NBER WORKING PAPER SERIES

THE RISK-FREE RATE AND THE RISK-ADJUSTED GROWTH RATE

Stavros Panageas

Working Paper 35260
http://www.nber.org/papers/w35260

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
May 2026

I thank Ravi Bansal, Lars Peter Hansen, Lorenzo Garlappi, Nicolae Garleanu, Francois Gourio, Martin Lettau, Erik Loualiche, Rodolfo Prieto, Dejanir Silva, Mindy Xiaolan and conference and seminar participants at Arizona State University, Cleveland Fed, City University of Hong Kong, Colorado Finance Summit, Hong Kong University, Hong Kong University of Science and Technology, Insead, London Business School, Macro-Finance Society, Oslo Center for Advanced Studies, SFS Cavalcade, University of British Columbia, University of California Los Angeles, Utah Winter Finance Conference, and Western Finance Association for useful comments. A previous version of the paper was circulated under the title ``Growth-Indexed Securities''. The views expressed herein are those of the author and do not necessarily reflect the views of the National Bureau of Economic Research.

The Risk-free Rate and the Risk-adjusted Growth Rate
Stavros Panageas
NBER Working Paper No. 35260
May 2026
JEL No. E60, G1, G12, H60

ABSTRACT

Several important questions — such as the possibility of debt-rollover without primary surpluses — turn on whether the present value of the aggregate endowment is finite, i.e., whether the economic growth rate under the "risk-neutral" measure, lies below the risk-free rate. It is tempting to argue that the endowment must be finitely valued, since there exist finitely-valued, non-depreciating assets whose cash flows are cointegrated with aggregate output. This paper shows why this argument is incorrect. A remarkable historical episode in which French government bonds were indexed to aggregate growth allows direct measurement of the risk-adjusted growth rate, which is found to exceed the risk-free rate.

Stavros Panageas
University of California, Los Angeles
Anderson School of Management
and NBER
stavros.panageas@anderson.ucla.edu

1 Introduction

The persistent deficits of the last decade have rekindled interest in some of the oldest questions in public finance. Is it possible to roll over government debt without primary surpluses? More broadly, is it possible for some assets (such as government bonds) to have a price higher than the present discounted value of their dividends (primary surpluses in the case of government bonds)?

As a theoretical matter, it is well understood that such outcomes are possible in deterministic, overlapping generations economies, in which the interest rate, r, is below the aggregate growth rate, g, along a balanced growth path. In such economies the condition $r - g < 0$ is equivalent to an infinite present discounted value of the aggregate endowment; this is simply an implication of the Gordon growth formula. As it turns out, the link between the finiteness of the present value of the aggregate endowment and the possibility of bubbles is broader and extends to stochastic economies.[1] Specifically, if the present value of the aggregate endowment is finite, then there can be no bubbles on any positive-supply asset.[2]

At first glance, it seems obvious that the present value of the aggregate endowment ought to be finite, because there exist finitely-valued, non-depreciating assets, such as land, with dividends that appear cointegrated with the aggregate endowment.[3] Because of cointegration, one might think that the finite value of the non-depreciating asset implies a finite value for the aggregate endowment. Appealing as it may be, this argument is incorrect. This paper shows that a cash-flow process C_t may be cointegrated with a cash-flow process X_t, and yet the present value of C_t is infinite, while the present value of X_t is finite. More broadly, if two

[1] In representative-agent models, the transversality condition of the representative agent typically ensures the finiteness of the value of the aggregate endowment. However, this finiteness is not guaranteed in overlapping generations economies.

[2] A seminal reference is Santos and Woodford (1997), which extends results of earlier papers (e.g., Scheinkman (1977) and Wilson (1981)) to a broader set of economies, including economies with incomplete markets.

[3] For this argument it is important that the asset be non-depreciating. If an asset depreciates over time, then it is possible that the asset is finitely-valued, while the value of the aggregate endowment is infinite. For example, in textbook overlapping generations models the capital stock depreciates at the rate δ and the production function is Cobb-Douglas in capital and labor. In this economy, the price of capital is the present value of profits discounted at the rate $r + \delta$, not r. Therefore, it may well be the case that $r + \delta > g > r$ so that the value of capital is finite (profits grow at the rate g, but are discounted at the rate $r + \delta > g$) and yet the present value of the aggregate goods produced is infinite ($r < g$).

cash flow processes are cointegrated, the yields on their long-term strips need not be equal.

This result shows that one cannot rely on some simple and broad theoretical argument, such as the finiteness of the value of land, to exclude bubbles. If a broad theoretical argument is unavailable, is there at least a simple way to gauge whether the finite-aggregate-endowment assumption is plausible?

The ideal dataset to answer this question would consist of yields derived from aggregate-endowment strips, that is, securities whose payoff at time T is indexed to the aggregate endowment at time T. If the yields of such strips were positive, this would indicate that, under the risk-neutral measure the (risk-adjusted) growth rate of the economy is below the discount rate. This would provide some reassurance that the present value of the aggregate endowment is finite. While growth-indexed securities were never traded in the US, the paper studies a historical episode in which growth-indexed bonds were traded in France between 1956 and 1971. The main finding is that, during this historical episode, the yields of these strips were negative, implying that the risk-adjusted growth rate exceeded the discount rate.

The paper's results should be interpreted narrowly. The paper does not claim that government debt can necessarily be rolled over without primary surpluses, nor that positive-supply assets necessarily contain bubbles. Rather, the point is that such possibilities cannot be ruled out on the basis of broad theoretical arguments alone, such as the existence of land. In particular, the historical episode studied in the paper suggests that the market-implied risk-adjusted growth rate was above the risk-free rate. This finding lends support to models (such as overlapping generations economies, or heterogeneous-agent economies) in which such outcomes may arise.

Likewise, the paper does not claim that the long-term strips of two cointegrated cash-flow streams must generally have different yields. It merely shows that cointegration alone is insufficient to guarantee equality of long-term strip yields. Additional conditions are required, including the finiteness of a certain unconditional expectation discussed later in Section 3. In some standard parametric environments these conditions hold automatically—for example, in models with i.i.d. lognormal increments of the stochastic discount factor and stochastic trend, together with AR(1) dynamics for the cointegrating vector. The contribution of the paper is to show that even modest departures from this benchmark—such as replacing linear with nonlinear mean reversion in the cointegrating vector—may invalidate the required finiteness condition. Importantly, the analysis does not rely on unbounded market prices of risk or on stationary distributions with unbounded moments.

The paper relates to several strands of the literature. The possibility of debt rollover in deterministic economies with $r < g$ dates back to Diamond (1965). Tirole (1985) underscores the importance of this condition for the emergence of bubbles. In deterministic models featuring a balanced growth path, the existence of bubbles is essentially equivalent to "dynamic inefficiency", i.e., capital over-accumulation (Cass (1972)). Stochastic economies, however, can simultaneously be dynamically efficient and exhibit bubbles. As a result, the dynamic-efficiency test of Abel et al. (1989), which indicates that the US economy is dynamically efficient, does not imply the absence of bubbles.[4] Santos and Woodford (1997) show that the finiteness of the aggregate endowment is the key condition to exclude bubbles on all positive-supply assets in both deterministic and stochastic economies. In deterministic economies with a balanced growth path the existence of land can exclude the possibility of bubbles.[5]

The secular decline in interest rates and the rising levels of deficits have rekindled interest in the issue of debt rollover without primary surpluses, i.e., a bubble on government debt. Blanchard and Weil (2001), Blanchard (2019), Barro (2023), Cochrane (2021), Reis (2021), Hellwig (2021), Kocherlakota (2023a), Kocherlakota (2023b), Jiang et al. (2024), Brunnermeier et al. (2021), Abel and Panageas (2025), Aguiar et al. (2024), Amol and Luttmer (2022) are recent examples of this growing literature.[6]

The notion that the long-term strips of cointegrated cash flows should command similar discount rates is widespread in the asset-pricing literature.[7] As emphasized earlier, this

[4]On the issue of dynamic efficiency in stochastic economies, see Zilcha (1990), Zilcha (1991). On the relation between dynamic efficiency and the possibility of bubbles, see Bertocchi (1991), Binswanger (2005), Barbie et al. (2007), Farhi and Tirole (2011), Martin and Ventura (2012), Ball and Mankiw (2023), Bloise and Reichlin (2023), and Abel and Panageas (2025).

[5]See, e.g., the discussion in Hirano et al. (2023) on this issue.

[6]Indicative earlier contributions to this important topic include Aiyagari and McGrattan (1998), Blanchard and Weil (2001), Ball et al. (1998), Bohn (1995).

[7]This intuition appears in various forms across several papers. Notable contributions include Alvarez and Jermann (2005) and Backus et al. (2018), among many others. More recent papers applying related ideas to

intuition may well hold in many applications, but cointegration alone is not sufficient. The seminal papers of Hansen (2012) and Hansen and Scheinkman (2009) carefully impose additional conditions in their analysis of long-horizon strips. The contribution of the present paper is to highlight the importance of these conditions and to show that they may fail even in simple, non-pathological examples.

There is a small academic and practitioner literature on the benefits of GDP-linked debt. See Borensztein and Mauro (2004) for a survey. See also Benford et al. (2016), Bowman and Naylor (2016), Blanchard et al. (2016), Kamstra and Shiller (2009), Shiller (1998) for indicative policy proposals. Barro (1995), Hatchondo and Martinez (2012), and Caballero and Panageas (2008) are indicative papers that discuss the potential benefits of indexation in macroeconomic models. Mitchener and Pina (2026) is a contemporaneous paper that also discusses the same historical episode as this paper, but without analyzing the implications for the risk-adjusted growth rate of the economy.

Sections 2 and 3 contain the theoretical results. Section 4 discusses the French experience with growth-indexed bonds. Section 5 concludes. An online appendix contains proofs and additional results.

2 Cointegrated cash-flow tracking funds

This section introduces the concept of a cash-flow tracking fund. To start, consider a given ("target" cash flow) process C_t with log-normal dynamics

$$\frac{dC_t}{C_t} = gdt + \sigma_C dB_t^C, \tag{1}$$

where g is a constant, and B_t^C is a standard Brownian motion.

Suppose that a fund can invest in N risky assets. Letting S_t^i denote the price of asset i at time t and D_t^i its flow of dividends, the instantaneous return of asset i is $dR_t^i = \frac{dS_t^i + D_t^i dt}{S_t^i}$.

monetary policy include Cieslak and Khorrami (2026) and Ai and Fu (2025). Jiang et al. (2024) applies this logic to the pricing of government debt. Dufresne et al. (2025) provides a critical assessment of the findings in Jiang et al. (2024), albeit unrelated to the issues emphasized in this paper.

Assume that the dynamics of the vector of returns dR_t are given by

$$\underbrace{dR_t}_{N\times 1} = \underbrace{\mu}_{N\times 1} dt + \underbrace{\sigma}_{N\times M} \underbrace{dB_t}_{M\times 1}, \text{ where } dB_t \equiv \begin{bmatrix} dB_t^C \\ d\overrightarrow{B}_t \end{bmatrix}, \tag{2}$$

where μ is an $N \times 1$ vector of expected returns, and σ is an $N \times M$ matrix of exposures to the M Brownian motions driving the returns. Assume that $N \leq M$ and that σ has rank N. Accordingly, the market could be either complete ($N = M$) or incomplete ($N < M$). In addition, there is a risk-free asset with an instantaneous rate of return equal to r.

The fund can make distributions equal to X_t per unit of time dt, so that the dynamic evolution of the fund's value, W_t, is

$$\frac{dW_t}{W_t} = \left(r - \frac{X_t}{W_t}\right) dt + w'(dR_t - r dt) = \left(r + w'\mu^e - \frac{X_t}{W_t}\right) dt + w'\sigma dB_t, \tag{3}$$

where $\mu^e \equiv \mu - r\mathbf{1}_N$ is an $N \times 1$ column vector of expected *excess* returns, and w is a time-invariant $N \times 1$ portfolio vector.

Let $z_t \equiv \log W_t - \log C_t$ denote the fund-value-to-cash-flow ratio. Ito's Lemma implies that

$$dz_t = \left(r + w'\mu^e - g - \frac{w'\sigma\sigma' w - \sigma_C^2}{2} - \frac{X_t}{W_t}\right) dt + w'\sigma dB_t - \sigma_C dB_t^C. \tag{4}$$

Suppose that the fund makes distributions according to the following rule:

$$\frac{X_t}{W_t} \equiv \max(\alpha + \beta z_t, \varepsilon), \text{ where } \beta > 0 \text{ and } \varepsilon > 0. \tag{5}$$

The truncation at some $\varepsilon > 0$ ensures that the fund always makes positive distributions, a property that is important for Proposition 2 below. Substituting (5) into (4) results in the dynamics:

$$dz_t = \left(r + w'\mu^e - g - \frac{w'\sigma\sigma' w - \sigma_C^2}{2} - \max(\alpha + \beta z_t, \varepsilon)\right) dt + w'\sigma dB_t - \sigma_C dB_t^C. \tag{6}$$

Equation (6) is a one-dimensional stochastic differential equation (SDE) with initial condition $z_0 = \log W_0 - \log C_0$. Using $z_t = \log W_t - \log C_t$, the level of wealth at time t is $W_t = C_t e^{z_t} > 0$.

