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Kirill Borusyak
Jiafeng Chen
Peter Hull
Lihua Lei

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ABSTRACT

We study identification of differentiated product demand from market-level data when product characteristics can be endogenous. Past work suggests nonparametric identification may be impossible: that is, in addition to standard price instruments, exogenous characteristic-based instruments are essentially necessary to identify sufficiently flexible demand models with standard index restrictions. We show, however, that price counterfactuals are nonparametrically identified using recentered instruments—which combine exogenous price instruments with possibly endogenous product characteristics—under a weaker index restriction and a new condition we term faithfulness. We argue that faithfulness, like the usual completeness condition for nonparametric instrumental variable identification, is best viewed as a technical requirement on the strength of identifying variation rather than a substantive economic or statistical restriction. We show the two conditions are closely related, though generally distinct. We conclude with several practical implications for the parametric estimation of demand counterfactuals.

Kirill Borusyak
University of California, Berkeley
Department of Agricultural and
Resource Economics
and NBER
k.borusyak@berkeley.edu

Jiafeng Chen
Stanford University
Department of Economics
jiafeng@stanford.edu

Peter Hull
Brown University
Department of Economics
and NBER
peter_hull@brown.edu

Lihua Lei
Stanford University
Graduate School of Business
lihualei@stanford.edu

1. Introduction

Conventional wisdom holds that exogenous supply-side instruments trace out demand curves and identify important counterfactuals: how quantities would change under hypothetical price changes (Wright, 1928). Indeed, when considering demand for a single product, Angrist et al. (2000) show such instruments are enough to nonparametrically identify certain average price elasticities across markets. Modern demand analyses, however, involve multiple differentiated goods and target counterfactuals that change prices in a *particular* market (Berry and Haile, 2021). These analyses usually incorporate other kinds of instruments: functions of the observed characteristics of competing products (e.g., Berry et al., 1995, 1999; Gandhi and Houde, 2019). It is well-known that such characteristic-based instruments can be invalid if firm entry is strategic or characteristics are otherwise endogenous (Berry et al., 1995; Petrin et al., 2022) and they can moreover reduce robustness to model misspecification (Andrews et al., 2025).

Can other instruments be found to avoid these issues? Are exogenous shocks to prices enough to identify market-specific counterfactuals? In influential work, Berry and Haile (2014) argue that the answer is generally no: i.e., that exogenous characteristic-based instruments are essentially necessary to flexibly estimate demand with market-level data. Specifically, they consider a nonparametric demand model of the form:

$$\sigma^{-1}(S,P) = X + \xi \equiv \delta, \tag{1}$$

where S and P are J-vectors of the observed quantity shares and prices of J goods in a market, X is a J-vector of an observed product characteristic, and ξ is an unobserved J-vector of demand shocks. The demand shocks and characteristics combine linearly in an index δ. The unknown $\sigma^{-1}(\cdot)$ is the inverse of a demand function $\sigma(d,p)$, which returns the shares that would arise if δ were set to d and P were set to p.[1]

Berry and Haile (2014) argue that observing a J-vector of exogenous price instruments Z—or even having exogenous prices—generally does not identify $\sigma(\cdot)$ or even price counterfactuals, i.e. the effects of changing P while holding δ fixed.[2] Intuitively, the inverse demand function $\sigma^{-1}(\cdot)$ depends separately on S and P but Z affects market shares only through prices; this suggests needing other instruments that shift S holding P fixed.[3] The characteristic X that enters the index uniquely fits this bill, provided it

[1] $\sigma(\cdot)$ may also depend on other observable characteristics $\tilde{X}$, which we suppress here for exposition.

[2] This is directly stated in a review, Berry and Haile (2021, p. 6): "having valid instruments for all J prices will not generally suffice for identification of [...] the *ceteris paribus* effects of price changes."

[3] Berry and Haile (2014) also consider a model in which P enters the index and is excluded from $\sigma^{-1}(\cdot)$, concluding that price instruments suffice for identification, though again assuming that

is also exogenous. All functions of (X, Z) can then serve as instruments and identify $\sigma(\cdot)$ so long as they induce enough variation in (S, P)—a standard technical condition known as completeness (Newey and Powell, 2003).

We reexamine this setting and reach a more optimistic conclusion: while exogenous price shocks are not generally enough for nonparametric identification of market-specific counterfactuals on their own, they *can* suffice when combined with possibly *endogenous* characteristics. This is because combinations of an as-good-as-randomly assigned Z and an endogenous X that are mean-zero given X—what Borusyak and Hull (2023) call recentered instruments—are valid instruments with identifying power. In particular, they yield a simple test: for any candidate $\check{\sigma}(\cdot)$, if the implied index $\check{\delta} = \check{\sigma}^{-1}(S, P)$ correlates with a recentered instrument, then $\check{\sigma}(\cdot)$ cannot be the true demand function.

We introduce a new technical condition—*faithfulness*—under which all price counterfactuals are identified by candidate inverse demand functions that survive this recentered instrument test. Under faithfulness, any surviving $\check{\sigma}^{-1}(\cdot)$ equals an invertible transformation of the true $\sigma^{-1}(\cdot)$. The transformation is unknown, making demand not fully identified; we cannot, for example, learn counterfactuals that change the endogenous characteristics. But this is not a problem for price counterfactuals, as the unknown transformation is normalized away in those calculations.

Faithfulness requires the variation in Z and X to be rich enough to make all price effects on any functions of the form $H(\delta, P)$ detectable. More precisely, it says that functions of the form $H(\delta, P)$ cannot be mean-independent of the exogenous Z, given the possibly endogenous X, unless they are constant in P. This is a useful condition because any candidate $\check{\sigma}^{-1}(\cdot)$ can be written as such a function: $\check{\sigma}^{-1}(S, P) = \check{\sigma}^{-1}(\sigma(\delta, P), P) \equiv \check{H}(\delta, P)$. Thus, if faithfulness holds, any $\check{\sigma}^{-1}(S, P)$ that survives the recentered instrument test (and is thus mean-independent of Z given X) must be a transformation of the true $\sigma^{-1}(S, P) = \delta$: i.e., $\check{H}(\delta, P) = \check{H}(\delta) = \check{H}(\sigma^{-1}(S, P))$.

Intuitively, faithfulness is satisfied when two conditions hold: *(i)* Z is a strong instrument for P, given X, and *(ii)* X is a strong *proxy* for δ. Under *(i)*, we have identification of all conditional-on-X average price effects on any $\check{H}(\delta, P)$. This is not generally enough to make all price effects on $\check{H}(\delta, P)$ detectable, however, since the identified effects average over the unobserved $\delta \mid X$ distribution. Complex interactions between δ and P could potentially average out to zero conditional on X, making a candidate $\check{\sigma}^{-1}(\cdot)$ survive the recentered instrument test despite not being a transformation of the true $\sigma^{-1}(\cdot)$. The proxy condition *(ii)* prevents this by ensuring the variation

characteristics are exogenous. Borusyak et al. (2025) show that recentered instruments identify price counterfactuals in this case, allowing endogenous characteristics.

in X is sufficiently predictive of δ such that it is impossible for such interactions to average out across all the strata of X. Importantly, this argument does not require X to causally vary the distribution of δ: the strong proxy condition can hold even when all characteristics are endogenous.[4]

This intuition highlights the conceptually distinct roles of Z (as an instrument) and X (as a proxy) in our identification results. Unlike with Z, researchers do not need to justify why the characteristics X are as-good-as-randomly assigned or otherwise independent of unobserved demand shocks; indeed, faithfulness is even more plausible when firms strategically choose characteristics using partial information on δ. This distinction is also reflected in how X and Z would be used in forming "technical instruments" for estimation: while X can be useful when combined with Z and recentered, it has no identifying power on its own (i.e., in conventional "BLP instruments") as recentered functions of X only are identically zero.

While faithfulness is a novel condition, we argue it is a close cousin to completeness. Completeness says that functions of the form $\check{\sigma}^{-1}(S, P)$ cannot be mean-independent of (X, Z) unless they are constant in (S, P). Like completeness, faithfulness is therefore best viewed as a requirement on the strength of identifying variation rather than a substantive economic or statistical assumption.

To explore the close relationship between faithfulness and completeness, we study when one follows from the other. In the limit where Z is a perfect instrument for price (i.e., $P = Z$), we show faithfulness and completeness are equivalent. Outside this limit, faithfulness follows from completeness under several conditions on either pricing or the utility index. On the pricing side, it suffices that either *(i)* X and Z combine in a nonparametric index λ such that P is independent of (X, Z) given (λ, δ) or *(ii)* the derivatives of P with respect to Z satisfy a particular separability condition. Both conditions are compatible with firms engaging in Bertrand–Nash pricing with certain forms of marginal costs. On the δ index side, we show faithfulness follows from completeness when *(i)* δ and X have finite support with the same number of values or *(ii)* $\delta \mid X$ can be transformed to follow a location-scale model satisfying certain smoothness conditions—regardless of how prices are determined. In the other direction, completeness of $(S, P) \mid (X, Z)$ follows from faithfulness when $\delta \mid X$ is complete.

Importantly, researchers do not need to take a stand on the specific sufficient conditions—our identification arguments are the same so long as faithfulness holds. Our menu of

[4] Allowing characteristics to be endogenous further relaxes the index restriction in Berry and Haile (2014) by allowing δ to be an arbitrary function of X and ξ and for ξ to have an arbitrary dimension.

conditions shows faithfulness can hold without strong economic or statistical restrictions. Given completeness, faithfulness is not a big additional leap.

Overall, our analysis clarifies the types of variation that are useful for identifying different demand counterfactuals. The model specifies potential outcomes in two "treatments": prices and characteristics (Chen, 2025). If counterfactuals in both treatments are of interest, then it is unsurprising that exogenous variation in both X and P is needed—leading to the familiar "$2J$" instrument requirement in Berry and Haile (2014). If only the average causal effects of price were of interest, then J instruments for price would intuitively suffice. Price counterfactuals in individual markets have an intermediate requirement: only average price effects are needed, but a sufficiently large and diverse set of them. This is satisfied by having J strong instruments and J strong proxies.

Our analysis also yields several practical insights for applied researchers estimating parametric demand models. Most importantly, it strengthens the recommendation—made previously in the parametric analyses of Borusyak et al. (2025) and Andrews et al. (2025)—that researchers use recentered instruments in order to make their price counterfactual estimates more robust to characteristic endogeneity and model misspecification concerns. Our nonparametric results reassure practitioners that such robustness is not tied to any particular functional form or distributional assumptions. Our results also suggest a new role for potentially endogenous variation in X, as proxying for unobserved model heterogeneity, which practitioners can assess alongside the conceptually distinct requirement of price instrument exogeneity.

This paper contributes to two main literatures. First, we add to an influential literature on the identification of differentiated product demand with market-level data started by Berry (1994) and Berry et al. (1995, 1999); most closely related is the nonparametric identification analysis of Berry and Haile (2014). We depart from most of this literature by allowing product characteristics to be endogenous and by focusing on the identification of price counterfactuals. In this way, our market-level identification results complement Proposition 1 in Borusyak et al. (2025) which shows nonparametric identification of price counterfactuals without exogenous characteristics in a model with an index restriction on price and a standard completeness condition. Our analysis also relates to papers like Ackerberg and Crawford (2009) which study the estimation of parametric demand models with endogenous characteristics; Borusyak et al. (2025) propose recentered instruments for this case.

From this literature, our focus on nonparametric identification of price counterfactuals aligns most closely with Berry and Haile (2024), who call such counterfactuals "conditional demand." Their analysis is also similar in allowing product characteristics X to be endogenous and using price instruments that are exogenous conditional on X. The key difference is that Berry and Haile (2024) leverage "micro data": i.e., variation in market shares across consumer characteristics. Our results show that price counterfactuals can be identified with market-level data only. This entails a very different identification strategy: while Berry and Haile (2024) essentially condition on X and do not require variation on it, we use X as a proxy for unobserved heterogeneity.

Second, we contribute to a large literature studying nonparametric identification with instrumental variables, including Brown and Matzkin (1998), Newey et al. (1999), Newey and Powell (2003), Altonji and Matzkin (2005), Chernozhukov et al. (2007), Chiappori et al. (2015), Imbens and Newey (2009), Torgovitsky (2015), D'Haultfœuille and Février (2015), and Blundell et al. (2017). The model we study imposes an index restriction in the spirit of Berry and Haile (2014), albeit a substantially weaker one. Thus, it is still restrictive relative to the fully-general potential outcomes model that Angrist et al. (2000) consider for linear instrumental variables (IV) estimation. The index restriction is important for identifying market-specific counterfactuals, which are generally not given by the local average demand elasticities that linear IV identifies (Berry and Haile, 2021; Chen, 2025). Beyond demand, our results can be restated to apply to triangular models in which unobserved heterogeneity enters the second stage through an index with a potentially endogenous covariate.[5] Our faithfulness condition appears new and may prove useful for identifying other nonparametric models with this structure.[6]

The rest of this paper is organized as follows. Section 2 develops the main identification results. Section 3 relates faithfulness to completeness via a series of sufficient and necessary conditions. Section 4 discusses key practical insights for parametric estimation. Section 5 concludes. For expositional ease we omit various technical details in the main text (e.g., regularity conditions on measurability, existence of moments, and support); Appendix A gives precise theoretical statements and proofs.

[5]Specifically, our results apply to the model characterized by $Y = g(W_1, \delta(W_2, \xi))$ and $W_1 = h(Z, W_2, \omega)$ with $Z \perp\!\!\!\perp (\xi, \omega) \mid W_2$ and $(Y, W_1, \delta, W_2) \in \mathbb{R}^J$.

[6]Our faithfulness condition derives its name from and relates conceptually to a condition in the causal discovery literature (Spirtes et al., 2000), which considers whether a joint distribution of variables is "faithful" to an underlying causal graph; Appendix B.1 details this connection. Appendix B.2.2 shows that some nonparametric identification results in Imbens and Newey (2009), Torgovitsky (2015), and D'Haultfœuille and Février (2015) can be interpreted as verifying an analog of faithfulness.

2. **Theory**

2.1. **Setup.** We consider a demand system in a market with J products. The product quantity shares (or some other quantity measures) $S \in \mathbb{R}^J$ are given by:

$$S = \sigma(\delta, \tilde{X}, P), \quad \delta = \delta(\overline{X}, \xi) \tag{2}$$

for prices $P \in \mathbb{R}^J$, observed characteristics $\overline{X} = (X, \tilde{X}) \in \mathbb{R}^J \times \mathcal{X}$, and unobserved demand shocks $\xi \in \Xi$ which capture latent consumer tastes or unobserved product characteristics. Focusing on identification, we suppress market subscripts.

Equation (2) follows Berry and Haile (2014) by imposing an index restriction in which the demand shocks and characteristics enter together through the unobserved $\delta \in \mathbb{R}^J$ with the "special characteristic" X otherwise excluded from σ.[7] We also follow Berry and Haile (2014) by assuming demand is invertible in this index:

Assumption 1. *For every $(\tilde{x}, p)$, the map $d \mapsto \sigma(d, \tilde{x}, p)$ is invertible.*

Invertibility follows here from a "connected substitutes" condition (Berry et al., 2013).

We are interested in identifying price counterfactuals: the quantities $\sigma(\delta, \tilde{X}, p')$ that would be observed if prices were set to some counterfactual p' while holding $\delta, \tilde{X}$ fixed at their realized values. These capture both local changes in prices (i.e. own- and cross-price elasticities) and global counterfactuals that can inform many positive and normative analyses—such as merger simulations, conduct testing, and studying the effects of product entry and exit.[8] It is straightforward to extend our analysis to counterfactuals in other product attributes by interpreting P as those non-price variables of interest.

To identify price counterfactuals, we assume that a set of price instruments $Z \in \mathbb{R}^{d_z}$, $d_z \geq J$, is observed in addition to S, $\overline{X}$ and P. We formalize the sense in which these instruments are exogenous below. The identification challenge stems from the unobservability of ξ and the unknown nonparametric functions σ and δ.

Berry and Haile (2014) consider identification of σ with the additional assumptions of *(i)* exclusion of $\tilde{X}$ from the utility index, $\delta(\overline{X}, \xi) = \delta(X, \xi)$, *(ii)* a linear index, $\delta(X, \xi) = X + \xi$ with $\xi \in \mathbb{R}^J$, and *(iii)* mean-independence of the unobserved demand shocks from the characteristics and instruments: $\mathbb{E}[\xi \mid \overline{X}, Z] = 0$. The last assumption

[7] Note that the meaning of the δ index differs from that in the literature on parametric demand models following Berry et al. (1995) where it usually denotes the mean utility vector over products. Most importantly, here δ does not include any effects of price on demand.

[8] Under standard assumptions on price-setting, firm optimization only involves demand at counterfactual prices—such that our counterfactuals are enough for computing marginal costs, markups, and welfare changes from mergers or entry/exit (modeled by prices moving from/to infinity). Borusyak et al. (2025) discuss how δ can be obtained for new products if characteristics are endogenous.

captures the exogeneity of both $\overline{X}$ and Z. Together, *(i)–(iii)* imply:

$$\mathbb{E}[\delta \mid \overline{X}, Z] = X. \tag{3}$$

Equation (3) identifies σ under a completeness condition (Newey and Powell, 2003; Ai and Chen, 2003; Andrews, 2011; D'Haultfoeuille, 2011; Miao et al., 2018):

Assumption 2. *The distribution of $(S, P, \tilde{X}) \mid (X, Z, \tilde{X})$ is complete: For all h with $\mathbb{E}[\|h(S, P, \tilde{X})\|] < \infty$, if $\mathbb{E}[h(S, P, \tilde{X}) \mid X, Z, \tilde{X}] = 0$ then $h(S, P, \tilde{X}) = 0$. Equivalently, under Assumption 1, the distribution of $(\delta, P, \tilde{X}) \mid (X, Z, \tilde{X})$ is complete.*

Identification follows from the fact that, under Assumption 1 and Equation (3),

$$X - \mathbb{E}[\sigma^{-1}(S, P, \tilde{X}) \mid \overline{X}, Z] = 0, \tag{4}$$

while Assumption 2 ensures the solution to this integral equation in σ^{-1} is unique.

Berry and Haile (2014)'s identification result leads to a standard intuition on the need for exogenous variation in the special characteristic X. Assumptions *(i)–(iii)* yield a structural equation, $X = \sigma^{-1}(S, P, \tilde{X}) - \xi$, with $2J$ endogenous variables (S, P) on the right-hand side after conditioning on $\tilde{X}$. This suggests a need for $2J$ exogenous instruments: J "for the prices" P and an additional J "for the shares" S. When X is exogenous, it can serve as the latter set of instruments since it is excluded from the right-hand side. The role of exogenous characteristics, then, is not to generate variation in prices but to generate variation in shares conditional on prices so that one can disentangle the two arguments of σ^{-1}. The completeness condition ensures that transformations of X and Z provide sufficient variation for this.

For price counterfactuals, we work under a partial relaxation of Equation (3) that only imposes conditional exogeneity of the price instruments:

Assumption 3. $Z \perp\!\!\!\perp \delta \mid \overline{X}$.

This restriction strengthens the mean independence in (3) to full independence of Z given $\overline{X}$ but crucially drops the exogeneity of $\overline{X}$.[9] Assumption 3 can be motivated by viewing Z as a set of supply-side shocks drawn in some true or natural experiment after product characteristics are determined (Borusyak et al., 2025).[10]

[9]Assumption 3 is implied by the more standard $Z \perp\!\!\!\perp \xi \mid \overline{X}$ but is weaker when ξ has a higher dimension than δ. We do not view the strengthening from mean- to full independence as meaningfully restrictive, since there is little reason to distinguish ξ from some transformation $g(\xi)$. Full independence amounts to mean-independence of any $g(\xi)$, treating all functions symmetrically.

[10]For example, in the U.S. automobile market, Z may contain exchange rate shocks in cars' countries of production that are excluded from demand and drawn randomly after cars are designed but before prices are set. In this case $Z \perp\!\!\!\perp (\xi, \overline{X})$, implying Assumption 3. See Ackerberg and Crawford (2009)

Our relaxation of Equation (3) is motivated by three related misspecification concerns. First, suppose the functional form $\delta(\overline{X}, \xi) = X + \xi$ is correctly specified but characteristics $\overline{X}$ are chosen strategically by firms with partial knowledge of demand conditions as captured by ξ. Then $\mathbb{E}[\xi \mid \overline{X}] = 0$ is unlikely to hold. Second, ξ may include physical product characteristics chosen by firms along with $\overline{X}$ but unobserved by the econometrician. Again, there is little reason they would be uncorrelated with $\overline{X}$. Finally, suppose that characteristics $\overline{X}$ are fully independent of ξ but that the functional form of δ is not linear or does not exclude $\tilde{X}$: i.e., $\delta(\overline{X}, \xi) \neq X + \xi$. Then (3) need not hold, and model-implied price counterfactuals relying on (3) may be incorrect (Andrews et al., 2025). This in particular accommodates the view of ξ as a "statistical disturbance" rather than an economically meaningful object, in which case $\overline{X} \perp\!\!\!\perp \xi$ by definition of ξ but exclusion and linearity are restrictive.[11] Relaxing the linear index functional form substantively weakens restrictions on the heterogeneity of causal effects of $\overline{X}$ on S (Chen, 2025); it also allows ξ to be multi-dimensional for each product, in contrast to most demand models in the literature.[12]

2.2. **Identification strategy.** Under Assumption 3,

$$\mathbb{E}[\delta \mid \overline{X}, Z] = \mathbb{E}[\delta \mid \overline{X}] \equiv k_0(\overline{X}) \tag{5}$$

for some unknown k_0. This motivates our identification strategy. The true inverse demand function, which returns δ, is mean-independent of Z given $\overline{X}$:

$$\mathbb{E}[\sigma^{-1}(S, P, \tilde{X}) \mid \overline{X}, Z] = \mathbb{E}[\sigma^{-1}(S, P, \tilde{X}) \mid \overline{X}]. \tag{6}$$

We can thus rule out some candidate inverse demand functions $\check{\sigma}^{-1}(S, P, \tilde{X})$: those that violate the conditional moment restriction (6).[13]

To develop this strategy, we first ease some notation. Our identification results fully condition on the "non-special" characteristics $\tilde{X}$. To simplify notation, we suppress $\tilde{X}$ and replace $\overline{X}$ with X throughout.

and Borusyak et al. (2025) for further discussion; Appendix A of Berry and Haile (2024) shows how Assumption 3 can arise for different types of price instruments in various economic models.

[11]Any two random vectors $(\delta, \overline{X})$ can be represented as $\delta = \tilde{\delta}(\overline{X}, \tilde{\xi})$ for some $\tilde{\xi} \perp\!\!\!\perp \overline{X}$ and some measurable function $\tilde{\delta}$—for instance by using Knothe–Rosenblatt (conditional quantile) transforms. In such a representation, $\tilde{\xi}$ need not have a straightforward economic interpretation.

[12]For instance, Appendix B of Berry and Haile (2014) contains a nonseparable model in which $\delta = \delta(X, \xi)$, but restricts the map coordinate-wise: $\delta_j = \delta_j(X_j, \xi_j)$ with the function increasing in ξ_j. This requires ξ to be of dimension J, in addition to other restrictions that we remove.

[13]Assumption 3 implies conditional independence of higher moments of $\sigma^{-1}(S, P, \tilde{X})$ too, which could in principle be used for identification. We focus on identification via first moments, which is more aligned with Berry and Haile (2014) as well as common parametric estimation strategies.

We next define a (potentially non-sharp) identified set for σ^{-1}:

$$\Theta_I \equiv \left\{h(s,p),\ \mathbb{R}^J\text{-valued and invertible in } s\colon \mathbb{E}[h(S,P) \mid X, Z] = k(X) \text{ for some } k\right\}.$$

This set is nonempty because (6) implies it contains at least the true σ^{-1}.

