NBER WORKING PAPERS SERIES

AUCTIONS WITH ENDOGENOUS VALUATIONS, THE SNOWBALL EFFECT, AND OTHER APPLICATIONS

Kala Krishna

Working Paper No. 3483

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
October 1990

This paper is part of NBER's research program in International Studies. Any opinions expressed are those of the author and not those of the National Bureau of Economic Research.

NBER Working Paper #3483
October 1990

AUCTIONS WITH ENDOGENOUS VALUATIONS, THE SNOWBALL EFFECT, AND OTHER APPLICATIONS

ABSTRACT

In most of the literature on auctions the valuations of agents are exogenously specified. This assumption may be inappropriate in a number of cases where valuations are better derived endogenously. Endogenous valuations are appropriate when there are many units being auctioned and their value is determined in a secondary market which is imperfectly competitive. The model is thus appropriate for studying the sale of quota licenses and scarce resources used in production when product markets are imperfectly competitive.

A series of examples are developed to show how these models work. Particular models are developed which cast light on a number of issues in applied micro-economics. These issues include the evolution of market structure, in particular, the "snowball effect", the effect on market structure of selling quota licenses, and the relationship between increasing returns to scale and the monopolization of markets. The models also provide another resolution of the "transponder puzzle".

Kala Krishna
Department of Economics
Harvard University
Littauer 215
Cambridge, MA 02138

1. INTRODUCTION

There exists a large and well developed literature on auctions and auction design.[1] The focus of most of this literature is on the implications of uncertainty in the valuation of the object or objects auctioned. For the most part, the literature exogenously specifies the distributions of the valuations of the bidders for the object or objects sold. However, in general when multiple objects are being auctioned off to multiple buyers and value is derived from a secondary market which is imperfectly competitive, these valuations cannot be specified exogenously.

The value of a scarce resource used in production is derived from the value of the product it helps produce. The value of this product is in turn determined by the operation of its market. If this market is imperfectly competitive, the value of the scarce resource depends on its allocation across the firms in the final product market. Hence, it is impossible to specify an exogenous distribution of valuations: valuations must be endogenously determined by the operation of the secondary market, in this case, that of the final product produced. Similarly, the value of an object of art is determined by the resale market for the object. If this market is imperfectly competitive, as seems likely given the uniqueness of each object of art, the value of such art by a bidder must depend on the allocation of other related objects of art.

If the secondary market is perfectly competitive, then it is quite appropriate to define the valuations of agents exogenously. With perfect competition, the price in the secondary market is taken as given. The valuation of an agent is then the marginal value product of the resource, which may be constant or not. Thus, valuations, even if they are exogenous, need not be constant and independent of the number of units purchased as is often assumed in the auction of multiple objects.[2] But when the secondary market is imperfectly competitive, exogenous valuations become untenable. This aspect of the auction of multiple objects seems to have been neglected by the literature. It seems odd that a literature devoted to studying the strategic behavior of agents in auctions in effect assumes away such behavior in the operation of the secondary markets in which value is determined.[3]

It is to this general area of auctions that this paper is directed. There are a large number of questions that need to be answered here. It would be impossible to address all of them in one paper. These include the standard questions in the literature on auctions, such as the comparison of various auction schemes and optimal auction design, all of which need to be re-examined when valuations are endogenous. In addition, models of auctions with endogenous valuations are closely related to a number of interesting questions in applied microeconomics. The latter, rather than the former, is the main thrust of this paper.

Section 2 relates this paper to existing work on auctions. In order to focus on endogenous valuations, I abstract from any uncertainty in the valuations themselves. I also focus on sequential auctions. A sequential auction is a natural way to capture a situation where there are a number of units which are to be sold, where each is owned by a different seller when sellers arrive sequentially at random. In this way, the model can be applied to study the evolution of market structure when the object being sold is thought of as capacity, or a scarce resource that is used in production.

Consider first the simplest case where one object is sold to one of two agents. Clearly, in the absence of any uncertainty the object goes to the agent with the higher marginal valuation who pays a price essentially equal to the other agent's marginal valuation. When the marginal valuations of the agents are constant, with one agent's marginal valuation being greater than the other agent's, then it is clear that the same result holds in the sequential auction with many units being sold. Thus making the usual assumptions about valuations leads to completely uninteresting results in the absence of uncertainty in those valuations.

However, what if marginal valuations are not constant? This is of course what would occur with imperfectly competitive secondary markets. Define by $f_n(x)$ the value of having x units when all n units have been sold, i.e., the other agent has $n-x$ units. For simplicity, temporal aspects of when the capacity is acquired are suppressed by assuming that payoffs depend only on the final allocation. This corresponds to assuming that capacity acquisition occurs in a brief period relative to the life of the industry. This function arises from the operation of the secondary market, and its form depends on the particular application in question.

Section 3 provides an introduction to the general area. A number of examples are provided that illustrate both how the model works and suggest areas for future work. I then look at a particular sequential auction model which sheds some light on the evolution of market structure and is related to recent work in applied micro economics. This model, analyzed in Section 4, is of a two player symmetric sequential auction game. I look at what conditions on the valuations ensure monopolization by one agent in the case where the two agents are ex-ante identical? I find that in this symmetric case convexity of the $f_n(x)$ function is sufficient to ensure that monopolization occurs. In addition I show that in this model the price paid in the equilibrium always falls. In fact, the price for all but the first unit is a constant. I also show that convavity of $f_n(x)$ ensures that monopolization is not an equilibrium.

This model is of considerable economic interest as it captures the essence of a number of problems in industrial economics and is related to some recent work on the endogenous determination of market structure. Ghemawat (1990) argues that if total capacity is limited and in a certain range, then even with two ex-ante identical agents capacity will go to only one of them in a model of price competition and homogeneous products. However, he does not consider a sequential auction and I show that his conditions are not sufficient to give his result when the sequential nature of capacity acquisition is taken into account.

