist who switches from academia to Wall Street. On Wall Street, his future wages will be closely tied to the health of financial markets. According to item 6, one would expect him to adopt a much more conservative investment strategy after the transition. Of course, the wealth effect of the transition (much higher salaries on Wall Street) could work in the opposite direction. But even this wealth effect is ambiguous. The economist could well have more labor flexibility (and as much <u>potential</u> human capital) in academia as on Wall Street.

This paper underscores the value of labor supply flexibility. This simple observation, however, suggests others. Economic agents can increase their labor flexibility by investing in education and training in an effort to make their skills more marketable. Thus, the value of this flexibility is crucial for determining the optimal investment in human capital. Similarly, employers can and will design contractual relations (including risk sharing and pay scales based on seniority) that affect the flexibility and riskiness of human capital. We will explore these issues in future research.

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