The next proposition provides a condition under which z_t has a stationary distribution. To prepare for the statement of the proposition, define the following quantity, which is the expected difference between the logarithmic rate of return on the portfolio and the logarithmic growth rate of C_t:

$$A = A(w) \equiv r - g + w'\mu^e - \frac{w'\sigma\sigma' w}{2} + \frac{\sigma_C^2}{2}. \tag{7}$$

The proposition shows that z_t has a stationary distribution whenever $A(w) > \varepsilon$.

Proposition 1 *Fix a vector* w, *define* $\phi' \equiv w'\sigma - [\sigma_C, 0, \ldots, 0]$, *and let* $\sigma_z \equiv \sqrt{\phi'\phi}$ *and* $z^* \equiv \frac{\varepsilon - \alpha}{\beta}$. *Assume that* $\sigma_z > 0$. *If* $A(w) > \varepsilon$, *then* z_t *has a stationary distribution* $l(z)$ *given by*

$$l(z) = \frac{\mathbf{1}_{z \le z^*} e^{\frac{2}{\sigma_z^2}(A-\varepsilon)z} + \mathbf{1}_{z > z^*} e^{\frac{2}{\sigma_z^2}\left[-\beta \frac{z^2 - (z^*)^2}{2} + (A-\alpha)z - (\varepsilon - \alpha)z^*\right]}}{\int_{-\infty}^{z^*} e^{\frac{2}{\sigma_z^2}(A-\varepsilon)z} dz + \int_{z^*}^{\infty} e^{\frac{2}{\sigma_z^2}\left[-\beta \frac{z^2 - (z^*)^2}{2} + (A-\alpha)z - (\varepsilon - \alpha)z^*\right]} dz}. \tag{8}$$

Figure D.1 in the online appendix illustrates $l(z)$, which is a concatenation of an exponential distribution for values of $z \le z^*$ and a normal distribution for $z > z^*$. Accordingly, all moments of the stationary distribution $l(z)$ are finite.

The two assumptions of Proposition 1 are that (a) $\sigma_z > 0$ and (b) $A(w) > \varepsilon$. Assumption (a) restricts attention to portfolios that are not replicating the cash flow $\frac{dC_t}{C_t}$.[8] Assumption (b) is the more important assumption. Since ε can be chosen to be arbitrarily small, this assumption effectively requires that $A(w) > 0$.

The intuition for the existence of a stationary distribution in Proposition 1 is simple: whenever z_t is below the cutoff z^*, the assumption $A(w) > \varepsilon$ ensures that the expected logarithmic growth rate of the fund, $r + w'\mu^e - \frac{w'\sigma\sigma' w}{2} - \varepsilon$, exceeds the expected logarithmic growth rate of the cash flow C_t, $g - \frac{\sigma_C^2}{2}$, resulting in a positive drift for z_t. Conversely, when

[8] This assumption is generically satisfied in an incomplete market ($N < M$) since the matrix σ has rank N.

z_t exceeds z^*, the drift of z_t becomes negative for high enough z_t, since the payout, $\frac{X_t}{W_t}$, is linear in z_t and eventually exceeds $r + w'\mu^e - \frac{w'\sigma\sigma' w}{2}$. Note that if $A(w)$ were negative, then $\lim_{t\to\infty} z_t = -\infty$ a.s. for any non-negative payout policy $X_t \geq 0$.[9]

Maintaining the assumption that $A(w) > \varepsilon > 0$, the difference $\log(X_t) - \log(C_t)$ is a function of the process z_t:

$$\log X_t - \log C_t = \log\left(\frac{X_t}{W_t}\right) - \log\left(\frac{C_t}{W_t}\right) = \log\left(\max(\alpha + \beta z_t, \varepsilon)\right) + z_t, \tag{9}$$

where the last equation follows from (5) and the definition of z_t. The plots of Figure D.1 in the online appendix depict the (simulated) stationary distributions of $\log(X_t) - \log(C_t)$ (top right), $\frac{W_t}{C_t}$ (bottom left), and $\frac{X_t}{C_t}$ (bottom right).

Remark 1 *Even though the cash-flow X_t is the result of an endogenous payout strategy, the process X_t could equivalently be specified as an exogenous process. In particular, one could have simply started by postulating an exogenous, Markov process z_t with dynamics given by* (6) *and defined the cash flow X_t exogenously as* $\log(X_t) = \log(C_t) + \log\left(\max(\alpha + \beta z_t, \varepsilon)\right) + z_t$. *While the "endogenous" and the "exogenous" specification of X_t are mathematically identical, the "endogenous" specification makes it clear that the cash flow X_t is financed by a feasible trading strategy. This has the straightforward implication that the present value of X_t is finite, as shown in the next section.*

3 Long-dated strips

This section shows that even though the cash-flow processes $\log X_t$ and $\log C_t$ are cointegrated, the dividend "strips" to the cash flow X_T are summable while the dividend strips to C_T are not.

To remove any ambiguity about how to compute the value of dividend strips, the remainder of this section focuses on the case $N = M$, so that the market is complete. By standard

[9] The easiest way to see this is to let $z_t^{(0)}$ denote the process for z_t assuming that $X_t = 0$. Note that for any $X_t \geq 0$ we have $z_t^{(0)} \geq z_t$. The properties of the Brownian motion with negative drift imply that $\lim_{t\to\infty} z_t^{(0)} = -\infty$ a.s.. Since $\lim_{t\to\infty} z_t^{(0)} \geq \lim_{t\to\infty} z_t$, it follows that $\lim_{t\to\infty} z_t = -\infty$ a.s..

results[10] , the unique stochastic discount factor (SDF) is

$$\frac{dm_t}{m_t} = -rdt - \underbrace{\kappa'}_{M\times 1}\begin{bmatrix} dB_t^C \\ d\overrightarrow{B}_t \end{bmatrix}, \tag{10}$$

where the vector $\kappa \equiv \sigma^{-1}\mu^e$ is the vector of the "market prices of risk."

The price $P_t^{(X_T)}$ of a dividend strip to the cash flow X_T, divided by the current level of X_t is defined as

$$\frac{P_t^{(X_T)}}{X_t} \equiv E_t\left(\frac{m_T}{m_t}\frac{X_T}{X_t}\right) = E_t^Q\left(e^{-r(T-t)}\frac{X_T}{X_t}\right), \tag{11}$$

where Q is the "risk-neutral" measure associated with the SDF (10).

Because the cash flow X_t is a positive cash flow financed by a self-financing trading strategy, the dividend strips, $P_t^{(X_u)}$, are summable, as the next Lemma shows.

Lemma 1 *The prices of the dividend strips to X_u satisfy: (a) $\int_t^\infty P_t^{(X_u)}du < \infty$, (b) $\lim_{T\to\infty}\int_T^\infty P_t^{(X_u)}du = 0$, and (c) $\liminf_{T\to\infty} P_t^{(X_T)} = 0$.*

The value of a dividend strip to the cash flow C_T divided by C_t is given by

$$\frac{P_t^{(C_T)}}{C_t} \equiv E_t\left(\frac{m_T}{m_t}\frac{C_T}{C_t}\right) = E_t^Q\left(e^{-r(T-t)}\frac{C_T}{C_t}\right) = e^{-\left(r-g^Q\right)(T-t)}, \tag{12}$$

where g^Q is defined as

$$g^Q = g - \kappa^{(1)}\sigma_C,$$

with $\kappa^{(1)}$ denoting the first element of the vector κ (i.e., the price of risk associated with the Brownian motion B_t^C). I will refer to g^Q as the "risk-adjusted growth rate" (or the growth rate under the risk-neutral measure) since it equals the growth rate g net of the risk premium $\kappa^{(1)}\sigma_C$. If $r < g^Q$, then $\lim_{T\to\infty}\frac{P_t^{(C_T)}}{C_t} = \infty$ and the dividend strips to C_t are not summable

[10] See Duffie (2001).

($\int_t^\infty P_t^{(C_u)} du = \infty$). The next proposition shows that $\log X_t - \log C_t$ may have a stationary distribution (i.e., the conditions of Proposition 1 hold), even though $r < g^Q$.

Proposition 2 *Assume that $\sigma_z > 0$ and also*

$$r - g^Q < 0 < r - g + \frac{1}{2}\kappa'\kappa + \frac{\sigma_C^2}{2}. \tag{13}$$

Then there exists a vector w and a positive scalar $\varepsilon > 0$ such that $\log X_t - \log C_t$ has a stationary distribution, and yet $\lim_{T\to\infty} P_t^{(C_T)} = \infty$, and $\int_t^\infty P_t^{(C_u)} du = \infty$, while $\int_t^\infty P_t^{(X_u)} du < \infty$.

The remainder of this section aims to provide some intuition as to why Proposition 2 holds. To start, define

$$f(z_t) \equiv \max(\alpha + \beta z_t, \varepsilon) e^{z_t} = \frac{X_t}{C_t}, \tag{14}$$

where the last equation follows from (9). Next, define the yield of the X_T-strip as $-\frac{1}{T-t}\log\left(\frac{P_t^{(X_T)}}{X_t}\right)$; similarly, let $-\frac{1}{T-t}\log\left(\frac{P_t^{(C_T)}}{C_t}\right)$ denote the yield of the C_T-strip.[11]

Note that equations (11), (12) and (14) imply that

$$\begin{aligned}
-\frac{1}{T-t}\log\left(\frac{P_t^{(X_T)}}{X_t}\right) &= -\frac{1}{T-t}\log E_t^Q\left(e^{-r(T-t)}\frac{X_T}{X_t}\right) = \\
&= -\frac{1}{T-t}\log\left\{e^{-r(T-t)}E_t^Q\left(\frac{C_T}{C_t}\right) \times E_t^Q\left(\frac{C_T}{E_t^Q(C_T)}\frac{f(z_T)}{f(z_t)}\right)\right\} \\
&= -\frac{1}{T-t}\left[\log\left(\frac{P_t^{(C_T)}}{C_t}\right) + \log E_t^Q\left(\frac{C_T}{E_t^Q(C_T)}\frac{f(z_T)}{f(z_t)}\right)\right].
\end{aligned}$$

Multiplying both sides with -1 and taking limits as $T - t \to \infty$ gives

$$\lim_{T-t\to\infty}\frac{1}{T-t}\log\left(\frac{P_t^{(X_T)}}{X_t}\right) = \lim_{T-t\to\infty}\frac{1}{T-t}\log\left(\frac{P_t^{(C_T)}}{C_t}\right) + \mathcal{R}, \tag{15}$$

[11] The division of the price of each strip by the current level of the respective cash-flow ensures that the yields are expressed on a "per-dollar" basis, and are therefore comparable.

where

$$\mathcal{R} \equiv \lim_{T-t\to\infty} \frac{1}{T-t} \log E_t^Q \left(\frac{C_T}{E_t^Q\left(C_T\right)} \frac{f(z_T)}{f\left(z_t\right)} \right). \tag{16}$$

Equation (15) implies that the (asymptotic) yields of the long-term dividend strips to X_T and C_T are equal if and only if the term $\mathcal{R}$ is equal to zero.

To examine whether $\mathcal{R}$ is zero or not, note that the ratio $\frac{C_T}{E_t^Q(C_T)} = e^{-\frac{1}{2}\sigma_C^2(T-t)+\sigma_C(B_T^C - B_t^C)}$ is a positive martingale (with T denoting running time) and has an expectation equal to one; therefore, by Girsanov's theorem, it can be used to define a new probability measure, $\widetilde{Q}$, equivalent to Q. The following Lemma provides the dynamics of z_t under the measure $\widetilde{Q}$:

Lemma 2 *Under the measure $\widetilde{Q}$ the term $\mathcal{R}$ can be expressed as*

$$\mathcal{R} = \lim_{T-t\to\infty} \frac{1}{T-t} \log E_t^{\widetilde{Q}} \left[f(z_T) \right], \tag{17}$$

and the dynamics of z_t under the measure $\widetilde{Q}$ are given by

$$dz_t = \left(\widetilde{A}(w) - \max(\alpha + \beta z_t, \varepsilon) \right) dt + \phi' dB_t^{\widetilde{Q}}, \tag{18}$$

where ϕ' is defined in Proposition 1, $dB_t^{\widetilde{Q}}$ is an $M-$dimensional Brownian motion under the probability measure $\widetilde{Q}$, and $\widetilde{A}(w)$ is given by

$$\widetilde{A}(w) = r - g^Q - \frac{\sigma_z^2}{2}. \tag{19}$$

The following corollary is a straightforward implication of Proposition 1 and Lemma 2.