The identified set can be more constructively defined via recentered instruments, which take the form $R(X,Z) - \mathbb{E}[R(X,Z) \mid X]$ for some function R (Borusyak and Hull, 2023). The set of functions in Θ_I is exactly the set of functions that are orthogonal to any recentered instrument:

Lemma 1. *Let h be $\mathbb{R}^J$-valued, invertible, and square-integrable. Then $h \in \Theta_I$ if and only if $\mathbb{E}[h(S,P)\{R(X,Z) - \mathbb{E}[R(X,Z) \mid X]\}] = 0$ for all square-integrable R.*

Intuitively, a candidate $\check{\sigma}^{-1}$ can only be conditionally mean-independent of Z, and thus in Θ_I, if it is uncorrelated with all conditionally mean-zero functions of Z.

The identified set is not a singleton that contains σ^{-1} only: under Assumption 3, all transformations of the true σ^{-1} are independent of Z given X and hence in Θ_I. However, this does not necessarily impede point-identification of price counterfactuals: in fact, it suffices for σ^{-1} to be identified only up to some transformation.[14]

Lemma 2. *Suppose all $\check{\sigma}^{-1} \in \Theta_I$ have the form $\check{\sigma}^{-1}(S,P) = T(\sigma^{-1}(S,P))$ for some $T : \mathbb{R}^J \to \mathbb{R}^J$. Then price counterfactuals are identified.*

Intuitively, counterfactual quantities given hypothetical prices p' under a candidate model $\check{\sigma}^{-1} = T(\sigma^{-1})$ are computed from the observed (S,P) as:

$$\check{\sigma}(\check{\sigma}^{-1}(S,P), p') = \sigma(T^{-1}(T(\sigma^{-1}(S,P))), p') = \sigma(\delta, p').$$

The transformation T cancels, so it does not affect the counterfactual $\sigma(\delta, p')$.[15]

Our identification strategy is therefore successful when Θ_I contains nothing but transformations of σ^{-1}. We next consider when this condition holds.

2.3. **A surprisingly insightful case: exogenous prices.** We start with the case of $Z = P$: i.e., where prices are themselves exogenous ($P \perp\!\!\!\perp \delta \mid X$). This is unrealistic in many settings, but it is an illuminating baseline case for developing our strategy (and notably one which Berry and Haile (2021) call "surprisingly difficult"). In particular,

[14]Statements like Lemma 2 are also intermediate steps in Matzkin (2008, p. 950), Torgovitsky (2015, p. 1190), and Berry and Haile (2024, p. 1143-1144). In particular, Berry and Haile (2024) normalize their index to avoid considering analogous indeterminacy as with T.

[15]Note that T is implicitly invertible because Θ_I only includes invertible $\check{\sigma}^{-1}$ candidates.

it clarifies why exogenous variation in P can be enough and what the endogenous variation in X nevertheless contributes. This case can also be viewed as the limit of our general setting with perfect price instruments.

With exogenous prices, price counterfactuals are identified under the same completeness condition as in Berry and Haile (2014)—despite our substantial relaxation of their model. Indeed, all characteristics can be fully endogenous in this case:

Proposition 1. *Suppose $P = Z$ and Assumptions 1 to 3 hold. Then price counterfactuals are identified.*

To see this result, consider any candidate $\breve{\sigma}^{-1} \in \Theta_I$. We can write $\breve{\sigma}^{-1}(S, P) = \breve{\sigma}^{-1}(\sigma(\delta, P), P) \equiv H(\delta, P)$ for an unknown H such that $\mathbb{E}[H(\delta, P) \mid P, X] = k(X)$. Fixing some price value p_0,[16] we then have:

$$\begin{aligned} 0 &= \mathbb{E}[H(\delta, P) \mid P, X] - \mathbb{E}[H(\delta, p_0) \mid P = p_0, X] \\ &= \mathbb{E}[H(\delta, P) \mid P, X] - \mathbb{E}[H(\delta, p_0) \mid P, X] \\ \implies 0 &= H(\delta, p) - H(\delta, p_0) \end{aligned}$$

where the first line uses the fact that P does not enter $k(X)$, the second line uses Assumption 3, and the third line uses Assumption 2. Thus $H(\delta, p) = H(\delta, p_0)$ does not depend on p: $H(\delta, P) = H(\delta)$. In turn, this implies that the candidate inverse demand function is a transformation of the true inverse demand function:

$$\breve{\sigma}^{-1}(S, P) = H(\delta, P) = H(\delta) = H(\sigma^{-1}(S, P)).$$

Since this holds for any $\breve{\sigma}^{-1} \in \Theta_I$, price counterfactuals are identified by Lemma 2.

Proposition 1 contains two key intuitions. First, for identifying price counterfactuals, it is enough to detect causal effects of prices only. Indeed, since it suffices to identify $\sigma^{-1}(S, P)$ only up to a transformation (Lemma 2), we only need to find some function $h(S, P)$ such that $h(\sigma(\delta, P), P) \equiv H(\delta, P)$ is a transformation of δ only with no dependence on P. From a causal inference perspective, this means finding some outcome $H = h(S, P)$ which is fully unaffected by P.

Importantly, here we are only interested in whether P has *any* causal effects on H and not in disentangling the effects that come through S versus P. We thus sidestep the need for separate exogenous variation in S given P, which is what compels characteristic exogeneity in Berry and Haile (2014). Hence, to trace out *average* price effects, we require only J-dimensional exogenous price variation—not $2J$ instruments.

[16] When P is continuously distributed, conditioning on a realization of it requires measure-theoretic care. See Proposition A.1 and Proposition A.4 for a formal argument.

The second key intuition is on the role of the additional J-dimensional variation in X as a *proxy* for δ.[17] Tracing out average effects of P is not generally enough: while (2) implies that the effect of P on S (and therefore on H) is fully governed by the index δ, this index is unobserved. Zero average price effects, which average across the unobserved δ, do not by itself imply zero price effects for a given δ.[18] This gap can be filled by leveraging variation in X, so long as it strongly correlates with δ—i.e., "proxies" for it. For each observed value of X, exogenous price variation reveals the conditional average price effects on H. Completeness of $(\delta, P) \mid (X, Z)$—which here, with $Z = P$ and $\delta \perp\!\!\!\perp P \mid X$, is equivalent to completeness of $\delta \mid X$—ensures that this set of identified effects is rich enough to span the set of price effects for each unobserved value of δ. Hence, under completeness, the finding of no conditional-on-X average price effects ensures no causal dependence of H on P.[19]

This proxy role of X is substantively different from the role that X plays in Berry and Haile (2014), as an instrument "for the shares." Here, X need not be exogenous with respect to the economically meaningful unobserved demand shock ξ, nor must it causally shift δ. For instance, it is perfectly fine if X correlates with δ because of selection only. As a result, while practitioners should justify why Z is as-good-as-randomly assigned and why it is not strategically chosen (in order to satisfy Assumption 3), no such arguments are needed for X.

Of course, one can always make X "technically exogenous" by rewriting (6) as:

$$0 = \sigma^{-1}(S, P) - k_0(X) + \tilde{\xi}, \qquad \mathbb{E}[\tilde{\xi} \mid X, P] = 0.$$

But since k_0 is unknown, X is not excluded from this model: variation in it informs both k_0 directly and σ^{-1} indirectly through S. This highlights that we cannot merely redefine residuals, make X exogenous without loss, and appeal to the argument of Berry and Haile (2014); fundamentally, our results rely on a distinct use of X.

The proxy role of X is also substantively different from the role of other characteristics $\tilde{X}$, which—being fully conditioned on—might be viewed here as "controls." While there are no restrictions on the amount or nature of variation in $\tilde{X}$ or its effects on

[17] A related argument by Ahn (2025), in a context with an exogenous binary treatment P, uses X in a similar way and calls it a "prognostic variable."

[18] Borrowing an example from Benkard and Berry (2006), if X is independent of δ and thus variation in X is not useful, then one can choose $\delta \sim \mathcal{N}(0, I_J)$ and $H(\delta, p)$ to be a p-dependent rotation of δ. This construction implies that $H(\delta, P) \perp\!\!\!\perp (X, P)$, and thus, absent other conditions that rule out such H, the set Θ_I would include inverse demand candidates that are not just transformations of δ—leading to incorrect price counterfactuals.

[19] Section 3 in Chen (2025) shows that many counterfactual predictions from structural models analogously extrapolate from a lack of average causal effects to a total lack of causal effects.

demand, X must be strongly correlated with δ and excluded from demand in the sense of the index restriction in (2). Indeed, the proxy role of X is what gives the index restriction bite: with a constant or completely randomly generated X, any demand model could be written in the form of (2).

We next generalize these results and intuitions to settings with endogenous prices.

2.4. **Faithfulness and identification with endogenous prices.** When $P \neq Z$, we continue to need sufficiently rich variation in X to proxy for the unobserved δ. We additionally need rich enough variation in the exogenous instruments Z to trace out the causal effects of the now-endogenous prices. Perhaps surprisingly, the usual completeness condition is not the appropriate formalization of these two requirements. Instead, we consider a new condition to replace Assumption 2:

Assumption 4 (**Faithfulness**). *The distribution of* $(\delta, P) \mid (X, Z)$ *is* faithful*: for all* $\mathbb{R}^J$*-valued* $H(\delta, P)$*, if* $\mathbb{E}[H(\delta, P) \mid X, Z] = k(X)$ *does not depend on* Z *then* H *does not depend on* P*, i.e.,* $H(\delta, P) = H(\delta)$*.*

Under faithfulness, price counterfactuals are identified even with endogenous prices:

Proposition 2. *Under Assumptions 1, 3, and 4, price counterfactuals are identified.*

The proof of Proposition 2 follows the same steps as for Proposition 1: write $\breve{\sigma}^{-1}(S, P) = \breve{\sigma}^{-1}(\sigma(\delta, P), P) \equiv H(\delta, P)$ and note that $\mathbb{E}[H(\delta, P) \mid X, Z] = \mathbb{E}[\breve{\sigma}^{-1}(S, P) \mid X, Z] = k(X)$ for some k if $\breve{\sigma}^{-1} \in \Theta_I$. Under faithfulness, this means that $\breve{\sigma}^{-1}(S, P) = H(\delta, P) = H(\delta) = H(\sigma^{-1}(S, P))$ is a transformation of the true inverse demand function. All price counterfactuals are thus identified by Lemma 2.

The two key intuitions from the exogenous price case are retained in Proposition 2. First, because we are only interested in whether P has any effects on $H(\delta, P) = h(S, P)$, and not in disentangling its direct effects from indirect effects through S, we avoid the need for separate exogenous variation in S given P. Second, though variation in X need not be exogenous, it is still critical for proxying for the unobserved δ. As in the exogenous price case, complex interactions between δ and P can potentially average out and threaten identification by obscuring true price effects on $H(\delta, P)$. Here faithfulness, rather than completeness, rules out such scenarios via rich variation in X that strongly predicts δ, together with strong price instruments Z.[20]

[20] Appendix B.2 shows how, under additional restrictions, analogs of faithfulness can hold without an X that satisfies the index restriction and proxies for δ. These include the case where P combines linearly with δ (as in Proposition 1 of Borusyak et al. (2025)), and where $J = 1$ with σ monotone in δ (as in Imbens and Newey (2009), Torgovitsky (2015), and D'Haultfœuille and Février (2015)).

How different is faithfulness from the usual completeness condition? They are similar in kind: both are high-level conditions that yield identification by directly asserting that certain operations are injective. Namely, they link the variation in some $H(\delta, P)$ that is detectable via conditional expectations in (X, Z) to the structural dependence of H on (δ, P). Completeness requires that if $H(\delta, P)$ varies then $\mathbb{E}[H(\delta, P) \mid X, Z]$ also varies. In this sense, conditional expectations "faithfully reflect" changes in (δ, P). Faithfulness instead says that if prices truly affect a function of demand primitives, these effects must show up in the instrument-induced price variation conditional on X. Conversely, functions with no instrument-induced conditional-on-X variation cannot depend on P.[21]

In the nonparametric instrumental variables literature, completeness is often treated as a technical condition which encodes a sense of instrument strength while not imposing substantive restrictions (see, e.g., Newey and Powell, 2003; Ai and Chen, 2003; Darolles et al., 2011).[22] Lower-level conditions are given by, e.g., D'Haultfoeuille (2011) and Andrews (2011). We argue that faithfulness should be similarly treated as a technical condition which encodes a sense of instrument strength for Z as well as proxy strength for X.

To bolster this argument, we next detail connections between the two conditions—showing, in particular, that faithfulness follows from completeness under a wide range of different lower-level conditions. Together these results suggest faithfulness and completeness, while non-nested in general, are close cousins. Thus, to the extent nonparametric identification under completeness reassures practitioners that demand can be flexibly estimated with exogenous characteristics and price instruments, our results under faithfulness should likewise reassure that price counterfactuals can be flexibly estimated with recentered instruments.

3. Relationship between completeness and faithfulness

We first provide a useful calibration: when Z is a perfect instrument (i.e., the exogenous price case), faithfulness and completeness are exactly equivalent.

Proposition 3. *Suppose $P = Z$ and Assumption 3 holds. Then Assumption 2 is equivalent to Assumption 4. Hence Proposition 1 is a special case of Proposition 2.*

[21] Note that faithfulness is different than conditional-on-X completeness—i.e., completeness of $(\delta, P, X) \mid (X, Z)$. This condition cannot hold here as Z generates no variation in δ.

[22] This is especially true when nonparametric identification is viewed as a theoretical argument for how particular parametric structure does not "drive" conclusions. In any given parametric model, one could posit a strong-instrument-type condition where all non-constant functions $w \in \mathcal{W}$ of endogenous variables $w(S, P)$ are assumed to correlate with some function of (X, Z), over some parametrized class $\mathcal{W}$. Completeness is the limit of such conditions as we enlarge $\mathcal{W}$ to include all (integrable) functions.

In this sense, faithfulness is not an exotic new assumption: it is the natural technical condition for leveraging the conditional moment restriction (6).

Outside of the $P = Z$ case, faithfulness and completeness are distinct conditions. Two counterexamples illustrate this: Appendix B.3 exhibits a class of demand models where faithfulness holds but $(\delta, P) \mid (X, Z)$ needs not be complete, while Appendix B.4 presents a distribution $(\delta, P) \mid (X, Z)$ that is complete but not faithful.

We next show that the two conditions are nevertheless closely related, in the sense that each condition implies the other under additional restrictions.

3.1. **When does completeness imply faithfulness?** We first present four non-nested conditions under which completeness implies faithfulness; these conditions are sufficient but not necessary. The sufficient conditions upper-bound the extent to which faithfulness is "stronger" than completeness.

3.1.1. *Restrictions on price-setting.* We start from two sufficient conditions that restrict how price depends on the observables and unobservables of the model. Both conditions extend the case of exogenous prices. For some function f, write

$$P = f(X, Z, \delta, \tilde{\omega}), \qquad \tilde{\omega} \perp\!\!\!\perp (X, Z, \delta). \tag{7}$$

Here $\tilde{\omega}$ is an unobservable of arbitrary dimension that captures residual variation in P that is independent of (X, Z, δ). So far, Equation (7) is without loss of generality.[23] In what follows we place different restrictions on f. At the end of this subsection, we show that these restrictions can be restated as similar conditions on marginal costs under Bertrand–Nash pricing.

The first sufficient condition is that X and Z enter price only through the utility index δ and an index $\lambda(X, Z) = \lambda$ that is invertible in Z:

Assumption 5. *$P \perp\!\!\!\perp (X, Z) \mid (\lambda(X, Z), \delta)$, for some $\lambda(x, z)$ that is invertible in z. Equivalently, in (7), $f(X, Z, \delta, \tilde{\omega}) = f(\lambda(X, Z), \delta, \tilde{\omega})$.*

This assumption is satisfied, in particular, when X enters price in (7) only through δ—paralleling how it enters $\sigma(\cdot)$. Under this index restriction for price, Assumption 5 is satisfied with $\lambda(X, Z) = Z$. By further setting $Z = P$, this special case also nests exogenous prices and generates Proposition 1 as a corollary. In general we have:

Proposition 4. *Assumptions 2, 3, and 5 imply Assumption 4.*

[23] Note that f is not a structural function because of the parametrization of $\tilde{\omega}$, but it is consistent with any structural formulation. Indeed, for any structural shock ω possibly correlated with (X, Z, δ) and $P = \tilde{f}(X, Z, \delta, \omega)$, one can represent $\omega = f_\omega(X, Z, \delta, \tilde{\omega})$ and $P = \tilde{f}(X, Z, \delta, f_\omega(X, Z, \delta, \tilde{\omega}))$.

To see how this result follows, note that for any realization λ_0 of λ:

$$
\begin{aligned}
k(X) &= \mathbb{E}[H(\delta, P) \mid X, Z] \\
&= \mathbb{E}[\mathbb{E}[H(\delta, P) \mid \delta, \lambda] \mid X, Z] && \text{(Iterated expectations, Assumption 5)} \\
&= \mathbb{E}[\mathbb{E}[H(\delta, P) \mid \delta, \lambda = \lambda_0] \mid X, Z = \lambda^{-1}(X, \lambda_0)] && (Z \text{ does not enter } k(X)) \\
&= \mathbb{E}[\mathbb{E}[H(\delta, P) \mid \delta, \lambda = \lambda_0] \mid X] && \text{(Assumption 3)} \\
&= \mathbb{E}[\mathbb{E}[H(\delta, P) \mid \delta, \lambda = \lambda_0] \mid X, Z]. && \text{(Assumption 3)}
\end{aligned}
$$

Hence for $H(\delta) = \mathbb{E}[H(\delta, P) \mid \delta, \lambda = \lambda_0]$ we have $\mathbb{E}[H(\delta) \mid X, Z] = k(X)$. Finally, by completeness (Assumption 2), $H(\delta, P) = H(\delta)$.

The second sufficient condition instead imposes a separability condition on the derivatives of price with respect to Z:

Assumption 6. *In (7), f is continuously differentiable in Z with Jacobian $D_z f$, which satisfies the following separability condition: for measurable functions A, B,*

$$\underbrace{D_z f(X, Z, \delta, \tilde{\omega})}_{J \times d_z} = \underbrace{A(f(X, Z, \delta, \tilde{\omega}), \delta)}_{J \times J} \cdot \underbrace{B(X, Z)}_{J \times d_z} \tag{8}$$

where $A(f(X, \delta, Z, \tilde{\omega}), \delta)$ and $B(X, Z)$ are full row rank with $\dim(Z) \equiv d_z \geq J$.

Assumption 6 holds, in particular, when

$$P = f_0\Big(f_1(X, Z) + f_2(X, \delta, \tilde{\omega}), \delta\Big) \tag{9}$$

(with regularity conditions on f_0 and f_1 given in Lemma A.4). The restriction in (9) is that Z enters P through an index $f_1(X, Z) + f_2(X, \delta, \tilde{\omega})$, which is a form of separability between observed and unobserved cost shifters. Clearly, it is satisfied when $P = Z$.[24]

Proposition 5. *Assumptions 2, 3, and 6 imply Assumption 4.*

[24]Moreover, under (9) and suitable regularity conditions on f_0, f_1, f_2, faithfulness is *equivalent* to completeness. To see this, suppose faithfulness holds but completeness fails. By Proposition 8, below, $\delta \mid X$ is not complete: there exists $h(\delta) \neq 0$ such that $\mathbb{E}[h(\delta) \mid X] = 0$. Consider $\mathbb{E}[h(\delta) f_0^{-1}(P, \delta) \mid X, Z] = \mathbb{E}[h(\delta) \mid X] f_1(X, Z) + \mathbb{E}[h(\delta) f_2(X, \delta, \tilde{\omega}) \mid X, Z]$. The first term is 0 and the second does not vary with Z. Yet, $h(\delta) f_0^{-1}(P, \delta)$ depends on P, contradicting faithfulness.

The logic for this result is as follows: under Equation (8), given any differentiable $H(\delta, P)$ with $H_p(\delta, P) \equiv \frac{\partial H}{\partial P}$,[25] we have

$$
\begin{aligned}
0_{J\times d_z} &= \frac{\partial}{\partial z}\mathbb{E}[H(\delta, P) \mid X, Z = z] = \frac{\partial}{\partial z}\mathbb{E}[H(\delta, f(X, Z, \delta, \omega)) \mid X, Z = z] \\
&= \mathbb{E}\left[H_p(\delta, P)A(P, \delta) \mid X, Z = z\right] B(X, z) && \text{(Assumption 6)} \\
\implies \quad 0_{J\times J} &= \mathbb{E}\left[H_p(\delta, P)A(P, \delta) \mid X, Z\right] && (B \text{ is full-rank}) \\
\implies \quad 0_{J\times J} &= H_p(\delta, P)A(P, \delta) && \text{(Assumption 2)} \\
\implies \quad 0_J &= H_p(\delta, P). && (A \text{ is nonsingular})
\end{aligned}
$$

By the fundamental theorem of calculus, $H_p(\delta, P) = 0$ implies $H(\delta, P) = H(\delta)$.

Assumptions 5 and 6 are non-nested. Assumption 5 allows richer interactions between how Z and $\tilde{\omega}$: e.g., $P = f(Z, \tilde{\omega})$ always satisfies Assumption 5 but not necessarily Assumption 6. Assumption 6 allows richer interactions between X and $\tilde{\omega}$: e.g., $P = Z + f(X, \tilde{\omega})$ always satisfies Assumption 6 but not necessarily Assumption 5.

Assumptions 5 and 6 can be economically grounded with more primitive conditions on marginal costs. Under Bertrand–Nash pricing with constant marginal costs that are represented without loss of generality as $c(X, Z, \delta, \tilde{\omega}) \equiv C \in \mathbb{R}^J$, one can always write the equilibrium prices as

$$P = g(C, \delta) \tag{10}$$

for some function g (see Appendix B.5). Thus, if one assumes that X and Z enter marginal costs via the index λ, i.e., $C = c(\lambda(X, Z), \delta, \tilde{\omega})$, then Assumption 5 follows. Similarly, Assumption 6 holds if $C = f_0(f_1(Z, X) + f_2(\delta, X, \tilde{\omega}), \delta)$ with g and f_0, f_1 satisfying certain regularity conditions.

3.1.2. *Restrictions on the δ index.* Our second set of sufficient conditions leverage statistical restrictions on the conditional distribution of $\delta \mid X$. With these conditions, it is possible to show that *every* function of X belonging to some known class $\mathcal{K}$ can be written as $\mathbb{E}[H(\delta) \mid X]$ for some function H. If this is true, we can then conclude from completeness that $H(\delta, P) = H(\delta)$ for some $H(\delta)$, so long as $\mathbb{E}[H(\delta, P) \mid X, Z] \in \mathcal{K}$. To be sure, these conditions are not fully general: our main goal is to demonstrate the existence of restrictions that are purely statistical; these restrictions should not necessarily be viewed as recommended modeling choices.

[25] Because we assume H is differentiable here, we have to modify faithfulness to restrict only differentiable functions. These complications are resolved in Proposition A.5, which redefines faithfulness and ensures that differentiation is exchangeable with expectation.

One simple case is if δ and X are discrete and known to take the same finite number of values. Here, completeness directly implies that any function $k(X)$ can be represented as a projection of some function $H(\delta)$ to (X, Z)-space.

Proposition 6. *Fix $M \in \mathbb{N}$. Assume the support of δ and X are both finite sets of size M. Then Assumptions 2 and 3 imply Assumption 4.*

The same strategy can be used in the case of continuously-distributed X and δ. Suppose that $\delta \mid X$ can be transformed into a location-scale model of the form:

$$a(\delta) = b(X) + \Sigma(X)\epsilon, \quad \epsilon \sim q(\cdot), \quad \epsilon \perp\!\!\!\perp X, \tag{11}$$

for some invertible $a(\cdot)$ and continuously distributed J-dimensional ϵ with density $q(\cdot)$; ϵ can be seen as reparametrizing the component of δ that is independent of X. Note that (11) is a purely statistical assumption as ϵ is generally distinct from ξ.

We consider the following assumptions on $a(\cdot)$, $b(\cdot)$, $\Sigma(\cdot)$, and $q(\cdot)$:

Assumption 7. *Fix some integers $s > J + 1$ and $M = J + 2s + 1$. Let*

$$\mathbb{K}^s = \left\{u : \mathbb{R}^J \to \mathbb{R}^J : u(x) = Ax + r(x), A \in \mathbb{R}^{J\times J}, r \in \mathbb{W}^{s,\infty}(\mathbb{R}^J)\right\}$$

where $\mathbb{W}^{s,p}(\mathbb{R}^J)$ is a Bessel potential space of smoothness parameter s and integrability parameter p, defined in (A.10). Let $\mathcal{K} = \mathbb{K}^M$. Assume

(1) a, b in (11) are invertible.

