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ABSTRACT

These notes discuss three aspects of dynamic factor pricing (i.e., APT) models. The first one is that diversifiable idiosyncratic risk is unpredictable in a no-arbitrage world. The second feature is that the conditional factor loadings or betas on the common factors are approximately constant when returns follow an unconditional factor structure. The third topic concerns the estimation of dynamic factor pricing models in large cross-sections when returns follow an unconditional factor structure. These results aid in the interpretation of existing applications and identify some of the issues in the formulation and estimation of dynamic factor pricing models.

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This paper concerns a simple and straightforward problem—the analysis of dynamic factor or Arbitrage Pricing Theory (APT) models. The original analysis in Ross(1976,1977) has typically been interpreted as a factor pricing model that is static in the sense that the Capital Asset Pricing Model is a static model with constant investment opportunities. Put differently, Ross did not explicitly distinguish between conditional and unconditional moments.

Of course, the analysis in Ross(1976,1977) is not really static but rather concerns a single period model. Single period models which assume the absence of arbitrage opportunities can place nontrivial dynamic restrictions on asset prices conditional on available information. Accordingly, they can be treated as making assumptions about diversification prospects implicit in the conditional covariance matrices of returns that follow conditional factor structures and as calculating their no-arbitrage implications for conditional mean returns.

Nevertheless, some aspects of the translation from the static to the dynamic setting are not entirely routine. Dynamic factor pricing models place implicit economic and technical restrictions on the time series properties of security returns. If idiosyncratic risk is diversifiable risk, idiosyncratic risk is approximately unpredictable in the absence of arbitrage opportunities as the number of securities grows without bound. If returns possess an unconditional factor structure, conditional factor loadings are approximately constant as the number of securities grows without bound.

These observations are useful for formulating, estimating, and testing dynamic factor models. They also provide a consistent framework for interpreting existing empirical work that assumes both that the conditional covariances of security returns with a small number of common factors determine conditional expected returns and that these conditional covariances are constant. Accordingly, these notes discuss the time series implications of dynamic factor pricing models with a view toward empirical applications. As such, my focus differs from related papers such as Huberman and Kandel(1987), Ingersoll(1984), Rothschild(1986), and Stambaugh(1985).

The paper is organized as follows. The next section records the pricing implications of a conditional factor structure in security returns, focussing on the implied unpredictability of idiosyncratic risk as the number of assets grows without bound. The third section shows that the assumption of an unconditional factor structure in security returns technically restricts conditional covariances between factors and security returns to be approximately constant in large cross-sections. The penultimate section sketches some of the econometric implications of these observations and a brief conclusion ends the paper.

1. Factor Pricing in a Dynamic Setting

Static factor pricing models predict that diversifiable risk is not 'priced' for most of a large menu of assets when their returns follow a factor structure. Not surprisingly, the same basic intuition applies in a dynamic setting. The main novelty introduced by dynamic considerations is that idiosyncratic risk is not 'priced' when it is unpredictable.

To fix matters, let $\underline{R}_t$ be an N vector of security returns that follow the conditional factor structure:

$$\underline{R}_t = \underline{E}_{ct} + B_{ct}\underline{\delta}_{ct} + \underline{\varepsilon}_{ct}; \quad E\left[\underline{\delta}_{ct}/I_{t-1}\right] = E\left[\underline{\varepsilon}_{ct}/I_{t-1}\right] = E\left[\underline{\delta}_{ct}\underline{\varepsilon}_{ct}'/I_{t-1}\right] = 0 \;\forall\; t \qquad (1)$$

where $\underline{E}_{ct}$ is the N vector of their conditional expected returns, $\underline{\delta}_{ct}$ is a K_c vector of zero conditional mean common factors, B_{ct} is the NxK_c matrix of coefficients from the conditional population regression of returns on the common factors, and $\underline{\varepsilon}_{ct}$ is an N vector of zero conditional mean idiosyncratic or residual disturbance terms. The information set I_{t-1} reflects unspecified information available to investors and the subscript 'c' refers to the assumption that the factor structure is conditioned on this information.[1] It is always possible to decompose security returns in this fashion because there are no assumptions whatsoever about market prices implicit in this description save for the existence of the relevant moments.

The only assumption about market prices made in the APT is that the common factors are the dominant source of covariation among these security returns. This places restrictions on both the factors and the idiosyncratic risks. The factors must be pervasive and affect 'most' security returns. A sufficient condition for this to occur is:

$$\lim_{N\to\infty} \xi_{min}(B_{ct}'B_{ct}) = \infty \ \forall\ t \qquad (2)$$

where $\xi_{min}(\bullet)$ is the smallest eigenvalue of its argument.[2] If some such condition were not met, idiosyncratic risk would be incorrectly classified as common factor risk which, in turn, would change the details of some of the arguments presented below.