Corollary 1 *Assume that $\sigma_z > 0$. The process z_t admits a stationary distribution under the probability measure $\widetilde{Q}$ iff $\widetilde{A}(w) > \varepsilon$.*

The quantities $\phi' \equiv w'\sigma - [\sigma_C, 0, \ldots, 0]$ and $\sigma_z \equiv \sqrt{\phi'\phi}$ depend on the portfolio, w, but $r - g^Q$ does not. Therefore, an immediate consequence of (19) is the following:

Corollary 2 *For any portfolio, w, with $\sigma_z > 0$ it holds that $\widetilde{A}(w) < r - g^Q$.*

Corollary 2 implies that if $r - g^Q < 0$ then it must be that $\widetilde{A}(w) < r - g^Q < 0 < \varepsilon$ for all portfolio vectors, w. However, $r - g^Q < 0$ does not preclude the possibility $A(w) > \varepsilon > 0$. To see this consider the growth-optimal portfolio, i.e., the portfolio that maximizes $A(w)$ over w. Carrying out the maximization and substituting into (7) implies

$$\max_w A(w) = r - g + \frac{1}{2}\kappa'\kappa + \frac{\sigma_C^2}{2}. \tag{20}$$

Since Corollary 2 holds for any portfolio, it holds for the portfolio that maximizes $A(w)$ over w. Therefore, as long as the parameters satisfy

$$r - g^Q < 0 < r - g + \frac{1}{2}\kappa'\kappa + \frac{\sigma_C^2}{2},$$

there exist w and $\varepsilon > 0$ such that $A(w) > \varepsilon > 0$ (so that z_t is stationary under the data-generating process), but $\widetilde{A}(w) < r - g^Q < 0 < \varepsilon$ (so that z_t is non-stationary under the probability measure $\widetilde{Q}$). As a result, there is no reason to expect that $\log E_t^{\widetilde{Q}}(f(z_T))$ converges to a finite number as $T \to \infty$ (and, by implication, that $\mathcal{R}$ is zero).

The possibility that $\lim_{T\to\infty} \log E_t^{\widetilde{Q}}(f(z_T))$ is infinite is the reason why there is no contradiction between the results of this paper and the analysis in Hansen (2012) and Hansen and Scheinkman (2009), which assume the finiteness of this limit (when discussing the yields on long-term strips). As mentioned in the introduction, Hansen (2012) and Hansen and Scheinkman (2009) do not claim or imply that cointegration between $\log X_t$ and $\log C_t$ is sufficient to imply the finiteness of $\lim_{T\to\infty} \log E_t^{\widetilde{Q}}(f(z_T))$. Further discussion on the relation between the example presented in this paper and the factorization methods in Hansen (2012) and Hansen and Scheinkman (2009) can be found in the newly added Section 8.8.2 of the online textbook Hansen et al. (2026).[12]

Remark 2 *It is useful to note that* $r - g^Q < 0$ *is sufficient to ensure* $\widetilde{A}(w) < 0 < \varepsilon$, *but not necessary. In particular, if* $r - g^Q > 0$ *(so that the dividend strips to both cash flows,*

[12] This discussion is summarized by the following excerpt: "Although stated in the context of a very special example, the analysis in Panageas (2025) reminds us that in a stochastic setting, unconditional moments can be infinite even when there is stochastic stability in the Markov process and the horizon-dependent conditional moments are finite."

C_t and X_t are summable), but $\widetilde{A}(w) < A(w)$, there exists $\varepsilon > 0$ so that $\widetilde{A}(w) < \varepsilon < A(w)$. Therefore, the process z_t is non-stationary under $\widetilde{Q}$ and the long-term yields of strips to C_T and X_T will generally differ and converge to different numbers even though the dividend streams to both X_T and C_T are summable.

Remark 3 *Figure D.2 in the online appendix depicts the yield differences between the dividend strips to X_T and C_T for finite values of T on the x-axis. While the yields of strips to C_T are the same for all T, the yields of strips to X_T are downward sloping, illustrating that two cointegrated cash flows can have very different dividend-strip term-structure shapes.*

4 The risk-adjusted growth rate: Empirical evidence

The negative theoretical result of the previous section suggests a different, empirical approach to gauging whether the risk-adjusted growth rate exceeds the interest rate. Suppose it were possible to observe the yields of strips whose payoffs are indexed to C_t. The yields of such strips would reveal the quantity of interest, namely $r - g^Q$, at least over the maturities of these strips.

4.1 The French experience in the mid-1950s and 1960s

It is hard to find historical episodes where countries issued bonds with coupons linked to GDP growth. Typically, the attempts to introduce GDP-indexed bonds occur in developing economies and usually in the aftermath of sovereign default. Not too surprisingly, market participants are reluctant to purchase such bonds, since the possibility of another sovereign default looms in their minds.

An interesting exception, where a large, developed economy introduced indexed bonds is the case of France in the mid 1950s.[13] In addition, there is readily available data on the market prices of these bonds spanning about 15 years.

Specifically, on June 1, 1956, the French Government issued bonds with coupons that were indexed on the path of industrial production ("Bons d'Équipement Industriel et Agri-

[13]See, e.g., Rozental (1959).

cole"). The online Appendix E provides the translated announcement describing the terms of these bonds from the official journal of the French government. In summary, these bonds had denominations of 10,000, 100,000, and 1,000,000 francs and a 15-year maturity. They represented 19% of public debt borrowing[14] in 1956. Redemption would be at 105% of par and would occur through "random lots": The bonds called for redemption were determined by a random lot every April with a redemption date of June 1. Specifically, 6% of the originally issued bonds were called for redemption in each of the first 5 years and 7% in each of the remaining 10 years (so the bonds are fully redeemed by year 15).[15] For the purposes of this paper, the interesting aspect was the determination of the annual coupon (payable on June 1 of each year), which was set at a 5% minimum; in addition, there was a "bonus": the coupon increased by 0.05% for each point that the annual index of industrial production exceeded its level in 1955 (base 119 in 1955). While industrial production is not equal to GDP, it is an important component of GDP, essentially reflecting the production of goods (but not services). As Figure D.4 in the online Appendix shows, the evolution of industrial production is essentially identical to the evolution of GDP over that period. As a matter of theory, if the strips to a component of GDP appear not summable, then strips to GDP itself must be non-summable (since GDP is the sum of nonnegative components, the present value of GDP strips is bounded below by the present value of strips to any one component).

The bonds were traded on the French stock exchange and their prices are available on the "Database for Financial History" (dfih.fr). The database contains observations on a biweekly frequency from August 1956 to April 1971.[16] I complement this time series with the following data series: (1) Annual data on French inflation and real GDP 1950-1971 (from FRED). (2) Data on "Production of Total Industry in France", available from 1956-1971 (from FRED and annual INSEE publications) (3) Annual interest rates on newly issued long-term bonds (10 years) from Chauveau (1975), Table 3, Column 1 for the years 1956-1971. I provide more

[14] See Mitchener and Pina (2026).

[15] Note that $5 \times 6 + 10 \times 7 = 100$.

[16] The database reports a "transaction #" field exceeding 1 on many days, indicating that prices correspond to actual transactions rather than indicative quotes. Arbitrage across the three denominations also appears to function well: prices are effectively identical across denominations on the same day, up to scaling by face value.

details on these data sources in online Appendix C.

As is frequently the case with historical data, there are some important limitations. The first and most important limitation is that there appears to have been only a single issue of those bonds. This makes it impossible to derive a term structure (since this would require multiple maturities on the same date).

The second limitation is that it is unclear when investors had access to information about the index of industrial production. The index of industrial production in year t is reported in the official announcement of the coupon in May of year $t+1$. However, because of the availability of monthly statistical journals, it is very likely that investors were able to form expectations of this index earlier than the official announcements, and it is not clear when these monthly journals were available to investors, or how revisions were handled.

The third limitation is that the (nominal) interest rates, i_t, that I use for discounting purposes are derived from newly issued bonds, see Chauveau (1975), not bonds on the secondary market. The (unobserved) discount rates on the secondary market appear to be higher. For example, during the first two years of the sample, when the value of the bonus coupon is small, the prices of the Bons d'Équipement are slightly below the expected[17] present discounted value of the minimum value of their coupons (5%) plus principal repayment, using i_t for discounting purposes. This implies that the discount rates on the secondary market must be higher than in the primary market, since the value of collecting the bonus coupon cannot be negative. The discount rates appear to be too low not only at the beginning of the sample, but also at the end. For example, in the final months of the sample, when the Bons d'Équipement becomes effectively a zero-coupon bond with known terminal payoff, its yield to maturity exceeds the yields of newly issued bonds. In the online Appendix (Section C) I provide additional evidence on why the series i_t is a lower bound to the secondary-market discount rates by comparing it to the yield of a traded, non-indexed bond described in Mitchener and Pina (2026).

The remainder of the section develops an econometric approach to obtain a *conservative* estimate of the risk-adjusted growth rate, g^Q, while respecting these limitations of the data.

[17] I use the word "expected" discounted value to indicate that I take an expectation over the random maturity dates using the random lot rules.

To start, assume that the dynamics of industrial production follow the process

$$\frac{dY_t}{Y_t} = g^Q dt + \sigma dB_t^Q \tag{21}$$

under the risk-neutral measure Q. Using (21) and letting T denote the (random) maturity date as determined by the "random lot" rules, the arbitrage-free price of the bond, $\widehat{P}(Y_t)$, at time t is equal to

$$\widehat{P}(Y_t) = \sum_{T \geq t} q_t(T) p(Y_t; T), \tag{22}$$

where $q_t(T)$ is the probability that the bond matures at date T, conditional on not having been called for redemption by date t. The quantity $p(Y_t; T)$ is the arbitrage-free price[18] conditional on the maturity date being T,

$$p(Y_t; T) = 100 \times \sum_{T_i \in [t,T]} e^{-i_t(T_i - t)} \left[0.05 + 0.0005 \times E_t^Q (Y_{T_i} - Y_{1955})^+ \right] + 105 e^{-i_t(T-t)}, \tag{23}$$

where T_i are the coupon (and also possible redemption) dates of the bond. Equation (22) allows one to infer the value of Y_t by solving the equation $P_t = \widehat{P}(Y_t)$, where P_t is the observed price of the security at time t. This leads to the implicit value of industrial production $\widetilde{Y}_t(P_t; g^Q, \sigma) = \widehat{P}^{-1}(P_t; g^Q, \sigma)$. In words, this equation says that conditional on a choice of g^Q, and σ, one can infer a time series of imputed industrial production, $\widetilde{Y}_t$, from the time-series of bond prices, P_t.

Next, fix a value $\widehat{\sigma}$, define the residual $u_{t_i} \equiv \log Y_{t_i} - \log \widetilde{Y}_{t_i}(P_{t_i}; g^Q, \widehat{\sigma}) - \alpha$, and estimate the risk-neutral growth parameter, g^Q, by minimizing the expression

$$(\widehat{g}^Q, \widehat{\alpha}) = \arg \min_{g^Q, \alpha} \sum_{i=1}^{N} u_{t_i}^2. \tag{24}$$

The value of industrial production, Y_{t_i}, that enters u_{t_i} is based on the (ex-post observed) value of monthly industrial production at time t_i. As mentioned above, this value may

[18] The expectation in equation (23) can be computed using the Black-Scholes formula. See Appendix B.3.

differ from the value of industrial production that is in the investors' information set, which is one reason why one should expect non-zero residuals. However, over time, one would expect $\log Y_{t_i}$ to be tracking $\log \widetilde{Y}_{t_i}$. The role of the constant is to absorb an average lead or lag between $\log Y_{t_i}$ and $\log \widetilde{Y}_{t_i}$. Using the derivatives of the Black-Scholes equation and the implicit function theorem, Appendix B.3 derives the (Newey-West) standard error of this estimator.

Appendix B.1 addresses the issue that the discount rates are lower bounds to the true secondary-market discount rates. That section shows analytically that if the true discount rates are higher than the observed ones, then the difference between the estimated value, $\widehat{g}^Q$, and the average value of the observed i_t understates the difference between g^Q and the true (unobserved) interest rate.

One final remark pertains to the fact that the i_t are nominal interest rates, while the index of industrial production is a real quantity. Appendix B.2 considers a setup where shocks to the (log) price level may be correlated with shocks to $\log Y_t$. Letting π_t denote the inflation rate and assuming a positive inflation risk premium, Appendix B.2 shows that the estimated $\widehat{g}^Q$ net of the observed $i_t - \pi_t$ provides a lower bound on the difference between g^Q and the real interest rate.