(2) (Density smoothness for ϵ) The Fourier transform of q and its derivatives have bounded tails obeying Assumption A.7(2).

(3) (Limited heteroskedasticity) $\Sigma(x)$ is uniformly close to some fixed Σ_0, in the sense that certain distances in Assumption A.7(3) are bounded above by a sufficiently small ψ.

(4) (Smooth b with Lipschitz inverse) It is known that $b \in \mathcal{K}$ and that $\sup_{x\in\mathbb{R}^J} \|(Db(x))^{-1}\|_{\text{op}} < \infty$ where $Db(x)$ is the Jacobian of b.

The key requirement in Assumption 7 is that b is known to fall in a "well-behaved" function class $\mathcal{K}$, which—loosely speaking—consists of functions that deviate from a linear map by a suitably smooth function with sth-order derivatives in L^p. Because of this, we can limit Θ_I to just those functions h with conditional expectations in $\mathcal{K}$:

$$\Theta_I(\mathcal{K}) = \{h : \mathbb{E}[h(S, P) \mid X, Z] \in \mathcal{K}\} \ni a(\sigma^{-1}(\cdot, \cdot)).$$

The other regularity conditions in Assumption 7 establish that $\mathcal{K}$ is sufficiently small and the operator $u \mapsto \mathbb{E}[u(\delta) \mid X]$ is sufficiently well-behaved so that $\mathcal{K}$ can be entirely populated by conditional expectations of functions of δ.[26] That is, for any $k \in \mathcal{K}$, there exists some $H(\delta)$ such that $\mathbb{E}[H(\delta) \mid X] = k(X)$. Completeness would then imply a version of faithfulness with respect to $\mathcal{K}$. That is, for any candidate $H(\delta, P)$ where $\mathbb{E}[H(\delta, P) \mid X, Z] \in \mathcal{K}$, it follows that for some $H(\delta)$, we have

$$\mathbb{E}[H(\delta, P) \mid X, Z] = \mathbb{E}[H(\delta) \mid X, Z],$$

and therefore $H(\delta, P) = H(\delta)$ by completeness. We summarize this argument in the following result and verify that the conclusion of Proposition 2 continues to hold.

Proposition 7. *Under Assumptions 1 to 3 and 7, for any integrable $H(\delta, P)$, if $\mathbb{E}[H(\delta, P) \mid X, Z] \in \mathcal{K}$, then $H(\delta, P) = H(\delta)$ for some $H(\delta)$. As a result, price counterfactuals are identified.*

Unlike Proposition 6, which requires an exact equivalence in the number of support points of δ and X, Assumption 7 is not knife-edge in nature—suggesting that it can be further relaxed to allow for more flexible models for $\delta \mid X$. Overall, the combination of results in Section 3.1 confirms that faithfulness can follow from completeness without strong economic or statistical assumptions. We expect that many other sufficient conditions for faithfulness exist as well.

3.2. **When does faithfulness imply completeness?** The above arguments verify faithfulness from completeness by, intuitively, using X as a proxy for δ. Since faithfulness only requires ruling out the possibility of certain δ and P interactions averaging out, it is possible—indeed shown by the example in Appendix B.3—that faithfulness does not fully use the completeness of $\delta \mid X$. In other words, completeness of $\delta \mid X$ is sufficient for faithfulness but may not be necessary. It turns out this is the only reason that faithfulness does not always imply completeness:

Proposition 8. *Suppose Assumption 3 holds and $\delta \mid X$ is complete. Then Assumption 4 implies Assumption 2.*

The proof is simple: Given $\mathbb{E}[H(\delta, P) \mid X, Z] = 0$, faithfulness implies that $H(\delta, P) = H(\delta)$ since 0 is constant in Z. Exogeneity of Z further implies that $\mathbb{E}[H(\delta) \mid X, Z] = \mathbb{E}[H(\delta) \mid X] = 0$. Finally, completeness of $\delta \mid X$ implies that $H(\delta) = 0$.

[26] Assumption 7(2) is satisfied by distributions with Gamma-like tails, though it rules out Gaussian distributions. See Lemma C.1 and Remark C.1. We strongly suspect that these restrictions are not essential and can be further relaxed.

4. **Practical Implications**

Our nonparametric identification results yield several insights for applied researchers estimating *parametric* demand models. Namely, they suggest key conditions that practitioners can scrutinize in order to use our results to argue (perhaps informally) that their parametric assumptions serve the usual role of filling gaps left by insufficient identifying variation. These conditions, discussed here, concern the counterfactuals of interest, the available variation in the price instruments and characteristics, and the estimation method. We also discuss how the conditions differ from what researchers appealing to Berry and Haile (2014) and Berry and Haile (2024) would need to argue.

First, in order to appeal to our results, a researcher should be interested in estimating counterfactuals that change a product attribute for which some plausibly exogenous variation is available. Most commonly, this attribute is the product's price. Price counterfactuals arise, for instance, in merger simulations, when studying product exit or entry (which can be understood as moving prices to or from infinity), and when price elasticities are fundamentally of interest. In each of these cases, one can plausibly observe a set of supply-side shocks Z which exogenously vary prices P.

Second, a researcher should argue there exists at least one "special" characteristic X which satisfies the index restriction in (2). For mixed logit models, this would mean X enters demand without a random coefficient; Berry and Haile (2014) argue this requirement is satisfied in most applications. Note that no assumptions are generally needed on how the other characteristics $\tilde{X}$ enter demand.

The researcher should further argue that the special characteristic is a strong proxy for the index δ. In practice, this means that X is strongly predictive of demand holding the other product attributes fixed. Again, no such requirement is generally imposed on the other observed characteristics $\tilde{X}$.

Importantly, and in contrast with the Berry and Haile (2014) results, researchers motivated by our identification results need not justify that either X or $\tilde{X}$ are exogenous. Several distinct forms of endogeneity are allowed: firms can strategically design products (i.e., choose characteristics both observed and unobserved by the researcher) or choose which markets to enter with a given product with some knowledge of local demand conditions. Moreover, the functional form restrictions that the researcher imposes on how characteristics enter demand need not be correct; for instance, the model is still able to predict price counterfactuals correctly if quadratic terms in X are incorrectly excluded, violating their exogeneity (Andrews et al., 2025).

Third, a researcher should scrutinize the exogeneity of the price instruments Z. These should be fully independent of the δ index conditional on $(X, \tilde{X})$ which in practice will generally involve certain timing assumptions. The cleanest scenario is when Z is a set of unconditionally as-good-as-random shocks that have some variation across products, are realized after the product design is chosen and entry decisions are made but before P is set, and do not affect demand except through P. Short-term unanticipated fluctuations in the exchange rate of countries producing different goods (Borusyak et al., 2025) or deviations of realized input prices from futures markets values (Ackerberg and Crawford, 2009) are possible examples of such price instruments. However, there are also scenarios where only the conditional exogeneity of Z is plausible; see Appendix A of Berry and Haile (2024) for a detailed discussion.

Fourth, to avoid any bias from the potentially endogenous $(X, \tilde{X})$, the moment conditions used in estimation must be based on recentered instruments. Functions of Z alone are not typically enough to identify parametric demand models beyond pure logit, so other instruments have to be chosen. This paper shows that recentered instruments can be sufficient for this goal nonparametrically, while Borusyak et al. (2025) propose specific constructions of powerful recentered instruments in parametric models. Importantly, other instruments—such as "BLP instruments" that are functions of characteristics only—are not generally valid without characteristic exogeneity.[27]

We conclude here by noting that with our identification strategy works with only market-level data, which are widely available. Other types of data—e.g., on how market shares vary by consumer characteristics within the market or on second choices of consumers—are known to be helpful for demand estimation. However, these types of data are not always available and our results show that they are not necessary for identification even nonparametrically. Moreover, even when microdata are available, the identification strategies developed for them impose additional homogeneity assumptions (Chen, 2025, Section 4) which our identification strategy does not require.

5. **Conclusion**

We have shown that price counterfactuals are identified without exogenous product characteristics in a nonparametric demand model with a weak index restriction, given exogenous instruments that induce sufficient variation in prices and a strong proxy

[27] In fact, functions of own and rivals' characteristics can be controlled for and potentially improve estimation efficiency by absorbing some residual variation (Borusyak et al., 2025).

for the index. The richness of this identifying variation is captured by a new faithfulness condition, which essentially requires that the instruments and the proxies make all causal price effects detectable. We show through a variety of non-nested sufficient conditions that faithfulness can follow from a standard completeness condition without strong statistical or economic restrictions. These results reassure practitioners that price counterfactuals can be reliably identified with recentered instruments, without needing to justify that observed characteristics are as-if-randomly assigned or chosen non-strategically. We suspect the new faithfulness condition and identification results may also prove useful in other nonparametric models.

We have not provided a theoretical analysis of nonparametric estimation based on these results, which would naturally require additional regularity conditions (Compiani, 2022; Chen and Pouzo, 2015). As in Berry and Haile (2014), we leave developing these conditions to future research. We nevertheless hope that our identification analysis will help guide empirical researchers towards more robust and credible estimation strategies, by clarifying the kinds of variation that can reveal counterfactuals in flexible demand models. Most importantly, in contrast to widely-held intuition, our results suggest researchers can generally avoid conventional characteristic-based instruments—and the potential biases associated with them—by looking for plausibly exogenous supply shocks and leveraging them via recentered instruments.

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Appendix A. Detailed Statements and Proofs

A.1. **Setup.** We first restate the assumptions and results in Sections 2 and 3 more formally. Let F^* denote the distribution of (S, P, δ, X, Z). We assume for each F^* there is some σ such that $S = \sigma(\delta, P)$ almost-surely. Let F be the distribution of (S, P, X, Z). The assumptions will restrict the class of $\mathcal{F}^* \ni F^*$, which we keep implicit. Identification of price counterfactuals is formally defined in the following sense: Counterfactuals are identified if there exists some function of the observed data that predicts them, and these predictions are correct on a set of F^*-probability 1.

Definition 1. *We say that price counterfactuals are identified at F if there exists a mapping $C(s, p, p')$ such that, for any F^* generating F with a corresponding $\sigma = \sigma_{F^*}$, there exists a set E with $\mathbb{P}_{F^*}(E) = 1$ where*

$$C(s, p, p') = \sigma(\sigma^{-1}(s, p), p')$$

for all values $(s, p, \delta, x, z) \in E$ and $(s', p', \delta, x, z) \in E$. We say that price counterfactuals are identified if they are identified at all F generated by some F^*.

We first restate Assumption 1:

Assumption A.1. *σ is invertible in δ: For some measurable σ^{-1}, $\delta = \sigma^{-1}(S, P)$ almost surely. Assume that $\mathbb{E}[\|\sigma^{-1}(S, P)\|] < \infty$.*

We next restate Assumptions 2 and 3. Here, completeness is defined relative to all integrable functions. Weakening completeness by changing integrable to square-integrable does not affect subsequent results, so long as we modify them accordingly.

Assumption A.2. *For all F, the conditional distribution $(S, P) \mid (X, Z)$ is complete with respect to integrable functions: For any integrable h,*

$$\mathbb{E}_F[h(S,P) \mid X, Z] = 0 \implies h(S,P) = 0 \text{ almost surely.}$$

Assumption A.3. *For each F^*, $Z \perp\!\!\!\perp \delta \mid X$ under F^*.*

Fix some class of functions $\mathcal{H}$ mapping (S, P) to $\mathbb{R}^J$. Suppose $0 \in \mathcal{H}$ and all $h \in \mathcal{H}$ are integrable at all F. Moreover, suppose all $h(s,p) \in \mathcal{H}$ are invertible in s: There exists h^{-1} such that, almost surely, $S = h^{-1}(h(S,P), P)$. Fix some class of functions $\mathcal{K}$ mapping X to $\mathbb{R}^J$. Define an identified set for σ^{-1} with respect to $\mathcal{H}$ and $\mathcal{K}$:

$$\Theta_I = \Theta_I(\mathcal{H}, \mathcal{K}, F) = \{h \in \mathcal{H} : \mathbb{E}_F[h(S,P) \mid X, Z] = \mathbb{E}_F[h(S,P) \mid X] \in \mathcal{K}\}.$$

We will examine throughout identification relative to these function classes $\mathcal{H}$ and $\mathcal{K}$, representing a priori restrictions (e.g., integrability, smoothness, etc.) of a researcher.

A.2. Identification with recentered instruments.

Lemma A.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be all square-integrable functions of (S, P) and of X, respectively. Let $\mathcal{R}$ collect all square-integrable, conditionally mean zero functions of (X, Z): For $R \in \mathcal{R}$, $\mathbb{E}[R(X,Z) \mid X] = 0$ and $\mathbb{E}[R(X,Z)^2] < \infty$. Then*

$$\Theta_I = \{h \in \mathcal{H} : \mathbb{E}[h(S,P)R(X,Z)] = 0 \ \forall R \in \mathcal{R}\}.$$

Proof. First, consider $h \in \Theta_I$—let $k(X) = \mathbb{E}[h(S,P) \mid X, Z]$—and $R \in \mathcal{R}$. Since R and h are both square integrable, hR is integrable. Therefore $\mathbb{E}[hR] = \mathbb{E}[\mathbb{E}[hR \mid X, Z]] = \mathbb{E}[k(X)R(X,Z)] = \mathbb{E}[k(X)\mathbb{E}[R(X,Z) \mid X]] = 0$. Conversely, let $h \in \mathcal{H}$ and $\mathbb{E}[hR] = 0$ for all $R \in \mathcal{R}$. With slight abuse of notation, we overload h to refer to an element of the vector-valued function h. Let $R(X,Z) = \mathbb{E}[h(S,P) \mid X, Z] - \mathbb{E}[h(S,P) \mid X] \in \mathcal{R}$. Then

$$\mathbb{E}[hR] = \mathbb{E}[R^2] = 0 \implies R = 0.$$

Hence $\mathbb{E}[h(S,P) \mid X, Z] = \mathbb{E}[h(S,P) \mid X] \in \mathcal{K}$, and thus $h \in \Theta_I$. □

A.3. Suffices to identify σ^{-1} up to transformation.

The following formalizes Lemma 2: That ensuring Θ_I contains solely functions of the form $T(\sigma^{-1}(\cdot))$ suffices for identifying price counterfactuals.

Lemma A.2. *Assume that for every F^*,*

(1) *Every $h \in \Theta_I$ is of the form $h(S,P) = T(\sigma^{-1}(S,P))$ almost surely for some measurable $T : \mathbb{R}^J \to \mathbb{R}^J$.*
(2) *Θ_I is nonempty.*

Then counterfactuals are identified by $C(s,p,p') = h^{-1}(h(s,p),p')$ for any $h \in \Theta_I$.

Proof. Fix F^* and its implied F. Fix any $h \in \Theta_I$. The assumptions imply that there exists a F^*-probability-one set E such that

$$s = h^{-1}(h(s,p),p) \quad h(s,p) = T(\sigma^{-1}(s,p)) \quad \delta = \sigma^{-1}(s,p) \quad s = \sigma(\delta,p)$$

for all $(s,p,\delta,x,z) \in E$. Thus $s = h^{-1}(T(\delta),p)$. Likewise, for some $(s',p',\delta,x,z) \in E$,

$$\sigma(\delta,p') = s' = h^{-1}(T(\delta),p') = h^{-1}(h(s,p),p'). \qquad \square$$

A.4. **Exogenous prices.** Proposition 1 is a consequence of Proposition A.4:

Proposition A.1. *Let Assumption A.5 hold with $\lambda(X,Z) = Z$ and $Z = P$. Then price counterfactuals are identified under Assumptions A.2 and A.3.*

Proof. Let

(1) $\mathcal{F}$ be all integrable functions of (δ, P)
(2) $\mathcal{H}$ be all integrable functions of (S,P) invertible in S
(3) $\mathcal{K}$ be all integrable functions of X

Then Proposition A.4 implies *(i)* faithfulness follows from completeness under $\mathcal{F},\mathcal{H},\mathcal{K}$ and *(ii)* Proposition A.2's assumptions hold. Identification follows by Proposition A.2. $\square$

A.5. **Suffices to impose faithfulness.** Let $\mathcal{F}$ denote a class of functions mapping from (δ,P) to $\mathbb{R}^J$. We first restate Assumption 4 taking into account $\mathcal{F}$ and $\mathcal{K}$:

Assumption A.4. *Under Assumption A.3, the conditional distribution $(\delta,P) \mid (X,Z)$ satisfies faithfulness with respect to $\mathcal{F}$ and $\mathcal{K}$: For any $H \in \mathcal{F}$ such that $\mathbb{E}_{F^*}[H(\delta,P) \mid X,Z] \in \mathcal{K}$, we have that $H(\delta,P) = H(\delta)$ almost surely for some measurable $H(\delta)$.*

We now formalize Proposition 2—that faithfulness is sufficient for identification—relative to these definitions and assumptions:

Proposition A.2. *Fix $(\mathcal{H},\mathcal{K},\mathcal{F})$. Suppose Assumptions A.1, A.3, and A.4 hold for every F^*. Assume further that for every F^*,*

(1) *There exists an invertible $T_0(\cdot)$ such that $\mathbb{E}_{F^*}[T_0(\delta) \mid X,Z] \in \mathcal{K}, T_0(\sigma^{-1}(\cdot)) \in \mathcal{H}$.*
(2) *For every $h \in \mathcal{H}$, $h(\sigma(\delta,p),p) \equiv H(\delta,p) \in \mathcal{F}$.*

Then price counterfactuals are identified.

Proof. By Lemma A.2, we need to check assumptions (1)–(2) in Lemma A.2. First, condition (1) assumed implies that $T_0(\sigma^{-1}(s,p)) \in \mathcal{H}$ and its conditional expectation lies in $\mathcal{K}$. Thus $T_0(\sigma^{-1}(s,p)) \in \Theta_I$, verifying (2) in Lemma A.2. Now fix $h \in \Theta_I$; by condition (2), $h(\sigma(\delta,p),p) \in \mathcal{F}$. By Assumption A.4, we have that

$$h(S,P) = H(\delta) = H(\sigma^{-1}(S,P))$$

almost surely. This verifies assumption (1) in Lemma A.2. $\square$

A.6. **Faithfulness verification from λ index.** Next, we state and verify the case (Proposition 4) where

$$P \perp\!\!\!\perp (X,Z) \mid \lambda(X,Z), \delta.$$

We show Proposition 3 as partly a corollary of Proposition 4 with $P = \lambda(X,Z) = Z$.

Assumption A.5. *Let $\lambda : \mathbb{R}^J \times \mathcal{Z} \to \mathcal{L} \subset \mathbb{R}^{d_z}$ be an index that is invertible in Z.*

(1) For such a λ, $P \perp\!\!\!\perp (X,Z) \mid \lambda(X,Z), \delta$.

(2) Let $\lambda = \lambda(X,Z)$. The joint distribution of (X,λ) has a density with respect to some product measure $\mu_X \otimes \mu_\lambda$ over $\mathcal{X} \times \mathcal{L}$, which is strictly positive for $\mu_X \otimes \mu_\lambda$-almost every $(x,\lambda) \in \mathcal{X} \times S_\lambda$ for some $S_\lambda \subset \mathcal{L}$ and $(\mu_X \otimes \mu_\lambda)(\mathbb{R}^J \times S_\lambda) > 0$.

Proposition A.3. *Let Assumption A.3 hold and let Assumption A.5 hold with $\lambda(X,Z) = Z$ and $Z = P$. Suppose also that P is not degenerate given δ: for any S_δ where $\mathbb{P}_{F^*}(S_\delta) > 0$, there are two disjoint sets S_1, S_2 such that $\mathbb{P}_{F^*}(\delta \in S_\delta \cap P \in S_1) > 0, \mathbb{P}_{F^*}(\delta \in S_\delta \cap P \in S_2) > 0$. Let*

(1) $\mathcal{F}$ be all integrable functions of (δ, P)

(2) $\mathcal{H}$ be all integrable functions of (S,P) invertible in S

(3) $\mathcal{K}$ be all integrable functions of X.

Then Assumption A.4 holds if and only if Assumption A.2 holds.

Proof. The "if" direction is a corollary of Proposition A.4. For the only if direction, consider the contrapositive. Suppose Assumption A.2 does not hold.

Then, there exists a nonzero $f(\delta,P) \in \mathcal{F}$ such that $\mathbb{E}[f(\delta,P) \mid X,Z] = 0$. If $f(\delta,P) \neq f(\delta)$ for some $f(\delta)$, then Assumption A.4 fails and we are done. Otherwise, we have $\mathbb{E}[f(\delta) \mid X,Z] = \mathbb{E}[f(\delta) \mid X] = 0$ but $f(\delta) \neq 0$. Let $S_\delta = \{\delta : f(\delta) \neq 0\}$ be such that $\mathbb{P}_{F^*}(S_\delta) > 0$. By the non-degeneracy condition, we can choose sets S_1, S_2 and $B(P) = \mathbf{1}(P \in S_1)$. Then $H(P,\delta) = B(P)f(\delta)$ is integrable and not constant in P. But $\mathbb{E}[H(P,\delta) \mid X,P] = B(P) \cdot 0 = 0$. Thus Assumption A.4 does not hold. $\square$

Proposition A.4. *Let*

(1) $\mathcal{F}$ be all integrable functions of (δ, P)

(2) $\mathcal{H}$ be all integrable functions of (S, P) invertible in S

(3) $\mathcal{K}$ be all integrable functions of X

Then,

(1) Assumption A.4 follows from Assumption A.2 and Assumption A.3.

(2) The assumptions of Proposition A.2 are satisfied.

Proof. (1) Take $H \in \mathcal{F}$ with

$$\mathbb{E}\left[H(\delta, P) \mid X, Z\right] = k(X) \in \mathcal{K}, \quad \text{a.s.}$$

By law of iterated expectations,

$$k(X) = \mathbb{E}[\mathbb{E}[H(\delta, P) \mid \delta, X, \lambda] \mid X, Z], \quad \text{a.s.} \tag{A.1}$$

since $(X, Z) = (X, \lambda^{-1}(X, \lambda))$ is measurable with respect to (X, λ) and hence to (δ, X, λ). By Assumption A.5 (1),

$$\mathbb{E}[H(\delta, P) \mid \delta, X, \lambda] = \mathbb{E}[H(\delta, P) \mid \delta, \lambda] := r_H(\delta, \lambda), \quad \text{a.s.} \tag{A.2}$$

Since $(X, \lambda) = (X, \lambda(X, Z))$ is measurable with respect to (X, Z),

$$k(X) = \mathbb{E}[k(X) \mid X, \lambda] = \mathbb{E}[\mathbb{E}[\overline{H}(\delta, \lambda) \mid X, Z] \mid X, \lambda] = \mathbb{E}[r_H(\delta, \lambda) \mid X, \lambda], \quad \text{a.s.}$$

By Assumption A.5 (2), for $(\mu_X \otimes \mu_\lambda)$-almost every (x, λ'),

$$k(x) = \mathbb{E}[r_H(\delta, \lambda') \mid X = x, \lambda = \lambda'].$$

By Assumption A.3, $\lambda \perp\!\!\!\perp \delta \mid X$. Thus, for $(\mu_X \otimes \mu_\lambda)$-almost every (x, λ'),

$$k(x) = \mathbb{E}[r_H(\delta, \lambda') \mid X = x]. \tag{A.3}$$

Fix any $\lambda_0 \in S_\lambda$ such that the above equation holds for μ_X-almost every x. Then, for μ_λ-almost every λ',

$$0 = \mathbb{E}[r_H(\delta, \lambda') - r_H(\delta, \lambda_0) \mid X = x], \quad \text{for } \mu_X\text{-almost every } x.$$

Since X has positive density on $\mathbb{R}^J$,

$$\mathbb{E}[r_H(\delta, \lambda') - r_H(\delta, \lambda_0) \mid X] = 0 \;\text{ a.s. in } X, \text{ for } \mu_\lambda\text{-almost every } \lambda'.$$

By Assumption A.2,

$$r_H(\delta, \lambda') = r_H(\delta, \lambda_0) := \tilde{H}(\delta) \;\text{ a.s. in } \delta, \text{ for } \mu_\lambda\text{-almost every } \lambda'.$$

Using Assumption A.5 again, λ has a positive density on S_λ and thus,

$$r_H(\delta, \lambda) = \tilde{H}(\delta) \quad \text{a.s. in } (\delta, \lambda).$$

Together with (A.2), we have

$$\mathbb{E}[H(\delta, P) - \tilde{H}(\delta) \mid \delta, X, \lambda] = 0, \quad \text{a.s.}$$

Taking conditional expectation given (X, Z) on both sides and using the same argument for (A.1), we obtain

$$\mathbb{E}[H(\delta, P) - \tilde{H}(\delta) \mid X, Z] = 0, \quad \text{a.s.}.$$

Completeness (Assumption A.2) then implies $H(\delta, P) = \tilde{H}(\delta)$, hence faithfulness.