Another application comes from international trade. It is often proposed that quota licenses be auctioned, with concern expressed, however, about monopolization of the market for licenses occurring. As the valuation of these licenses arises from the operation of a secondary market--that for the product--a model of endogenous valuations is the correct one to use. Another interpretation of capacity is as a license to sell in the market. This permits this model to be thought of as the correct one for modeling the auction of quota licenses directly to producers.[4]

The model also relates to the question of how the intensity of competition in the post-acquisition game is related to behavior in the acquisition game. When does ex-ante competition eliminate profits? When does it not? When does ex-post competition encourage concentration in the acquisition game?

It is also related to the question of whether increasing returns to scale (IRS) creates monopoly. The model suggests that this link is less than tight as IRS need not generate convexity of the $f_n(\cdot)$ function.

In addition, this model provides a simple framework which allows for a declining equilibrium price in a sequential auction and so sheds light on the "transponder puzzle" (which refers to the fact that in a sequential auction of transponders their price fell) and to the observations in Ashenfelter (1989) on falling prices in wine auctions. This, it is argued, cannot reflect equilibrium behavior as agents could do better by waiting.[5] With endogenous valuations, however, a falling price in a sequential auction is no puzzle. An open question of some interest concerns a characterization of when prices must rise in equilibrium and when they must fall.

2. RELATION TO THE LITERATURE

The work on auction theory is clearly one of the showpieces of economic theory.[4] Much of the focus of the literature is on single object auctions and on uncertainty in the valuation of the object auctioned.

I focus on work on multiple object auctions here. There are a number of strands in the literature. One strand looks at ways of extending the single object auction results to multiple object auctions. Harris and Raviv (1981) derive the form of optimal multiple-unit auctions. Buyers are assumed to be risk neutral and their preferences are parametrized by a scalar V. The distribution of V, $F(V)$, is known to the seller and all buyers. The seller has a fixed number of units and is also risk neutral. Harris and Raviv (1981) consider the case where each buyer demands one unit and the distribution of V is uniform. They prove a revenue equivalence theorem and characterize the optimal auction. Maskin and Riley (1981) extend Harris and Raviv (1981) in two directions. First, they look at optimal auctions when $F(V)$ is not uniform and show that, in contrast to Harris and Raviv, it may be necessary to prohibit not just bids below a minimum, but also over other predetermined intervals. Second, they analyze the case with each agent having a downward sloping demand curve and characterize the optimal selling strategy.

Maskin and Riley (1987) further builds on their earlier work to show that the Harris and Raviv (1981) result on revenue equivalence with multiple objects, where each buyer demands one unit and $F(V)$ is uniform, also holds for all $F(V)$. In addition, they characterize the optimal auction scheme both when $F(V)$ satisfies some regularity properties and when it does not. When each agent has a downward sloping demand curve, they show that standard auctions are not optimal and characterize the optimal auction.

Other work on multiple object auctions includes that of Engelbrecht-Wiggans and Weber (1979) who consider the effect of non-linear valuation functions and point out that in such cases it is inappropriate to analyze the bidding on one of a number of simultaneously auctioned objects as if its sale were independent of the sale of the remaining objects. They then illustrate this by looking at a particular example of a multi-object auction.

Another set of papers looks at multiple object auctions when bidders have constraints on their budgets. These include Engelbrecht-Wiggans (1987), Palfrey (1980), and most recently, Pitchik and Schotter (1989) and Pitchik (1989). Lastly, Swinkels (1990) looks at a model where the total availability of objects is not known and focuses on conditions under which equilibria are efficient and shows that efficient equilibria are revenue equivalent.

The work closest to mine is that of Bernheim and Whinston (1986). They consider a model of complete information among the bidder and many objects in a "menu" auction. As they specify the valuation of all agents across the allocations of n objects, their model can capture endogenous valuations. Their model is more general in that the objects are not assumed to be identical. They consider a simultaneous move game when buyers bid over all the allocations. They show that these auctions always implement efficient actions. Although the Nash equilibria of first price menu auctions need not be efficient, their refinement to "truthful" equilibria yield efficient outcomes. They do not consider sequential auctions, and for a number of problems, this is the more natural assumption.

Wilson (1979) on share auctions is related to both Bernheim and Whinston (1986) and to this work. However, in Wilson's share auction mechanism bidders submit demand schedules as a function of the price per share and the auctioneer picks a price to clear the market. This is not a menu auction as bids are not over allocations. Wilson (1979) is interested in whether a share auction does better or worse than a unit auction. Analogous to this, I am interested, among other things, in whether a sequential auction does better than a unit auction.

The focus of the existing literature has not been on endogenous valuations in auctions. The one exception is the work of Bernheim and Whinston (1986). However, their formulation is not the natural one for many multiple unit auctions and lends itself to different economic applications.

3. SOME ILLUSTRATIVE EXAMPLES

With endogenous valuations it is not clear that one should expect the standard results of auction theory to obtain. However, it is also not clear as yet what results are likely to hold in general and what to conjecture. There are a large number of variations of the model possible: symmetric versus asymmetric, single unit versus multiple unit, two person versus n person, simultaneous versus sequential and so on. The most fruitful approach seems to be to look at economically interesting problems where this approach might shed some light and analyze them in turn, thereby building up the understanding of the model in general. This is the approach taken in this paper. Once valuations are perceived as endogenous, it is hoped that other applications and results will abound.