Table 1 presents the results of estimating (24). I start the analysis in 1959 to ensure that Y_t is sufficiently larger than Y_{1955} so that the following approximation holds: $E_t^Q \left(Y_{T_i} - Y_{1955}\right)^+ \approx E_t^Q \left(Y_{T_i} - Y_{1955}\right) = Y_t e^{g^Q (T_i - t)} - Y_{1955}$. Therefore, the choice of volatility σ is immaterial since σ does not enter the expression $Y_t e^{g^Q (T_i - t)}$. The irrelevance of σ is evident in Table 1. While I use the Black-Scholes formula to compute the right-hand side of (23), Table 1 makes it clear that the choice of σ does not materially affect the estimated value, $\widehat{g}^Q$. Focusing on the part of the sample where $Y_t \gg Y_{1955}$, also ensures that the log-normal specification (21) is unimportant, so that using the Black-Scholes equation or any other option pricing model is immaterial: As long as the increments to $\log Y_t$ are i.i.d., the approximation $E_t^Q \left(Y_{T_i} - Y_{1955}\right)^+ \approx E_t^Q \left(Y_{T_i} - Y_{1955}\right) = Y_t e^{g^Q (T_i - t)} - Y_{1955}$ continues to hold.[19]

[19]For example, if one modified the stochastic process of Y_t to include, say, jumps, the option pricing model would change, but as long as $Y_t \gg Y_{1955}$, it would still be the case that $E_t^Q \left(Y_{T_i} - Y_{1955}\right)^+ \approx E_t^Q \left(Y_{T_i} - Y_{1955}\right) = Y_t e^{g^Q (T_i - t)} - Y_{1955}$, with g^Q reflecting a risk premium for both diffusion and jump

One should also keep in mind that modern option pricing was not developed until the mid-1970s. By focusing on the sample where the option is "deep in the money," we ensure that investors would have been able to approximately price the option using a forward-pricing formula

Table 1 shows that the baseline estimate of g^Q is equal to 3.26% with a standard error of 0.74%. The average yield on long-term nominal bonds is 6.31% over the sample period and has a small standard deviation (0.88%). The yield on a long-term nominal bond can be viewed as an approximation to the long-run mean of the nominal interest rate under the risk-neutral measure. Subtracting average annual inflation from the nominal yield, the resulting real rate has a mean equal to 2.19% and a standard deviation of 1.03%. Assuming a positive inflation risk premium (so that the risk-neutral expectation of inflation exceeds realized inflation on average) implies that the difference between the estimate of $\widehat{g}^Q$ and the realized real rate (3.26% - 2.19% = 1.07 %) provides a conservative estimate of $g^Q - r$. (For a more detailed discussion on this issue see Online Appendix Sections B.1 and B.2).

Figure 1 gives a visual impression of the goodness of fit of the model, by depicting the actual and the model-implied index of industrial production. Not too surprisingly, the model-implied series is somewhat more volatile than the actual index, since it is based on asset prices that may be subject to liquidity effects, leads and lags in the information set of investors, etc. Overall the two series share a common trend with deviations that are quite short-lived.

One concern with the estimation procedure is that it only allows a single estimate of g^Q. Some economic models imply that the yields of strips to Y_T may depend on the maturity, T. As mentioned earlier, the fact that there was a single issue of the growth-indexed bonds prevents the estimation of such a term structure of yields. However, it is possible to estimate g^Q over different subsamples and examine whether the estimate of g^Q changes as the bond approaches its date of termination. The last two columns of Table 1 estimate g^Q separately for observations in the first part and the second part of the sample. The columns show that the estimate of g^Q remains essentially identical across the two subsamples, even though the

risks.

expected date of termination of the bonds is quite different in the two subsamples. It is also worth noting that the real interest rate and the growth rate of industrial production seem quite similar in the first and second halves of the sixties in France (see Figures D.3 and D.4 in the online Appendix). Therefore, it is plausible to assume that the main difference impacting the estimation of g^Q between the first and the second subsample is predominantly the shortening of the maturity of the bonds. Yet, this shortening does not cause a change in the estimate of g^Q, which suggests a flat term structure of risk-adjusted growth rates.

5 Conclusion

On the theory side, this paper has examined whether the present value of the aggregate endowment can be inferred to be finite simply from the existence of finitely valued assets with cash-flows that are cointegrated with GDP. The answer to this question is negative. Even with a constant interest rate and constant market prices of risk, a cash flow X_t may be finitely valued, and yet another cash flow, C_t, which is cointegrated with X_t, may be infinitely valued. The implication is that there is no broad and simple argument to dismiss the possibility that the value of the aggregate endowment is infinite. Accordingly, the value of some assets may not equal the expected, appropriately discounted value of their dividends (primary surpluses in the case of public debt).

This negative theoretical result motivates the empirical exercise. The paper studies a historical episode in which the French government issued bonds with coupons indexed to industrial production, traded between 1956 and 1971. Two features make this episode attractive: the bonds were issued by a major industrialized country, and they were not issued in the aftermath of sovereign default, as is typically the case in the few other historical examples of GDP-indexed bonds. The main empirical finding is that the risk-neutral growth rate of industrial production—an important component of GDP—exceeded the risk-free rate over this sample.

A natural and broad concern with any paper that uses a finite sample and finite asset maturities is that the ultimate object of interest is the behavior of yields at infinity, which is inherently unobservable. I offer two responses. First, knowing that the yields on growth-indexed strips are negative over intermediate maturities still helps gauge the plausibility of competing models. For example, such negative yields would seem easier to reconcile with calibrated overlapping-generations models than with representative-agent ones.[20] Second, setting models aside, the negative yield in a growth-indexed bond has a direct practical reading: it equals the deficit that can be financed while *deterministically* holding the debt-to-GDP ratio fixed over the maturity of the bond. Consider the following thought experiment: suppose that at time t the government has 100 dollars of maturing debt and that GDP is also 100, so the debt-to-GDP ratio is one. The government issues a one-period bond that promises to repay $100 \cdot G_{t+1}/G_t$ at $t+1$. If $r - g^Q = -1\%$, this bond sells for 101 dollars at t.[21] The government uses 100 dollars to roll over its maturing debt and finances one dollar of expenditure with the remainder. At $t+1$, its liability equals $100 \cdot G_{t+1}/G_t = G_{t+1}$—the new GDP—so the government finances expenditure equal to 1% of GDP, and the debt-to-GDP ratio remains one regardless of the realization of G_{t+1}/G_t.

[20] Indicative examples of perpetual-youth asset pricing models that could account for this phenomenon include Gârleanu and Panageas (2021), Gârleanu and Panageas (2023).

[21] The bond's expected payoff under the risk-neutral measure is $100 \cdot e^{g^Q}$, which, discounted at r, gives $100 \cdot e^{g^Q - r} \approx 101$.

References

Abel, A. B., N. G. Mankiw, L. H. Summers, and R. J. Zeckhauser (1989). Assessing dynamic efficiency: Theory and evidence. *The Review of Economic Studies 56*(1), 1–19.

Abel, A. B. and S. Panageas (2025). Running primary deficits forever in a dynamically efficient economy: Feasibility and optimality. *Econometrica 93*(5), 1601–1633.

Aguiar, M., M. Amador, and C. Arellano (2024, November). Micro risks and (robust) pareto-improving policies. *American Economic Review 114*(11), 3669–3713.

Ai, H. and L. Fu (2025). Using asset prices to measure the long-run impact of monetary policy. Working Paper.

Aiyagari, S. and E. McGrattan (1998). The optimum quantity of debt. *Journal of Monetary Economics 42*(3), 447–469.

Alvarez, F. and U. J. Jermann (2005). Using asset prices to measure the persistence of the marginal utility of wealth. *Econometrica 73*(6), 1977–2016.

Amol, A. and E. G. Luttmer (2022). Permanent primary deficits, idiosyncratic long-run risk, and growth. Working Paper 794, Federal Reserve Bank of Minneapolis.

Backus, D., N. Boyarchenko, and M. Chernov (2018). Term structures of asset prices and returns. *Journal of Financial Economics 129*(1), 1–23.

Ball, L., D. W. Elmendorf, and N. G. Mankiw (1998). The deficit gamble. *Journal of Money, Credit and Banking 30*(4), 699–720.

Ball, L. and N. G. Mankiw (2023). Market power in neoclassical growth models. *The Review of Economic Studies 90*(2), 572–596.

Barbie, M., M. Hagedorn, and A. Kaul (2007). On the interaction between risk sharing and capital accumulation in a stochastic olg model with production. *Journal of Economic Theory 137*(1), 568–579.

Barro, R. J. (1995). Optimal debt management. Working Paper 5327, National Bureau of Economic Research.

Barro, R. J. (2023). r minus g. *Review of Economic Dynamics 48*, 1–17.

Benford, J., M. Joy, and M. Kruger (2016). Sovereign gdp-linked bonds. Technical report. Financial Stability Paper No. 39, Bank of England.

Bertocchi, G. (1991). Bubbles and inefficiencies. *Economics Letters 35*(2), 117–122.

Binswanger, M. (2005). Bubbles in stochastic economies: Can they cure overaccumulation of capital? *Journal of Economics 84*(2), 179–202.

Blanchard, O., P. Mauro, and J. Acalin (2016). The case for growth-indexed bonds in advanced economies today. Technical report. Peterson Institute for International Economics.

Blanchard, O. J. (2019). Public debt and low interest rates. *American Economic Review 109*(4), 1197–1229.

Blanchard, O. J. and P. Weil (2001). Dynamic efficiency, the riskless rate, and debt ponzi games under uncertainty. *Advances in Macroeconomics 1*(2), 1–21.

Bloise, G. and P. Reichlin (2023). Low safe interest rates: A case for dynamic inefficiency? *Review of Economic Dynamics 51*, 633–656.

Bohn, H. (1995). The sustainability of budget deficits in a stochastic economy. *Journal of Money, Credit and Banking 27*(1), 257–271.

Borensztein, E. and P. Mauro (2004). The case for gdp-indexed bonds. *Economic Policy 19*, 165–216.

Bowman, J. and P. Naylor (2016). Gdp-linked bonds. Technical report. Reserve Bank of Australia Bulletin.

Brunnermeier, M. K., S. Merkel, and Y. Sannikov (2021). Debt as safe asset: Mining the bubble. Working Paper.

Caballero, R. and S. Panageas (2008). Hedging sudden stops and precautionary contractions. *Journal of Development Economics 85*(1-2), 28–57.

Cass, D. (1972). On capital overaccumulation in the aggregative, neoclassical model of economic growth: A complete characterization. *Journal of Economic Theory 4*(2), 200–223.

Chauveau, T. (1975). La demande d'obligations émanant des ménages. *Revue Economique 26*(5), 806–833.

Cieslak, A. and P. Khorrami (2026). Monetary shocks and long-run neutrality in asset pricing. Working Paper.

Cochrane, J. H. (2021). *The Fiscal Theory of the Price Level*. Princeton University Press.

Diamond, P. A. (1965). National debt in a neoclassical growth model. *The American Economic Review 55*(5), 1126–1150.

Duffie, D. (2001). *Dynamic Asset Pricing Theory* (Third ed.). Princeton University Press.

Dufresne, P. C., J. Hugonnier, and E. Perazzi (2025). Admissible surplus dynamics and the government debt puzzle. Working Paper.

Farhi, E. and J. Tirole (2011). Bubbly Liquidity. *The Review of Economic Studies 79*(2), 678–706.

Gârleanu, N. and S. Panageas (2021). What to expect when everyone is expecting: Self-fulfilling expectations and asset-pricing puzzles. *Journal of Financial Economics 140*(1), 54–73.

Gârleanu, N. and S. Panageas (2023). Heterogeneity and asset prices: An intergenerational approach. *Journal of Political Economy 131*(4), 839–876.

Hansen, L. P. (2012). Dynamic valuation decomposition within stochastic economies. *Econometrica 80*(3), 911–967.

Hansen, L. P., T. J. Sargent, and J. Borovička (2026). Risk, uncertainty and value. Available at `https://lphansen.github.io/QuantMFR/book/0_opening_remarks.html`.

Hansen, L. P. and J. A. Scheinkman (2009). Long-term risk: An operator approach. *Econometrica 77*(1), 177–234.

Hatchondo, J. C. and L. Martinez (2012). On the benefits of gdp-indexed government debt: Lessons from a model of sovereign defaults. *Economic Quarterly 98*(2), 139–157.

Hellwig, M. (2021). Safe assets, risky assets, and dynamic inefficiency in overlapping-generations economies. Working Paper.

Hirano, T., R. Jinnai, and A. A. Toda (2023). Leverage, endogenous unbalanced growth, and asset price bubbles. Working Paper.

Jiang, Z., H. Lustig, S. Van Nieuwerburgh, and M. Z. Xiaolan (2024). The U.S. public debt valuation puzzle. *Econometrica 92*(4), 1309–1347.

Kamstra, M. and R. J. Shiller (2009). The case for trills: giving the people and their pension funds a stake in the wealth of the nation. Technical report. Cowles Foundation Working Paper CFDP 1717.

Kocherlakota, N. R. (2023a). Infinite debt rollover in stochastic economies. *Econometrica 91*(5), 1629–1658.