In addition, the theory requires that diversification eliminates idiosyncratic or residual risk from the returns of large well-diversified portfolios. This amounts to assuming that the idiosyncratic disturbances satisfy a weak law of large numbers. The version of the law of large numbers employed here, introduced by Chamberlain and Rothschild(1983), concerns the conditional covariance matrix of the idiosyncratic disturbances:

$$\Omega_{ct} = E\left[\varepsilon_{ct}\varepsilon_{ct}'/I_{t-1}\right]; \qquad \lim_{N\to\infty} \xi_{max}(\Omega_{ct}) < \infty \ \forall\ t \qquad (3)$$

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[1]This information set need not be possessed by any investor and I will not characterize the nature or distribution of information across investors. What is important here is the implicit restriction that investors agree on the linear dependence of the returns of well-diversified portfolios when security returns follow a factor structure. The restriction that investors agree on the behavior of zero variance portfolios is analogous to the restriction that investors must agree on both variances and the relations between derivative securities and their underlying assets in continuous time diffusion models.

[2]Note that this limit is taken by adding rows to B_{ct} as N grows without bound.

where $\xi_{max}(\bullet)$ is the largest eigenvalue of its argument.[3] This condition is sufficient because the idiosyncratic variance of a well-diversified portfolio $\underline{w}_p \in I_{t-1}$ converges to zero since:

$$\sigma_p^2 = \underline{w}_p'\Omega_{ct}\underline{w}_p \leq \underline{w}_p'\underline{w}_p \, \xi_{max}(\Omega_{ct}) \rightarrow 0 \text{ a.s.} \tag{4}$$

because the portfolio weights are of order $1/N$ (i.e., $\underline{w}_p'\underline{w}_p \rightarrow 0$ as $N \rightarrow \infty$).

This factor model places obvious restrictions on the pricing of limiting well-diversified portfolios. Any K_c+1 well-diversified portfolios have linearly dependent returns in the limit since they have no exposure to idiosyncratic risk. Accordingly, their expected returns must be linearly dependent in the absence of taxes, transactions costs, or other impediments to trade if there are no arbitrage opportunities in the marketplace. Hence, the limiting expected return of a well-diversified portfolio $\underline{w}_p \in I_{t-1}$ is:

$$\lim_{N \rightarrow \infty} E_{pt} = \lambda_{0t} + \underline{b}_{pct}'\underline{\lambda}_t; \quad \underline{b}_{pct} = B_{ct}'\underline{w}_p \tag{5}$$

for some constants λ_{0t} and $\underline{\lambda}_t$ that are the same for all assets. Assets with identical factor loadings are perfect substitutes in well-diversified portfolios.

How does the intuition about the valuation of well-diversified portfolios codified in (5) translate to individual security valuation? Consider the weighted least squares projection of conditional expected returns on a vector of ones $\underline{\iota}$ and the conditional factor loadings:

$$\underline{E}_{ct} = \underline{\iota}\lambda_{0t} + B_{ct}\underline{\lambda}_t + \underline{\alpha}_{ct} \tag{6}$$

using the inverse of the conditional covariance matrix of the idiosyncratic disturbances Ω_{ct} as weights. Note that the parameters $\underline{\lambda}_t$ have two natural interpretations—they are both cross-sectional regression coefficients and the conditional expected excess returns of limiting portfolios that are perfectly correlated

[3] Once again, this limit is taken by adding N covariances and a variance as each additional security is added.

with the common factors. Similarly, the pricing intercept λ_{0t} is either the riskless rate or the conditional expected return of a conditional zero factor beta portfolio.

The pricing implications of this dynamic factor pricing model for individual securities can be determined by the standard method introduced in Huberman(1982). Consider the costless (i.e., zero net investment) portfolio with weights:

$$\underline{w}_{ct} = \left(\underline{\alpha}_{ct}'\Omega_{ct}^{-1}\underline{\alpha}_{ct}\right)^{-1}\underline{\alpha}_{ct}'\Omega_{ct}^{-1} \tag{7}$$

The conditional mean profit of this portfolio is:

$$\begin{aligned} E\left[\underline{w}_{ct}'\underline{R}_t/I_{t-1}\right] &= \left(\underline{\alpha}_{ct}'\Omega_{ct}^{-1}\underline{\alpha}_{ct}\right)^{-1}\underline{\alpha}_{ct}'\Omega_{ct}^{-1}\left(\iota\lambda_{0t} + B_{ct}\underline{\lambda}_t + \underline{\alpha}_{ct}\right) \\ &= 1 \end{aligned} \tag{8}$$

since this is a zero net investment portfolio with no factor risk exposure (i.e., the weighted residuals $\Omega_{ct}^{-1}\underline{\alpha}_{ct}$ from the cross-sectional regression (6) are orthogonal to the factor loadings and a vector of ones). Similarly, the conditional variance of this arbitrage portfolio's payoff is:

$$\begin{aligned} V\left[\underline{w}_{ct}'\underline{R}_t/I_{t-1}\right] &= \left(\underline{\alpha}_{ct}'\Omega_{ct}^{-1}\underline{\alpha}_{ct}\right)^{-1}\underline{\alpha}_{ct}'\Omega_{ct}^{-1}V\left(\underline{R}_t/I_{t-1}\right)\Omega_{ct}^{-1}\underline{\alpha}_{ct}\left(\underline{\alpha}_{ct}'\Omega_{ct}^{-1}\underline{\alpha}_{ct}\right)^{-1} \\ &= \left(\underline{\alpha}_{ct}'\Omega_{ct}^{-1}\underline{\alpha}_{ct}\right)^{-1}\underline{\alpha}_{ct}'\Omega_{ct}^{-1}\Omega_{ct}\Omega_{ct}^{-1}\underline{\alpha}_{ct}\left(\underline{\alpha}_{ct}'\Omega_{ct}^{-1}\underline{\alpha}_{ct}\right)^{-1} \\ &= \left(\underline{\alpha}_{ct}'\Omega_{ct}^{-1}\underline{\alpha}_{ct}\right)^{-1} \end{aligned} \tag{9}$$

By the usual **APT** pricing argument, the weighted sum of squared conditional pricing errors is bounded:

$$\lim_{N\to\infty} \underline{\alpha}_{ct}'\Omega_{ct}^{-1}\underline{\alpha}_{ct} < \infty \tag{10}$$

in the absence of riskless arbitrage opportunities if there are no taxes, transactions costs, or other market frictions. If the weighted sum of squared conditional pricing errors was unbounded, the arbitrage portfolio would pay off a dollar for certain as the number of assets grew without bound. Note that the sum of squared conditional pricing errors is also bounded since $\underline{\alpha}_{ct}'\Omega_{ct}^{-1}\underline{\alpha}_{ct} \geq \underline{\alpha}_{ct}'\underline{\alpha}_{ct}/\xi_{min}(\Omega_{ct})$. Assets with identical factor loadings are close substitutes in arbitrary portfolios in this approximate sense.[4]

This restriction on conditional expected returns has an interesting implication for the time series properties of security returns. Recall that the parameter λ_{0t} in the pricing relation is either the riskless rate or the conditional expected return of a conditional zero factor beta portfolio and $\underline{\lambda}_t$ is the vector of conditional expected excess returns of limiting portfolios that are perfectly correlated with the common factors. Accordingly, the residual risks $\underline{\varepsilon}_{ct}$ have the natural interpretation as the unexpected returns of synthetic securities with conditional mean return vector $\underline{\alpha}_{ct}$ and conditional covariance matrix Ω_{ct}. These synthetic securities are constructed by going long the original N securities, short $1-\underline{b}_{ict}'\iota$ of the riskless or zero beta security (where $\underline{b}_{ict}$ is a vector consisting of the K factor loadings of security i with elements b_{ick}), and short b_{ick} dollars of each of the portfolios with returns perfectly correlated with the common factors.

This identification of the pricing errors $\underline{\alpha}_{ct}$ with the predictable component of idiosyncratic risk restricts the time series behavior of security returns. Since the sum of squared pricing errors is bounded, most of the elements of $\underline{\alpha}_{ct}$ are negligible as the number of securities grows without bound. Accordingly, this implies that the idiosyncratic risk process is unpredictable for most assets as well. This implication is analogous to the condition that the unconditional mean of the idiosyncratic risk process is approximately zero for most securities in static factor pricing models. Put differently, the serial correlation in most security returns must be mediated through the common factors and, hence, through λ_{0t} and $\underline{\lambda}_t$ in a factor pricing world.

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[4]While the elements of $\underline{\alpha}_{ct}$ are approximately zero for most securities, we often assume that they are equal to zero for all securities so that the APT is exactly true for individual securities. This involves a stronger theory—that investors treat securities with identical factor loadings as perfect substitutes in their portfolios. In existing equilibrium factor pricing models, this typically requires that investors perceive that idiosyncratic risk is uncorrelated with their marginal utility of wealth.

2. Conditional and Unconditional Factor Structures

The previous section studied factor pricing models when security returns followed a factor structure conditional on some unspecified information set I_{t-1}. The analysis showed that the absence of arbitrage opportunities placed restrictions on the time series properties of the idiosyncratic risk process. This section further explores the time series implications of factor models for security returns by distinguishing conditional from unconditional factor structures.

An unconditional factor structure arises when the unconditional covariance matrix of security returns follows a factor structure. This occurs when:

$$V = BB' + \Omega; \qquad \lim_{N\to\infty} \xi_{max}(\Omega) < \infty \tag{11}$$

where V is the unconditional covariance matrix of security returns, B is the NxK matrix of unconditional factor loadings, and Ω is the unconditional idiosyncratic risk covariance matrix. In these circumstances, security returns have the representation:

$$\underline{R}_t = \underline{E} + B\underline{\delta}_t + \underline{\varepsilon}_t; \qquad E[\underline{\delta}_t] = E[\underline{\varepsilon}_t] = E[\underline{\delta}_t\underline{\varepsilon}_t'] = 0;\ E[\underline{\varepsilon}_t\underline{\varepsilon}_t'] = \Omega \tag{12}$$

where $\underline{E}$ is the N vector of unconditional expected returns.[5] Note that the common factors $\underline{\delta}_t$ (which number K with K usually larger than K_c) and the residuals $\underline{\varepsilon}_t$ are generally different from those that appeared in the previous section.[6]

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[5]The unconditional factor model does not restrict idiosyncratic risk to be unpredictable. The idiosyncratic disturbances of the individual firms must be only weakly correlated at all leads and lags but the serial correlations of the disturbances themselves are unrestricted so long as their unconditional variances exist.