In this section I present a few examples that illustrate the kinds of results such models give and suggest conjectures to be explored in the future. Let $f_{n-k}(x)$ be the payoff to the agent having x when $n-k$ out of n units are allocated, where $x \le n-k$. Let $\Delta f_{n-k}(x) = f_{n-k}(x+1) - f_{n-k}(x)$, the agent's marginal valuation of obtaining an additional unit at this stage. Once $f_n(\cdot)$ is defined the game can be solved. If agents are symmetric, no superscript to $f_n(\cdot)$ is given as the superscript indexes the agent. Note that when there are any number of objects and only two buyers, endogenous valuations can be specified independent of the allocation, as if they were exogenous. This can be done by assuming that all objects are sold so that the valuation function for each agent is interpreted as being the valuation of x when the other agent has the remainder of the goods. However, even with two buyers and many objects the literature has in general assumed that marginal valuations are constant or decreasing, while they could in reality be increasing. With many buyers and many objects the standard assumption has been that each buyer buys only one unit, and has a constant valuation for it, or that each agent has a downward sloping demand curve.

In general, each unit goes to the agent with the higher marginal valuation at a price given by the second highest marginal valuation. However, what complicates matters is that, except for the last unit, the marginal valuation of a unit is defined by the difference in the subgame perfect equilibrium payoffs in the subgames both when the agent gets the unit and when he does not! One implication of this is that even if one agent has a higher marginal valuation for every unit at the last stage of the game, it does not mean that he will obtain all the units. The following is an asymmetric example where $n = 2$ and one agent's marginal valuation at the last stage always exceeds the other's. Yet, monopolization does not occur.

<u>Example 1</u>
($n=2$)

$$f_2^A(0) = 0, \quad f_2^A(1) = 10, \quad f_2^A(2) = 20, \quad \Delta f_2^A(0) = 10, \quad \Delta f_2^A(1) = 10$$

$$f_2^B(0) = 0, \quad f_2^B(1) = 9, \quad f_2^B(2) = 10, \quad \Delta f_2^B(0) = 9, \quad f_2^B(1) = 1$$

The extensive form of the game is given below. Arrows denote the allocation in each subgame and prices are given in brackets next to the paths.

FIGURE 1 HERE

Although A always gets the last unit because his marginal valuation exceeds B's, the first unit goes to B! This is because:

$$f_1^A(1) = 11, \quad f_1^A(0) = 9, \quad \Delta f_1^A = 2$$
$$f_1^B(1) = 9, \quad f_1^B = 0, \quad \Delta f_1^B = 9$$

so that B's marginal valuation of the first unit exceeds A's! Note also that the total revenue raised is only 3. If both units were sold together, A would obtain them and revenues would be 10! Even if one agent's marginal valuation at the last stage lies everywhere above that of the other, he need not obtain all the units! Clearly, enough of an asymmetry in valuations would ensure monopolization. If, for example, in Example 1, $f_2^A(1) = 20$ and $f_2^A(2) = 30$ both units would go to A. It would be of interest to specify the extent of asymmetry needed to ensure monopolization.

Example 2 looks at the two unit sequential auctions with two symmetric agents. With one unit it is clear that the agent with the highest valuation of the object obtains it at a price equal to the second highest agent's valuation. If agents are identical, then the entire surplus is paid.

Example 2

With two units, $f_n(\cdot)$ is either linear, concave or convex. If it is linear, the agent with the highest marginal valuation gets both units and if agents are symmetric, then any allocation is an equilibrium and all the surplus is paid. If $f_n(\cdot)$ is concave, and there are two symmetric agents, as depicted in Figure 2A, then the second unit goes to the agent who did not get the first unit so that in equilibrium both agents get one unit. Note that even though firms are symmetric, ex-ante profits are positive! Also that revenue is less than that under a unit auction.

FIGURE 2A HERE

If $f_n(\cdot)$ is convex, and there are two symmetric agents, then one agent gets both units though the total price paid equals his total surplus and equals that under a unit auction.[7] This is depicted in Figure 2B.

FIGURE 2B HERE

Thus, one might conjecture that with two symmetric agents, convexity of $f_n(\cdot)$ ensures monopolization and concavity ensures that an equal allocation is always an equilibrium. While the former is true, the latter is not! All that is true with concavity is that monopolization cannot be an equilibrium as the last unit goes to the agent who has less units. This does not imply that the allocation is equal when n exceeds 2. A counter-example is the following:

<u>Example 3</u>
$(n=4)$

$$f_n(0) = 0,\ f_n(1) = 5,\ f_n(2) = 9.9,\ f_n(3) = 14,\ f_n(4) = 15.$$

There are two equilibria as depicted in Figure 3. Both consist of the first three units going to one agent and the last to the other. The price for the first unit is 6.4, the second and third is 1.8, and the last is 1.

Note that prices fall in equilibrium and that a unit auction would raise more revenue in this example. The latter seems to be true in the two person symmetric case when $f_n(\cdot)$ is concave, whatever be n. However, with more than two agents this is not true. Consider the example with concave but non-endogenous valuations below.*

<u>Example 4</u>
$(n=2)$

Let $f_n(\cdot) = 0$, $f_n(n/2) = f(n/2)$, and $f_n(n) = f(n)$.

Let $f(n)$ be concave. The extensive form and solution are depicted in Figure 4.

FIGURE 4 HERE

With more than two players and two units, the second unit goes to one of the agents who did not get the first unit, at a price of $f(n/2)$. The value of the first unit therefore also equals $f(n/2)$ to all agents and so this is its price. Total revenue is thus $2f(n/2)$ which exceeds $f(n)$, that under a unit auction.

With concavity of $f_n(\cdot)$, having more agents than units ensures that there is always more than one agent with a marginal valuation of $f_n(n) - f_n(n-1)$ and it is this which makes a sequential auction dominate a unit auction, since with concavity $n(f_n(n) - f_n(n-1))$ exceeds $f_n(n) - f_n(0)$. This suggests that the optimal sequential auction when $f_n(\cdot)$ is concave will have the number of units set less than the number of agents. Some simple results exist on sequential versus unit auctions for some special cases.

<u>Example 5</u>
$(n=2)$

Let x units be sold in stage 1 and $n-x$ in stage 2. The extensive form is given in Figure 5.