Kocherlakota, N. R. (2023b). Public debt bubbles in heterogeneous agent models with tail risk. *International Economic Review 64*(2), 491–509.

Martin, A. and J. Ventura (2012). Economic growth with bubbles. *American Economic Review 102*(6), 3033–3058.

Mitchener, K. J. and G. A. Pina (2026). The state-contingent debt premium: Evidence from french public bonds. NBER Working Paper 35136.

Newey, W. K. and K. D. West (1987). A simple, positive semi-definite, heteroskedasticity and autocorrelation consistent covariance matrix. *Econometrica 55*(3), 703–708.

Reis, R. (2021). The constraint on public debt when r < g but g < m. Working Paper.

Rozental, A. A. (1959). Variable-return bonds—the french experience. *The Journal of Finance 14*(4), 520–530.

Santos, M. S. and M. Woodford (1997). Rational asset pricing bubbles. *Econometrica 65*(1), 19–57.

Scheinkman, J. A. (1977). Notes on asset pricing. Mimeo, University of Chicago.

Shiller, R. J. (1998). *Macro markets:creating institutions for managing society's largest economic risks*. Oxford University Press.

Tirole, J. (1985). Asset bubbles and overlapping generations. *Econometrica 53*(6), 1499–1528.

Wilson, C. A. (1981). Equilibrium in dynamic models with an infinity of agents. *Journal of Economic Theory 24*, 95–111.

Zilcha, I. (1990). Dynamic efficiency in overlapping generations models with stochastic production. *Journal of Economic Theory 52*(2), 364–379.

Zilcha, I. (1991). Characterizing efficiency in stochastic overlapping generations models. *Journal of Economic Theory 55*(1), 1–16.

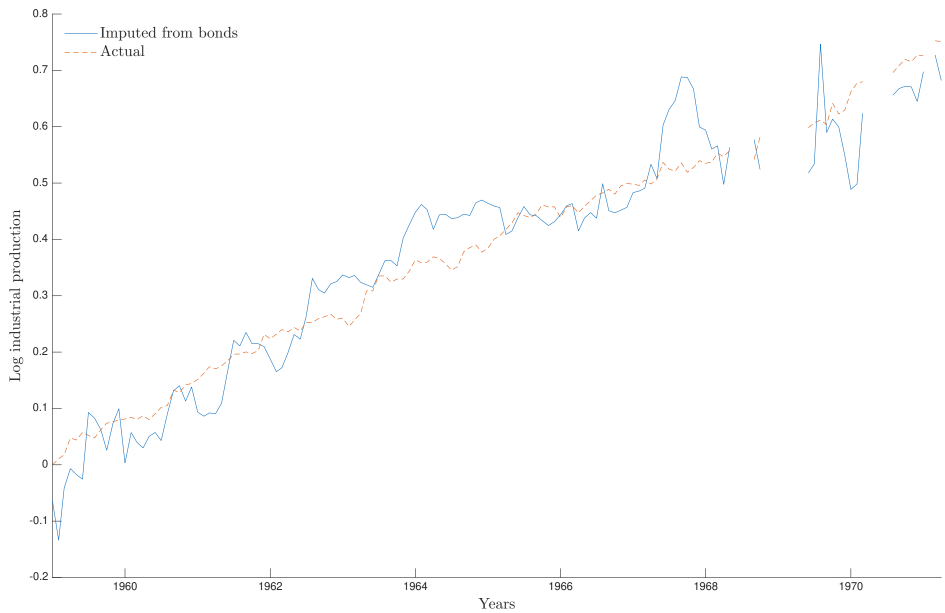

Figure 1: Log-Industrial production index inferred from bond prices (solid line), and the index of log industrial production (dotted line). The monthly observations in March 1963 and May-June 1968, which represent transient drops in the monthly industrial production series (which reverse in the subsequent month), are omitted from the figure. The gaps in the solid line reflect missing bond price data in the DFIH database.

	One sample	Two subsamples	
	g^Q	$g^Q_{\text{first half}}$	$g^Q_{\text{second half}}$
σ =0.042	0.0326	0.0331	0.0344
	(0.0074)	(0.0091)	(0.0165)
σ =0.020	0.0329	0.0335	0.0352
	(0.0073)	(0.0089)	(0.0163)
σ =0.030	0.0328	0.0334	0.0349
	(0.0074)	(0.0090)	(0.0164)
σ =0.050	0.0324	0.0329	0.0341
	(0.0075)	(0.0091)	(0.0166)
σ =0.060	0.0322	0.0326	0.0337
	(0.0076)	(0.0092)	(0.0167)

Table 1: Estimates of g^Q for different subsamples and different volatility assumptions. The column labeled "One sample" reports the estimates of the nonlinear least squares estimation in equation (24). The first row reports the estimate of g^Q using the estimated annualized standard deviation of (log) monthly first differences in industrial production. The other rows consider various volatility choices as a robustness check. Standard errors of the estimated g^Q are reported in parentheses below the estimates and computed using the Newey and West (1987) formula with four lags. (See Appendix B.3 for more details). The columns "subsamples" split the sample (approximately) in the middle and estimate the nonlinear regression of equation (24), except that $\log \widetilde{Y}_{t_i}(P_{t_i}; g^Q, \widehat{\sigma})$ is replaced with $\log \widetilde{Y}_{t_i}(P_{t_i}; \mathbf{1}_{t_i<1965} \times g^Q_{\text{first half}} + \mathbf{1}_{t_i\geq 1965} \times g^Q_{\text{second half}}, \widehat{\sigma})$. On dates with multiple observations across face values (typically identical up to rescaling), I collapse these observations to a single one by taking the mean. This results in 261 observations. Sample years: 1959-1971.

Online Appendix

A Proofs

Proof of Proposition 1. Since w is a fixed portfolio, I write A as shorthand for $A(w)$ throughout the proof of Proposition 1 to simplify notation. Using the definitions of σ_z and z^*, the dynamics for z_t can be written as

$$dz_t = \begin{cases} (A - \alpha - \beta z_t)\, dt + \sigma_z d\widetilde{B}_t & \text{if } z_t > z^*, \\ (A - \varepsilon)\, dt + \sigma_z d\widetilde{B}_t & \text{if } z_t \leq z^*, \end{cases}$$

where $\widetilde{B}_t$ is a standard Brownian motion. Note that dz_t behaves like an Ornstein-Uhlenbeck process whenever $z_t > z^*$ and dz_t behaves like a Brownian motion with (positive) drift $A - \varepsilon > 0$ whenever $z_t \leq z^*$. The Fokker-Planck equation implies that if one can find a solution to the differential equation

$$\frac{\sigma_z^2}{2} \frac{\partial^2 l\,(z)}{\partial z^2} = \frac{\partial \left[l\,(z)\,(A - \alpha - \beta z)\right]}{\partial z} \text{ if } z > z^*, \tag{A.1}$$

and

$$\frac{\sigma_z^2}{2} \frac{\partial^2 l\,(z)}{\partial z^2} = \frac{\partial \left[l\,(z)\,(A - \varepsilon)\right]}{\partial z} \text{ if } z \leq z^*, \tag{A.2}$$

with the additional property that $l(z) > 0$ and $\int_{-\infty}^{+\infty} l(z)dz = 1$, then the process z_t possesses a stationary density.

To construct such a solution, integrate both sides of (A.1) and (A.2) with respect to z and set the integration constant to zero by a standard argument[22] to arrive at the pair of

[22] If the constant of integration were not zero, then the resulting solution of the ODE would not satisfy $\int_{-\infty}^{+\infty} l(z)\, dz < \infty$.

differential equations

$$\frac{\sigma_z^2}{2}\frac{\partial l(z)}{\partial z} = \begin{cases} l(z)(A-\alpha-\beta z) & \text{if } z > z^*, \\ l(z)(A-\varepsilon) & \text{if } z \leq z^*. \end{cases}$$

This differential equation has solution

$$l(z) = \begin{cases} D_1 e^{\frac{2}{\sigma_z^2}\left[(A-\alpha)z-\beta\frac{z^2}{2}\right]} & \text{if } z > z^*, \\ D_2 e^{\frac{2}{\sigma_z^2}(A-\varepsilon)z} & \text{if } z \leq z^*. \end{cases} \tag{A.3}$$

for two positive constants D_1 and D_2, which will be determined next. The requirement that $l \in C^1$ implies the following continuity and differentiability properties at z^*:

$$D_1 e^{\frac{2}{\sigma_z^2}\left[(A-\alpha)z^*-\beta\frac{(z^*)^2}{2}\right]} = D_2 e^{\frac{2}{\sigma_z^2}(A-\varepsilon)z^*}, \tag{A.4}$$

and

$$D_1 e^{\frac{2}{\sigma_z^2}\left[(A-\alpha)z^*-\beta\frac{(z^*)^2}{2}\right]}\left[(A-\alpha)-\beta z^*\right] = D_2(A-\varepsilon)e^{\frac{2}{\sigma_z^2}(A-\varepsilon)z^*}. \tag{A.5}$$

The definition of z^* implies that if equation (A.4) is satisfied, then (A.5) is automatically satisfied. Therefore, D_2 can be arbitrary and D_1 must be set so that

$$D_1 = D_2 \exp\left\{\frac{2}{\sigma_z^2}\left[\beta\frac{(z^*)^2}{2} - (\varepsilon-\alpha)z^*\right]\right\}. \tag{A.6}$$

Substituting (A.6) into the top equation of (A.3) and choosing D_2 so that $\int_{-\infty}^{+\infty} l(z)\,dz = 1$ leads to (8). ■

Proof of Lemma 1. Note that under the probability measure Q, equation (3) becomes

$$dW_t = (rW_t - X_t)\,dt + W_t w'\sigma dB_t^Q. \tag{A.7}$$

Multiplying both sides of (A.7) with e^{-rt}, defining $G_t \equiv e^{-rt}W_t + \int_0^t e^{-rs}X_s ds$, and applying

Ito's Lemma to G_t shows that G_t is a local martingale under Q. Since both W_t and X_t are bounded below by zero, so is G_t, and accordingly G_t is a supermartingale. Therefore $G_t \geq E_t^Q G_T$, which implies

$$W_t \geq E_t^Q \int_t^T e^{-r(u-t)} X_u du + E_t^Q e^{-r(T-t)} W_T.$$

Since $W_T > 0$ for all T, it follows that

$$W_t \geq E_t^Q \int_t^T e^{-r(u-t)} X_u du = \int_t^T P_t^{(X_u)} du.$$

Since $X_t > 0$, it follows that $P_t^{(X_u)} > 0$. Accordingly, $\int_t^T P_t^{(X_u)} du$ is an increasing function of T, which is bounded above by W_t. Therefore it converges to a finite limit, $\int_t^\infty P_t^{(X_u)} du < \infty$, which is statement (a). Statements (b) and (c) are corollaries of statement (a). ■

Proof of Proposition 2. Carrying out the maximization in (7) and using the definition of κ implies

$$\max_w A(w) = r - g + \frac{1}{2}\kappa'\kappa + \frac{\sigma_C^2}{2}. \tag{A.8}$$

The assumption of the proposition implies $r - g + \frac{1}{2}\kappa'\kappa + \frac{\sigma_C^2}{2} > 0$. Therefore, there exists a portfolio w and an $\varepsilon > 0$ such that $A(w) > \varepsilon$. Accordingly, the assumptions of Proposition 1 hold. Therefore, $\log \frac{X_t}{C_t}$ possesses a stationary density (which can be computed explicitly by using (9), applying a Jacobi transformation and then using Proposition 1). In addition, Lemma 1 implies $\lim_{T\to\infty} \int_T^\infty P_t^{(X_u)} du = 0$, and $\int_t^\infty P_t^{(X_u)} du < \infty$. The assumption of the proposition also implies that $r - g^Q < 0$, which leads to the conclusion $\lim_{T\to\infty} P_t^{(C_T)} = \infty$ and $\lim_{T\to\infty} \int_T^\infty P_t^{(C_u)} du = \infty$. ■

Proof of Lemma 2. Using Girsanov's theorem, the dynamics of z_t under the probability measure $\widetilde{Q}$ are given by

$$\begin{aligned} dz_t = &\left(r - g^Q + \frac{\sigma_C^2}{2} - \frac{1}{2}w'\sigma\sigma'w - \max(\alpha + \beta z_t, \varepsilon)\right)dt \\ &+ w'\sigma\left(dB_t^{\widetilde{Q}} + \left[\begin{array}{c} \sigma_C \\ 0_{(M-1)\times 1} \end{array}\right]dt\right) - \sigma_C\left(dB_t^{C,\widetilde{Q}} + \sigma_C dt\right). \end{aligned} \tag{A.9}$$

Collecting terms, and using the definitions of ϕ' and σ_z leads to (18) and (19). ■

B Estimation: Additional results and discussion

B.1 The impact of higher interest rates

This section assumes that $Y_t \gg Y_{1955}$. The first step is to derive the marginal change in the estimated value of g^Q when the discount rates are increased by $\epsilon > 0$.