[6]For example, suppose that security returns follow the conditional one factor structure:

$$\underline{R}_t = \underline{E}_t + \underline{b}_t\delta_t + \underline{\varepsilon}_t$$

and the factor loading vector follows the single factor model:

$$\underline{b}_t = \underline{b} + \underline{\beta}\eta_t + \underline{\nu}_t; \qquad \eta_t \in I_{t-1}$$

with $\underline{\nu}_t$ diversifiable. Then returns follow the unconditional two factor model:

$$\underline{R}_t = \underline{E} + \underline{b}\delta_t + \underline{\beta}\eta_t\delta_t + \underline{\zeta}_t$$

where $\underline{E}$ is the N vector of unconditional mean security returns and the disturbance

What is the link between the unconditional factor structure (12) and the conditional factor structure (1)? This can be seen by examining the conditional projection of returns on the common factors:

$$\underline{R}_t = \underline{E}_t + B_t\left[\underline{\delta}_t - E(\underline{\delta}_t/I_{t-1})\right] + \underline{\eta}_t; \tag{13}$$

where the residuals $\underline{\eta}_t$ involve the idiosyncratic risks $\underline{\varepsilon}_t$, their conditional expectations, and the conditional factor loading matrix B_t.

Algebraic manipulation of the unconditional and conditional representations (12) and (13) restricts the time series behavior of the conditional factor loading matrix B_t in an obvious way.[7] Comparing the conditional expectation of (12) with the conditional projection (13) yields:

$$\begin{aligned} B_t &= \mathrm{cov}\left[\underline{R}_t, \underline{\delta}_t/I_{t-1}\right] V\left[\underline{\delta}_t/I_{t-1}\right]^{-1} \\ &= \mathrm{cov}\left[B\left[\underline{\delta}_t - E(\underline{\delta}_t/I_{t-1})\right] + \underline{\varepsilon}_t - E(\underline{\varepsilon}_t/I_{t-1}), \underline{\delta}_t/I_{t-1}\right] V\left[\underline{\delta}_t/I_{t-1}\right]^{-1} \\ &= B + \mathrm{cov}\left[\underline{\varepsilon}_t, \underline{\delta}_t/I_{t-1}\right] V\left[\underline{\delta}_t/I_{t-1}\right]^{-1} \end{aligned} \tag{14}$$

so that the residuals $\underline{\eta}_t$ from the conditional projection (13) are given by:

$$\begin{aligned} \underline{\eta}_t &= \underline{\varepsilon}_t - E\left[\underline{\varepsilon}_t/I_{t-1}\right] - \mathrm{cov}\left[\underline{\varepsilon}_t, \underline{\delta}_t/I_{t-1}\right] V\left[\underline{\delta}_t/I_{t-1}\right]^{-1}\left[\underline{\delta}_t - E(\underline{\delta}_t/I_{t-1})\right] \\ &= \underline{\varepsilon}_t - E\left[\underline{\varepsilon}_t/I_{t-1}\right] - (B_t - B)\left[\underline{\delta}_t - E(\underline{\delta}_t/I_{t-1})\right] \end{aligned} \tag{15}$$

Now consider the largest eigenvalue of the conditional covariance matrix of the difference between the conditional and unconditional factor loadings B_t-B:

$$\begin{aligned} \xi_{max}\left[V(B_t - B/I_{t-1})\right] &= \xi_{max}\left[\mathrm{cov}(\underline{\varepsilon}_t, \underline{\delta}_t/I_{t-1}) V(\underline{\delta}_t/I_{t-1})^{-2} \mathrm{cov}(\underline{\delta}_t, \underline{\varepsilon}_t/I_{t-1})\right] \\ &\leq \xi_{max}\left[V(\underline{\varepsilon}_t/I_{t-1})\right] \\ &\leq \xi_{max}(\Omega) < \infty \end{aligned} \tag{16}$$

and, hence:

$\underline{\zeta}_t$ include the diversifiable risks $\underline{\varepsilon}_t$, $\underline{\upsilon}_t\delta_t$, and $\underline{E}_t - \iota\lambda_{0t} - \underline{b}_t\lambda_t - \underline{E}$.