FIGURE 5 HERE

There are two symmetric agents, and $f_n(\cdot)$ denotes the terminal payoffs. In this example, more revenue is raised in a unit auction of n objects than in any sequential two-stage auction with x units sold in the first stage and $n-x$ units sold in the second.

<u>Proof</u>: The price of the second block of units is denoted by P_2.

$$P_2 = \mathrm{Min}[f_n(n) - f_n(x),\ f_n(n-x) - f_n(0)]$$
$$= \mathrm{Min}[\alpha,\beta]$$

if $\alpha > \beta$, then $P_2 = \beta$ and the last block goes to the agent with the first block. In this case:

$$f_x(x) = f_n(n) - [f_n(n-x) - f_n(0)]$$

$$f_x(0) = f_n(0).$$

Thus

$$P_1 = f_x(x) - f_x(0) = f_n(n) - f_n(n-x)$$

$$P_1 + P_2 = f_n(n) - f_n(n-x) + f_n(n-x) - f_n(0)$$

$$= f_n(n) - f_n(0)$$

which is exactly what would be obtained by a unit auction.

If $\beta > \alpha$, then the last bloc goes to the agent who did not get the first block. Also, $P_2 = \alpha$ so that

$$f_x(x) = f_n(x) \quad \text{and} \quad f_x(0) = f_n(n-x) - P_2.$$

$$P_1 = f_x(x) - f_x(0)$$

$$= f_n(n) - f_n(n-x),$$

$$P_1 + P_2 = 2f_n(n) - (f_n(x) + f_n(n-x)).$$

However, as $\beta > \alpha$

$$f_n(n-x) + f_n(x) > f_n(0) + f_n(n).$$

Hence:

$$P_1 + P_2 < 2f_n(n) - [f_n(0) + f_n(n)]$$

$$= f_n(n) - f_n(0).$$

so a unit auction weakly dominates a sequential auction. ▪

This is not true when agents are asymmetric. In this case it is also possible for a sequential auction to give more revenue than a unit auction even with two agents as in Example 6.

Example 6
(n=2)

There are two agents, A and B. Their valuation functions are given by:

$$f_2^A(2) = 20, \quad f_2^A(1) = 19, \quad f_2^A(0) = 10, \quad f_2^B(2) = 30, \quad f_2^B(1) = 10, \quad f_2^B(0) = 0 .$$

The extensive form and solution is depicted in Figure 6.

FIGURE 6 HERE

In equilibrium, Agent B obtains both units and pays 9 for each. The total revenue of 18 exceeds that of 10 which is what would be obtained if both the units were sold together.

This series of examples illustrate how such models work. The sale of one unit alters the marginal valuations of both agents so that the intuition valid for unit auctions or for linear valuations need not hold. The next section deals with a multiple object auction with two symmetric agents and non-linear valuations.

4. A MODEL OF DUOPOLY

There are two agents bidding for n identical goods which are auctioned sequentially. These "goods" can be thought of as capacity, in which case the model applies to the evolution of an industry with only two producers. Alternatively, they can be thought of as licenses to import when there are two foreign producers, in which case the model defines the effect of auctioning quota licenses directly to the producers. Finally, as there are only two agents, the model also applies to exogenous but non-linear valuations.

The agents are assumed to be ex-ante identical. The valuations of the agents are endogenously determined by the operation of a secondary market, as yet unspecified. For every allocation of the n units, there is an outcome in the secondary market which determines the payoffs to the agents for the allocation. This valuation function is denoted by $f_n(x)$, $x = 0,\ldots n$, which gives the payoffs to the agent obtaining x units when all n units have been allocated. Hence, the other agent gets $f_n(n-x)$. These valuation functions are all that is needed to determine the allocation of the n objects in the sequential auction.

Let $f_{n-k}(x)$ be the value of having x units when $n-k$ have been allocated, and k remain. Then $f_{n-k}(x)$ is obtained by folding the game backwards. It is easy to see that $f_{n-1}(x)$ is defined by $f_n(x)$. The marginal valuation for the agent with x is $\Delta f_n(x) \equiv f_n(x+1) - f_n(x)$. The marginal valuation of the other agent is $\Delta f_n(n-1-x) = f_n(n-x) - f_n(n-1-x)$. If $\Delta f_n(x) > \Delta f_n(n-1-x)$, the unit goes to the agent with x who pays the other agent's marginal valuation. Therefore:

$$f_{n-1}(x) = f_n(x+1) - \Delta f_n(n-1-x) \qquad \text{if } \Delta f_n(x) \geq \Delta f_n(n-1-x)$$
$$= f_n(x) \qquad \text{if } \Delta f_n(x) \leq \Delta f_n(n-1-x) \ .$$

However, note that:

$$\Delta f_n(x) \gtreqless \Delta f_n(n-1-x) \Longleftrightarrow f_n(x+1) - \Delta f_n(n-1-x) \gtreqless f_n(x) \ .$$

Hence:

$$f_{n-1}(x) = \max \begin{cases} f_n(x+1) - \Delta f_n(n-1-x) \\ f_n(x) \ . \end{cases}$$

In general, $f_{n-k-1}(x)$; $x=0,\ldots n-k-1$, denotes the value of having x when $n-k-1$ of the n units have been allocated. It can be defined recursively as:

$$f_{n-k-1}(x) = \max \begin{cases} f_{n-k}(x+1) - \Delta f_{n-k}(n-k-1-x) \\ f_{n-k}(x) \ . \end{cases}$$

In this section I show that if $f_n(x)$ is convex in x, then the equilibrium allocation consists of one agent obtaining all the units in the sequential auction specified. It is easy to see that when $f_n(x)$ is convex:

$$f_{n-1}(x) = f_n(x+1) - \Delta f_n(n-1-x) \qquad \text{if } x \geq \frac{n-1}{2} \ ,$$
$$= f_n(x) \qquad \text{if } x \leq \frac{n-1}{2} \ .$$

This is because $\Delta f_n(n-1-x) \gtreqless \Delta f_n(x)$ as $n-1-x \gtreqless x$ or $x \gtreqless \frac{n-1}{2}$.