To start, define the price of a bond conditional on a redemption date $T \geq t$ as

$$\begin{aligned} p\left(Y_t; T, i_t + \epsilon, g^Q\right) \;\equiv\; & 100 \times \left\{\sum_{T_i \in [t,T]} e^{-(i_t+\epsilon)(T_i - t)}\left[\begin{array}{c} 0.05+ \\ 0.0005E_t^Q\left(Y_{T_i} - Y_{1955}\right) \end{array}\right]\right\} \\ & +105 \times e^{-(i_t+\epsilon)(T-t)}. \end{aligned} \tag{B.1}$$

Taking the expectation over T results in the price of the bond

$$\widehat{P}\left(Y_t; i_t + \epsilon, g^Q\right) \equiv \sum_{T \geq t} q_t(T) p\left(Y_t; T, i_t + \epsilon, g^Q\right). \tag{B.2}$$

The main difference between equation (B.1) and equation (23) is that the option payoff $(Y_{T_i} - Y_{1955})^+$ is replaced with the payoff of a forward contract $(Y_{T_i} - Y_{1955})$, an approximation that is justified for values of $Y_t \gg Y_{1955}$.

With this approximation, the computation of $\widetilde{Y}$ simplifies considerably. Equations (B.1), (B.2), along with $E^Q\left(Y_{T_i}\right) = Y_t e^{g^Q(T_i - t)}$ and the definition $P_t = \widehat{P}\left(\widetilde{Y}_t; i_t + \epsilon, g^Q\right)$ yield

$$\log \widetilde{Y}\left(P_t; i_t + \epsilon, g^Q\right) = \log\left(P_t - A_t(i_t + \epsilon)\right) - \log B_t\left(i_t + \epsilon - g^Q\right) - \log(0.05),$$

where

$$B_t\left(i_t+\epsilon-g^Q\right) = \sum_{T\geq t}\left\{q_t(T)\times\sum_{T_i\in[t,T]}e^{-\left(i_t+\epsilon-g^Q\right)(T_i-t)}\right\}, \tag{B.3}$$

$$A_t\left(i_t+\epsilon\right) = \sum_{T\geq t}\left\{q_t(T)\times\left[\begin{array}{c}\sum_{T_i\in[t,T]}e^{-(i_t+\epsilon)(T_i-t)}\times[5-0.05Y_{1955}]\\ +105\times e^{-(i_t+\epsilon)(T-t)}\end{array}\right]\right\}.$$

Next, consider the minimization problem $\min_{\alpha,g^Q} F\left(\alpha,g^Q;\epsilon\right)$, where

$$F\left(\alpha,g^Q;\epsilon\right)\equiv\frac{1}{2}\frac{1}{N}\sum_{i=1}^{N}\left(\log\widetilde{Y}\left(P_{t_i};i_{t_i}+\epsilon,g^Q\right)+\alpha-\log Y_{t_i}\right)^2. \tag{B.4}$$

The first-order conditions of this problem are

$$F_\alpha = \frac{1}{N}\sum_{i=1}^{N}\left\{\alpha+\log\widetilde{Y}\left(P_{t_i};i_{t_i}+\epsilon,g^Q\right)-\log Y_{t_i}\right\}=0, \tag{B.5}$$

$$F_{g^Q} = \frac{1}{N}\sum_{i=1}^{N}\left\{\left(\alpha+\log\widetilde{Y}\left(P_{t_i};i_{t_i}+\epsilon,g^Q\right)-\log Y_{t_i}\right)\frac{\partial\log\widetilde{Y}\left(P_{t_i};i_{t_i}+\epsilon,g^Q\right)}{\partial g^Q}\right\}=0. \tag{B.6}$$

Next consider the effects of varying the parameter ϵ. The implicit function theorem implies

$$\frac{\partial g^Q}{\partial\epsilon}=-\frac{F_{\alpha\alpha}F_{g^Q\epsilon}-F_{\alpha g^Q}F_{\alpha\epsilon}}{F_{g^Qg^Q}F_{\alpha\alpha}-\left(F_{\alpha g^Q}\right)^2}. \tag{B.7}$$

Differentiating (B.5) with respect to α implies $F_{\alpha\alpha}=1$. Also, using the fact that B_t is a function of $i_t+\epsilon-g^Q$, and accordingly $\frac{\partial\log B_t}{\partial\epsilon}=-\frac{\partial\log B_t}{\partial g^Q}$, yields

$$\begin{aligned}F_{\alpha\epsilon} &= \frac{1}{N}\sum_{i=1}^{N}\left[\frac{\partial\log\left(P_{t_i}-A_{t_i}\left(i_{t_i}+\epsilon\right)\right)}{\partial\epsilon}-\frac{\partial\log B_t\left(i_{t_i}+\epsilon-g^Q\right)}{\partial\epsilon}\right]\\ &= \frac{1}{N}\sum_{i=1}^{N}\left[\frac{\partial\log\left(P_{t_i}-A_{t_i}\left(i_{t_i}+\epsilon\right)\right)}{\partial\epsilon}+\frac{\partial\log B_t\left(i_{t_i}+\epsilon-g^Q\right)}{\partial g^Q}\right]\\ &= \Phi_1-F_{\alpha g^Q},\end{aligned} \tag{B.8}$$

where $F_{\alpha g^Q}$ is shorthand notation for $\frac{\partial^2 F}{\partial\alpha\partial g^Q}$ and

$$\Phi_1 = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{\partial \log\left(P_{t_i} - A_{t_i}\left(i_{t_i}+\epsilon\right)\right)}{\partial\epsilon}\right]. \tag{B.9}$$

By a similar argument

$$F_{g^Q\epsilon} = -F_{g^Q g^Q} - \Phi_2, \text{ with } \Phi_2 = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{\partial \log\left(P_{t_i} - A_{t_i}\left(i_{t_i}+\epsilon\right)\right)}{\partial\epsilon}\frac{\partial \log B_{t_i}\left(i_{t_i}+\epsilon-g^Q\right)}{\partial g^Q}\right]. \tag{B.10}$$

Combining $F_{\alpha\alpha}=1$, (B.7), (B.8), and (B.10) yields

$$\frac{\partial g^Q}{\partial\epsilon} = -\frac{F_{\alpha\alpha}F_{g^Q\epsilon} - F_{\alpha g^Q}F_{\alpha\epsilon}}{F_{g^Q g^Q}F_{\alpha\alpha} - \left(F_{\alpha g^Q}\right)^2} = \frac{F_{g^Q g^Q} + \Phi_2 + F_{\alpha g^Q}\left(\Phi_1 - F_{\alpha g^Q}\right)}{F_{g^Q g^Q} - \left(F_{\alpha g^Q}\right)^2} = 1 + \frac{\Phi_2 + F_{\alpha g^Q}\Phi_1}{F_{g^Q g^Q} - \left(F_{\alpha g^Q}\right)^2}.$$

Differentiating (B.5) with respect to g^Q yields

$$F_{\alpha g^Q} = -\frac{1}{N}\sum_{i=1}^{N}\left[\frac{\partial \log B_{t_i}\left(i_{t_i}+\epsilon-g^Q\right)}{\partial g^Q}\right]. \tag{B.11}$$

Combining (B.11) with (B.9) and (B.10) gives

$$\begin{aligned}
\Phi_2 + F_{\alpha g^Q}\Phi_1 &= \frac{1}{N}\sum_{i=1}^{N}\left[\frac{\partial \log\left(P_{t_i} - A_{t_i}\left(i_{t_i}+\epsilon\right)\right)}{\partial\epsilon}\frac{\partial \log B_{t_i}\left(i_{t_i}+\epsilon-g^Q\right)}{\partial g^Q}\right] \\
&\quad -\frac{1}{N}\sum_{i=1}^{N}\left[\frac{\partial \log B_{t_i}\left(i_{t_i}+\epsilon-g^Q\right)}{\partial g^Q}\right] \times \frac{1}{N}\sum_{i=1}^{N}\left[\frac{\partial \log\left(P_{t_i} - A_{t_i}\left(i_{t_i}+\epsilon\right)\right)}{\partial\epsilon}\right] \\
&= \widehat{cov}\left(\frac{\partial \log\left(P_{t_i} - A_{t_i}\left(i_{t_i}+\epsilon\right)\right)}{\partial\epsilon}, \frac{\partial \log B_{t_i}\left(i_{t_i}+\epsilon-g^Q\right)}{\partial g^Q}\right),
\end{aligned}$$

where $\widehat{cov}(x,y)$ is the (empirical) covariance between x and y. Accordingly,

$$\frac{\partial g^Q}{\partial\epsilon} = 1 + \frac{\widehat{cov}\left(\frac{\partial \log\left(P_{t_i} - A_{t_i}\left(i_{t_i}+\epsilon\right)\right)}{\partial\epsilon}, \frac{\partial \log B_{t_i}\left(i_{t_i}+\epsilon-g^Q\right)}{\partial g^Q}\right)}{F_{g^Q g^Q} - \left(F_{\alpha g^Q}\right)^2}. \tag{B.12}$$

The sign of $F_{g^Q g^Q} - \left(F_{\alpha g^Q}\right)^2$ is positive around a (local) minimum. Therefore assuming that the covariance in the numerator of (B.12) is positive, it follows that $\frac{\partial g^Q}{\partial \epsilon} > 1$.

In our sample we can confirm that this last conclusion indeed holds. Figure D.5 depicts the values of $g^Q(\epsilon)$ for different values of ϵ and shows that the resulting values lie above the 45-degree line starting at $\epsilon = 0$. Therefore, if one used higher discount rates in the estimation of g^Q, the difference between $\widehat{g}^Q$ and these higher discount rates would increase compared to using lower discount rates.

B.2 The impact of inflation

To start, postulate the following, simple dynamics for the price level, Π_t,

$$\frac{d\Pi_t}{\Pi_t} = \pi dt + \sigma_\Pi dB_t^\Pi,$$

where $dB_t^\Pi = \rho dB_t + \sqrt{1-\rho^2} dB_t^\perp$ is a Brownian motion with correlation ρ with the Brownian motion B_t, which drives the dynamics of $\frac{dY_t}{Y_t}$. $B_t^\perp$ is a Brownian motion that is orthogonal to B_t, and π, σ_Π are constants. Without loss of generality, assume $\sigma_\Pi > 0$. Letting m_t denote the (real) stochastic discount factor, the price, $P_t^{Y(T_i)}$, of a security that pays Y_{T_i} at time T_i is

$$P_t^{Y(T_i)} = Y_t E_t \left[\frac{m_{T_i}}{m_t} \frac{\Pi_t}{\Pi_{T_i}} \frac{Y_{T_i}}{Y_t} \right].$$

Equivalently, using the risk-neutral measure Q,

$$P_t^{Y(T_i)} = Y_t E_t^Q \left[e^{-r(T_i - t)} \frac{\Pi_t}{\Pi_{T_i}} e^{\left(g^Q - \frac{\sigma^2}{2}\right)(T_i - t) + \sigma\left(B_{T_i}^Q - B_t^Q\right)} \right], \tag{B.13}$$

where the dynamics of the price level under the measure Q are given by

$$\frac{d\Pi_t}{\Pi_t} = \pi^Q dt + \sigma_\Pi dB_t^{\Pi,Q} = \pi^Q dt + \sigma_\Pi \left(\rho dB_t^Q + \sqrt{1-\rho^2} dB_t^{\perp,Q}\right), \tag{B.14}$$

where π^Q is the drift of the price level under the risk-neutral measure. The price of a nominal bond paying one dollar at time T_i is

$$P_t^{nom,T_i} = E_t^Q\left[e^{-r(T_i-t)}\frac{\Pi_t}{\Pi_{T_i}}\right] = e^{-i(T_i-t)}, \text{ where } i = r + \pi^Q - \sigma_\Pi^2. \tag{B.15}$$

Using (B.15) in conjunction with (B.13) gives

$$\begin{aligned} P_t^{Y(T_i)} &= Y_t E_t^Q\left[e^{-r(T_i-t)} e^{-\left(\pi^Q - \frac{\sigma_\Pi^2}{2}\right)(T_i-t) - \sigma_\Pi\left(B_{T_i}^{\Pi,Q} - B_t^{\Pi,Q}\right)} e^{\left(g^Q - \frac{\sigma^2}{2}\right)(T_i-t) + \sigma\left(B_{T_i}^Q - B_t^Q\right)}\right] \\ &= Y_t e^{\left(g^Q - i\right)(T_i-t)} e^{-\rho\sigma\sigma_\Pi(T_i-t)}. \end{aligned} \tag{B.16}$$

Accordingly, if $Y_t \gg Y_{1955}$, the right-hand side of (23) becomes

$$p\left(Y_t; T\right) = 100 \sum_{T_i \in [t,T]} e^{-i(T_i-t)} \left[\begin{array}{c} (0.05 - 0.0005 Y_{1955}) \\ +0.0005 Y_t e^{\left(g^Q - \rho\sigma\sigma_\Pi\right)(T_i-t)} \end{array}\right] + 105 \times e^{-i(T-t)}. \tag{B.17}$$

Equation (B.17) makes it obvious that the estimated value of $\widehat{g}^Q$ satisfies $\widehat{g}^Q = g^Q - \rho\sigma\sigma_\Pi$, or equivalently

$$g^Q = \widehat{g}^Q + \rho\sigma\sigma_\Pi.$$

In particular, if $\rho \geq 0$, so that positive shocks to output raise unexpected inflation (the "Phillips curve" in the terminology of macroeconomics), the true g^Q exceeds the estimated $\widehat{g}^Q$.