[7]While it may appear that the conditional factor loading matrix B_t must be constant by taking conditional expectations of (12), this conclusion does not follow directly since $\mathrm{cov}[\underline{\varepsilon}_t, \underline{\delta}_t/I_{t-1}]$ is generally nonzero without additional restrictions on (12).

$$\lim_{N \to \infty} \xi_{max}\left[\operatorname{cov}\left(\varepsilon_t, \delta_t / I_{t-1}\right)\operatorname{cov}\left(\delta_t, \varepsilon_t / I_{t-1}\right)\right] < \infty \tag{17}$$

As a consequence, conditional factor loadings are approximately constant since:

$$\begin{aligned} B_t &= B + \operatorname{cov}\left[\varepsilon_t, \delta_t / I_{t-1}\right] V\left[\delta_t / I_{t-1}\right]^{-1} \\ &\approx B \end{aligned} \tag{18}$$

This fact is of some relevance to empirical applications such as Gibbons and Ferson(1985), Campbell(1987), Chan(1988), Campbell and Hamao(1989), Ferson(1989,1990), and Ferson, Foerster, and Keim(1990). These authors follow the common practice of asserting both that some collection of unobserved factors determine conditional expected equity returns and that conditional factor loadings are constant. These results show that one distributional assumption that is compatible with both assertions is that security returns follow both conditional and unconditional factor structures and exact factor pricing obtains.

Hence, an additional set of restrictions on the time series properties of returns arises from the assumption that returns possess an unconditional factor structure. When a large number of returns possess a conditional factor structure, the common factors must be the dominant source of both contemporaneous and serial correlation in asset returns. When this same collection of returns follows an unconditional factor structure, these same implications follow with one addition—the coefficients from the conditional regression of returns on the common factors are roughly constant. This purely technical consequence of the unconditional factor structure assumption simplifies the problem of estimating dynamic factor models, a subject dealt with in the next section.

3. Econometric Issues

The previous sections have emphasized the restrictions on the time series properties of returns that follow from the pricing restrictions in Section 1 and from the unconditional factor structure studied in Section 2. These restrictions are that the common factors are the dominant source of serial and contemporaneous correlation among security returns (or, equivalently, that idiosyncratic risk is unpredictable) and that conditional factor loadings are approximately constant. Both sorts of restrictions are useful in the estimation of dynamic factor models.

The econometric identification issues in dynamic factor pricing models differ somewhat from those in the static setting. Under the joint null hypothesis that a conditional factor model exactly prices securities (i.e., $\underline{\alpha}_{ct}=\underline{0}\ \forall\ t$) and returns follow an unconditional factor structure, security returns satisfy the projection equation:

$$\underline{R}_t = P\left(\underline{R}_t/\underline{R}_s,\ s<t\right) + B\left[\underline{\delta}_t - P\left(\underline{\delta}_t/\underline{R}_s,\ s<t\right)\right] + \underline{\nu}_t \tag{19}$$

$$= B\underline{\delta}_t + \underline{\nu}_t; \qquad \underline{\delta}_t = P\left(\underline{\delta}_t/\underline{R}_s,\ s<t\right) + \underline{\psi}_t; \qquad \underline{\psi}_t = \left[\underline{\delta}_t - P\left(\underline{\delta}_t/\underline{R}_s,\ s<t\right)\right]$$

where $P(\bullet/\bullet)$ denotes the projection operator and the residual $\underline{\nu}_t$ is a vector of diversifiable random variables that differ from $\underline{\eta}_t$ in (15) because unconditional and conditional projections differ. The composite residual $B\underline{\psi}_t+\underline{\nu}_t$ represents the innovations to security returns while the projection of returns on lagged returns depends on the factor risk premiums, the factor loadings, and the postulated time series model for the factors as indicated above.

As in standard factor analytic applications, the factors are identified up to an arbitrary linear transformation or rotation Θ_t so long as idiosyncratic risk is diversifiable.[8] Hence, a given factor loading matrix B and realization of the vector of common factors $\underline{\delta}_t$ is observationally equivalent to the alternative pair B^*_t and $\underline{\delta}^*_t$ where $B^* = B\Theta_t$ and $\underline{\delta}^*_t = \Theta_t^{-1}\underline{\delta}_t$ in the absence of *a priori* restrictions on the factor loading matrix or the unobservable factor process where Θ_t is any KxK nonsingular matrix. The transformation matrix Θ_t is generally time varying in the absence of

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[8]As is common in factor models, this implies that we are confining attention to first and second moments in attempting to achieve identification. As is well-known, nonnormality of the common factors is typically sufficient to achieve identification. See Lehmann(1983) for additional references and analysis.

additional restrictions. The dynamic factor structure places similar restrictions on the time series model for the common factors in that $P(\underline{\delta}_t^*/\underline{R}_s, s<t) = \Theta_t^{-1}P(\underline{\delta}_t/\underline{R}_s, s<t)$ and $V(\underline{\psi}_t^*/\underline{R}_s, s<t) = \Theta_t^{-1}V(\underline{\psi}_t/\underline{R}_s, s<t)\Theta_t^{-T}$ where the exponent T denotes transpose.