The last unit goes to the agent with none of the units already allocated, as convexity implies that he has the higher marginal valuation of the remaining unit. However, it remains to be shown that this is true when k units remain to be allocated, i.e. that:

$$f_{n-k}(x) = f_{n-k+1}(x+1) - \Delta f_{n-k+1}(n-k-x) \qquad \text{if } x \geq \frac{n-k}{2},$$

$$= f_{n-k+1}(x) \qquad \text{if } x \leq \frac{n-k}{2}.$$

I do not prove this directly as convexity is not preserved in the induction step. Rather, I show that $f_{n-k}(x)$ can be written directly in terms of $f_n(\cdot)$. This is done in Lemma 1. I first specify the relationship between $f_{n-k}(x)$ and $f_n(x)$ on the assumption that at any stage the agent with strictly more of the units sold to date gets all the remaining units, and so pays the marginal valuation of the other agent conditional on this assumption. This gives the statement of Lemma 1. This guess is then verified, which constitutes a proof.

Lemma 1:

For all k,

$$f_{n-k}(x) = \begin{cases} f_n(x+k) - k\Delta f_n(n-(x+k)) & \text{if } x > \frac{n-k}{2}, \\ f_n(x) & \text{if } x \leq \frac{n-k}{2}. \end{cases}$$

Proof: See Appendix. ∎

Theorem 1:

In the symmetric two-agent sequential auction of n units.

(a) One agent obtains all the units in the auction when $f_n(x)$ is convex.

(b) Moreover, the price of the ith unit, P_i, is given by:

$$P_n = P_{n-1} = \ldots P_2 = \Delta f_n(0) \quad \text{and}$$

$$P_1 = [f_n(n) - f_n(0)] - (n-1)\,\Delta f_n(0) \ .$$

(c) The total revenue from the sequential auction equals that under the auction where all units are sold together and equals $f_n(n) - f_n(0)$.

(d) In each subgame, with $n-k$ units sold, if $x > \frac{n-k}{2}$, the agent with x gets all remaining units. If $x = \frac{n-k}{2}$, the unit could go to either agent. The agent with less than $\frac{n-k}{2}$ gets none of the remaining units.

Proof:

(a) This follows from using Lemma 1 to verify that when $n-k$ units have all been allocated to one agent his marginal valuation exceeds that of the other agent:

$$\Delta f_{n-k+1}(n-k) - \Delta f_{n-k+1}(0) = (k-1)[\Delta f_n(1) - \Delta f_n(0)] + [\Delta f_n(n-1) - \Delta f_n(0)] > 0$$

so that (a) follows.

(b) From (a) it follows that $P_i = \Delta f_n(0)$ for $i > 1$, as the winner pays the loser's marginal valuation. Also,

$$\begin{aligned} P_i &= f_1(1) - f_1(0) \\ &= f_n(n) - (n-1)\,\Delta f_n(0) - f_n(0) \ . \end{aligned}$$

(c) $\sum_{i=1}^{n} P_i = f_n(n) - f_n(0)$ which equals the price for the entire n units if sold together.[9]

(d) Follows from Lemma 1. ∎

The natural question to ask at this point is whether concavity of the $f_n(x)$ function in this model ensures that equal allocation of the n objects is an equilibrium of this sequential auction. Example 3 above disproves this conjecture. All that can be shown is that monopolization of the objects can never be an equilibrium in this game as concavity ensures that the last unit must go to the agent with the fewer units.

Interpreting the n units available as capacity says that if $f_n(\cdot)$ is convex the larger firm will obtain all additional units of capacity and that even if firms are ex-ante identical, ex-post one of them will obtain all additional units of capacity. This is termed the "snowball effect" by Ghemawat (1989). He argues that this effect will occur in a model of duopoly with price competition, a homogeneous product, and capacity constraints. The essence of his argument is that for certain levels of total capacity, n, the sum of duopoly profits is maximized at the most asymmetric allocation of capacity. In other words, that $f_n(x) + f_n(n-x)$ is maximized at $x = (0,N)$ and that this sum is decreasing for $x < n/2$ and increasing for $x > n/2$. This implies that $f'_n(x) - f'_n(n-x)$ is negative for $x < n/2$ and positive for $x > n/2$ and equals zero at $x = n/2$. Hence, the marginal valuation of the agent with $x > n/2$ exceeds that of the other agent. From this he argues that any additional capacity will go to the agent with initially more capacity, and that this in turn implies that starting from zero, one of the firms will acquire all the units. However, he does not model this as a sequential auction and his result is not true when this is done. The following is a counter-example.

<u>Example 7</u>
(n=4)

$f_n(0) = 0,\ f_n(1) = 4,\ f_n(2) = 5,\ f_n(3) = 10,\ f_n(4) = 15.$

Note $f(0) + f(4) = 15 > f(1) + f(3) = 14 > f(2) + f(2) = 10$ so that Ghemawat's conditions are met. However, in the sequential auction depicted in Figure 7 there are two equilibria, which are mirror images of each other. The first, second, and fourth unit go to one agent, and the third to the other. Their prices are $P_1 = 5$, $P_2 = 2$, $P_3 = 2$, and $P_4 = 1$, so that prices fall along the equilibrium path. While convexity, which ensures monopolization, implies that $f_n(x) + f_n(n-x)$ is maximized at the boundary his conditions do not imply convexity, and his assumptions are not sufficient to ensure monopolization as this example shows.

FIGURE 7 HERE

Note also that if $f_n(\cdot)$ is convex, ex-ante profits are zero, though ex-post profits are positive! If $f_n(\cdot)$ is not convex, competition to acquire capacity need not lead to zero profits ex-ante as is evident from Example 7.