(2) The proof is analogous to Proposition A.6(2). □

A.7. **Faithfulness verification from price separability.** We next restate Assumption 6 and Proposition 5, which rely on restrictions about derivatives of P in z. (A.4) is slightly different from—but is implied by—its counterpart in Assumption 6.

Assumption A.6. *We have the following:*

1. *(Support) For almost every δ, there is a connected open set $U_\delta \subset \mathbb{R}^J$ such that $P \mid \delta$ has a density $f(p \mid \delta) > 0$ on U_δ and $\mathbb{P}(P \in U_\delta \mid \delta) = 1$. The distribution $(S, P, Z) \mid X$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{J \times J \times d_z}$. There exists an open set $U \subset \mathbb{R}^{J \times J \times d_z}$ where the density of $(S, P, Z) \mid X$ is positive on U and $P((S, P, Z) \in U \mid X) = 1$ a.s.*
2. *(Price equation) The random variable P can be represented as*
$$P = f(X, \delta, Z, \tilde{\omega}) \text{ for } (\tilde{\omega}, \delta) \perp\!\!\!\perp Z \mid X \tag{A.4}$$
for some function f continuously differentiable in Z with Jacobian $D_z f$. The joint distribution $(\tilde{\omega}, \delta) \mid X$ has density $f(\tilde{\omega}, \delta \mid x)$ with respect to some dominating measure λ_x.
3. *(Price separability) The Jacobian of f in Z satisfies the following: For measurable functions A, B,*
$$\underbrace{D_z f(X, \delta, Z, \tilde{\omega})}_{J \times d_z} = \underbrace{A(f(X, \delta, Z, \tilde{\omega}), \delta)}_{J \times J} \cdot \underbrace{B(X, Z)}_{J \times d_z} \tag{A.5}$$
where $A(f(X, \delta, Z, \tilde{\omega}), \delta)$ and $B(X, Z)$ are both full-rank a.s. with $d_z \geq J$

(4) *(Domination) There exists* $G_1(x,\delta,\tilde{\omega}) \geq 0$ *and* $G_2(x,\delta,\tilde{\omega}) \geq 0$ *such that a.s.,*[28]

$$\|\sigma_p(\delta, P) f_z(X, Z, \delta, \tilde{\omega})\| \leq G_1(X, \delta, \tilde{\omega})$$
$$\|f_z(X, Z, \delta, \tilde{\omega})\| \leq G_2(X, \delta, \tilde{\omega})$$

and $\mathbb{E}[G_1^4 + G_2^4 \mid X] = \mathbb{E}[G_1^4 + G_2^4 \mid X, Z] < \infty$.

Proposition A.5. *Let*

(1) $\mathcal{F}$ *denote the class of functions* $H(\delta, P)$ *where (i)* $\mathbb{E}[H^2 \mid X, Z] < \infty$, *(ii) for* F^**-almost every* δ, $H(\delta, \cdot)$ *is continuously differentiable with derivative* $H_p(\delta, \cdot)$ *on* U_δ, and *(iii)*

$$\frac{\partial}{\partial z}\mathbb{E}[H(\delta, f(X,\delta,z,\tilde{\omega})) \mid X, Z = z] = \mathbb{E}[H_p(\delta, f(X,\delta,z,\tilde{\omega})) D_z f(X,\delta,z,\tilde{\omega}) \mid X].$$
(Interchange of differentiation and expectation)

(2) $\mathcal{H} \subset L^2_F(S, P)$ *be the set of* $\mathbb{R}^J$*-valued functions* $h(s,p)$ *such that (i)* h *is continuously differentiable in* (s,p) *on* U, *(ii) the derivatives are locally Lipschitz: There exists* $\epsilon > 0$ *such that for each* $j = 1, \ldots, J$ *and all values* s, s', p, p' *where* $\|(s,p) - (s',p')\| < \epsilon$,

$$\max\left(\left\|\frac{\partial h_j}{\partial s}(s,p) - \frac{\partial h_j}{\partial s}(s',p')\right\|, \left\|\frac{\partial h_j}{\partial p}(s,p) - \frac{\partial h_j}{\partial p}(s',p')\right\|\right) \leq L(s',p')\|(s,p) - (s',p')\|$$

where $\mathbb{E}[L^2(S,P) \mid X, Z] < \infty$, *and (iii) The derivatives* h_s, h_p *are integrable:*

$$\mathbb{E}[\|h_s(S,P)\|^2 + \|h_p(S,P)\|^2 \mid X, Z] < \infty.$$

(iv) h *is invertible in* S.

(3) $\mathcal{K}$ *be all square-integrable functions of* X.

Assume $\sigma^{-1} \in \mathcal{H}$. *Suppose Assumptions A.1 to A.3* and *A.6* hold. *Then,*

(1) *Assumption A.4* holds

(2) *The assumptions in Proposition A.2 hold*

Proof. (1) Let $H \in \mathcal{F}$. Write

$$k(x) = \mathbb{E}[H(\delta, P) \mid X = x, Z = z] = \int H(\delta, f(x,\delta,z,\tilde{\omega}))\, f(\tilde{\omega}, \delta \mid x) d\lambda_x.$$

Since $\mathcal{H} \in \mathcal{F}$, differentiating both sides in z yields

$$0 = \mathbb{E}[H_p(\delta, P) A(P, \delta) \mid X, Z] \cdot B(X, Z).$$

[28] For concreteness, we choose the Frobenius norm as the matrix norm.

Since $d_z \geq J$ and $B(X, Z)$ is full rank, we have that $\mathbb{E}[H_p(\delta, P)A(P, \delta) \mid X, Z] = 0$ almost surely. By completeness, since A is full rank,

$$H_p(\delta, P)A(P, \delta) = 0 \implies H_p(\delta, P) = 0 \text{ a.s.}$$

Since H_p is continuous in p and P is supported with positive density on U_δ, $H_p(\delta, p) = 0$ on U_δ. As a result, $H(\delta, \cdot)$ is constant on U_δ for almost every δ. Hence there exist some $H(\delta)$ where $H(\delta) = H(\delta, P)$ almost surely, verifying Assumption A.4.

(2) Lemma A.3 verifies that assumption (2) of Proposition A.2 holds. Finally, assumption (1) of Proposition A.2 holds by choosing $T_0(d) = d$, since

$$\sigma^{-1} \in \mathcal{H}, \quad \mathbb{E}[\sigma^{-1} \mid X, Z] = k(X) \in L^2_F(X) = \mathcal{K},$$

under the assumption that σ^{-1} is square-integrable. □

Lemma A.3. *Suppose Assumption A.6 holds. Let $\mathcal{H}, \mathcal{F}$ be defined as in Proposition A.5 and suppose $\sigma^{-1} \in \mathcal{H}$. Define $H(\delta, p) = h(\sigma(\delta, p), p)$. Then $H \in \mathcal{F}$.*

Proof. It suffices to show that differentiation in z and expectation are interchangeable for H, since the other properties of $\mathcal{F}$ are assumed in $\mathcal{H}$. Since it suffices to show the claim entrywise over entries of h, we abuse notation and denote $h(s, p)$ as an entry and treat it as a scalar function.

We let f_z denote $D_z f$. Define $h^*(\delta, x, z, \tilde{\omega}) = h(\sigma(\delta, f(x, \delta, z, \tilde{\omega})), f(x, \delta, z, \tilde{\omega}))$. Fix some z_0. Let

$$h_z^*(\delta, X, Z, \tilde{\omega}) = (h_s(S, P)'\sigma_p(\delta, P) + h_p(S, P)')\, f_z(X, \delta, Z, \tilde{\omega})$$

be the (transposed) gradient at $(\delta, X, Z, \tilde{\omega})$. We show that if $h \in \mathcal{H}$ then

$$\frac{\partial}{\partial z}\mathbb{E}[h^*(\delta, X, Z, \tilde{\omega}) \mid X, Z] = \mathbb{E}[h_z^*(\delta, X, Z, \tilde{\omega}) \mid X, Z].$$

It suffices to show that

$$\lim_{t \to 0} \mathbb{E}\left[\sup_{\|v\|=1} \left| \frac{h^*(\delta, X, z_0 + tv, \tilde{\omega}) - h^*(\delta, X, z_0, \tilde{\omega})}{t} - h_z^*(\delta, X, z, \tilde{\omega})v \right| \mid X \right] = 0. \quad \text{(A.6)}$$

(A.6) implies

$$\lim_{t \to 0} \sup_{\|v\|=1} \left| \mathbb{E}\left[\frac{h^*(\delta, X, z_0 + tv, \tilde{\omega}) - h^*(\delta, X, z_0, \tilde{\omega})}{t} \mid X \right] - \mathbb{E}[h_z^*(\delta, X, z, \tilde{\omega}) \mid X]v \right| = 0,$$

which implies that $z \mapsto \mathbb{E}[h^*(\delta, X, z, \tilde{\omega}) \mid X]$ is (Frechet) differentiable at z_0 and its gradient is equal to $\mathbb{E}[h_z^*(\delta, X, z, \tilde{\omega}) \mid X]$.

Towards (A.6), at a fixed $(\delta, x, \tilde{\omega})$, observe that since h^* is differentiable at z_0, we have pointwise convergence

$$\lim_{t \to 0} \sup_{\|v\|=1} \left| \frac{h^*(\delta, x, z_0 + tv, \tilde{\omega}) - h^*(\delta, x, z_0, \tilde{\omega})}{t} - h_z^*(\delta, x, z_0, \tilde{\omega})v \right| = 0.$$

Thus it suffices to show the following and apply the dominated convergence theorem: For all sufficiently small t,

$$\sup_{\|v\|=1} \left| \frac{h^*(\delta, X, z_0 + tv, \tilde{\omega}) - h^*(\delta, X, z_0, \tilde{\omega})}{t} - h_z^*(\delta, X, z_0, \tilde{\omega})v \right| \leq G_0(\delta, X, \tilde{\omega}) \tag{A.7}$$

where $\mathbb{E}[G_0(\delta, X, \tilde{\omega}) \mid X] < \infty$.

Observe that

$$\begin{aligned} \|h_z^*(\delta, x, z_0, \tilde{\omega})\| &\leq \|h_s(s,p)\| \|\sigma_p(\delta, p) f_z(x, z_0, \delta, \tilde{\omega})\| + \|h_p(s,p)\| \|f_z(x, z_0, \delta, \tilde{\omega})\| \\ &\leq \|h_s(s,p)\| G_1(x, \delta, \tilde{\omega}) + \|h_p(s,p)\| G_2(x, \delta, \tilde{\omega}) \end{aligned}$$

for $s = \sigma(\delta, x, z_0, \tilde{\omega}), p = f(x, \delta, z_0, \tilde{\omega})$. Since the derivatives of h and G_1, G_2 are all square-integrable, by Cauchy-Schwarz $\|h_s(s,p)\| G_1(x, \delta, \tilde{\omega}) + \|h_p(s,p)\| G_2(x, \delta, \tilde{\omega})$ is an integrable dominating function.

Therefore it suffices to show that the difference quotient can be dominated

$$\sup_{\|v\|=1} \left| \frac{h^*(\delta, X, z_0 + tv, \tilde{\omega}) - h^*(\delta, X, z_0, \tilde{\omega})}{t} \right| \leq G_0(\delta, X, \tilde{\omega}) \tag{A.8}$$

Towards (A.8), define $p_{tv} = f(\delta, x, z_0 + tv, \tilde{\omega})$ and $s_{tv} = \sigma(\delta, p_{tv})$. By the mean-value theorem in h, we have

$$h^*(\delta, x, z_0 + tv, \tilde{\omega}) - h^*(\delta, X, z_0, \tilde{\omega}) = h_s(\tilde{s}_{t,v}, \tilde{p}_{t,v})(s_{tv} - s_0) + h_p(\tilde{s}_{t,v}, \tilde{p}_{t,v})(p_{tv} - p_0) \tag{A.9}$$

where $(\tilde{s}_{t,v}, \tilde{p}_{t,v})$ is some point on the line segment connecting (s_0, p_0) with (s_{tv}, p_{tv}).

We can write $h_s(\tilde{s}_{tv}, \tilde{p}_{tv}) = h_s(s_0, p_0) + R_s(s_0, p_0)$ where

$$\|R_s(s_0, p_0)\| \leq L(s_0, p_0) t (G_1(x, \delta, \tilde{\omega}) + G_2(x, \delta, \tilde{\omega}))$$

for all $t < \epsilon$. Similarly for h_p. Thus

$$\text{(A.9)} = h_s(s_0, p_0)(s_{tv} - s_0) + R_s(s_0, p_0)(s_{tv} - s_0) + h_p(s_0, p_0)(p_{tv} - p_0) + R_p(s_0, p_0)(p_{tv} - p_0)$$

By the mean-value theorem applied to $z \mapsto \sigma(\delta, f(\delta, x, z, \tilde{\omega}))$, we have that

$$s_{tv} - s_0 = \{\sigma_p(\delta, \check{p}_{tv}) f_z(x, \check{z}, \delta, \tilde{\omega})\} tv$$

where $\check{z}$ is on the line segment between $z_0, z_0 + tv$ and $\check{p}_{tv} = f(x, \check{z}, \delta, \tilde{\omega})$. We thus have

$$\|s_{tv} - s_0\| \le tG_1(x, \delta, \tilde{\omega}).$$

Similarly, we have

$$\|p_{tv} - p_0\| \le tG_2(x, \delta, \tilde{\omega}).$$

Thus, for all $t < \epsilon$,

$$\sup_{\|v\|=1} \left\| \frac{(\text{A.9})}{t} \right\| \le \|h_s(s_0, p_0)\| G_1(x, \delta, \tilde{\omega}) + \|h_p(s_0, p_0)\| G_2(x, \delta, \tilde{\omega})$$
$$+ tL(s_0, p_0) \left\{ G_1(x, \delta, \tilde{\omega}) + G_2(x, \delta, \tilde{\omega}) \right\}^2$$

The right-hand side is a function of $(\tilde{\omega}, \delta, x)$ that is integrable by Cauchy–Schwarz. This concludes the proof. □

Lemma A.4. *Let $(\tilde{\omega}, \delta) \perp\!\!\!\perp Z \mid X$ and suppose P is generated by (9). Assume the dimension of z is at least J. Assume that*

(1) The codomains of f_0, f_1, f_2 are all $\mathbb{R}^J$.
(2) $f_0(\cdot, \delta)$ is invertible where $f_0^{-1}(\cdot, \delta)$ is continuously differentiable for every δ with invertible Jacobian $Q_0(p, \delta) \in \mathbb{R}^{J \times J}$.
(3) f_1 is continuously differentiable in z with full-rank Jacobian $Q_1(z, x) \in \mathbb{R}^{J \times d_z}$

Then items 2 and 3 of Assumption A.6 are satisfied.

Proof. Write $f_0^{-1}(p, \delta) = f_1(z, x) + f_2(\delta, x, \tilde{\omega})$. Differentiate in z implicitly to obtain

$$Q_0(p, \delta) D_z f = Q_1(z, x).$$

Since Q_0 is assumed to be invertible, we have

$$D_z f = \underbrace{Q_0^{-1}(p, \delta)}_{A} \underbrace{Q_1(z, x)}_{B}.$$

By assumption A and B are full rank. □

A.8. **Faithfulness verification from discrete support.** We verify completeness implies faithfulness if X and δ are discrete with same-sized supports.

Proposition A.6. *For every F^*, suppose the support of δ and X are both finite sets of size $M \in \mathbb{N}$. Let*

(1) $\mathcal{F}$ be all integrable functions of (δ, P),
(2) $\mathcal{H}$ be all integrable functions of (S, P) invertible in S
(3) $\mathcal{K}$ be all $\mathbb{R}^J$-valued functions of X.

Then, with these choices,

(1) *Assumption A.4 follows from Assumption A.2 and Assumption A.3.*

(2) *The assumptions of Proposition A.2 are satisfied.*

Proof. Note any real-valued $h(\delta)$ and $g(x)$ can be represented as $\mathbb{R}^M$ vectors. Consider

$$\mathbb{E}[h(\delta) \mid X, Z] = \mathbb{E}[h(\delta) \mid X] \equiv g(X).$$

If we represent both as vectors in $\mathbb{R}^M$, then we have $Q'_{\delta|X} h = g$, for a matrix $Q_{\delta|X}$ whose $(i,j)^{\text{th}}$ entry is $\mathbb{P}_{F^*}(\delta = \delta_i \mid X = x_j)$, where we number values in the support of δ as δ_i and likewise for values of X. The matrix $Q_{\delta|X}$ is square by assumption. Completeness implies that $Q'_{\delta|X}$ is full-rank, and thus invertible.

Now, consider any function where $\mathbb{E}[H(\delta, P) \mid X, Z] = k(X) \in \mathcal{K}$. For each coordinate j, there is a function $H_{0j}(\delta)$ which can be represented as $H_{0j} = (Q'_{\delta|X})^{-1} k_j$. Construct $H_0 = (H_{01}, \ldots, H_{0J})'$. By construction $\mathbb{E}[H_0(\delta) \mid X, Z] = k(X)$. Completeness implies that $H(\delta, P) = H_0(\delta)$. This shows (1).

For (2), we can easily check that $\sigma^{-1} \in \mathcal{H}$ and $\mathbb{E}[\sigma^{-1} \mid X, Z] = \mathbb{E}[\delta \mid X] \in \mathcal{K}$. Thus (1) in Proposition A.2 is satisfied with $T_0(\delta) = \delta$. If $h(S, P) \in \mathcal{H}$ is integrable, then so is $H(\delta, P) = h(\sigma(\delta, P), P)$ since $S = \sigma(\delta, P)$ a.s. Thus $H \in \mathcal{F}$. This verifies (2) in Proposition A.2. □

A.9. **Faithfulness verification from location-scale model for δ.**

A.9.1. *Mathematical preliminaries.* Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{S}^J_+$ denote the set of $J \times J$ positive semidefinite matrices. For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_J) \in \mathbb{N}^J$ and function $f(x) : \mathbb{R}^J \mapsto \mathbb{R}^s$, where $x \in \mathbb{R}^J$ and s may be different from J, we write $|\alpha| = \sum_{j=1}^J \alpha_j$, $\alpha! = \prod_{j=1}^J \alpha_j!$, and

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_J^{\alpha_J}, \quad D_x^\alpha f(x) = D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} \cdots D_{x_J}^{\alpha_J} f(x).$$

Furthermore, for any function $g : \mathbb{R}^J \mapsto \mathbb{R}$, we define its Fourier transform as

$$\hat{g}(\omega) = \int_{\mathbb{R}^J} e^{-i\langle \omega, u \rangle} g(u) du.$$

For a function class $\mathcal{L}$, $f \in \mathcal{L}^{\otimes J}$ iff $f = (f_1, \ldots, f_J)'$ where $f_j \in \mathcal{L}$.

Given a function $v : \mathbb{R}^J \mapsto \mathbb{R}$ for which v is positive everywhere, let $\mathbb{W}^{s,p}(\mathbb{R}^J)$ denote the Bessel potential space of indices (s, p):

$$\mathbb{W}^{s,p}(\mathbb{R}^J; \mathbb{R}^J) = \left\{ u : \mathbb{R}^J \to \mathbb{R}^J : (1 + \|\omega\|_2^2)^{s/2} \hat{u}_j(\omega) \in L^p(\mathbb{R}^J),\ j \in [J] \right\}. \quad \text{(A.10)}$$

For any $s = m + \sigma$ where $m = \lfloor s \rfloor$ is the integer part and $\sigma \in [0, 1)$ is the fractional part, let $\|\cdot\|_{\mathbb{W}^{s,p}}$ denote the associated norm with

$$\|u\|_{\mathbb{W}^{s,p}} = \sum_{|\alpha| \le m} \|D^\alpha u\|_{L^p} + \sum_{\alpha:|\alpha|=m} [D^\alpha u]_{W^{\sigma,p}},$$

where

$$[h]_{\mathbb{W}^{\sigma,p}} := \left(\int_{\mathbb{R}^J} \int_{\mathbb{R}^J} \frac{\|h(x) - h(y)\|^p}{\|x - y\|^{\sigma p + J}} dxdy \right)^{1/p}, \quad [h]_{\mathbb{W}^{\sigma,\infty}} := \sup_{x \ne y} \frac{\|h(x) - h(y)\|}{\|x - y\|^\sigma}.$$

Define the following space:

$$\begin{aligned} \mathbb{K}^s(\mathbb{R}^J; \mathbb{R}^J) &\equiv \mathrm{lin}(\mathbb{R}^J; \mathbb{R}^J) \oplus \mathbb{W}^{s,\infty}(\mathbb{R}^J; \mathbb{R}^J) \\ &\equiv \{u : \mathbb{R}^J \to \mathbb{R}^J : u(x) = Ax + r(x), A \in \mathbb{R}^{J \times J}, r \in \mathbb{W}^{s,\infty}(\mathbb{R}^J, \mathbb{R}^J)\} \end{aligned} \tag{A.11}$$

where $\mathrm{lin}(\mathbb{R}^J; \mathbb{R}^J)$ is the space of all bounded linear functions that map $\mathbb{R}^J$ to $\mathbb{R}^J$. In the following, for notational convenience, we suppress the spaces $\mathbb{R}^J$ or $\mathbb{R}^J; \mathbb{R}^J$ in $\mathbb{W}^{s,\infty}$ and $\mathbb{K}^s$. For any $s_0 \le s_1$, since $\mathbb{W}^{s_1,\infty} \subset \mathbb{W}^{s_0,\infty}$, we have $\mathbb{K}^{s_1} \subset \mathbb{K}^{s_0}$.

Last, for a linear operator $\mathcal{T}$ that maps a Banach space $\mathcal{U}$ to a Banach space $\mathcal{V}$, let

$$\|\mathcal{T}\|_{\mathcal{U} \mapsto \mathcal{V}} = \sup_{u \in \mathcal{U}} \frac{\|\mathcal{T} u\|_{\mathcal{V}}}{\|u\|_{\mathcal{U}}}.$$

A.9.2. *Assumptions.*

Assumption A.7. *There exists $s, M \in \mathbb{N}$ with $M = J + 2s + 1$ and $s > J + 1$ such that the following holds:*

(1) There exist invertible $a, b : \mathbb{R}^J \to \mathbb{R}^J$ with

$$a(\delta) = b(X) + \Sigma(X)\epsilon \tag{A.12}$$

where $\epsilon \perp\!\!\!\perp X$ and $\Sigma : \mathbb{R}^J \mapsto \mathbb{S}^J_+$.