Remark 4 *Whatever the true value of ρ, the magnitude of $\rho\sigma\sigma_\Pi$ is bounded above by $\sigma\sigma_\Pi$, the product of the volatility of industrial production, σ, and the volatility of unexpected inflation, σ_Π. Given that both these volatilities are only a few percentage points, the magnitude of $\rho\sigma\sigma_\Pi$ is probably negligible.*

Combining $g^Q \geq \widehat{g}^Q$ and (a) using that the observed $\widehat{g}^Q - i + \pi$ is non-negative in the data, and (b) assuming a positive inflation risk premium $\left(\pi^Q - \sigma_\Pi^2 \geq \pi\right)$ yields the inequalities

$$\begin{aligned} 0 &\leq \widehat{g}^Q - i + \pi \leq g^Q - i + \pi = g^Q - \left(r + \pi^Q - \sigma_\Pi^2\right) + \pi \\ &= g^Q - r - \left(\pi^Q - \sigma_\Pi^2 - \pi\right) \leq g^Q - r. \end{aligned}$$

For simplicity the analysis assumed constant r, π. If instead both the real rate and the inflation rate are time-varying, then, letting $P_t^{nom,T_i} = e^{-i_t(T_i-t)}$, equation (B.16) becomes

$$P_t^{Y(T_i)} = Y_t e^{\left(g^Q - i_t\right)(T_i - t)} \mathcal{A}_t,$$

where $\log \mathcal{A}_t(T_i) = \log E_t^{\mathcal{Q}}\left[e^{-\int_t^{T_i} r_s ds} \frac{\Pi_t}{\Pi_{T_i}}\right] - \log E_t^{Q}\left[e^{-\int_t^{T_i} r_s ds} \frac{\Pi_t}{\Pi_{T_i}}\right]$ and the likelihood ratio between $\mathcal{Q}$ and Q is given by $\frac{d\mathcal{Q}}{dQ} = e^{-\frac{\sigma^2}{2}(T_i - t) + \sigma\left(B_{T_i}^Q - B_t^Q\right)}$. Clearly, if the Brownian motions driving the dynamics of r_t, π_t, Π_t are independent of the Brownian motion that drives the dynamics of Y_t, then $\log \mathcal{A}_t(T_i) = 0$. More generally, when the volatility, σ, of $d \log Y_t$ is small and assuming that the volatility of the shocks to the real interest rate and the price level are also small, $\frac{\log \mathcal{A}_t(T_i)}{T_i - t}$ will be close to zero.

B.3 Standard errors

This section derives the standard error of the estimated $\widehat{g}^Q$. To start, note that the right-hand side of equation (23) can be computed using the Black-Scholes formula. Specifically,

$$p(Y_t; T) = 100 \times \sum_{T_i \in [t,T]} \left[e^{-i_t(T_i - t)} 0.05 + 0.0005 \times C^{BS}\left(Y_t, Y_{1955}, T_i - t, i_t, \sigma, i_t - g^Q\right)\right] + 105 e^{-i_t(T-t)},$$

where $C^{BS}(S_t, K, \tau, i, \sigma, q)$ denotes the Black-Scholes Call option formula, with S_t denoting the price of the underlying, K the strike price, τ the time to maturity, i_t the interest rate at time t, σ the volatility of the underlying, and q the continuous dividend yield on the underlying. Note that Y_t is not a directly traded security, but by assuming a dividend yield equal to $q = i_t - g^Q$, the drift of $\frac{dY_t}{Y_t}$ in the Black-Scholes formula under the risk-neutral measure is given by $i_t - q = i_t - \left(i_t - g^Q\right) = g^Q$, which is precisely (21). The chain rule and the implicit function theorem imply that

$$\begin{aligned} Z_t &\equiv \frac{\partial \log \widetilde{Y}_t(P_t; g^Q, \sigma)}{\partial g^Q} \\ &= \frac{1}{\widetilde{Y}_t} \frac{E_T \left[\sum_{T_i \in [t,T]} \frac{\partial C^{BS}}{\partial q} \left(\widetilde{Y}_t, Y_{1955}, T_i - t, i_t, \sigma, i_t - g^Q\right)\right]}{E_T \left[\sum_{T_i \in [t,T]} \frac{\partial C^{BS}}{\partial \widetilde{Y}_t} \left(\widetilde{Y}_t, Y_{1955}, T_i - t, i_t, \sigma, i_t - g^Q\right)\right]}, \end{aligned}$$

where E_T denotes expectation over the random termination times (conditional on $T \geq t$). Using standard properties of the Black-Scholes formula $C_q^{BS} = -\tau \widetilde{Y} C_{\widetilde{Y}}^{BS}$ and therefore

$$Z_t = -\frac{E_T \left[\sum_{T_i \in [t,T]} (T_i - t) \times \frac{\partial C^{BS}\left(\widetilde{Y}_t, Y_{1955}, T_i - t, i_t, \sigma, i_t - g^Q\right)}{\partial \widetilde{Y}_t}\right]}{E_T \left[\sum_{T_i \in [t,T]} \frac{\partial C^{BS}\left(\widetilde{Y}_t, Y_{1955}, T_i - t, i_t, \sigma, i_t - g^Q\right)}{\partial \widetilde{Y}_t}\right]}.$$

Standard results in nonlinear least squares imply that the standard error of $\widehat{g}^Q$ is given by the square root of the lower right element of the matrix

$$\frac{1}{N} \left(\frac{X'X}{N}\right)^{-1} \widehat{S} \left(\frac{X'X}{N}\right)^{-1}, \tag{B.18}$$

where X is the $N \times 2$ matrix whose first column is a column of ones and whose second column collects the values of Z_t across the N observations, and $\widehat{S}$ is the Newey-West HAC estimator,

$$\widehat{S} = \widehat{\Gamma}_0 + \sum_{j=1}^{L} \left(1 - \frac{j}{L+1}\right) \left(\widehat{\Gamma}_j + \widehat{\Gamma}_j'\right), \qquad \widehat{\Gamma}_j \equiv \frac{1}{N} \sum_{t=j+1}^{N} X_t u_t u_{t-j} X_{t-j}', \tag{B.19}$$

with lag-length L chosen according to $N^{1/4}$.

Remark 5 *Note that when* $Y_t \gg Y_{1955}$, $\frac{\partial C^{BS}\left(\widetilde{Y}_t, Y_{1955}, T_i - t, i_t, \sigma, i_t - g^Q\right)}{\partial \widetilde{Y}_t} \approx e^{-\left(i_t - g^Q\right)(T_i - t)}$ *and there-*

fore Z_t is approximately equal to a duration-like expression

$$Z_t \approx -\frac{E_T\left[\sum_{T_i\in[t,T]}(T_i-t)\times e^{-\left(i_t-g^Q\right)(T_i-t)}\right]}{E_T\left[\sum_{T_i\in[t,T]}e^{-(i_t-g^Q)(T_i-t)}\right]}.$$

Remark 6 *When the regression allows for separate estimates of g^Q across subsamples, the matrix X is adjusted in the obvious way: the second column Z_t is replaced by a block of columns formed by interacting Z_t with the indicator variables for each subsample.*

C Data

The following is the list of the data sources. FRED is the acronym for the Federal Reserve Economic Data, maintained by the Research Department at the Federal Reserve Bank of St. Louis.

1. Real Gross Domestic Product for France, Annual, 1950-onwards, Source: FRED NGDPRXDCFRA.

2. Inflation, consumer prices for France, Percent, Annual, 1960-onwards, Source: FRED FPCPITOTLZGFRA.

3. Prices for "Bons d'Équipement", August 1956 – April 1971, Biweekly. Source: DFIH — Data for Financial History (https://dfih.fr). In total, there are 875 observations (across the three bond denominations described in Section E). On any given observation date there are multiple trades across the three denominations, which are typically all identical per unit of face value. On dates with multiple observations across face values, I collapse these observations to a single one by taking the mean. As a result, there are effectively 320 bond-price observations (261 post 1959).

4. Industrial production (Production, Sales, Work Started and Orders: Production Volume: Economic Activity: Industry Except Construction) for France, monthly, seasonally adjusted, January 1956 - onwards. Source: FRED FRAPROINDMISMEI. The

monthly series (readily available on FRED) excludes construction. However, the index of industrial production that is used for the determination of the bonus coupon of the "Bons d'Équipement" includes construction. To ensure that the value of industrial production on the date of coupon payments (June 1 of each year) matches exactly the value that enters the coupon calculation, I hand collected the annual numbers of the industrial production index (from INSEE "Annuaire statistique" publications for the corresponding years) and interpolated the monthly values of the industrial production index according to

$$y_t = y_{[t]} + \left(x_t - x_{[t]}\right) + (t - [t]) \left(x_{[t]} - x_{[t+1]} + y_{[t+1]} - y_{[t]}\right), \tag{C.1}$$

where $[t]$ is the most recent June 1 on or before t, x_t the (log) monthly index of industrial production (from FRED) at time t, $x_{[t]}$ is the (log) value of the monthly index (from FRED) on the most recent June 1 on or before t, and $y_{[t]}$ is the (log) value of the index of industrial production entering the calculation of the coupon on the most recent June 1 on or before t. The series resulting from (C.1) implies a continuous series for y_t, which coincides with the annual values used in the coupon determination every June 1.[23] While $x_{[t]}$ and $y_{[t]}$ are quite similar in the data, equation (C.1) captures more precisely the relevant industrial production index for the bond coupons.

5. Interest rates, Annual, 1953–1972. Source: Chauveau (1975), Table 3, Column 1 (Net rate at issuance of "first category" bonds). The data in Chauveau (1975) is sourced from INSEE (Institut national de la statistique et des études économiques) and CDC (Caisse des dépôts). Monthly interest rates are linearly interpolated from the corresponding annual values.[24]

[23]The monthly series for x_t from FRED includes two outlier episodes (March 63 and May-June 68) that appear to be the result of national strikes in those months. The monthly index of industrial production collapses during those two episodes and fully rebounds in the month thereafter. Importantly, the annual index does not exhibit a noticeable jump during those years. (It is worth noting that only the annual value of the index of industrial production is relevant for the coupon payment). I ignore these two outlier events by linearly interpolating the value of x_t during those two episodes between the corresponding, neighboring monthly observations.

[24]In performing the linear interpolation, I assume that the annual value corresponds to the value of the interest rate at the midpoint of year t.

To confirm that the yields of bonds in the primary market are lower bounds to the discount rates in the secondary market, I performed the following exercise: I compared the yields reported in Chauveau (1975) to the yields depicted in Figure 5 of Mitchener and Pina (2026) for the bonds issued by the PTT (Postes, Télégraphes et Téléphones). The PTT bonds were issued at the same time as the Bons d'Équipement and had similar coupon and redemption provisions, except that they were conventional bonds with no indexed payment. I combined the nominal interest rates depicted in Figure D.3 of this paper with Figure 5 of Mitchener and Pina (2026), which describes the evolution of the yields to maturity (YTM) of the PTT bonds.[25] The YTM of the PTT bonds are higher than the discount rates I use in this paper except for 9 of the approximately 190 observations. A caveat is that the YTM of the PTT bonds become implausibly noisy and volatile towards the end of the sample (starting in 1967). This is unsurprising: the PTT bond had a relatively small issuance, and the periodic redemption provisions imply that the monetary value of outstanding bonds was probably low by the end of the sample. (The yields in Chauveau (1975) don't exhibit implausible volatility.)

[25] I used Matlab code to exactly match the axes of the graphs so as to ensure their comparability.