The analysis is simplified when the projection model for the common factors and the corresponding innovations covariance matrix have constant coefficients.[9] In these circumstances, the transformation matrix is $\Theta_t = \Theta$, a constant for all t. Accordingly, identification can be achieved by normalizing the common factors so that their innovations covariance matrix $V(\underline{\psi}_t/\underline{R}_s, s<t)$ is the identity matrix. This is analogous to the conventional practice in the static factor analysis model of normalizing the common factors to be unconditionally uncorrelated random variables with unit unconditional variances. Setting the conditional variations to unity simply scales the common factors and setting their conditional correlations to zero solves the problem of distinguishing correlation between returns and the common factors from conditional correlation among the common factors.[10]

Any statistical procedure for estimating and testing factor pricing models implicitly involves the measurement of the underlying common factors. In static factor pricing models, the choice of procedures for measuring the common factors is relatively unimportant in large cross-sections—any K imperfectly correlated, well-diversified portfolios will do. However, this intuition fails in general dynamic settings for an obvious reason: arbitrary fixed weight, well-diversified portfolios have time-varying conditional factor loadings when returns follow a conditional factor structure. The returns of such portfolios confound the effects of the time-

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[9]Alternatively, this can be viewed as identification in models that assume constant coefficients even when this parameterization is false.

[10]Other restrictions can be used to achieve identification. These include restrictions on the factor loading matrix and the time series factor model implicit in the projection of the common factors on lagged returns. Once again, see Lehmann(1983) for additional references and analysis.

varying factor loadings and the factors themselves and, hence, are not suitable for estimation and inference in conditional factor pricing models.[11]

This complication vanishes when security returns possess an unconditional factor structure. In these circumstances, conditional factor loadings are approximately constant so that well-diversified, fixed weight portfolios possess constant conditional factor loadings as the number of securities grows without bound. Accordingly, standard statistical procedures for estimating the common factors employed in static factor pricing applications translate to the dynamic setting under these conditions.[12] These procedures will produce consistent estimates of some rotation of the common factors and unbiased estimates of the factor loadings as the number of assets grow without bound.

Estimates of the common factors and factor loadings obtained from the unconditional covariance matrix of security returns ignore the information about the common factors contained in the serial correlations of security returns. Employing such information greatly complicates the estimation problem here because the covariance and autocovariance matrices of security returns are nonlinear functions of the factor loadings and the time series model for the factors. Fortunately, these nonlinearities are simplified considerably in large cross-sections.

To make matters concrete, consider a quasi-likelihood approach where the

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[11]Consider the arbitrary well-diversified, fixed weight portfolio $\underline{w}_p$. The return of this portfolio is given by:

$$R_{pt} = \underline{w}_p'\left(\underline{E}_t + B_t\underline{\delta}_t + \underline{\varepsilon}_t\right) = E_{pt} + \underline{b}_{pt}'\underline{\delta}_t$$

and the covariances of returns with R_{pt} convolve the factor risk $\underline{\delta}_t$ with the covariance risk $\underline{b}_{pt}$. Put differently, a well-diversified portfolio with a beta of one on one factor and zero on the others would have time-varying weights.

[12]Examples of such procedures include the maximum likelihood factor analysis approach used in Lehmann and Modest(1988), the instrumental variables procedure employed in Chen(1983) and Lehmann and Modest(1985,1987), and the principal components method advocated in Chamberlain and Rothschild(1983) and Connor and Korajczyk(1988).

innovations to security returns in (19) are treated as normally and independently distributed random variables.[13] If the covariance matrix of the innovations is denoted as Σ, the quasi-log-likelihood function is given by:

$$\mathcal{L}\left(\underline{R}_t,\ t\leq T;\ \underline{\alpha}_0\right) = \frac{-T}{2}\ln|\Sigma| - \frac{1}{2}\sum_{t=1}^{T} \underline{e}_t'\Sigma^{-1}\underline{e}_t; \qquad \underline{e}_t = B\underline{\psi}_t + \underline{\upsilon}_t \tag{20}$$

$$\Sigma = BV(\underline{\psi}_t)B' + V(\underline{\upsilon}_t)$$

where $\underline{\alpha}_0$ is the unknown parameter vector.[14] The derivatives of (20) are:

$$\frac{\partial\mathcal{L}\left(\underline{R}_t,\ t\leq T;\ \underline{\alpha}\right)}{\partial\alpha_i} = \sum_{t=1}^{T} \frac{\partial\underline{e}_t'}{\partial\alpha_i}\Sigma^{-1}\underline{e}_t + \frac{\partial\mathrm{vec}(\Sigma)'}{\partial\alpha_i}\left(\Sigma^{-1}\otimes\Sigma^{-1}\right)\mathrm{vec}\left[\underline{e}_t\underline{e}_t'-\Sigma\right] \tag{21}$$

where $\mathrm{vec}(\bullet)$ denotes vectorization of a matrix by columns and $\otimes$ is the Kronecker product operator.