(2) Let q denote the density function of $-\epsilon$ with respect to the Lebesgue measure, which we assume to exist. There exist $0 < C_- < C_+ < \infty$ such that, for any $\upsilon \in \mathbb{R}^J$,

$$C_-(1 + \|\upsilon\|_2^2)^{-s/2} \le |\hat{q}(\upsilon)| \le C_+(1 + \|\upsilon\|_2^2)^{-s/2},$$

and, for any $\alpha \in \mathbb{N}^J$ with $|\alpha| \le M$,

$$\|D^\alpha_\upsilon \hat{q}(\upsilon)\| \le C_+(1 + \|\upsilon\|_2^2)^{-s/2-|\alpha|/2}$$

(3) *There exist* $\Sigma_0 \in \mathbb{S}^J_+$, *and* $\psi \in (0, \lambda_{\min}(\Sigma_0)/2)$ *such that,*

$$\sup_{x\in\mathbb{R}^J} \|\Sigma(x) - \Sigma_0\|_{\mathrm{op}} + \int_{\mathbb{R}^J} \|\Sigma(x) - \Sigma_0\|_{\mathrm{op}} dx \le \psi,$$

and, for any $\gamma \in \mathbb{N}^J$ *with* $|\gamma| \le M$,

$$\sup_{x\in\mathbb{R}^J} \|D^\gamma_x \Sigma(x)\|_{\mathrm{op}} + \int_{\mathbb{R}^J} \|D^\gamma_x \Sigma(x)\|_{\mathrm{op}} dx \le \psi.$$

(4) $b \in \mathbb{K}^M(\mathbb{R}^J)$ *and* $\sup_{x\in\mathbb{R}^J} \|(Db(x))^{-1}\|_{\mathrm{op}} < \infty$.

Intuitively,

- Assumption A.7(1) assumes that $\delta \mid X$ follows a location-scale model.
- Assumption A.7(2) assumes that this familly has shape corresponding to some density $q(\cdot)$, and $|\hat{q}(v)| \asymp \|v\|^{-s}$ for large v in signal space, with derivatives up to M of the Fourier transform satisfying $\|D^\alpha_v \hat{q}(v)\| \lesssim \|v\|^{-s-|\alpha|}$.
- Assumption A.7(3) imposes that $\Sigma(x)$ is close in operator norm to some constant Σ_0 and similarly for the derivatives of $\Sigma(x)$.
- Assumption A.7(4) imposes that $b \in \mathbb{K}^M$ and its inverse b^{-1} is Lipschitz.

A.9.3. *Faithfulness verification.*

Proposition A.7. *Assume that Assumption A.7 holds with sufficiently small* ψ (A.13). *Let* M, s *be as in Assumption A.7. Let*

(1) $\mathcal{F}$ *be all integrable functions of* (δ, P)
(2) $\mathcal{H}$ *be all integrable functions of* (S, P) *invertible in* S
(3) $\mathcal{K} = \mathbb{K}^{2s}$

Then, with these choices,

(1) *Assumption A.4 follows from Assumptions A.2 and A.3*
(2) *The assumptions of Proposition A.2 are satisfied.*

Proof. (1) follows immediately from Theorem A.1 and Assumption A.2, applied entrywise. For Proposition A.2(1), observe that $b(X) = \mathbb{E}[a(\delta) \mid X, Z] \in \mathbb{K}^M \subset \mathbb{K}^{2s} = \mathcal{K}$ and $\sigma^{-1} \in \mathcal{H}$ with $T_0(\delta) = a(\delta)$. Likewise, Proposition A.2(2) holds immediately. □

We now outline the remaining argument:

(1) Theorem A.1 is the key result. Its proof argues that it is without loss to work with $\tilde{\delta} = \Sigma_0^{-1}\delta$, $\tilde{X} = \Sigma_0^{-1} b(X)$ such that $\tilde{\delta} = \tilde{X} + \tilde{\Sigma}(\tilde{X})\epsilon$ and $\tilde{\Sigma}(\tilde{X})$ is centered around I. Moreover, it suffices to show that

$$\mathbb{E}[u(\delta) \mid X] = k(X) \quad k \in \mathbb{W}^{2s,\infty}$$

is solvable by some u. With these normalizations, Theorem A.2 shows that there exists some $u(\delta)$ such that $\mathbb{E}[u(\delta) \mid X] = k(X)$, for scalar-valued functions (u, k). The bulk of the argument justifies Theorem A.2.

(2) Suppose $\Sigma(x) = I$. Then $\mathbb{E}[u(\delta) \mid X] = (u \star q)(x)$ is a convolution. Thus we can take u such that its Fourier transform is a ratio $\hat{u} = \frac{\hat{k}}{\hat{q}}$. Lemma A.5 verifies that u is a proper function.

(3) Next, with $\Sigma(x) \neq I$, one could consider the deviation of the conditional expectation operator from the $\Sigma = I$ case:

$$(\mathcal{T}u)(x) \equiv (\mathcal{T}_0 u)(x) + (\mathcal{E}u)(x) = \left\{\mathcal{T}_0\left(\mathrm{Id} + \mathcal{T}_0^{-1}\mathcal{E}\right)u\right\}(x)$$

for $\mathcal{T}$ the conditional expectation operator $\delta \mid X$, $\mathcal{T}_0$ the conditional expectation operator under $\Sigma(x) = I$, and $\mathcal{E}$ their difference. Now, we have the geometric expansion

$$(\mathrm{Id} + \mathcal{T}_0^{-1}\mathcal{E})^{-1} = \sum_{\ell=0}^{\infty}\left(-\mathcal{T}_0^{-1}\mathcal{E}\right)^{\ell}$$

upon verifying that the right-hand side converges. Lemma A.6 derives the key condition for convergence. Given a k, one can then construct u as

$$u = \left(\sum_{\ell=0}^{\infty}\left(-\mathcal{T}_0^{-1}\mathcal{E}\right)^{\ell}\right)\mathcal{T}_0^{-1}k.$$

(4) The key condition for convergence turns out to be a bound on the discrepancy in Fourier space of the integration kernels: $\hat{q}(\Sigma(x)\omega) - \hat{q}(\omega)$. The proof of Theorem A.2 verifies that so long as $\Sigma(x)$ is sufficiently close to I under Assumption A.7(3), this bound is attainable.

Theorem A.1. *Assume that Assumption A.7 holds with*

$$\psi < \psi^* \equiv \psi^*\left(J, s, C_{\pm}, \lambda_{\min}(\Sigma_0), \|Db\|_{\mathbb{W}^{M-1,\infty}}, \|Db^{-1}\|_{L^\infty}\right) \tag{A.13}$$

for some constant ψ^* that only depends on the parameters inside the parentheses. Then for any $k \in \mathbb{K}^{2s}(\mathbb{R}^J, \mathbb{R}^J)$, there exists a proper function $u : \mathbb{R}^J \mapsto \mathbb{R}^J$ such that $\mathbb{E}[u(\delta) \mid X] = k(X)$ almost surely.

Proof. We first note that it suffices to prove the claim for scalar-valued $k \in \mathbb{K}^{2s}(\mathbb{R}^J; \mathbb{R})$ since we can concatenate the solutions. We argue that it is sufficient to prove Theorem A.2. The reason is that we can change variables and write $\tilde{X}$ for $b(X)$, $\tilde{\epsilon}$ for $\Sigma_0\epsilon$, and $\tilde{\Sigma}(\tilde{X}) = (\Sigma \circ b^{-1}(\tilde{X}))\Sigma_0^{-1}$. Likewise, it is without loss to redefine δ as $a(\delta)$.

We need to show that, for sufficiently small choices of ψ, $\tilde{\Sigma}(\cdot) - I$ is suitably bounded by some $\tilde{\psi}$ so that Assumption A.7(3) is satisfied for Theorem A.2. Lemma C.6 bounds $\tilde{\Sigma}(\cdot) - I$ in terms of $\Sigma(\cdot) - \Sigma_0$, where the bound depends only on the additional quantities $\|Db\|_{\mathbb{W}^{M-1,\infty}}$, $\|Db^{-1}\|_{L^\infty}$, $\lambda_{\min}^{-1}(\Sigma_0)$. Thus, there exists

$$\psi^* := \psi^* \left(J, s, C_\pm, \lambda_{\min}(\Sigma_0), \|Db\|_{\mathbb{W}^{M-1},\infty}, \|Db^{-1}\|_{L^\infty}\right)$$

such that, if $\psi \leq \psi^*$, we have that

$$\|\tilde{\Sigma}(\cdot) - I\|_{\mathbb{W}^{M,\infty}} + \|\tilde{\Sigma}(\cdot) - I\|_{\mathbb{W}^{M,1}} \leq \tilde{\psi}^*(J, s, C_\pm).$$

required by Theorem A.2.

Now, having checked the conditions, we can apply Theorem A.2 to $\tilde{X}$. Given $k \in \mathbb{K}^{2s}$, since b is invertible, we can write $k(X) = (k \circ b^{-1} \circ b)(X) = (k \circ b^{-1})(\tilde{X})$. By Lemma C.5, $k \circ b^{-1} \in \mathbb{K}^{2s}$. By definition,

$$(k \circ b^{-1})(\tilde{x}) = A\tilde{x} + \tilde{k}(\tilde{x}) \quad \tilde{k}(\tilde{x}) \in \mathbb{W}^{2s,\infty}.$$

By Theorem A.2, we can then find $\tilde{u}(\delta)$ for which

$$\mathbb{E}[\tilde{u}(\delta) \mid \tilde{X}] = \tilde{k}(\tilde{X}).$$

Let $u(\delta) = A\delta + \tilde{u}(\delta)$, we then have

$$\mathbb{E}[u(\delta) \mid \tilde{X}] = \mathbb{E}[u(\delta) \mid X] = (k \circ b^{-1})(\tilde{X}) = k(X).$$

□

Theorem A.2. *Suppose Assumption A.7 holds with $a(\delta) = \delta, b(x) = x, \Sigma_0 = I$ and*

$$\psi < \psi^* \equiv \psi^* \left(J, s, C_\pm\right) \tag{A.14}$$

for some constant ψ^* that only depends on the parameters inside the parentheses, defined in (A.15). Then for any $k \in \mathbb{W}^{2s,\infty}$, there exists a proper function u such that $\mathbb{E}[u(\delta) \mid X] = k(X)$ almost surely.

Proof. By Lemmas A.5 and A.6, it remains to prove (A.19).

Step 1: Bound L^1 norm of $\Delta_\omega(x)$ and its derivatives by $(1 + \|\omega\|^2)^{-s/2}$.

By the Taylor expansion,

$$\Delta_\omega(x) = \langle \nabla\hat{q}((\lambda\Sigma(x) + (1 - \lambda)I)\omega), (\Sigma(x) - I)\omega \rangle$$

for some $\lambda \in (0, 1)$ that depends on x. By Assumption A.7(3), $\psi < 1/2$ and

$$\|(\lambda\Sigma(x) + (1 - \lambda)I)\omega\|_2 \geq (1 - \psi)\|\omega\|_2 \geq \|\omega\|_2/2.$$

By Assumption A.7(2),

$$
\begin{aligned}
\|\nabla \hat{q}((\lambda\Sigma(x) + (1-\lambda)I)\omega)\|_2 &\le C_+(1+\|\omega\|_2^2/4)^{-s/2-1/2} \\
&\le C_+ 2^{s+1}(1+\|\omega\|_2^2)^{-s/2-1/2}.
\end{aligned}
$$

Then

$$
\begin{aligned}
|\Delta_\omega(x)| &\le C_+ 2^{s+1}(1+\|\omega\|_2^2)^{-s/2-1/2} \cdot \|\Sigma(x) - I\|_{\mathrm{op}}\|\omega\|_2 \\
&\le C_+ 2^{s+1}(1+\|\omega\|_2^2)^{-s/2} \cdot \|\Sigma(x) - I\|_{\mathrm{op}}
\end{aligned}
$$

By Assumption A.7(3), $\|\Delta_\omega\|_{L^1} \le \psi C_+ 2^{s+1}(1+\|\omega\|_2^2)^{-s/2}$. By Assumption A.7(2) and Lemma A.7, for any $\alpha \in \mathbb{N}^J$ with $0 < |\alpha| \le M$

$$
\|D_x^\alpha \Delta_\omega\|_{L^1} = \|D_x^\alpha \hat{q}(\Sigma(x)\omega)\|_{L^1} \le \psi C_+ C_{J,s}(1+\|\omega\|_2^2)^{-s/2}.
$$

Step 2: Bound the Fourier transform $|\hat{\Delta}_\omega|$ with L^1 norm of Δ_ω derivatives.

By Lemma C.3,

$$
\begin{aligned}
|\hat{\Delta}_\omega(\upsilon)| &\le \psi \tilde{C}(1+\|\omega\|_2^2)^{-s/2}(1+\|\upsilon\|_2)^{-M} \\
&\le \psi \tilde{C}(1+\|\omega\|_2^2)^{-s/2}(1+\|\upsilon\|_2^2)^{-M/2},
\end{aligned}
$$

where $\tilde{C} = c_{J,M} C_+ C_{J,s}$.

Step 3: Bound the key condition (A.19).

By Assumption A.7(2),

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^J} |\hat{\Delta}_\omega(\omega - \upsilon)| \frac{|\hat{q}(\omega)|}{|\hat{q}(\upsilon)|^2} d\omega \right| \\
&\le \psi \frac{\tilde{C} C_+}{C_-^2} \int_{\mathbb{R}^J} \frac{(1+\|\omega-\upsilon\|_2^2)^{-M/2}(1+\|\omega\|_2^2)^{-s}}{(1+\|\upsilon\|_2^2)^{-s}} d\omega \\
&= \psi \frac{\tilde{C} C_+}{C_-^2} \int_{\mathbb{R}^J} \frac{(1+\|\upsilon\|_2^2)^{s}}{(1+\|\omega-\upsilon\|_2^2)^{M/2}(1+\|\omega\|_2^2)^{s}} d\omega.
\end{aligned}
$$

Note that

$$
1 + \|\upsilon\|_2^2 \le 1 + 2(\|\omega\|_2^2 + \|\omega - \upsilon\|_2^2) \le 2(1+\|\omega\|_2^2)(1+\|\omega-\upsilon\|_2^2).
$$

Thus,

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^J} |\hat{\Delta}_\omega(\omega - \upsilon)| \frac{|\hat{q}(\omega)|}{|\hat{q}(\upsilon)|^2} d\omega \right| \\
&\le \psi \frac{2^s \tilde{C} C_+}{C_-^2} \int_{\mathbb{R}^J} (1+\|\omega-\upsilon\|_2^2)^{-(\frac{M}{2}-s)} d\omega
\end{aligned}
$$

$$
\begin{aligned}
&= \psi \frac{2^s \tilde{C} C_+}{C_-^2} \int_{\mathbb{R}^J} (1 + \|\upsilon\|_2^2)^{-(\frac{M}{2}-s)} d\upsilon && \text{(Change of variables)} \\
&= \psi \frac{2^s \tilde{C} C_+}{C_-^2} \int_{\mathbb{R}^J} (1 + \|\upsilon\|_2^2)^{-(J+1)/2} d\upsilon \\
&= \psi \frac{2^s \tilde{C} C_+}{C_-^2} \frac{\pi^{(J+1)/2}}{\Gamma((J+1)/2)}.
\end{aligned}
$$

This can be made less than 1 because we can choose

$$
\psi < \tilde{\psi}^*(J, s, C_\pm) = \frac{C_-^2}{2^s c_{J,M} C_+ C_{J,s} C_+} \frac{\Gamma((J+1)/2)}{\pi^{(J+1)/2}}. \tag{A.15}
$$

The bound is uniform in ω. Thus, (A.19) is proved. The proof is then completed. $\square$

Lemma A.5. *For $g : \mathbb{R}^J \mapsto \mathbb{R}^J$, let $\mathcal{T}$ be the operator defined by*

$$
(\mathcal{T}_0 g)(x) \equiv \int_{\mathbb{R}^J} q(x - \delta) g(\delta) d\delta.
$$

Under Assumption A.7(2) and Assumption A.7(3), $\mathbb{W}^{2s,\infty}$ has a unique preimage in $\mathbb{W}^{s,\infty}$ under $\mathcal{T}_0$.

Proof. By Lemma C.4,

$$
(\mathcal{T}_0 g)(x) = (2\pi)^{-J} \int_{\mathbb{R}^J} e^{i\langle x, \omega \rangle} \hat{q}(\omega) \hat{g}(\omega) d\omega. \tag{A.16}
$$

Note that $\mathcal{T}_0$ is a multiplication operator, i.e.,

$$
\widehat{\mathcal{T}_0 g}(\omega) = \hat{q}(\omega) \hat{g}(\omega).
$$

For any $k \in \mathbb{W}^{2s,\infty}$, let u be the function with

$$
\hat{u}(\omega) = \frac{\hat{k}(\omega)}{\hat{q}(\omega)}. \tag{A.17}
$$

Assumption A.7(2) implies

$$
|\hat{u}(\omega)| = \underbrace{\frac{|\hat{k}(\omega)|}{|\hat{q}^2(\omega)|}}_{\in L^\infty(\mathbb{R}^J)} \cdot \underbrace{|\hat{q}(\omega)|}_{\in \mathbb{W}^{s,\infty}}.
$$

Thus, $u \in \mathbb{W}^{s,\infty}$. This also implies that u is a tempered distribution (which can be identified with a function). By Theorem 7.1.10 of Hörmander (2003), $\widehat{\mathcal{T}_0 u} = \hat{k} \iff \mathcal{T}_0 u = k$. Thus, $\mathcal{T}_0^{-1} k \in \mathbb{W}^{s,\infty}, \quad \forall k \in \mathbb{W}^{2s,\infty}$. $\square$

Lemma A.6. *Let $\mathcal{T}$ be the operator*

$$(\mathcal{T}g)(x) = \int_{\mathbb{R}^J} q\left(\Sigma(x)^{-1}(x-t)\right) g(t)\, dt = \mathbb{E}_{\delta = X + \Sigma(X)\epsilon, \epsilon \sim q(\cdot)}[g(\delta) \mid X = x]. \tag{A.18}$$

For any $\omega \in \mathbb{R}^J$, let $\Delta_\omega(x) = \hat{q}(\Sigma(x)\omega) - \hat{q}(\omega)$. Assume that

$$\max_{\upsilon \in \mathbb{R}^J} \int_{\mathbb{R}^J} |\hat{\Delta}_\omega(\omega - \upsilon)| \frac{|\hat{q}(\omega)|}{|\hat{q}(\upsilon)|^2} d\omega \leq 1 - \eta \tag{A.19}$$

for some constant $\eta > 0$. Under Assumption A.7(1)–(3), $\mathbb{W}^{2s,\infty}$ has a unique preimage in $\mathbb{W}^{s,\infty}$ under $\mathcal{T}$.

Proof. Define $\mathcal{E} = \mathcal{T} - \mathcal{T}_0$. By Lemma C.4 and (A.16), for any $g \in \mathbb{W}^{s,\infty}$,

$$(\mathcal{E}g)(x) = (2\pi)^{-J} \int_{\mathbb{R}^J} e^{i\langle x, \omega\rangle} \Delta_\omega(x) \hat{g}(\omega) d\omega.$$

Then,

$$\begin{aligned}
\widehat{(\mathcal{E}g)}(\upsilon) &= \int_{\mathbb{R}^J} e^{-i\langle x, \upsilon\rangle} \left((2\pi)^{-J} \int_{\mathbb{R}^J} e^{i\langle x, \omega\rangle} \Delta_\omega(x) \hat{g}(\omega) d\omega \right) dx \\
&= \int_{\mathbb{R}^J} \int_{\mathbb{R}^J} (2\pi)^{-J} e^{i\langle x, \omega - \upsilon\rangle} \Delta_\omega(x) \hat{g}(\omega) dx d\omega \\
&= \int_{\mathbb{R}^J} \hat{\Delta}_\omega(\omega - \upsilon) \hat{g}(\omega) d\omega.
\end{aligned} \tag{A.20}$$

By Assumption A.7(2) and the standard definition of $\mathbb{W}^{s,\infty}$, for any $k \in \mathbb{W}^{s,\infty}(\mathbb{R}^J)$,

$$\|k\|_{\mathbb{W}^{s,\infty}} < \infty \iff \left\| \frac{\hat{k}}{\hat{q}} \right\|_{L^\infty} \equiv \|k\|_{\hat{q}} < \infty.$$

Then we have

$$\mathbb{W}^{s,\infty}(\mathbb{R}^J) = \left\{ k : \|k\|_{\hat{q}} < \infty \right\}.$$

By (A.17), we have that $\widehat{\mathcal{T}_0^{-1}\rho} = \hat{\rho}/\hat{q}$ for any $\rho \in \mathbb{W}^{2s,\infty}$. We first verify that $\mathcal{E}g \in \mathbb{W}^{2s,\infty}$. By (A.20),

$$\begin{aligned}
\sup_{\upsilon \in \mathbb{R}^J} \frac{|\widehat{(\mathcal{E}g)}(\upsilon)|}{|\hat{q}(\upsilon)|^2} &= \sup_{\upsilon \in \mathbb{R}^J} \int_{\mathbb{R}^J} \frac{|\hat{\Delta}_\omega(\omega - \upsilon)\hat{g}(\omega)|}{|\hat{q}^2(\upsilon)|} d\omega \\
&= \sup_{\upsilon \in \mathbb{R}^J} \int_{\mathbb{R}^J} |\hat{\Delta}_\omega(\omega - \upsilon)| \frac{|\hat{q}(\omega)|}{|\hat{q}(\upsilon)|^2} \frac{|\hat{g}(\omega)|}{|\hat{q}(\omega)|} d\omega \\
&= \sup_{\upsilon \in \mathbb{R}^J} \int_{\mathbb{R}^J} |\hat{\Delta}_\omega(\omega - \upsilon)| \frac{|\hat{q}(\omega)|}{|\hat{q}(\upsilon)|^2} d\omega \cdot \left\| \frac{\hat{g}}{\hat{q}} \right\|_{L^\infty} \\
&\leq (1 - \eta) \|g\|_{\hat{q}}
\end{aligned} \tag{A.21}$$

$$< \infty,$$

where (A.21) uses (A.19) and the last line follows from $g \in \mathbb{W}^{s,\infty}$. Applying (A.17),

$$\begin{aligned}
\|\mathcal{T}_0^{-1}\mathcal{E}g\|_{\hat{q}} &= \left\| \frac{(\widehat{\mathcal{T}_0^{-1}\mathcal{E}g})}{\hat{q}} \right\|_{L^\infty} \\
&= \sup_{\upsilon \in \mathbb{R}^J} \int_{\mathbb{R}^J} \frac{|\hat{\Delta}_\omega(\omega - \upsilon)\hat{g}(\omega)|}{|\hat{q}^2(\upsilon)|} d\omega \\
&\le (1-\eta)\|g\|_{\hat{q}},
\end{aligned}$$

where the last line follows from (A.21). This implies that

$$\|\mathcal{T}_0^{-1}\mathcal{E}\|_{\|\cdot\|_{\hat{q}} \leftrightarrow \|\cdot\|_{\hat{q}}} = \|\mathcal{T}_0^{-1}\mathcal{E}\|_{\|\cdot\|_{\mathbb{W}^{s,\infty}} \leftrightarrow \|\cdot\|_{\mathbb{W}^{s,\infty}}} \le 1 - \eta.$$

As a result, the following Neumann series converges on $\mathbb{W}^{s,\infty}$:

$$\sum_{\ell \ge 0} (-\mathcal{T}_0^{-1}\mathcal{E})^\ell.$$

By Lemma A.5, for any $k \in \mathbb{W}^{2s,\infty}$, $\mathcal{T}_0^{-1}k \in \mathbb{W}^{s,\infty}$. Let

$$g = \sum_{\ell \ge 0} (-\mathcal{T}_0^{-1}\mathcal{E})^\ell \mathcal{T}_0^{-1}k.$$

Then $\|g\|_{\mathbb{W}^{s,\infty}} \le \eta^{-1}\|\mathcal{T}_0^{-1}k\|_{\mathbb{W}^{s,\infty}} < \infty$ and

$$(\mathrm{Id} + \mathcal{T}_0^{-1}\mathcal{E})g = \mathcal{T}_0^{-1}k.$$

We thus have that

$$\mathcal{T}g = (\mathcal{T}_0 + \mathcal{E})g = \mathcal{T}_0(\mathrm{Id} + \mathcal{T}_0^{-1}\mathcal{E})g = \mathcal{T}_0\mathcal{T}_0^{-1}k = k.$$