Interpreting the n units as licenses sold to producers suggests that even with only two foreign firms, monopolization of the market when quota licenses are auctioned off is not guaranteed. It is ensured only if ex-post competition leads to convexity of $f_n(\cdot)$.

Another theme that runs through much of economics is that increasing returns to scale causes monopolization. Convexity of the $f_n(x)$ function in symmetric environments is, in a way, the analogue of this. However, this has little to do with increasing returns in production. The cost effects of obtaining additional units have to feed through the operation of the secondary market. Increasing returns in production do not generate convexity of $f_n(x)$!

Example 8

Consider the symmetric Cournot duopoly model with increasing returns to scale. There are n units of a scarce resource being auctioned. Let the production function with k units of the scarce resource be $\theta(k)L$, with $\theta'(\cdot) > 0$. The amount of labor, L, needed to make one unit of output is $\frac{1}{\theta(k)}$. The constant marginal cost of production is $\frac{w}{\theta(k)}$. Let $w = 1$ for convenience. Marginal cost $c(k)$ are thus $\frac{1}{\theta(k)}$.

The Cournot equilibrium profits for every allocation of k, n-k are given by:

$$f_n(k) = \frac{1}{b}\left[\frac{a - 2c(k) + c(n-k)}{3}\right]^2$$

$$f_n(n-k) = \frac{1}{b}\left[\frac{a - 2c(n-k) + c(k)}{3}\right]^2$$

when inverse demand for the homogenous good being produced is:

$$P = a - bQ .$$

It is easy to verify that $c'(\cdot) < 0$ is not sufficient to ensure convexity of $f_n(\cdot)$. If $c(\cdot)$ is linear, i.e. $c(\cdot) = \alpha - \beta k$, or quadratic in k, i.e., $C(k) = \alpha - \beta k - \gamma k^2$, it is ensured.

Another application, as mentioned earlier, is a possible solution to the "transponder puzzle". With endogenous valuations, declining prices are no puzzle! The model can also help understand how competition ex-post (given the allocation of capacity as the scarce resource) affects competition to acquire the resource ex-ante and ex-ante profits. This model shows that with convexity of $f_n(\cdot)$, ex-ante profits are competed away. An open question is how the number of firms, competition among them, and demand conditions affect ex-ante profits and the allocation of scarce inputs in question. This model suggests that the results are likely to be less clear-cut than suggested by the analysis of the one-stage model commonly used.

5. CONCLUSION

The notion that valuations may be endogenous is a very basic one. It helps understand, among other things, the evolution of market structure. The models presented here shed light on a number of issues in applied micro-economics. This notion might also help understand other issues, such as the value of debt in the secondary market. Ozler and Huizinga (1990) show that the value of debt in the secondary market for debt depends on its distribution. This is precisely what one would expect if valuations are endogenous! It might also help in better understanding the relationship between market power and R&D, another central issue in the literature, and help develop models of takeover activity and mergers. Caves (1990), for example, argues that greater international economic integration could trigger a wave of corporate mergers, which again evokes a model of endogenous valuations. The appropriate model for analyzing each of these problems will, of course, be different so that a lot of room for work, both on the theoretical and applied side, remains to be done.

APPENDIX

Lemma 1

For all k,

$$f_{n-k}(x) = \begin{cases} f_n(x+k) - k\Delta f_n(n-(x+k)) & \text{if } x > \frac{n-k}{2}, \\ f_n(x) & \text{if } x \le \frac{n-k}{2}. \end{cases}$$

Proof:

We have already shown that this holds for $k = 1$. The proof proceeds by induction. Assume it is true for k, i.e., when k units remain to be sold. Then we show that it is true for $k+1$, i.e., when $k+1$ units remain to be sold. By definition:

$$f_{n-(k+1)}(x) = \max \begin{cases} f_{n-k}(x+1) - \Delta f_{n-k}(n-k-1-x) \\ f_{n-k}(x) . \end{cases} \quad (1)$$

We will show that the induction step holds for all values of x, i.e.,

$$\begin{aligned} f_{n-(k+1)}(x) &= f_n(x+k+1) - (k+1)\Delta f_n(n-x-k-1) && \text{if } x > \frac{n-k-1}{2}, \\ &= f_n(x) && \text{if } x \le \frac{n-k-1}{2}. \end{aligned}$$

We do this by considering all the five possible cases.

Case A. $x > \frac{n-k}{2}$: Here, x and $x+1$ strictly exceed $\frac{n-k}{2}$ so

$$f_{n-k}(x) = f_n(x+k) - k\,\Delta f_n(n-x-k) \quad \text{and}$$

$$f_{n-k}(x+1) = f_n(x+1+k) - k\Delta f_n(n-x-1-k) .$$

Also, as $n-k-x$ and $n-k-x-1$ are strictly less than $\frac{n-k}{2}$,

$$f_{n-k}(n-k-x) = f_n(n-k-x) \quad \text{and}$$
$$f_{n-k}(n-k-x-1) = f_n(n-k-x-1).$$

Substituting these into (1) gives:

$$f_{n-(k+1)}(x) = \max \begin{cases} f_n(x+1+k) - (k+1)\Delta f_n(n-k-1-x) \\ f_n(x+k) - k\Delta f_n(n-k-x) . \end{cases}$$

The first term exceeds the second as their difference equals:

$$\Delta f_n(x+k) - (k+1)\,\Delta f_n(n-k-1-x) + k\Delta f_n(n-k-x)$$
$$= [\Delta f_n(x+k) - \Delta f_n(n-(k+x))] + (k+1)[\Delta f_n(n-k-x) - \Delta f_n(n-k-x-1)] > 0 .$$

Since $f_n(\cdot)$ is convex and $x > \frac{n-k}{2}$, $x+k > \frac{n+k}{2} > \frac{n}{2}$ and $n-(x+k) < \frac{n}{2}$, the first term in brackets is positive. The second is also positive by convexity.