D Additional Figures

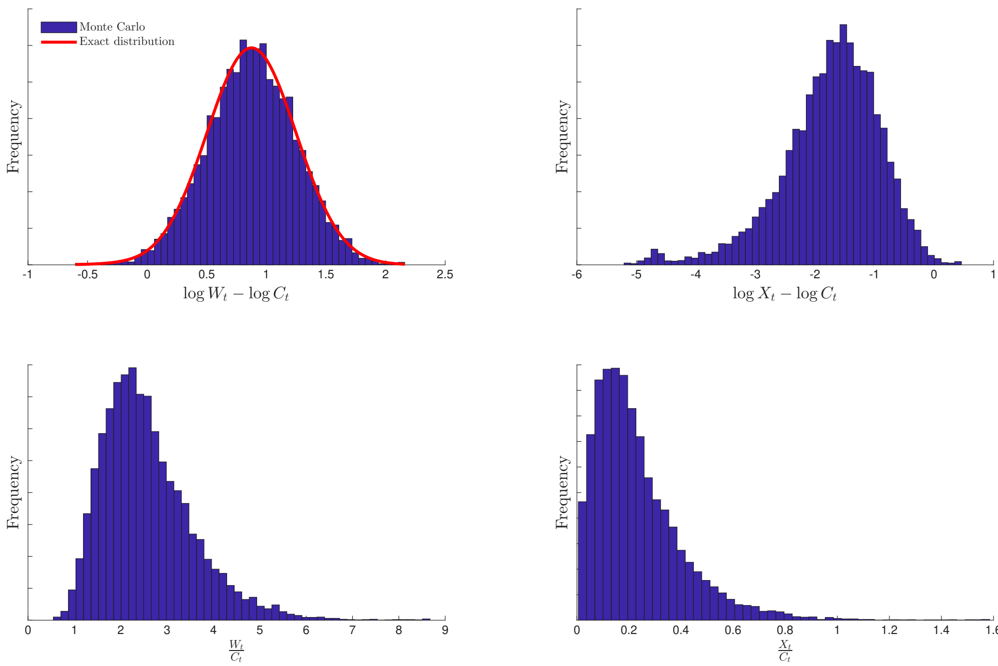

Figure D.1: Simulated and exact distribution of the log fund-to-cash-flow value, $z_t = \log(W_t) - \log(C_t)$ (top left), and simulated values of $\log(X_t) - \log(C_t)$ (top right), $\frac{W_t}{C_t}$ (bottom left) and $\frac{X_t}{C_t}$ (bottom right). Parameters are $A(w) = 0.08, \varepsilon = 0.01, \alpha = 0.01, \beta = 0.08, \sigma_z = 0.15$. The simulations are based on 10,000 independent paths of length 600 years each. The simulations for z_t are computed using an Euler-Maruyama scheme with $dt = 0.01$.

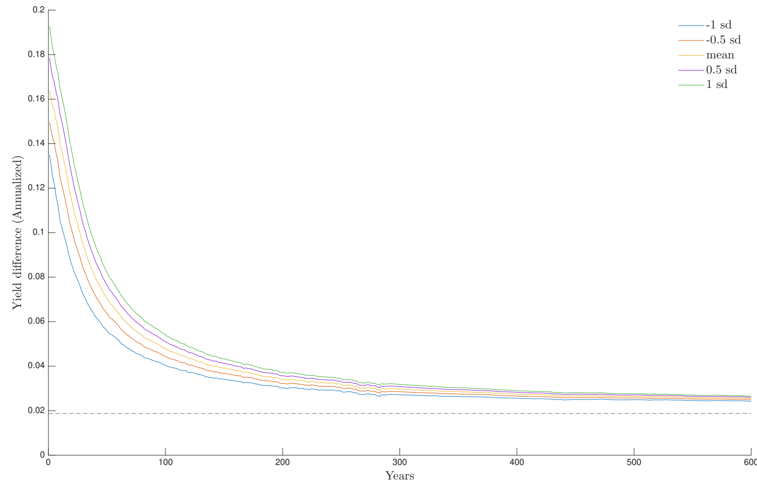

Figure D.2: Difference in the yields of dividend strips to X_T and C_T. The different lines correspond to initializing z_t at the stationary mean of z $\pm 0, \pm 0.5, \pm 1$ stationary standard deviations. The parameters are the same as in Figure D.1. The value of $\widetilde{A}(w)$ is -0.02. Note that the yields of the dividend strips to C_T are constant, so the declining lines in this figure reflect the decline in the yields of X_T-strips. Note also that the lines don't converge to zero, illustrating that the yields of the long-term strips to C_T and X_T are not identical in the long run.

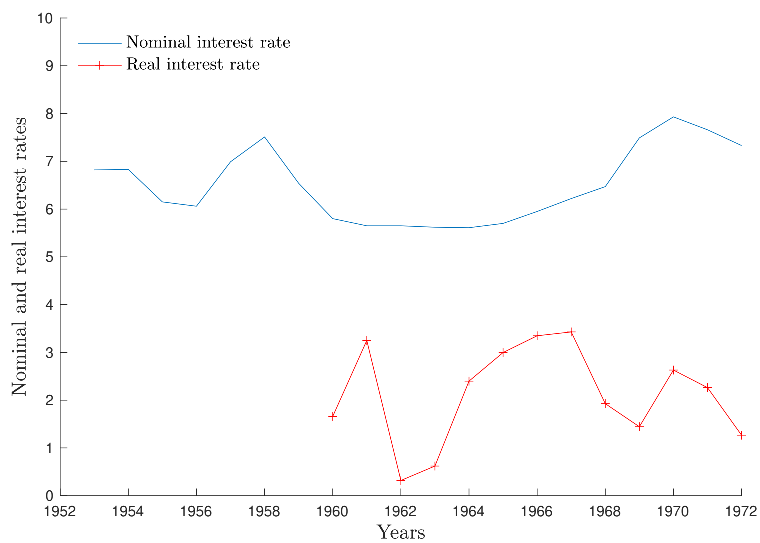

Figure D.3: Nominal Interest rates (from long-term bonds), and the real interest rate computed as the difference between the nominal interest rate and inflation. (Data on inflation are available from 1960 onward.)

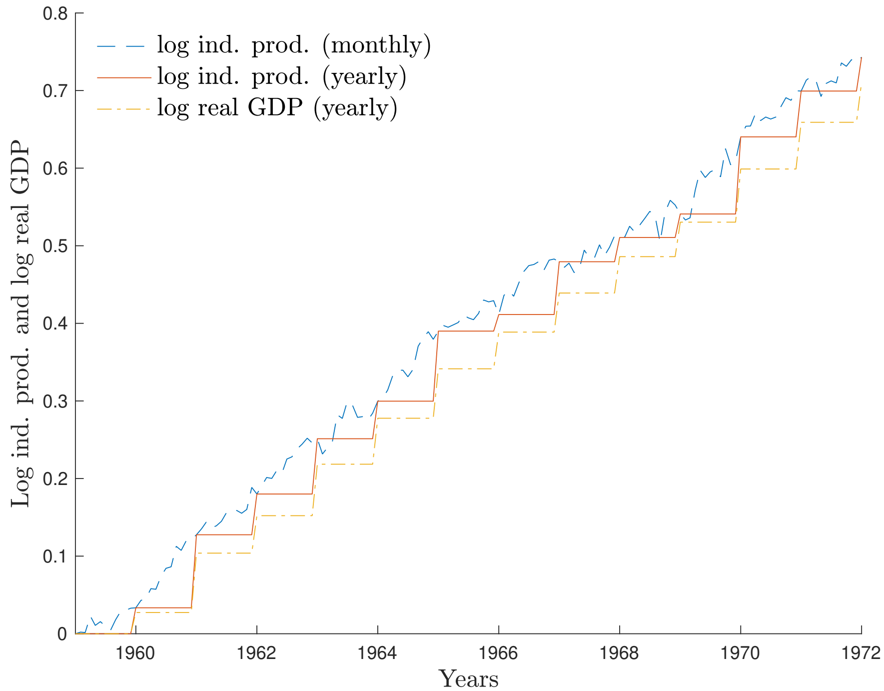

Figure D.4: Log real GDP (yearly) and log industrial production (monthly and yearly). Base year=1959. The monthly observations in March 1963 and May-June 1968, which represent a transient drop in the monthly industrial production series, are omitted from the figure. See Appendix section C on the data sources.

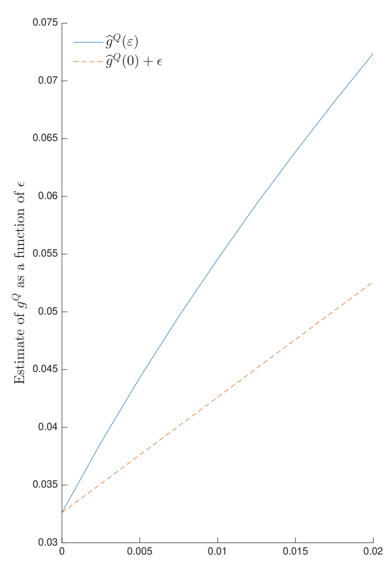

Figure D.5: Estimates of $\widehat{g}^Q$ when discount rates are raised by ϵ. The solid line depicts the estimate of $\widehat{g}^Q(\epsilon)$. The dotted line depicts the 45-degree line starting at $\widehat{g}^Q(0)$.

E The decree for the "Bons d'Équipement"

The following passage (translated using ChatGPT) describes the terms of the "Bons d'Équipement" from the "Journal officiel" (retrievable from https://www.legifrance.gouv.fr/)

The President of the Council of Ministers On the report of the Minister of Economic and Financial Affairs decrees

1. The Minister of Economic and Financial Affairs is authorized to issue Industrial and Agricultural Equipment Bonds redeemable over fifteen years.

2. The industrial and agricultural equipment bonds shall bear interest at a rate of 5 percent, payable in arrears on June 1 of each year, for the first time on June 1, 1957. They shall accrue interest as of June 1, 1956.

3. An interest premium, variable according to the development of French industrial production, shall be paid at each coupon maturity in addition to the nominal interest of 5 percent.

   The index used as the basis for calculating the interest premium shall be the annual industrial production index (overall index including construction, base 100 in 1952) as published by the National Institute of Statistics and Economic Studies.

   The interest premium payable each year shall be calculated according to the level reached by the annual industrial production index during the year immediately preceding.

   The rate of the interest premium is set at 0.05 percent of the nominal capital of the industrial and agricultural equipment bonds for each point of the annual industrial production index of the year concerned that exceeds the level reached by that index in 1955.

   An order of the Minister of Finance, to be published in the Official Journal, shall determine each year, in accordance with the provisions of this article, the amount of the coupon payable on June 1.

4. The industrial and agricultural equipment bonds shall be redeemed at 105 percent of their nominal value.

   Amortization shall be carried out by redemption following drawings by lot.

   The drawings shall take place each year starting in 1957, within the two months preceding the coupon maturity date. They shall involve 100 slips bearing one of the two-digit numbers in the ascending sequence from 00 to 99 inclusive. Six slips shall be drawn at each of the first five drawings, and seven slips at each of the subsequent drawings. The seven remaining slips not drawn in the previous drawings shall correspond to the final redemption date.

   All securities whose number ends with one of the numbers appearing on any of the slips drawn shall be called for redemption at the first coupon maturity following the drawing.

   Interest on the securities designated by lot for principal redemption shall cease to accrue as of the redemption date, and the principal shall be made available to the beneficiary, subject to deduction of the amount of any subsequent coupons not presented.

5. The industrial and agricultural equipment bonds may not be redeemed early.

6. The interest and the interest premium on the industrial and agricultural equipment bonds shall be exempt from all present and future taxes specifically levied on securities.

7. The industrial and agricultural equipment bonds shall be treated as short- and medium-term Treasury securities with respect to endorsement and transfer operations and shall be managed as medium-term Treasury securities. The provisions of Article 32 of Law No. 53-75 of February 6, 1953, relating to the deferred redemption of Treasury securities not entered in the public debt ledger, shall apply to them.
8. The industrial and agricultural equipment bonds shall be issued in bearer or order form. However, registered certificates may be issued, whose nominal value may in no case be less than 10 million francs.
9. The industrial and agricultural equipment bonds shall be issued in denominations of 10,000 francs, 100,000 francs, and 1 million francs of nominal capital.
10. Subscriptions may be paid in cash, by check, or by transfer. They must be paid in full in a single payment.
11. The issue price is set at par.
12. The issue shall open on May 22, 1956, and shall close without prior notice.
13. Subscriptions shall be received at the following offices:
    - Directorate of the Public Debt (Issuance Service), excluding cash subscriptions;
    - Offices of direct Treasury accountants in metropolitan France and overseas departments;
    - Offices of the Treasurer General of France in the Saar, and of the Paymaster General of France in Germany and Austria;
    - Offices of registration, indirect taxation, and customs in metropolitan France;
    - Post, telegraph, and telephone offices in metropolitan France;
    - Savings banks;
    - Municipal credit institutions;
    - Bank of France (head office, branches, and auxiliary offices);
    - Counters of banks and financial institutions, agricultural credit institutions, stockbrokers, securities brokers, and notaries.
14. Receipts, acknowledgments, or discharges issued in connection with the operations provided for by this decree shall be exempt from the special stamp duty on receipts.
15. The Minister of Economic and Financial Affairs is responsible for the execution of this decree, which shall be published in the Official Journal of the French Republic.

Done in Paris, May 14, 1956.