□

Lemma A.7. *Under Assumption A.7(2)–(3), for any $\alpha \in \mathbb{N}^J$ with $|\alpha| \le M$ and $\omega \in \mathbb{R}^J$,*

$$\sup_{x \in \mathbb{R}^J} |D_x^\alpha \hat{q}(\Sigma(x)\omega)| + \int_{\mathbb{R}^J} |\partial_x^\alpha \hat{q}(\Sigma(x)\omega)| dx \le \psi C_+ C_{J,s}(1 + \|\omega\|_2^2)^{-s/2},$$

where

$$C_{J,s} = \max_{\alpha : |\alpha| \le M} 2^{s+1+|\alpha|} \sum_{\kappa \in \mathbb{N}^J, 1 \le |\kappa| \le |\alpha|} \sum_{\substack{(k_{\gamma,j}) \in \mathbb{N} \\ \sum_\gamma k_{\gamma,j} = \kappa_j, \forall j \\ \sum_{\gamma,j} k_{\gamma,j}\, \gamma = \alpha}} \frac{\alpha!}{\prod_\gamma k_{\gamma,j}!\, (\gamma!)^{k_{\gamma,j}}}.$$

Proof. Fix $\omega \in \mathbb{R}^J$. Let $g(x) = \Sigma(x)\omega$. For any γ with $1 \leq |\gamma| \leq |\alpha|$,

$$D_x^\gamma g(x) = (D_x^\gamma \Sigma(x))\,\omega.$$

Thus,

$$\|D_x^\gamma g(x)\|_{\mathrm{op}} \leq \|D_x^\gamma \Sigma(x)\|_{\mathrm{op}} \|\omega\|$$

Furthermore, by Assumption A.7(3),

$$|D_v^\kappa \hat{q}(\Sigma(x)\omega)| \leq C_+(1 + \|\Sigma(x)\omega\|_2^2)^{-s/2-|\kappa|/2}.$$

By Assumption A.7(2),

$$\|\Sigma(x)\|_{\mathrm{op}} \geq 1 - \psi \geq \frac{1}{2}.$$

Then

$$\begin{aligned}
|D_v^\kappa \hat{q}(\Sigma(x)\omega)| &\leq C_+(1 + \|\omega\|_2^2/4)^{-s/2-|\kappa|/2} \\
&\leq C_+ 2^{s+|\kappa|}(1 + \|\omega\|_2^2)^{-s/2-|\kappa|/2} \\
&\leq C_+ 2^{s+|\alpha|}(1 + \|\omega\|_2^2)^{-s/2-|\kappa|/2} \\
&\triangleq C_1(1 + \|\omega\|_2^2)^{-s/2-|\kappa|/2}
\end{aligned}$$

By the multivariate Faà di Bruno formula (Lemma C.2),

$$\begin{aligned}
&|D_x^\alpha \hat{q}(\Sigma(x)\omega)| \\
&\leq \sum_{\kappa \in \mathbb{N}^J, 1 \leq |\kappa| \leq |\alpha|} |D_v^\kappa \hat{q}(\Sigma(x)\omega)| \sum_{\substack{(k_{\gamma,j}) \in \mathbb{N} \\ \sum_\gamma k_{\gamma,j} = \kappa_j,\, \forall j \\ \sum_{\gamma,j} k_{\gamma,j}\, \gamma = \alpha}} \frac{\alpha!}{\prod_{\gamma,j} k_{\gamma,j}!\,(\gamma!)^{k_{\gamma,j}}} \prod_{\gamma,j} |D_x^\gamma g_j(x)|^{k_{\gamma,j}} \\
&\leq C_+ 2^{s+|\alpha|} \sum_{\kappa \in \mathbb{N}^J, 1 \leq |\kappa| \leq |\alpha|} (1 + \|\omega\|_2^2)^{-s/2-|\kappa|/2} \\
&\qquad \sum_{\substack{(k_{\gamma,j}) \in \mathbb{N} \\ \sum_\gamma k_{\gamma,j} = \kappa_j,\, \forall j \\ \sum_{\gamma,j} k_{\gamma,j}\, \gamma = \alpha}} \frac{\alpha!}{\prod_{\gamma,j} k_{\gamma,j}!\,(\gamma!)^{k_{\gamma,j}}} \prod_{\gamma,j} (\|D_x^\gamma \Sigma(x)\|_{\mathrm{op}} \|\omega\|_2)^{k_{\gamma,j}} \\
&\leq C_+ 2^{s+|\alpha|}(1 + \|\omega\|_2^2)^{-s/2} \sum_{\kappa \in \mathbb{N}^J, 1 \leq |\kappa| \leq |\alpha|} (1 + \|\omega\|_2^2)^{-|\kappa|/2} \|\omega\|_2^{|\kappa|} \\
&\qquad \cdot \sum_{\substack{(k_{\gamma,j}) \in \mathbb{N} \\ \sum_\gamma k_{\gamma,j} = \kappa_j,\, \forall j \\ \sum_{\gamma,j} k_{\gamma,j}\, \gamma = \alpha}} \frac{\alpha!}{\prod_{\gamma} k_{\gamma,j}!\,(\gamma!)^{k_{\gamma,j}}} \prod_{\gamma,j} \|D_x^{\gamma,j} \Sigma(x)\|_{\mathrm{op}}^{k_{\gamma,j}}
\end{aligned}$$

$$\leq C_+ 2^{s+|\alpha|}(1+\|\omega\|_2^2)^{-s/2} \sum_{\kappa\in\mathbb{N}^J, 1\leq|\kappa|\leq|\alpha|} \sum_{\substack{(k_{\gamma,j})\in\mathbb{N} \\ \sum_\gamma k_{\gamma,j}=\kappa_j,\forall j \\ \sum_{\gamma,j} k_{\gamma,j}\,\gamma=\alpha}} \frac{\alpha!}{\prod_{\gamma,j} k_{\gamma,j}!\,(\gamma!)^{k_{\gamma,j}}} \prod_{\gamma,j} \|D_x^\gamma \Sigma(x)\|_{\mathrm{op}}^{k_{\gamma,j}}.$$

Let

$$C_1 = \sum_{\kappa\in\mathbb{N}^J, 1\leq|\kappa|\leq|\alpha|} \sum_{\substack{(k_{\gamma,j})\in\mathbb{N} \\ \sum_\gamma k_{\gamma,j}=\kappa_j,\forall j \\ \sum_{\gamma,j} k_{\gamma,j}\,\gamma=\alpha}} \frac{\alpha!}{\prod_{\gamma} k_{\gamma,j}!\,(\gamma!)^{k_{\gamma,j}}}.$$

By Assumption A.7(3),

$$\prod_{\gamma,j} \|D_x^\gamma \Sigma(x)\|_{\mathrm{op}}^{k_{\gamma,j}} \leq \psi^{|\kappa|} \leq \psi.$$

Thus,

$$|D_x^\alpha \hat{q}(\Sigma(x)\omega)| \leq \psi C_+ 2^{s+|\alpha|} C_1 (1+\|\omega\|_2^2)^{-s/2}.$$

Similarly, to bound the L^1 norm of $D_x^\alpha \hat{q}(\Sigma(x)\omega)$, we just need to show that

$$\left\| \prod_{\gamma,j} \|D_x^\gamma \Sigma(x)\|_{\mathrm{op}}^{k_{\gamma,j}} \right\|_{L^1} \leq \psi.$$

This is because at least one $k_\gamma \geq 1$ and $\|ab\|_{L^1} \leq \|a\|_{L^1}\|b\|_{L^\infty}$. The proof is then completed by setting $C_{J,s} = 2^{s+|\alpha|+1} C_+ C_1$. □

A.10. **Verifying completeness from faithfulness.**

Proposition A.8. *Let $\mathcal{F}$ be all integrable functions of (δ, P) and $\mathcal{K}$ be all integrable functions of X. Suppose that for all $\mathbb{E}[|f(\delta)|] < \infty$, $\mathbb{E}[f(\delta) \mid X] = 0$ implies $f = 0$. Then Assumption A.4 implies Assumption A.2.*

Proof. Let h be an integrable and (without loss) scalar function of δ, P. Thus Assumption A.4 implies that

$$\mathbb{E}[h \mid X, Z] = 0 \in \mathcal{K} \implies h(\delta, P) = h(\delta).$$

We further have that $\mathbb{E}[h(\delta) \mid X] = 0$ implies $h(\delta) = 0$. Thus $h(\delta, P) = 0$ and hence $(\delta, P) \mid (X, Z)$ is complete. Since any integrable function of S, P can be rewritten as an integrable function of (δ, P), we have that $(S, P) \mid (X, Z)$ is complete as well. □

Online Appendix for "Nonparametric Identification of Demand without Exogenous Product Characteristics"

Borusyak, Chen, Hull, and Lei

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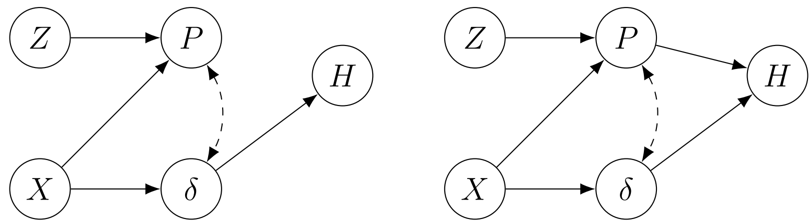
Appendix B. Additional results and discussions

B.1. **Faithfulness in causal graphs.** Given a candidate $H(\delta, P) = H$, consider the induced directed acyclic graph (DAG) of the variables (X, P, Z, δ, H). Depending on whether or not $H(\delta, P) = H(\delta)$, there are two possible DAGs shown in Figure B.1.[29] In the causal discovery literature (Spirtes et al., 2000), the joint distribution of $(X, P, Z, \delta, H) \sim Q_H$ is said to be *faithful* to a graph if, whenever two variables V_1 and V_2 are conditionally independent given some set of variables C, the correponding vertices in the DAG are *d-separated* by a set of vertices corresponding to C.

Suppose Q_H is known to be faithful to one of the two graphs, (a) or (b). Then, upon observing $Z \perp\!\!\!\perp H \mid X$, we can infer that Q_H is faithful to (a) and not (b), meaning H is constant in P. This is similar to the type of inference that Assumption 4 allows.

[29] Note that H is here a deterministic function of (δ, P) while each node in a DAG is typically assumed to have its own independent random variation.



(a) H does not depend on P (b) H depends on P

FIGURE B.1. DAG for $(X, P, Z, \delta, H(\delta, P))$.

In this sense, Assumption 4 essentially requires that every Q_H is faithful to one of (a) and (b), depending on whether H varies with P—although Assumption 4 only speaks to whether the edge $P \to H$ is faithfully reflected by Q_H.[30]

B.2. **Faithfulness verification without proxies.**

B.2.1. *Faithfulness verification from functional form.* Here we consider identification under the added restriction that P combines linearly with the utility index, and show that this suffices for faithfulness (given a sufficiently strong instrument Z) even absent variation in X. We interpret these results as faithfulness verification without a proxy for δ. The results follow identically to Proposition 1 in Borusyak et al. (2025).[31]

Consider a version of Equation (2) where δ and P enter additively

$$S = \sigma(\delta(\overline{X}, \xi) + P, \tilde{X}).$$

We consider an extreme case where X has no variation and drop it (in addition to $\tilde{X}$, as before) from the model: $S = \sigma(\delta + P)$. This is equivalent to an analysis which conditions fully on $\overline{X}$. Under this model, we have that $\delta = \sigma^{-1}(S) - P$. We can then consider all functions $h(S, P) = h(S) - P$ that are mean-independent of Z. For any such h, we have that $h(S, P) = H(\delta, P) = f(P + \delta) - P$ for $f = h(\sigma(\cdot))$.

Faithfulness here says that for all $H(\delta, P) = f(P + \delta) - P$,

$$\mathbb{E}[H(\delta, P) \mid Z] = 0 \implies H(\delta, P) = f(P + \delta) - P = H(\delta).$$

Suppose that Z is a strong instrument in the sense that $(P + \delta) \mid Z$ is complete.[32] Then

$$\mathbb{E}[f(P + \delta) \mid Z] = \mathbb{E}[P \mid Z]$$

[30]Assumption 4 is stronger in that it precludes the dependence of H on P from its *mean*-independence with Z given X only, though our general identification argument accommodates the analogue of Assumption 4 with full independence instead.

[31]The difference relative to Borusyak et al. (2025) is that we maintain the index restriction on X.

[32]Since we condition on $\overline{X}$, this should be read as conditional completeness given $\overline{X}$.

has a unique solution in f. Since $f(t) = \sigma^{-1}(t) - \mathbb{E}[\delta]$ is such a solution, we conclude that it is the unique solution. The corresponding H satisfies

$$H(\delta, P) = \sigma^{-1}(P + \delta) - P - \mathbb{E}[\delta] = \delta - \mathbb{E}[\delta],$$

and is thus solely a function of δ. Thus, in this example, faithfulness follows without any variation in X. This is because the functional form restriction on how δ and P interact in σ suffices to rule out the kind of complex interactions between δ and P that a strong proxy X is otherwise helpful for ruling out.

B.2.2. *Identification in triangular models as faithfulness verification.* Here we connect faithfulness to the literature on nonparametric identification of triangular models: i.e., Imbens and Newey (2009), Torgovitsky (2015), and D'Haultfœuille and Février (2015). We show that arguments in these papers can be viewed as verifying faithfulness without a strong proxy X by imposing instead that outcomes are scalar and monotone in δ.

Consider a special case of our model where $J = 1$ and let

$$S = \sigma(\delta, P), \quad P = f(Z, \omega).$$

Suppose that $\sigma(\cdot, P)$ is strictly increasing and $f(Z, \cdot)$ is strictly increasing (and their respective arguments are one-dimensional). Normalize the marginal distribution of ω to Unif$[0, 1]$. Since ω is the conditional quantile of $P \mid Z$, we can assume that ω is observed. Lastly, assume that δ is continuously distributed and $\delta \mid \omega$ is also continuously distributed.

Under these restrictions, the identified set for σ is (Theorem 1, Torgovitsky (2015)):

$$\Theta_I = \left\{h^{-1}(\delta, p) \text{ strictly increasing in } \delta : (h(S, P), \omega) \perp\!\!\!\perp Z\right\}.$$

Analogous to Proposition 2, for identifying price counterfactuals, it suffices to show that, for h^{-1} strictly increasing in δ and $H(\delta, P) \equiv h(\sigma(\delta, P), P)$, the following analogue of faithfulness holds:

$$(H(\delta, P), \omega) \perp\!\!\!\perp Z \implies H(\delta, P) \text{ does not depend on } P. \tag{B.1}$$

Let $H = H(\delta, P)$. Imbens and Newey (2009) observe that

$$F_{H|P,\omega}(t \mid p, \omega) = \int \mathbf{1}(H(\delta, p) \leq t)\, dF_{\delta|\omega}(\delta \mid \omega).$$

If the support of $\omega \mid P$ is $[0,1]$ for all P (Assumption 2, Imbens and Newey (2009)), we can then integrate the above with respect to the marginal distribution of ω:

$$\int_0^1 F_{H|P,\omega}(t \mid p, \omega) d\omega = \int \mathbf{1}(H(\delta, p) \le t)\, dF_\delta(\delta) = \mathbb{P}_{\delta \sim F_\delta}(H(\delta, p) \le t) = F_\delta\left(H^{-1}(t, p)\right). \tag{B.2}$$

Note that since $(H, \omega) \perp\!\!\!\perp Z$ and $P = f(Z, \omega)$, we have that

$$H \perp\!\!\!\perp P \mid \omega.$$

Thus $F_{H|P,\omega}(t \mid p, \omega) = F_{H|\omega}(t \mid \omega)$ does not depend on p. Thus, the left-hand side of Equation (B.2) does not depend on p. The only way for this to occur is if $H^{-1}(t, p)$ does not depend on p either, since F_δ is strictly increasing. This argument is in equation (5)–(7) of Imbens and Newey (2009) preceding their Theorem 3. We can thus view this argument as verifying the version of faithfulness (B.1).

Torgovitsky (2015) avoids the need in (B.2) to integrate over ω,[33] thus allowing for instruments with few support points.[34] Suppose that Z instead takes two values $\{z_0, z_1\}$. Let $\pi(p) = f(z_1, f^{-1}(p, z_0))$ such that for some identical value of ω,

$$\pi(p) = f(z_1, \omega), \quad p = f(z_0, \omega).$$

Torgovitsky (2015) observes that, since $Z \perp\!\!\!\perp (\delta, \omega)$, $H \perp\!\!\!\perp P \mid \omega$ and hence

$$\mathbb{P}(\delta \le H^{-1}(t, \pi(p)) \mid \omega) = F_{H|P,\omega}(t \mid \pi(p), \omega) = F_{H|P,\omega}(t \mid p, \omega) = \mathbb{P}(\delta \le H^{-1}(t, p) \mid \omega).$$

This implies that for any p, for all t,

$$H^{-1}(t, \pi(p)) = H^{-1}(t, p).$$

Torgovitsky (2015) imposes further assumptions that ensure that there is some value p_0 such that for any value p, the sequence $\pi(p), \pi(\pi(p)), \ldots$ approaches p_0.[35] Thus,

$$H^{-1}(t, \pi(p)) = H^{-1}(t, \pi(\pi(p))) = H^{-1}(t, \pi(\pi(\pi(p)))) = \cdots$$

equals $H^{-1}(t, p_0)$ given continuity. This ensures $H^{-1}(t, p) = H^{-1}(t, p_0)$, proving (B.1).

B.3. **Example of faithfulness without completeness.** Combining restrictions on P and δ analogous to those in Appendices A.6 and A.9, we can construct a broad class

[33] A similar argument appears in D'Haultfœuille and Février (2015).

[34] ω is the quantile of P in the $P \mid Z$ distribution. If there are only finitely many values for Z, then each price value has only finitely many quantiles that can be candidates for ω, meaning that $\omega \mid P$ cannot have full support.

[35] One example is to assume that p_0 is the left support end point of P and that $f(z_1, \omega) < f(z_0, \omega)$ for all $\omega > 0$ with $p_0 = f(z_0, 0) = f(z_1, 0)$. Then we have $\pi(p) < p$ approaching p_0.

of DGPs under which faithfulness holds yet $\delta \mid X$ can be incomplete. In particular, this implies the failure of joint completeness of (P, δ) given (X, Z) (i.e., Assumption 2).

We use Assumptions 3 and A.5, and the following additional assumptions. To simplify notation, we assume that $\tilde{X}$ is conditioned on and treated as fixed. We also write Λ for $\lambda = \lambda(X, Z)$ defined in Assumption A.5 to distinguish between the random variable λ and the value that λ takes.

Assumption B.1. *There exists a measurable $a : \mathbb{R}^J \to \mathbb{R}^J$ and a noise vector $\epsilon \in \mathbb{R}^J$ such that*

$$\delta = a(X) + \epsilon, \qquad \epsilon \perp (X, \Lambda).$$

Assume $\check{X} = a(X)$ has a positive density everywhere and $(-\epsilon)$ has a density p_ϵ with the characteristic function $\varphi_\epsilon(\omega) := \mathbb{E}[e^{i\langle \omega, \epsilon \rangle}]$, $\omega \in \mathbb{R}^J$.

Assumption B.2. *Let $q(p \mid \lambda, d)$ denote the conditional density of $P \mid (\Lambda = \lambda, \delta = d)$. Fixing $\lambda_0 \in \mathbb{R}^k$,*

(1) There exists a reference density $\mu(p) > 0$ and a constant $c_- > 0$ such that

$$q(p \mid \lambda, d) \;\geq\; c_- \, \mu(p) \qquad \forall (p, \lambda', d) \in \mathbb{R}^{3J}.$$

(2) There exists a measurable function $\zeta : \mathbb{R}^J \to [0, \infty)$ such that

$$\big| q(p \mid \lambda, d) - q(p \mid \lambda_0, d) \big| \;\leq\; \zeta(d) \, \mu(p) \qquad \forall (p, \lambda, d).$$

(3) Let p_δ be the marginal density of δ. There exists $a_0 > 0$ such that

$$\sup_{d \in \mathbb{R}^J} \frac{e^{a_0 \|d\|} \zeta(d)}{p_\delta(d)} \;<\; \infty.$$

Assumption B.3 (Completeness of $P \mid \Lambda$ given δ)**.** *For any function $g : \mathbb{R}^{2J} \mapsto \mathbb{R}$,*

$$\mathbb{E}[g(\delta, P) \mid \delta, \Lambda] = 0 \quad a.s. \implies g(\delta, P) = 0 \quad a.s.$$

Remark B.1. Under Assumption A.5, Assumption B.3 is necessary for faithfulness, such that we retain the need for Z to be a strong instrument for price. We prove this result in Appendix C.2.

Theorem B.1. *Under Assumptions 3, A.5, and B.1 to B.3, $(\delta, P) \mid (X, \Lambda)$ is faithful. Furthermore, if these exists $\omega_0 \in \mathbb{R}^J$ such that $\varphi_\epsilon(\omega_0) = 0$, then $\delta \mid X$ is not complete and hence Assumption A.2 fails.*

The proof is deferred to Appendix C.3.

We next provide a concrete class of examples that satisfies Assumptions B.1 to B.3, with the proof presented in Appendix C.4.

Proposition B.1. *Assume that*

(a) *The conditional density of P is given by*

$$q(p \mid \lambda, d) = \exp\left\{\langle\theta(\lambda, d), S(p)\rangle - A(\theta(\lambda, d))\right\}\mu(p), \;\; \theta(\lambda, d) = \zeta(d)t(\lambda) \tag{B.3}$$

where $S(p)$ is bounded, non-constant, and continuous, $\mu(p)$ is a carrier measure with positive density everywhere,

$$\zeta(d) = e^{-c\|d\|^2}, \quad c > 1/4,$$

and $A(\theta)$ is the log-normalizer

$$A(\theta) = \log \int_{\mathbb{R}^J} \exp\left(\langle\theta, S(u)\rangle\right)\mu(u)\, du.$$

(b) *$X \sim N(0, I_J), \nu \sim N(0, I_J)$ and $\eta = (\eta_1, \dots, \eta_J)$ with $\eta_j \overset{\text{indep.}}{\sim} \mathrm{Unif}([-b_j, b_j])$ for some $b_j > 0$. Assume (ν, η) are independent of (X, Λ) and*

$$\epsilon = \nu + \eta, \qquad \delta = X + \epsilon.$$

Then Assumptions B.1 to B.3 hold. Moreover, $\delta \mid X$ is incomplete.

Remark B.2. Note that b_j can be arbitrarily small in (b). Proposition B.1 then shows that, under the λ-index and additive δ model, completeness is sensitive to infinitesimal perturbations in δ while faithfulness is not.

B.4. **Example of completeness without faithfulness.** We assume $J = 1$ for this example, though analogous constructions exist for $J > 1$. Suppose $Z, X > 0$ and

$$P \mid \delta, Z, X \sim \mathcal{N}(r(\delta, X)\tau(\delta, Z, X), \tau^2(\delta, Z, X)).$$

This construction is robust to taking monotone transformations of (δ, P), and thus we may view P as log price instead—for instance—if we would like price to be strictly positive. Let δ be supported on $[1, 2]$ with PDF

$$f(\delta \mid x) = \exp(r(\delta, x)^2/2)a(x)\mathbf{1}(\delta \in [1, 2])$$

for

$$a(x)^{-1} = \int_1^2 e^{r(\delta,x)^2/2}\, d\delta.$$

Proposition B.2. *Let $r(\delta, x) = x\delta$ and $\tau(\delta, z, x) = z$. Then the above construction specifies $(\delta, P) \mid (Z, X)$ for which Assumptions 2 and 3 hold but Assumption 4 fails.*

We present the proof in Appendix C.5.

B.5. **Bertrand–Nash pricing.** Here we show how Bertrand–Nash pricing implies Equation (10); see also Appendix A in Berry and Haile (2014). Consider the case of single-product firms.[36] As is well-known, the first-order condition for profit maximization of firm j with a constant marginal cost C_j is

$$\sigma_j(\delta, P) + (P_j - C_j)\frac{\partial \sigma_j(\delta, P)}{\partial P_j} = 0.$$

Provided the equilibrium is unique, the solution for the vector of P can therefore be written as a function of δ and C: i.e., Equation (10) holds.

Appendix C. Auxiliary proofs

C.1. Auxiliary lemmas for Theorem A.1.