This is a complicated nonlinear estimation since both $\underline{e}_t$ and Σ depend on the underlying parameters in a nonlinear manner. This arises because the factors are not observed but are instead inferred from the time series and covariance behavior of returns. Fortunately, the nonlinearity vanishes in large cross-sections since $\underline{\psi}_t$, the innovation to the unobserved common factors, is observed in the limit as the

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[13]The prefix 'quasi' implies that the innovations need not satisfy this requirement. The basic strategy described below will apply to other generalized method of moments estimators.

[14]*A priori* restrictions must be placed on the idiosyncratic risk covariance matrix $V(\underline{\upsilon}_t)$ when the number of securities N is larger than the number of time series observations T. Setting $V(\underline{\upsilon}_t) = \sigma_\upsilon^2 I_N$ yields a dynamic analogue of the principal components methods of Chamberlain and Rothschild(1983) and Connor and Korajczyk(1988). Requiring $V(\underline{\upsilon}_t)$ to be diagonal yields a dynamic analogue of maximum likelihood factor analysis as used by Roll and Ross(1980) and Lehmann and Modest(1988). Note that there is another major difference between the static and dynamic settings—$V(\underline{\upsilon}_t)$ can be unrestricted so long as the values taken on by $P\left(\underline{R}_t/\underline{R}_s,\ s<t\right)$ vary over time whereas it must be diagonal to proceed with estimation in static (i.e., $P\left(\underline{R}_t/\underline{R}_s,\ s<t\right)$ constant) models.

number of securities grows without bound. This eliminates the need to compute the derivatives of Σ with respect to the factor innovations covariance matrix $V(\underline{\psi}_t)$ since its maximum likelihood estimate is simply its sample covariance matrix (unless it is constrained to be the identity matrix to achieve identification). It also simplifies the derivative calculations since:

$$\lim_{N\to\infty} \frac{\partial \underline{e}_t}{\partial \alpha_i} = B\frac{\partial P(\underline{\delta}_t/\underline{R}_s, s<t)}{\partial \alpha_i} + \frac{\partial B}{\partial \alpha_i}\left[P(\underline{\delta}_t/\underline{R}_s, s<t)+\underline{\psi}_t\right] \tag{22}$$

$$= B\frac{\partial P(\underline{\delta}_t/\underline{R}_s, s<t)}{\partial \alpha_i} + \frac{\partial B}{\partial \alpha_i}\underline{\delta}_t$$

because of the limiting observability $\underline{\delta}_t$ and $\frac{\partial vec(\Sigma)'}{\partial \alpha_i}$ can be replaced by $\frac{\partial vec(V(\underline{\psi}_t))'}{\partial \alpha_i}$ in (21) because of the large cross-section simplification of the derivatives in (22).

Accordingly, the estimation of dynamic factor pricing models is a little more complicated than for static models but is simplified considerably by three considerations. The first is the pricing restriction that idiosyncratic risk is approximately unpredictable while the second is the unconditional factor model constraint that conditional factor loadings are approximately constant (with the approximations treated as though they hold exactly). The third simplification is in the derivatives (21) in large cross-sections arising from the limiting observability of the common factors. Given these considerations, the model can be estimated by maximizing the quasi-log-likelihood function (20) by setting the derivatives (21) to zero given the large cross-section simplification (22). Initial consistent estimates of the parameters can be obtained by treating the preliminary estimates of the common factors obtained from some procedure for estimating static factor pricing models as measured without error.

Tests of dynamic factor pricing models can then proceed in a straightforward fashion. For example, the derivatives for the predictability of idiosyncratic risk implicit in the derivatives of $P(\underline{R}_t/\underline{R}_s, s<t)$ can be tested for their proximity to zero

with a Lagrange Multiplier test. Alternatively, the model can be estimated under the assumption that idiosyncratic risk is predictable (i.e., by adding the relevant parameters and setting the associated derivatives of $P(\underline{R}_t/\underline{R}_s, s<t)$ to zero) and the hypothesis that idiosyncratic risk is unpredictable can be tested using a Wald test.[15] Both tests can be made asymptotically (for large T) robust to the approximations inherent in this procedure by using a robust asymptotic covariance matrix.[16]

4. Conclusion

For much of the past twenty years, financial economists have sought to simplify the development of testable asset pricing relations by making distributional assumptions about security returns. The assumption that large cross-sections of security returns follow a linear factor structure has generated considerable interest

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[15]This is not technically feasible in large cross-sections where the sample asymptotic covariance matrices are typically singular when there are more securities than time series observations. In the Wald and Lagrange Multiplier frameworks, this defect can be remedied by testing combinations and subsets of these moment conditions.