Hence:

$$f_{n-(k+1)}(x) = f_n(x+1+k) - (k+1)\Delta f_n(n-k-1-n)$$

so the induction step holds when $x > \frac{n-k}{2}$.

Case B. $x < \frac{n-k}{2} - 1$: In this case, as $x+1 < \frac{n-k}{2}$, and $x < \frac{n-k}{2}$, so that

$$f_{n-k}(x+1) = f_n(x+1), \quad \text{and}$$
$$f_{n-k}(x) = f_n(x) .$$

Also, as $x+1 < \frac{n-k}{2}$, $n-k-x-1 > \frac{n-k}{2}$ and $n-k-x > \frac{n-k}{2}$ so:

$$f_{n-k}(n-k-1-x) = f_n(n-x-1) - k\Delta f_n(x+1), \quad \text{and}$$
$$f_{n-k}(n-k-x) = f_n(n-x) - k\Delta f_n(x) \ .$$

Hence:

$$f_{n-(k+1)}(x) = \max \begin{cases} f_n(x+1) - \Delta f_n(n-x-1) + k[\Delta f_n(x) - \Delta f_n(x+1)] \ , \\ f_n(x) \ . \end{cases}$$

Their difference equals:

$$[\Delta f_n(x) - \Delta f_n(n-x-1)] - k[\Delta f_n(x+1) - \Delta f_n(x)] < 0,$$

Since $x < n-x-1$, convexity implies the first term in brackets is negative. The second term in brackets is positive by convexity. Therefore, the induction step holds and :

$$f_{n-(x+1)}(x) = f_n(x) \text{ if } x < \frac{n-k}{2} - 1 \ .$$

<u>Case C</u>. $\frac{n-k}{2} - 1 < x < \frac{n-k}{2}$: Since x is an integer, this is either an empty set (when $n-k$ is even) or equals $\frac{n-k-1}{2}$ (when $n-k$ is odd). In this event:

$$f_{n-k}(x) = f_{n-k}(n-k-1-x) = f_{n-k}\left(\frac{n-k-1}{2}\right) = f_n\left(\frac{n-k-1}{2}\right)$$

$$f_{n-k}(x+1) = f_{n-k}(n-k-x) = f_{n-k}\left(\frac{n-k+1}{2}\right) = f_n\left(\frac{n+k+1}{2}\right) - k\Delta f_n\left(\frac{n-k-1}{2}\right)$$

so

$$f_{n-(k+1)}(x) = \max \begin{cases} f_{n-k}(n-k-x-1) \\ f_{n-k}(x) \end{cases}$$

$$= f_n(x) \qquad \text{when} \qquad x = \frac{n-k-1}{2}$$

so the induction step holds.

Case D. $x = \frac{n-k}{2}$. Here,

$$f_{n-k}(x) = f_{n-k}(n-k-x) = f_n(x)$$

$$f_{n-k}(x+1) = f_n(x+1+k) - k\Delta f_n(n-k-x-1)$$

$$f_{n-k}(n-k-x-1) = f_n(n-k-1-x)$$

Thus:

$$f_{n-k-1}(x) = \max \begin{cases} f_n(x+1+k) - k\Delta f_n(x-1) - \Delta f_n(x-1) \ , \\ f_n(x) \ . \end{cases}$$

Their difference equals:

$$[f_n(x+1+k) - f_n(x)] - (k+1)\ \Delta f_n(x-1) > 0$$

as the average slope between x and $(x+1+k)$ exceeds that between $x-1$ and x. As $n-x-k-1 = x - 1$ for $x = \frac{n-k}{2}$, the induction step holds and:

$$f_{n-k-1}(x) = f_n(x+1+k) - (k+1)\ \Delta f_n(n-x-k-1) \text{ if } x = \frac{n-k}{2} \ .$$

$\underline{\text{Case E}}$. $x = \frac{n-k}{2} - 1$. In this event,

$$f_{n-k}(x) = f_n(x) ,$$

$$f_{n-k}(x+1) = f_n(x+1) ,$$

$$f_{n-k}(n-k-x) = f_{n-k}\left(\frac{n-k}{2} + 1\right)$$

$$= f_n(n-k-x+k) - k\Delta f_n(n-k-n+k+x)$$

$$= f_n(n-x) - k\Delta f_n(x) ,$$

$$f_{n-k}(n-k-x-1) = f_{n-k}\left(\frac{n-k}{2}\right)$$

$$= f_n(n-k-x-1)$$

$$= f_n(x+1)$$

$$f_{n-k-1}(x) = \max \begin{cases} f_n(x+1) - [f_n(n-x) - k\Delta f_n(x)] + f_n(n-x-k-1) , \\ f_n(x) . \end{cases}$$

Their difference equals:

$$(k+1)\Delta f_n(x) - [f_n(n-x) - f_n(n-x-(k+1))]$$

$$= (k+1)\left\{\Delta f_n(x) - \frac{1}{(k+1)} [f_n(n-x) - f_n(n-x-(k+1))]\right\} < 0$$

as $(n-x-(k+1)) = x+1$, and $n-x = (x+1+k+1)$ the average slope between $x+1$ and $(x+1+k+1)$ exceeds that between x and $x+1$.

So, $f_{n-k-1}(x) = f_n(x)$ when $x = \frac{n-k}{2} - 1$.

This completes the proof.

FOOTNOTES

* I am grateful to Vijay Krishna for valuable comments, to Richard Gilbert for references, to Phillip Swagel for research assistance, to the National Science Foundation for research support under Grant No. SE8822204, and to the Institute for International Economics in Stockholm where work on this paper began.

1. See Milgrom (1985) for an excellent survey.

2. At best, agents are assumed to have a downward sloping demand curve. However, little attention is focused on the secondary market, and how this curve is generated.

3. The use of the common value model is motivated by such issues. However, it deals with them by making assumptions on the distribution of valuations, rather than by explicitly dealing with the operation of secondary markets.