Lemma C.1. *If $\hat{q}(\upsilon) = (1 + \|\upsilon\|_2^2)^{-s/2}$ for some $s > J$, then Assumption A.7(2) holds.*

Remark C.1. Since $\hat{q}(0) = 1$, $\hat{q}$ is the characteristic function of a density. In fact, for any $s > J$, q has the following expression

$$q(x) = \Theta_{J,s}\|x\|_2^{(s-J)/2} K_{(s-J)/2}(\|x\|_2)$$

where K_v is the modified Bessel function of the second kind and $\Theta_{J,s}$ is the normalizing constant. For large x,

$$q(x) \sim \|x\|_2^{(s-J-1)/2} \exp\{-\|x\|_2\}.$$

Thus, q behaves like a radial Gamma distribution.

Proof. We will prove that

$$D_\upsilon^\alpha \hat{q}(\upsilon) = \sum_{\ell=1}^{|\alpha|} (1 + \|\upsilon\|_2^2)^{-s/2-\ell} F_{\alpha,\ell}(\upsilon),$$

where $F_{\alpha,\ell}$ is a homogeneous polynomial of υ of order ℓ. We prove this claim by induction on $|\alpha|$. When $\ell = 1$,

$$D_\upsilon^{e_j} \hat{q}(\upsilon) = (1 + \|\upsilon\|_2^2)^{-s/2-1} \cdot (-s\upsilon_j)$$

when e_j is the j-th canonical basis. Suppose the claim holds for $|\alpha| - 1$. For any given α, assume WLOG that $\alpha_1 \geq 1$. Let $\tilde{\alpha} = \alpha - e_1$. Then

$$D_\upsilon^\alpha \hat{q}(\upsilon) = D_\upsilon^{e_1} D_\upsilon^{\tilde{\alpha}} \hat{q}(\upsilon) = D_\upsilon^{e_1} \sum_{\ell=1}^{|\tilde{\alpha}|} (1 + \|\upsilon\|_2^2)^{-s/2-\ell} F_{\tilde{\alpha},\ell}(\upsilon)$$

[36] The argument extends to multi-product firms if the “ownership matrix” is conditioned upon in the same way characteristics $\tilde{X}$ are.

$$
\begin{aligned}
&= \sum_{\ell=1}^{|\tilde{\alpha}|}(1+\|\upsilon\|_2^2)^{-s/2-\ell-1} \cdot -(s+2\ell)\upsilon_1 F_{\tilde{\alpha},\ell}(\upsilon) \\
&\qquad + \sum_{\ell=1}^{|\tilde{\alpha}|}(1+\|\upsilon\|_2^2)^{-s/2-\ell} \cdot D_\upsilon^{e_1} F_{\tilde{\alpha},\ell}(\upsilon) \\
&= \sum_{\ell=1}^{|\tilde{\alpha}|}(1+\|\upsilon\|_2^2)^{-s/2-\ell} \cdot \left(D_\upsilon^{e_1} F_{\tilde{\alpha},\ell}(\upsilon) - (s+2\ell-1)\upsilon_1 F_{\tilde{\alpha},\ell-1}(\upsilon)\right),
\end{aligned}
$$

where $F_{\tilde{\alpha},0}(\upsilon)=0$ for notational convenience. Clearly, $D_\upsilon^{e_1} F_{\tilde{\alpha},\ell}(\upsilon)-(s+2\ell-1)\upsilon_1 F_{\tilde{\alpha},\ell-1}(\upsilon)$ is a homogeneous polynomial of order ℓ. Thus, the claim holds for $|\alpha|$. By induction the proof is completed. ☐

Lemma C.2 (Multivariate Faà di Bruno formula; see e.g., Theorem 6.8 of Schumann (2019))**.** *Let $g(x)=(g_1(x),\dots,g_n(x)):\mathbb{R}^m \mapsto \mathbb{R}^n$ and $f(y):\mathbb{R}^n \mapsto \mathbb{R}$ be two functions that have M derivatives over $\mathbb{R}^n$. Then*

$$
D_x^\alpha (f\circ g)(x) = \sum_{\kappa\in\mathbb{N}^n, 1\le|\kappa|\le|\alpha|} D_y^\kappa f(g(x)) B_{\alpha,\kappa}((D_x^\gamma g_j(x))_{\gamma\in\mathbb{N}^n\setminus\{\mathbf{0}\},1\le j\le n}),
$$

where $B_{\alpha,\kappa}$ is the multivariate Bell polynomial defined as

$$
B_{\alpha,\kappa}\big((z_{\gamma,j})_{\gamma\in\mathbb{N}^n\setminus\{\mathbf{0}\},\,1\le j\le n}\big) = \sum_{\substack{(k_{\gamma,j})\in\mathbb{N} \\ \sum_\gamma k_{\gamma,j}=\kappa_j\ \forall j \\ \sum_{\gamma,j} k_{\gamma,j}\,\gamma=\alpha}} \frac{\alpha!}{\prod_{\gamma,j} k_{\gamma,j}!\,(\gamma!)^{k_{\gamma,j}}} \prod_{\gamma,j} z_{\gamma,j}^{k_{\gamma,j}}.
$$

Lemma C.3 (Corollary 3.3.10 of Grafakos et al. (2008))**.** *Let $f:\mathbb{R}^n\mapsto\mathbb{R}$ with $D^\alpha f\in L^1$ for any $\alpha\in\mathbb{N}^n$ with $|\alpha|\le M$. Then, for any $\upsilon\in\mathbb{R}^J$,*

$$
|\hat{f}(\upsilon)| \le c_{n,M}\max\left\{\|f\|_{L^1}, \max_{|\alpha|=M}\|D^\alpha f\|_{L^1}\right\}(1+\|\upsilon\|)^{-M},
$$

where $c_{n,M}$ is a constant that only depends on n and M.

Lemma C.4. *Let q be a density that satisfies Assumption A.7(2). For some $\Sigma(x)$ taking values in $\mathbb{S}_+^J$, let $\mathcal{T}$ be defined as in (A.18) and let $q_x(\zeta)=\det(\Sigma(x))^{-1}q\left(\Sigma(x)^{-1}\zeta\right)$. Then, $\mathcal{T}$ is a pseudo-differential operator with symbol $\hat{q}_x(\omega)$ in Kohn–Nirenberg quantization, i.e.,*

$$
(\mathcal{T}g)(x) = \frac{1}{(2\pi)^J}\int_{\mathbb{R}^J} e^{i\langle x,\omega\rangle}\hat{q}_x(\omega)\hat{g}(\omega)d\omega.
$$

Proof. By the assumptions on $q(\cdot)$,

$$
(\mathcal{T}g)(x) = \int_{\mathbb{R}^J} q_x(x-\delta)g(\delta)d\delta
$$

$$= \frac{1}{(2\pi)^J}\int_{\mathbb{R}^J}\int_{\mathbb{R}^J} e^{i\langle x-\delta,\omega\rangle}\hat{q}_x(\omega)g(\delta)d\omega d\delta \qquad \text{(Inverse Fourier transform)}$$
$$= \frac{1}{(2\pi)^J}\int_{\mathbb{R}^J}\int_{\mathbb{R}^J} e^{i\langle x,\omega\rangle}\hat{q}_x(\omega)e^{-i\langle \delta,\omega\rangle}g(\delta)d\delta d\omega$$
$$= \frac{1}{(2\pi)^J}\int_{\mathbb{R}^J}\int_{\mathbb{R}^J} e^{i\langle x,\omega\rangle}\hat{q}_x(\omega)\hat{g}(\omega)d\omega.$$

□

Lemma C.5. *Assume $M \geq 2s \geq 1$. If $k \in \mathbb{K}^{2s}$, then and $k \circ b^{-1} \in \mathbb{K}^{2s}$.*

Proof. By (C.1), $b^{-1} \in \mathbb{K}^M$. Since $M \geq 2s$, Lemma C.8 implies $k \circ b^{-1} \in \mathbb{K}^{2s}$. □

Lemma C.6. *Under Assumption A.7,*

$$\|\tilde{\Sigma}(\cdot) - I\|_{\mathbb{W}^{M,\infty}} + \|\tilde{\Sigma}(\cdot) - I\|_{\mathbb{W}^{M,1}}$$
$$\leq B_M\Big(\|\Sigma(\cdot) - \Sigma_0\|_{\mathbb{W}^{M,\infty}} + \|\Sigma(\cdot) - \Sigma_0\|_{\mathbb{W}^{M,1}}, \|Db\|_{\mathbb{W}^{M-1,\infty}}, \|Db^{-1}\|_{L^\infty}, \lambda_{\min}^{-1}(\Sigma_0)\Big)$$

for some function B_M that only depends on M. In particular,

$$\lim_{y\to 0} B_M\left(y, \|Db\|_{\mathbb{W}^{M-1,\infty}}, \|Db^{-1}\|_{L^\infty}, \lambda_{\min}^{-1}(\Sigma_0)\right) = 0.$$

Proof. First, we note that

$$\|\tilde{\Sigma}(\cdot) - I\|_{\mathbb{W}^{M,\infty}} + \|\tilde{\Sigma}(\cdot) - I\|_{\mathbb{W}^{M,1}}$$
$$\leq \left(\|\Sigma\circ b^{-1} - \Sigma_0\|_{\mathbb{W}^{M,\infty}} + \|\Sigma\circ b^{-1} - \Sigma_0\|_{\mathbb{W}^{M,1}}\right)\lambda_{\min}^{-1}(\Sigma_0)$$
$$= \left(\|(\Sigma(\cdot) - \Sigma_0)\circ b^{-1}\|_{\mathbb{W}^{M,\infty}} + \|(\Sigma(\cdot) - \Sigma_0)\circ b^{-1}\|_{\mathbb{W}^{M,1}}\right)\lambda_{\min}^{-1}(\Sigma_0)$$

By Assumption A.7(3), $\Sigma(\cdot) - \Sigma_0 \in \mathbb{W}^{M,\infty}$. By Lemma C.7 and Assumption A.7(4),

$$b^{-1} \in \mathbb{K}^M \Longrightarrow \|Db^{-1}\|_{\mathbb{W}^{M-1,\infty}} < \infty. \tag{C.1}$$

Furthermore,

$$b \in \mathbb{K}^M \Longrightarrow Db \in L^\infty. \tag{C.2}$$

By Lemma C.9, (C.1), and (C.2),

$$\left\|(\Sigma(\cdot) - \Sigma_0)\circ b^{-1}\right\|_{\mathbb{W}^{M,\infty}} \leq B_{M,\infty}(\|\Sigma(\cdot) - \Sigma_0\|_{\mathbb{W}^{M,\infty}}, \|Db^{-1}\|_{\mathbb{W}^{M-1,\infty}}, \|Db\|_{L^\infty}),$$

and

$$\left\|(\Sigma(\cdot) - \Sigma_0)\circ b^{-1}\right\|_{\mathbb{W}^{M,1}} \leq B_{M,1}(\|\Sigma(\cdot) - \Sigma_0\|_{\mathbb{W}^{M,1}}, \|Db^{-1}\|_{\mathbb{W}^{M-1,\infty}}, \|Db\|_{L^\infty}).$$

The proof is completed by defining

$$
\begin{aligned}
&B_M\Big(\|\Sigma(\cdot)-\Sigma_0\|_{\mathbb{W}^{M,\infty}}+\|\Sigma(\cdot)-\Sigma_0\|_{\mathbb{W}^{M,1}},\|Db\|_{\mathbb{W}^{M-1,\infty}},\|Db^{-1}\|_{L^\infty},\|\Sigma_0^{-1}\|_{\mathrm{op}}\Big)\\
&=\Big\{B_{M,\infty}(\|\Sigma(\cdot)-\Sigma_0\|_{\mathbb{W}^{M,\infty}},\|Db^{-1}\|_{\mathbb{W}^{M-1,\infty}},\|Db\|_{L^\infty})\\
&\qquad+B_{M,1}(\|\Sigma(\cdot)-\Sigma_0\|_{\mathbb{W}^{M,1}},\|Db^{-1}\|_{\mathbb{W}^{M-1,\infty}},\|Db\|_{L^\infty})\Big\}\lambda_{\min}^{-1}(\Sigma_0).
\end{aligned}
$$

□

Lemma C.7. *Let $s\geq 1$ and let $f(x)=Ax+u(x)\in\mathbb{K}^s$. Assume*

$$\left\|(Df)^{-1}\right\|_{L^\infty}<\infty. \tag{C.3}$$

Then $f^{-1}\in\mathbb{K}^s$.

Proof. We split the proof into a few steps.

Step 1: the linear part A is invertible

To contradiction, if A were not full rank, then there exists $0\neq e\in\mathrm{Ran}(A)^\perp$ (equivalently $A^Te=0$). For all $x\in\mathbb{R}^J$,

$$e\cdot f(x)=e\cdot(Ax+u(x))=e\cdot u(x).$$

Since $u\in\mathbb{W}^{s,\infty}\subset L^\infty$, the right-hand side is bounded; hence $e\cdot f(\mathbb{R}^J)$ is bounded. If f is surjective, $f(\mathbb{R}^J)=\mathbb{R}^J$, but $e\cdot y$ is unbounded on $\mathbb{R}^J$, a contradiction. Thus A is full rank.

Step 2: isolate the linear part of the inverse

Let $g=f^{-1}$. Using $y=f(g(y))=Ag(y)+u(g(y))$ we obtain

$$g(y)=A^{-1}y-A^{-1}u(g(y))=:A^{-1}y+v(y),\qquad v(y):=-A^{-1}u(g(y)).$$

Since $u\in L^\infty$, we have $v\in L^\infty$ and

$$\|v\|_{L^\infty}\leq\|A^{-1}\|\,\|u\|_{L^\infty}.$$

Hence $g\in\mathrm{lin}\oplus L^\infty$. It remains to show $v\in\mathbb{W}^{s,\infty}$.

Step 3: prove the result for integer s

Differentiating $f(g(y))=y$ gives

$$Df(g(y))\,Dg(y)=I, \tag{C.4}$$

Then

$$Dg(y)=\big(Df(g(y))\big)^{-1}.$$

The assumption $\|(Df)^{-1}\|_{L^\infty} < \infty$ implies $Dg \in L^\infty$. Since $g(y) = A^{-1}y + v(y)$, we have $Dv = Dg - A^{-1} \in L^\infty$. Thus, $v \in \mathbb{W}^{1,\infty}$.

We now prove $v \in \mathbb{W}^{m,\infty}$ for all $1 \leq m \leq s$ by induction. Suppose for some $2 \leq m \leq s$ we prove that $v \in \mathbb{W}^{m-1,\infty}$. Then

$$D^j v \in L^\infty, \quad j = 1, \dots, m-1. \tag{C.5}$$

Differentiating (C.4) m times we obtain, by the Leibniz rule and the multivariate Faà di Bruno formula,

$$\{(Df) \circ g\} \cdot D^m g = R_m((D^1 f) \circ g, \dots, (D^m f) \circ g, D^1 g, \dots, D^{m-1} g),$$

where $D^k g, (D^k f) \circ g \in (\mathbb{R}^J)^{\otimes k}$ is the k-th differentials and R_m is a tensor in $(\mathbb{R}^J)^{\otimes m}$ for which each coordinate is a polynomial of the entries of $(D^1 f) \circ g, \dots, (D^m f) \circ g, D^1 g, \dots, D^{m-1} g$. Since $f \in \mathbb{K}^s$, $D^k \in L^\infty$. Thus, by the induction hypothesis (C.5), $R_m \in L^\infty$. By (C.3),

$$D^m g = (Df(g))^{-1} R_m \in L^\infty.$$

Since $Dv = Dg - A^{-1}$, $D^m v = D^m g \in L^\infty$. Thus, $v \in \mathbb{W}^{m,\infty}$. The induction argument then implies $v \in \mathbb{W}^{s,\infty}$.

Step 4: prove the result for non-integer s

Let $s = m + \sigma$ with $m = \lfloor s \rfloor \in \mathbb{N}$ and $\sigma \in (0,1)$. For any $1 \leq k \leq m$, $f \in \mathbb{K}^s$ implies

$$D^k f \in \mathbb{W}^{\sigma,\infty}.$$

In Step 3, we have proved that $g \in \mathbb{W}^{m,\infty} \in \mathbb{W}^{1,\infty}$ and thus g is Lipschitz. By Lemma C.10,

$$(D^k f) \circ g \in \mathbb{W}^{\sigma,\infty}, \quad k = 1, \dots, m.$$

By (C.3) and Lemma C.11,

$$((Df) \circ g)^{-1} \in \mathbb{W}^{\sigma,\infty}.$$

Recall that each coordinate of R_m is a polynomial of the entries of $(D^1 f) \circ g, \dots, (D^m f) \circ g, D^1 g, \dots, D^{m-1} g$. Since $g \in \mathbb{W}^{m,\infty}$, $D^1 g, \dots, D^{m-1} g$ are all Lipschitz and hence in $\mathbb{W}^{\sigma,\infty}$. Since each monomial is a product, Lemma C.12 implies

$$R_m \in \mathbb{W}^{\sigma,\infty}.$$

By Lemma C.12 again, we conclude that

$$D^m v = D^m g - A^{-1} I(m = 1) \in \mathbb{W}^{\sigma,\infty}.$$

Therefore, $g \in \mathbb{K}^s$. $\square$

Lemma C.8. *If $f, g \in \mathbb{K}^s$ for some $s \geq 1$, then $f \circ g \in \mathbb{K}^s$.*

Proof. We split the proof into a few steps.

Step 1: reduction to the nonlinear part

Let

$$f(x) = Ax + u(x), \qquad g(x) = Bx + v(x),$$

with $u, v \in \mathbb{W}^{s,\infty}$. Then

$$(f \circ g)(x) = A(Bx + v(x)) + u(Bx + v(x)) = (AB)x + Av(x) + u \circ g(x).$$

Since $Av \in \mathbb{W}^{s,\infty}$, it suffices to show $u \circ g \in \mathbb{W}^{s,\infty}$.

Step 2: proof for integer s

Since $g \in \mathbb{K}^s$, $D^k g \in \mathbb{W}^{s-k,\infty}$. By the multivariate Faà di Bruno formula, for each integer $k \leq s$,

$$D^k(u \circ g) = S_k((D^1 u) \circ g, \ldots, (D^k u) \circ g, D^1 g, \ldots, D^k g), \tag{C.6}$$

where S_k is a tensor in $(\mathbb{R}^J)^{\otimes k}$ for which each coordinate is a polynomial of the entries of $(D^1 f) \circ g, \ldots, (D^k f) \circ g, D^1 g, \ldots, D^k g$. In particular, each monomial involves at least one coordinate of $D^j f \circ g$ for some j.

Since $k \leq s$,

$$(D^j u) \circ g \in L^\infty, \;\; D^j g \in L^\infty, \quad j = 1, \ldots, k.$$

Thus,

$$D^k(u \circ g) \in L^\infty.$$

Since this holds for all $k \leq s$, we conclude that $u \circ g \in \mathbb{W}^{s,\infty}$.

Step 3: proof for non-integer s

Let $s = m + \sigma$ with $m = \lfloor s \rfloor \in \mathbb{N}$ and $\sigma \in (0, 1)$. In Step 2, we have already proved that

$$D^m(u \circ g) = S_m((D^1 u) \circ g, \ldots, (D^m u) \circ g, D^1 g, \ldots, D^m g).$$

Since $u, g \in \mathbb{W}^{m+\sigma,\infty}$,

$$(D^1 u) \circ g, \ldots, (D^m u) \circ g, D^1 g, \ldots, D^m g \in \mathbb{W}^{\sigma,\infty}.$$

By Lemma C.12, $D^m(u \circ g) \in \mathbb{W}^{\sigma,\infty}$. This implies $u \circ g \in \mathbb{W}^{s,\infty}$. □

Lemma C.9. *If $u \in \mathbb{W}^{M,\infty}$ for some integer M and $g \in \mathbb{K}^M$ and g is invertible. Then*

$$\|u \circ g\|_{M,\infty} \leq B_M(\|u\|_{M,\infty}, \|Dg\|_{M-1,\infty}, \|Dg^{-1}\|_{L^\infty}),$$

for some function B_M that depends on M.

Proof. Since $g \in \mathbb{K}^M$, $Dg \in \mathbb{W}^{M-1,\infty}$. By definition,

$$\|u \circ g\|_{\mathbb{W}^{M,\infty}} = \sum_{\alpha:|\alpha|\leq M} \|D^\alpha(u \circ g)\|_{L^\infty}.$$

Since $g \in \mathbb{K}^M$ and g is invertible, Lemma C.7 implies $g^{-1} \in \mathbb{K}^M \in L^\infty$. In addition,

$$\|(D^j u) \circ g\|_{L^\infty} \leq \|D^j u\|_{L^\infty}.$$

For any $k \leq M$, each monomial in S_k defined by (C.6) involves at least one coordinate of $(D^j u) \circ g$. Note that $a \in L^\infty, b \in \mathbb{W}^{1,\infty}$ imply $\|ab\|_{L^\infty} \leq \|a\|_{L^\infty}\|b\|_{L^\infty}$. Letting $a = D^j u, b = g$, each monomial is in L^∞. Therefore, we can find $\overline{S}_k$ such that

$$\left|S_k((D^1 u) \circ g, \ldots, (D^k u) \circ g, D^1 g, \ldots, D^k g)\right| \leq \overline{S}_k(\|u\|_{M,\infty}, \|Dg\|_{\mathbb{W}^{M-1,\infty}}, \|Dg^{-1}\|_{L^\infty}).$$

In particular, $\overline{S}_k \to 0$ when $\|u\|_{M,\infty} \to 0$. Thus,

$$\begin{aligned}\|u \circ g\|_{\mathbb{W}^{M,\infty}} &\leq M^J \max_{0\leq k\leq M} \overline{S}_k(\|u\|_{\mathbb{W}^{M,\infty}}, \|Dg\|_{\mathbb{W}^{M-1,\infty}}) \\ &:= B_M(\|u\|_{\mathbb{W}^{M,\infty}}, \|Dg\|_{\mathbb{W}^{M-1,\infty}}, \|Dg^{-1}\|_{L^\infty}).\end{aligned}$$

□

Lemma C.10 (Composition with a Lipschitz map preserves $\mathbb{W}^{\sigma,\infty}$ regularity)**.** *If $h \in \mathbb{W}^{\sigma,\infty}$ for some $\sigma \in [0,1)$ and $\phi : \mathbb{R}^J \to \mathbb{R}^J$ is Lipschitz with Lipschitz constant* $\mathrm{Lip}(\phi)$*, then $h \circ \phi \in \mathbb{W}^{\sigma,\infty}$ and*

$$[h \circ \phi]_{\mathbb{W}^{\sigma,\infty}} \leq [h]_{\mathbb{W}^{\sigma,\infty}} \,\mathrm{Lip}(\phi)^\sigma,$$

Proof. For $x \neq y$,

$$\|h(\phi(x)) - h(\phi(y))\| \leq [h]_{\mathbb{W}^{\sigma,\infty}} \|\phi(x) - \phi(y)\|^\sigma \leq [h]_{\mathbb{W}^{\sigma,\infty}} \,\mathrm{Lip}(\phi)^\sigma \,\|x - y\|^\sigma.$$

Taking the supremum over $x \neq y$ gives the claim. □

Lemma C.11 (Inversion preserves $\mathbb{W}^{\sigma,\infty}$ regularity)**.** *Let $M : \mathbb{R}^J \to \mathbb{S}^J_+$ be such that $M \in \mathbb{W}^{\sigma,\infty}$ for some $\sigma \in [0,1)$ and $\|M^{-1}\|_{L^\infty} \leq C$. Then $M^{-1} \in \mathbb{W}^{\sigma,\infty}$ and*

$$[M^{-1}]_{\mathbb{W}^{\sigma,\infty}} \leq C^2 \,[M]_{\mathbb{W}^{\sigma,\infty}}.$$

Proof. Use the identity

$$M^{-1}(x) - M^{-1}(y) = M^{-1}(x)\big(M(y) - M(x)\big)M^{-1}(y).$$

Taking norms, dividing by $\|x - y\|^\sigma$, and taking suprema yields the estimate. □

Lemma C.12 ($\mathbb{W}^{\sigma,\infty}$ is closed in products)**.** *If $a, b \in \mathbb{W}^{\sigma,\infty} \cap L^\infty(\mathbb{R}^J)$ for some $\sigma \in [0, 1)$, then $ab \in \mathbb{W}^{\sigma,\infty}$ and*

$$[ab]_{\mathbb{W}^{\sigma,\infty}} \leq \|a\|_{L^\infty} [b]_{\mathbb{W}^{\sigma,\infty}} + \|b\|_{L^\infty} [a]_{\mathbb{W}^{\sigma,\infty}}.$$

Proof. Write

$$a(x)b(x) - a(y)b(y) = a(x)(b(x) - b(y)) + b(y)(a(x) - a(y)).$$

Divide by $\|x - y\|^\sigma$ and take suprema over $x \neq y$. □

C.2. **Proof of the claim in Remark B.1.** Consider any function g with $\mathbb{E}[g(\delta, P) \mid \delta, \Lambda] = 0$ a.s. Assumption A.5 implies $(X, Z) = (X, \lambda^{-1}(X, \Lambda))$. Thus, (X, Z) is measurable with respect to (X, Λ). By the tower property,

$$\mathbb{E}[g(\delta, P) \mid X, Z] = \mathbb{E}[\mathbb{E}[g(\delta, P) \mid X, \Lambda] \mid X, Z] = \mathbb{E}[\mathbb{E}[g(\delta, P) \mid \delta, X, \Lambda] \mid X, Z].$$

By Assumption A.5 again,

$$\mathbb{E}[g(\delta, P) \mid \delta, X, \Lambda] = \mathbb{E}[g(\delta, P) \mid \delta, \Lambda] = 0, \text{ a.s.}$$

This implies $\mathbb{E}[g(\delta, P) \mid X, Z] = 0$. Since the RHS does not depend on Z, faithfulness implies $g(\delta, P) = g(\delta)$. Then

$$0 = \mathbb{E}[g(\delta, P) \mid \delta, \Lambda] = g(\delta) = g(\delta, P), \text{ a.s.}$$

C.3. **Proof of Theorem B.1.** Through this section, we consider any function H with

$$\mathbb{E}[H(\delta, P) \mid X, Z] = k(X) \quad \text{a.s.,} \quad \mathbb{E}[|H(\delta, P)|] < \infty, \tag{C.7}$$

for some measurable $k : \mathbb{R}^J \to \mathbb{R}$. Define

$$r_H(\lambda, d) := \mathbb{E}[H(\delta, P) \mid \Lambda = \lambda, \delta = d]. \tag{C.8}$$

We start by proving a useful high-level result.