[16]That is, inference can proceed be examining the (inverted) standard expansion of the average score:

$$\sqrt{T}\left(\hat{\underline{\alpha}}-\underline{\alpha}_0\right) \xrightarrow{w.p.1} \left[\frac{-1}{T}\,\frac{\partial^2 \mathcal{L}\left(\bullet;\, \tau\hat{\underline{\alpha}}+(1-\tau)\underline{\alpha}_0\right)}{\partial\underline{\alpha}\,\partial\underline{\alpha}'}\right]^{-1} \frac{1}{\sqrt{T}}\,\frac{\partial \mathcal{L}(\bullet;\underline{\alpha}_0)}{\partial\underline{\alpha}}$$

A law of large numbers will apply to the elements of $T^{-1/2}\,\frac{\partial \mathcal{L}(\bullet;\underline{\alpha}_0)}{\partial\underline{\alpha}}$ if the projection $P(\underline{R}_t/\underline{R}_s, s<t)$ is correctly specified (or the projection errors are not too serially dependent), implying:

$$\sqrt{T}\left(\hat{\underline{\alpha}}-\underline{\alpha}_0\right) \xrightarrow{\mathcal{D}} N\left[\underline{0}, \frac{1}{T}\,\mathcal{L}''(\bullet;\hat{\underline{\alpha}})^{-1}\mathcal{S}(\bullet;\hat{\underline{\alpha}})\mathcal{L}''(\bullet;\hat{\underline{\alpha}})^{-1}\right];$$

$$\mathcal{S}(\bullet;\hat{\underline{\alpha}}) = E\left\{\frac{\partial \mathcal{L}(\bullet;\hat{\underline{\alpha}})}{\partial\underline{\alpha}}\,\frac{\partial \mathcal{L}(\bullet;\hat{\underline{\alpha}})}{\partial\underline{\alpha}'}\right\};\quad \mathcal{L}''(\bullet;\hat{\underline{\alpha}}) = \frac{\partial^2 \mathcal{L}(\bullet;\hat{\underline{\alpha}})}{\partial\underline{\alpha}\,\partial\underline{\alpha}'}$$

and a formulation of the robust asymptotic covariance matrix as $\hat{\underline{\alpha}}$ converges to $\underline{\alpha}_0$. If the innovations to security returns are normally and independently distributed random variables with mean zero and covariance matrix Σ, this calculation is simplified since $\mathcal{L}''(\bullet;\underline{\alpha}_0)^{-1}\mathcal{S}(\bullet;\underline{\alpha}_0)$ converges to an identity matrix.

in the resulting approximate linear asset pricing model, the **APT**. This model is distinguished by a reliance on the assumed absence of arbitrage opportunities and *a priori* distributional assumptions about relations among security returns in contrast to the equilibrium analysis in simple static models like the Capital Asset Pricing Model.[17]

Much recent research has been devoted to dynamic equilibrium asset pricing relations and it seems reasonable to extend factor pricing models to a dynamic setting as well. These notes provide the main ingredients of such an extension, some aspects of which have been implicitly or explicitly discussed by others. The natural questions that arise in dynamic factor pricing models involve the pricing implications and time series restrictions associated with the distinction between conditional and unconditional factor models.

The answers are simple. The main pricing implication is that diversifiable idiosyncratic risk is approximately unpredictable as the number of securities grows without bound in a no-arbitrage world. Put differently, the common factors, which are assumed to be the dominant source of conditional correlation among security returns, must be the dominant source of serial correlation as well in a factor pricing world. An additional time series restriction arises when returns follow a factor structure both unconditionally and conditional on some nontrivial information set. In these circumstances, conditional factor loadings are approximately constant and, hence, conditional expected returns are approximately linear in the approximately

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[17]As is obvious, this distinction is somewhat artificial in that the **CAPM** is often developed under the assumption of normally distributed discrete time or continuous time (i.e., diffusion) returns with constant means, variances, and covariances. Nevertheless, no-arbitrage reasoning typically requires weaker assumptions than equilibrium models and implies weaker predictions about expected returns. In particular, **APT** models do not predict the values of the factor risk premiums while equilibrium models typically do.

constant conditional factor betas. It is commonplace for empirical researchers to assume that such relations hold exactly for some unspecified set of factors so that this analysis provides a convenient framework for interpreting such assumptions.

The final observations contained in these notes concern the general econometrics of dynamic factor pricing models. Econometric identification is similar in both the dynamic and static settings. It is natural to assume that common factors have unit unconditional variances and zero unconditional correlations in the static setting. The analogous requirement in dynamic models is that these conditions hold for the corresponding conditional factor variances and covariances. Estimation issues are similar as well—estimation in both static and dynamic models is simplified by the limiting observability of the common factors as the number of securities grows without bound.

These notes help operationalize the development of dynamic factor pricing models that parallel the kinds of intertemporal asset pricing models that currently occupy much research attention. The costs and benefits of constructing dynamic factor pricing models are similar to the static case. Weak distributional assumptions lead to approximate restrictions on the time series behavior of security returns. The costs of the approach are that the predictions are approximate and not exact and that the distributional assumptions may not be compatible with equilibrium asset prices. The benefits are that these distributional assumptions substitute for the measurement of state variables like aggregate wealth that are difficult to measure in practice and that no-arbitrage is a weaker requirement than full dynamic equilibrium. Hence, dynamic factor pricing models provide a complementary framework for studying intertemporal asset pricing relations.

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