Fix any $\lambda_0 \in S_\lambda$ for which (A.3) in the proof of Proposition A.4 in Appendix A.6 holds. Define

$$H_\lambda(d) = r_H(d, \lambda) - r_H(d, \lambda_0), \tag{C.9}$$

and

$$\mathcal{H}_\lambda = \{H_\lambda(d) : \mathbb{E}[|H(\delta, P)|] < \infty\}.$$

Note that $\mathcal{H}_\lambda$ is a set of functions of δ.

Lemma C.13. *Assume that, for Lebesgue-almost every $\lambda \in \mathbb{R}^k$,*

$$\mathcal{H}_\lambda \cap \mathrm{Ker}(\mathcal{T}) = \{\mathbf{0}\}, \tag{C.10}$$

where $\mathbf{0}$ *denotes the class of functions of* δ *that are Lebesgue-almost everywhere* 0. *Under Assumptions 3, A.5, and B.1 to B.3,* $(\delta, P) \mid (X, Z)$ *is faithful.*

Remark C.2. The condition (C.10) reveals the minimal condition for faithfulness under the λ-index assumption. In particular, (C.10) becomes trivial under completeness of $(\delta, P) \mid X, Z$ (Assumption A.2), which implies $\mathrm{Ker}(\mathcal{T}) = \{\mathbf{0}\}$.

Proof of Lemma C.13. By (A.3), there exists a full-measure subset $\mathcal{S} \subset S_\lambda$ such that (A.3) holds on $\mathcal{S}$. Fix any $\lambda \in \mathcal{S}$. Then we have

$$\mathbb{E}[H_\lambda(\delta) \mid X] = \mathbb{E}[r_H(\delta, \lambda) - r_H(\delta, \lambda_0) \mid X] = 0 \quad \text{a.s.} \tag{C.11}$$

This implies

$$r_H(\delta, \lambda) - r_H(\delta, \lambda_0) \in \mathrm{Ker}(\mathcal{T}).$$

By (C.10),

$$r_H(\delta, \lambda) - r_H(\delta, \lambda_0) = 0, \quad \text{a.s.}$$

This can be written as

$$\mathbb{P}(r_H(\delta, \Lambda) \neq r_H(\delta, \lambda_0) \mid \Lambda = \lambda) = 0, \quad \text{a.s..}$$

As a result,

$$\mathbb{P}(r_H(\delta, \Lambda) \neq r_H(\delta, \lambda_0)) = 0 \Longleftrightarrow r_H(\delta, \Lambda) = r_H(\delta, \lambda_0) \quad \text{a.s.}$$

Write $h_0(\delta)$ for $r_H(\delta, \lambda_0)$ to highlight it is just a function of δ. By definition of r_H in (C.8),

$$\mathbb{E}[H(\delta, P) \mid \Lambda, \delta] = r_H(\delta, \Lambda) = h_0(\delta) \quad \text{a.s.}$$

This implies

$$\mathbb{E}[H(\delta, P) - h_0(\delta) \mid \Lambda, \delta] = 0 \quad \text{a.s.}$$

By Assumption B.3, we conclude that

$$H(\delta, P) = h_0(\delta) \quad \text{a.s..}$$

The proof is then completed. ☐

Proof of Theorem B.1. Fix H with $\mathbb{E}[|H(\delta, P)|] < \infty$ and assume

$$\mathbb{E}[H(\delta, P) \mid X, \Lambda] = k(X) \quad \text{a.s.} \tag{C.12}$$

Let r_H and H_λ be defined in (C.8) and (C.9), respectively. Define the class of exponentially integrable functions as

$$\mathcal{S}_{a_0} = \left\{ \int_{\mathbb{R}^J} e^{a_0\|d\|} \big|H_\lambda(d)\big| \mathrm{d}d < \infty \right\}. \tag{C.13}$$

Step 1. By definition and Assumption B.2 (2),

$$\begin{aligned} |H_\lambda(d)| &= \left| \int H(d,p)\big(q(p \mid \lambda, d) - q(p \mid \lambda_0, d)\big)\, dp \right| \\ &\leq \int |H(d,p)|\, |q(p \mid \lambda, d) - q(p \mid \lambda_0, d)|\, dp \\ &\leq \zeta(d) \int |H(d,p)| \mu(p)\, dp. \end{aligned}$$

By Assumption B.2 (1), for any $\lambda' \in \mathbb{R}^k$,

$$\int |H(d,p)|\mu(p)\, dp \leq \frac{1}{c_-} \int |H(d,p)|\, q(p \mid \lambda', d)\, dp = \frac{1}{c_-} \mathbb{E}\big[|H(\delta, P)| \mid \Lambda = \lambda', \delta = d\big].$$

Averaging over $\lambda' \sim \Lambda \mid \delta = d$ yields

$$\int |H(d,p)|\mu(p)\, dp \leq \frac{1}{c_-} \mathbb{E}\big[|H(\delta, P)| \mid \delta = d\big].$$

Therefore,

$$|H_\lambda(d)| \leq \frac{1}{c_-} \zeta(d) \mathbb{E}\big[|H(\delta, P)| \mid \delta = d\big].$$

Multiplying by $e^{a_0\|d\|}$, integrating over d, inserting $p_\delta(d)$, and using Assumption B.2 (3) gives

$$\begin{aligned} &\int e^{a_0\|d\|} |H_\lambda(d)| \mathrm{d}d \\ &\leq \frac{1}{c_-} \int \frac{e^{a_0\|d\|}\zeta(d)}{p_\delta(d)}\, \mathbb{E}\big[|H(\delta, P)| \mid \delta = d\big]\; p_\delta(d) \mathrm{d}d \\ &\leq \frac{1}{c_-} \sup_{d \in \mathbb{R}^J} \frac{e^{a_0\|d\|}\zeta(d)}{p_\delta(d)} \mathbb{E}|H(\delta, P)| < \infty. \end{aligned}$$

As a result,

$$H_\lambda \in \mathcal{S}_{a_0}. \tag{C.14}$$

Step 2. By Assumption B.1 and (C.11),

$$0 = \mathbb{E}[H_\lambda(\delta) \mid X = x] = \mathbb{E}[H_\lambda(a(x) + \epsilon)] = (H_\lambda * p_\epsilon)(a(x)),$$

where $*$ denotes convolution. Since $\check{X} = a(X)$ has a density positive everywhere, it follows that

$$(H_\lambda * p_\epsilon)(\check{x}) = 0 \quad \text{for Lebesgue-almost every } \check{x} \in \mathbb{R}^J. \tag{C.15}$$

Because $H_\lambda \in L^1(\mathbb{R}^J)$, implied by (C.14), and $p_\epsilon \in L^1(\mathbb{R}^J)$, we may take Fourier transforms:

$$\widehat{H_\lambda * p_\epsilon}(\omega) = \widehat{H}_\lambda(\omega)\, \widehat{p_\epsilon}(\omega) = \widehat{H}_\lambda(\omega)\, \varphi_\epsilon(\omega).$$

From (C.15), $\widehat{H_\lambda * p_\epsilon}(\omega) \equiv 0$, hence

$$\widehat{H}_\lambda(\omega)\, \varphi_\epsilon(\omega) = 0 \qquad \forall \omega \in \mathbb{R}^J. \tag{C.16}$$

Step 3. We now prove a lemma that H_λ is real-analytic. The proof is deferred to the end of the proof.

Lemma C.14. *Assume that for some $a > 0$,*

$$\int_{\mathbb{R}^J} e^{a\|d\|} |h(d)| \mathrm{d}d < \infty. \tag{C.17}$$

Define $\widehat{h}(\omega) = \int_{\mathbb{R}^J} e^{-i\langle \omega, d\rangle} h(d) \mathrm{d}d$ for $\omega \in \mathbb{R}^J$. Then $\widehat{h}$ is real-analytic on $\mathbb{R}^J$.

Step 4. From (C.16) and continuity of φ_ϵ with $\varphi_\epsilon(0) = 1$, there exists $\varepsilon > 0$ such that

$$\varphi_\epsilon(\omega) \neq 0, \quad \forall \|\omega\| < \varepsilon.$$

Therefore $\widehat{H}_\lambda(\omega) = 0$ for all $\|\omega\| < \varepsilon$. Since $\widehat{H}_\lambda$ is real-analytic, by the identity theorem, vanishing on the nonempty open ball $\{\|\omega\| < \varepsilon\}$ implies

$$\widehat{H}_\lambda(\omega) \equiv 0 \text{ for any } \omega \in \mathbb{R}^J.$$

Since $H_\lambda \in \mathcal{S}_{a_0} \subset L^1(\mathbb{R}^J)$, injectivity of the Fourier transform on L^1 yields $H_\lambda(\omega) = 0$ for Lebesgue-almost every λ. Thus, the condition (C.10) is proved. By Lemma C.13, the proof of faithfulness is completed.

Proof of incompleteness. If there exists $\omega_0 \in \mathbb{R}^J$ such that $\varphi_\epsilon(\omega_0) = 0$,

$$\mathbb{E}[\exp\{i\langle \omega_*, \delta\rangle\} \mid X] = \exp\{i\langle \omega_*, a(X)\rangle\}\mathbb{E}[\exp\{i\langle \omega_*, \epsilon\rangle\}] = 0.$$

Thus, $\mathbb{E}[\cos(\langle \omega_*, \delta\rangle) \mid X] = 0$. As a result, $\delta \mid X$ is incomplete.

Proof of Lemma C.14. First, we prove that, for any $b < a$ and any integer $m \geq 0$,

$$\int_{\mathbb{R}^J} \|d\|^m e^{b\|d\|}\, |h(d)|\, \mathrm{d}d \;\leq\; \frac{m!}{(a-b)^m}\, M_a. \tag{C.18}$$

In fact, using the elementary inequality $t^m \le m!e^t$ with $t = b - a$ (since $e^t = \sum_{k\ge0} t^k/k! \ge t^m/m!$), we obtain that

$$\|d\|^m \le \frac{m!}{(a-b)^m}\, e^{(a-b)\|d\|}.$$

Multiplying by $e^{b\|d\|}|h(d)|$ and integrating yields (C.18).

Let $z = \xi + i\chi$ with $\|\chi\| < a$ and

$$\tilde{h}(z) := \int_{\mathbb{R}^J} h(d)\, e^{-i\langle d,z\rangle}\,\mathrm{d}d \;=\; \int_{\mathbb{R}^J} h(d)\, e^{-i\langle d,\xi\rangle}\, e^{\langle \chi,d\rangle}\,\mathrm{d}d, \qquad z = \xi + i\chi. \tag{C.19}$$

Since $\langle \chi, d\rangle \le \|\chi\|\,\|d\|$,

$$|h(d)e^{-i\langle d,z\rangle}| = |h(d)|e^{\langle \chi,d\rangle} \le |h(d)|e^{\|\chi\|\|d\|} \le |h(d)|e^{a\|d\|}.$$

The RHS is integrable by (C.17), so (C.19) is absolutely convergent.

Fix $z_0 = \xi_0 + i\chi_0 \in T_a$ and set $\delta := a - \|\chi_0\| > 0$. For $w \in \mathbb{C}^J$,

$$\tilde{h}(z_0 + w) = \int_{\mathbb{R}^J} h(d)e^{-i\langle d,z_0\rangle}\, e^{-i\langle d,w\rangle}\,\mathrm{d}d.$$

Use the scalar power series $e^{-i\langle d,w\rangle} = \sum_{m=0}^{\infty} \frac{(-i)^m}{m!}(\langle d,w\rangle)^m$. Fix $r \in (0,\delta)$ and assume $\|w\| \le r$. Then $|\langle d,w\rangle|^m \le (\|d\|\cdot\|w\|)^m \le (\|d\|r)^m$, hence

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{1}{m!} \int_{\mathbb{R}^J} |h(d)|\,|e^{-i\langle d,z_0\rangle}|\,|\langle d,w\rangle|^m\,\mathrm{d}d \\
&\le \sum_{m=0}^{\infty} \frac{r^m}{m!} \int_{\mathbb{R}^J} |h(d)|\, e^{\langle \chi_0,d\rangle}\, \|d\|^m\,\mathrm{d}d \\
&\le \sum_{m=0}^{\infty} \frac{r^m}{m!} \int_{\mathbb{R}^J} |h(d)|\, e^{\|\chi_0\|\,\|d\|}\, \|d\|^m\,\mathrm{d}d.
\end{aligned}$$

Applying (C.18) with $b = \|\chi_0\|$ to bound the inner integral by $\frac{m!}{\delta^m}M_a$, we have

$$\sum_{m=0}^{\infty} \frac{r^m}{m!} \int_{\mathbb{R}^J} |h(d)|\, e^{\|\chi_0\|\,\|d\|}\, \|d\|^m\,\mathrm{d}d \le M_a \sum_{m=0}^{\infty} (r/\delta)^m < \infty,$$

so the integrated series is absolutely summable and the convergence is uniform over $\|w\| \le r$. By dominated convergence, we have

$$\tilde{h}(z_0 + w) = \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \int_{\mathbb{R}^J} h(d)e^{-i\langle d,z_0\rangle}\, (\langle d,w\rangle)^m\mathrm{d}d \tag{C.20}$$

For each m, the map $w \mapsto \int h(d)e^{-i\langle d,z_0\rangle}(\langle d,w\rangle)^m\,\mathrm{d}d$ is a polynomial in w. Since the series (C.20) converges uniformly on compact subsets $\{w : \|w\| \le r\}$ for every $r < \delta$, $\tilde{h}$

is holomorphic in a neighborhood of z_0. Because $z_0 \in T_a$ was arbitrary, $\tilde{h}$ is holomorphic on T_a. By definition, for $z \in \mathbb{R}^J$,

$$\tilde{h}(z) = \hat{h}(z).$$

We can then conclude that $\hat{h}$ is real-analytic.

C.4. **Proof of Proposition B.1.** To verify the incompleteness of $\delta \mid X$, we simply prove φ_ϵ has zeros. Since $\nu, \eta_1, \ldots, \eta_J$ are independent,

$$\varphi_\epsilon(\omega) = \varphi_\nu(\omega) \prod_{j=1}^{J} \varphi_{\eta_j}(\omega_j) = \varphi_\nu(\omega) \prod_{j=1}^{J} \frac{\sin(b_j\omega)}{b_j\omega}.$$

Thus,

$$\{\omega : \varphi_\epsilon(\omega) = 0\} = \left\{\omega : \omega_j = \frac{k\pi}{b_j} \text{ for some } j \in [J] \text{ and } k \in \mathbb{Z}\right\}.$$

Next, we verify Assumption B.2. Since $S(p)$ and $t(\lambda)$ are bounded,

$$C_1^{-1}\mu(p) \;\le\; q(p \mid \lambda, d) \;\le\; C_1\mu(p), \tag{C.21}$$

for some constant $C_1 > 0$. Thus, Assumption B.2 (1) is proved.

Write

$$q_\theta(p) = \exp(\langle\theta, S(p)\rangle - A(\theta))\mu(p).$$

For each i,

$$\partial_{\theta_i} q_\theta(p) = \big(S_i(p) - \partial_{\theta_i} A(\theta)\big)\, q_\theta(p) = \big(S_i(p) - \mathbb{E}_\theta[S_i(P)]\big)\, q_\theta(p),$$

where the last line uses the well-known property of exponential families $\mathbb{E}[S_i(P)] = \partial_{\theta_i} A(\theta)$. Thus,

$$|\partial_{\theta_i} q_\theta(p)| \le C_2 q_\theta(p),$$

for some constant $C_2 > 0$. By (C.21),

$$|\partial_{\theta_i} q_\theta(p)| \le C_1 C_2 \mu(p).$$

Then

$$|q_\theta(p) - q_{\theta_0}(p)| \le C_1 C_2 \|\theta - \theta_0\|_1\, \mu(p).$$

By definition,

$$\theta(\lambda, d) - \theta(\lambda_0, d) = \zeta(d)\big(t(\lambda) - t(\lambda_0)\big).$$

Thus,

$$|q(p \mid \lambda, d) - q(p \mid \lambda_0, d)| \le C_3 \zeta(d)\mu(p), \tag{C.22}$$

where $C_3 = 2C_1C_2 \sup_\lambda |t(\lambda)|$. As a result,

$$\begin{aligned}
|H_\lambda(d)| &= \left| \int H(d,p)\big(q(p \mid \lambda, d) - q(p \mid \lambda_0, d)\big)\, dp \right| \\
&\leq \int |H(d,p)| \cdot |q(p \mid \lambda, d) - q(p \mid \lambda_0, d)|\, dp \\
&\leq C_3 \zeta(d) \int |H(d,p)|\, \mu(p)\, dp \\
&\leq C_1 C_3 \zeta(d) \int |H(d,p)| q(p \mid d, \lambda') dp,
\end{aligned}$$

for any $\lambda' \in \mathbb{R}^J$ that may differ from λ, where the last line follows from (C.21). As a result,

$$|H_\lambda(d)| \leq C_1 C_3 \zeta(d) \mathbb{E}[|H(\delta, P)| \mid \Lambda = \lambda', \delta = d].$$

Taking integral of λ' over the distribution of $\Lambda \mid \delta = d$, we obtain that

$$|H_\lambda(d)| \leq C_1 C_3 \zeta(d) \mathbb{E}[|H(\delta, P)| \mid \delta = d].$$

Let $p_\delta(d)$ be the marginal density of δ and a_0 be any constant less than $c - 1/4$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^J} e^{a_0\|d\|} |H_\lambda(d)| \mathrm{d}d \\
&\leq C_1 C_3 \int_{\mathbb{R}^J} \frac{e^{a_0\|d\|}\zeta(d)}{p_\delta(d)} \cdot \mathbb{E}[|H(\delta, P)| \mid \delta = d] p_\delta(d) \mathrm{d}d \\
&\leq C_1 C_3 \mathbb{E}[|H(\delta, P)|] \cdot \sup_{d \in \mathbb{R}^J} \frac{e^{a_0\|d\|}\zeta(d)}{p_\delta(d)}. \qquad \text{(C.23)}
\end{aligned}$$

Under condition (b), $\delta = (X + \nu) + \eta$ where $X + \nu \sim N(0, 2I_J)$ and $\eta \in [-1, 1]^J$. Thus,

$$p_\delta(d) = \mathbb{E}_\eta[\phi_J((d - \eta)/2)],$$

where $\phi_J(x)$ denote the $N(0, I_J)$ density. Note that $p_\delta(d)$ is strictly positive and continuous everywhere. Moreover $p_\delta(d)$ has Gaussian tails $\asymp \exp(-\|d\|^2/4)$, so the ratio

$$e^{a_0\|d\|}\zeta(d)/p_\delta(d) \to 0, \quad \|d\| \to \infty$$

and is continuous on $\mathbb{R}^J$. Therefore it has a finite supremum. By (C.23), the proof os Assumption B.2 is then completed.

Lastly, condition (a) implies that the law of $P \mid \Lambda, \delta = d$ is an exponential family with sufficient statistic $S(p)$ given any d. Since S is continuous and non-constant, the range of S contains an open set. By the Lehmann-Scheffè completeness theorem, Assumption B.3 holds.

C.5. **Proof of Proposition B.2.** This choice is such that

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}(P>0)\mid \delta, Z, X\right] &= \mathbb{P}_{Q\sim\mathcal{N}(0,1)}\left(r(\delta,X)\tau(\delta,Z,X)+\tau(\delta,Z,X)Q>0\right)\\ &= \Phi\left(r(\delta,X)\right)\end{aligned}$$

Thus

$$\mathbb{E}[\mathbf{1}(P>0)\mid Z, X] = \mathbb{E}[\Phi\left(r(\delta,X)\right)\mid X] \equiv k(X).$$

Hence faithfulness does not hold. Assumption 3 holds by construction. Thus the remainder of the proof verifies that Assumption 2 holds.

The density of P is

$$\begin{aligned}f(p\mid x,\delta,z) &= \frac{1}{\sqrt{2\pi}}\exp\left(\log\frac{1}{\tau(\delta,Z,X)}-\frac{(p-r\tau)^2}{2\tau^2}\right)\\ &= \frac{1}{\sqrt{2\pi}}\exp\left(\log(1/\tau)-\frac{p^2}{2\tau^2}-\frac{r^2}{2}+\frac{yr}{\tau}\right)\end{aligned}$$

Under our choices for r,τ,

$$f(p\mid x,\delta,z) = \frac{1}{\sqrt{2\pi}z}\exp\left(-\frac{p^2}{2z^2}-\frac{r^2}{2}+p\delta\frac{x}{z}\right)$$

Thus

$$f(\delta,p\mid x,z) = \frac{1}{\sqrt{2\pi}z}a(x)\mathbf{1}(x\in[1,2])\exp\left(-p^2\frac{1}{2z^2}+p\delta\frac{x}{z}\right).$$

It is an exponential family supported on $[1,2]\times\mathbb{R}$ with natural parameters

$$\eta(x,z) = \left(-\frac{1}{2z^2},\frac{x}{z}\right)$$

It is possible to choose X,Z such that the support of $\eta(X,Z)$ contains an open set of $(-\infty,0)\times(0,\infty)$. The sufficient statistics are

$$(T_1,T_2) = \left(P^2,\delta P\right)$$

Note that

$$\delta = |T_2/\sqrt{T_1}|, P = \operatorname{sgn}(T_2)\sqrt{T_1}$$

so that (δ,P) and (T_1,T_2) are bijective. Thus, take any function $h(\delta,P)$, we can write it in terms of the sufficient statistics $h(T_1,T_2)$. Since the support of $\eta(x,z)$ contains an open set in $(-\infty,0)\times(0,\infty)$, T_1,T_2 are complete sufficient statistics. Thus

$$\mathbb{E}[H(\delta,P)\mid Z,X]=0 \implies \mathbb{E}[\tilde{H}(T_1,T_2)\mid\eta]=0 \implies \tilde{H}=H=0.$$