4. In previous work in this area (Krishna, 1988, 1989, 1989a), I assume that there is a competitive market for licenses and that the allocation of licenses is pre-determined thereby circumventing the issue of endogenous valuations and strategic behavior in the market for licenses.

5. Other explanations for this phenomenon include that of Swinkels (1990) whose explanation involves uncertainty about the total number of units available. A high early price reflects a premium paid for getting the object for sure. Another explanation comes from assuming that bidders are budget constrained as in Pitchik and Schotter (1989) which considers a model of complete information and Pitchik (1989) which deals with incomplete information. The reason for their results is that by raising the price for early units, the budget constraint is made more binding for later units, allowing them to be obtained at a lower price.

6. Feenstra, et. al (1990) discuss the lessons of this literature for auctions of quota licenses. Their paper focuses on exogenous valuation as does the literature on auctions.

7. If the units are interpreted as capacity, then $f_n(\cdot)$ is convex as long as monopoly profits exceed twice duopoly profits.

8. As valuations for the three agents are not dependent on the allocation, this is an example of non-linear valuations. With two agents, there is no need to make the distinction between endogenous valuations and non-linear ones.

9. Note that this means a sequential auction maximizes the revenue of the seller and so is an optimal scheme.

REFERENCES

Ashenfelter, O. (1989). "How Auctions Work for Wine and Art." Journal of Economic Perspectives (Summer), Vol. 3, 3.

Bernheim, B.D., and Whinston, M. (1986). "Menu Auctions, Resource Allocation and Economic Influence." Quarterly Journal of Economics, (February), Vol. CI, pp. 1-31.

Caves, R.E. (1990). "Corporate Mergers in International Economic Integration,". Paper presented at CEPR/Institute Mobiliare Italiano Conference on European Financial Integration, Rome (January), mimeo.

Engelbrecht-Wiggans, R. (1987). "Optimal Constrained Bidding." International Journal of Game Theory, Vol. 16, 2, pp. 115-121.

----------, and Weber R.J. (1979). "An Example of a Multi-object Auction Game." Management Science, (December), Vol. 25, 12, pp. 1272-1279.

Ghemawat, P. (1989). "The Snowball Effect." Forthcoming Journal of Industrial Organization.

Harris, M., and Raviv, A. (1981). "A Theory of Monopoly Pricing Schemes with Uncertain Demand." American Economic Review, 71, pp. 347-365.

Maskin, E., and Riley, J., (1981). "Multi-unit Auctions, Price Discrimination and Bundling." UCLA, Dept. of Economics Discussion Paper No. 201.

----------, and Riley, J. (1987). "Optimal Multi-unit Auctions." Forthcoming in F. Hahn (ed.), The Economics of Missing Markets, Games and Information. Oxford: Oxford University Press.

Ozler, S. and Huizinga, H. (1990). "Secondary Market Prices for Developing Country Debt: The Role of Creditor Country Factors." Stanford University, mimeo.

Palfrey, T.R. (1980). "Multiple-object, Discriminatory Auctions with Budget Constraints: A Game Theoretic Analysis." Management Science, Vol. 26, No. 9, pp. 935-96.

Pitchik, C. (1989). "Budget Constrained Sequential Auctions with Incomplete Information." University of Toronto, mimeo.

Pitchik, C., and Schotter, A. (1989). "Perfect Equilibria in Budget-Constrained Sequential Auctions: An Experimental Study." Rand Journal of Economics, (Winter).

Swinkels, J. (1990). "Multi-Unit, Multi-Period Auctions, Supply Uncertainty, and the Efficiency of Symmetric Equilibria." Princeton University, mimeo.

Wilson, R. (1979). "Auctions of Shares." Quarterly Journal of Economics, XCIII, pp. 675-89.

Figure 1
n = 2

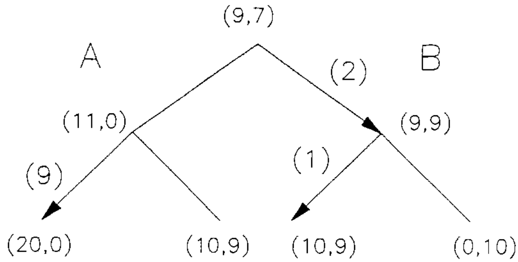
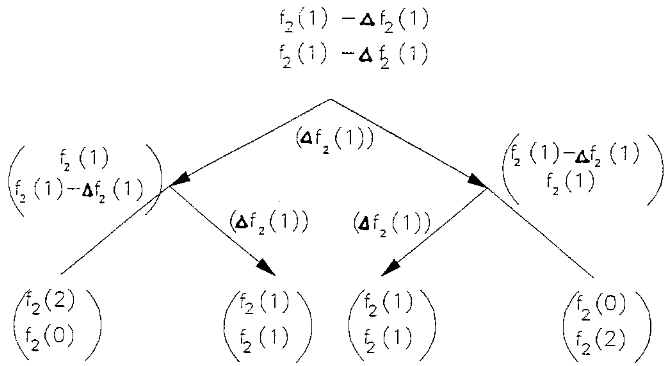
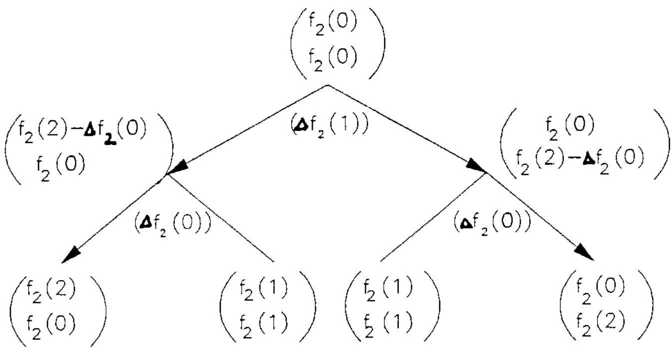


Figure 2a



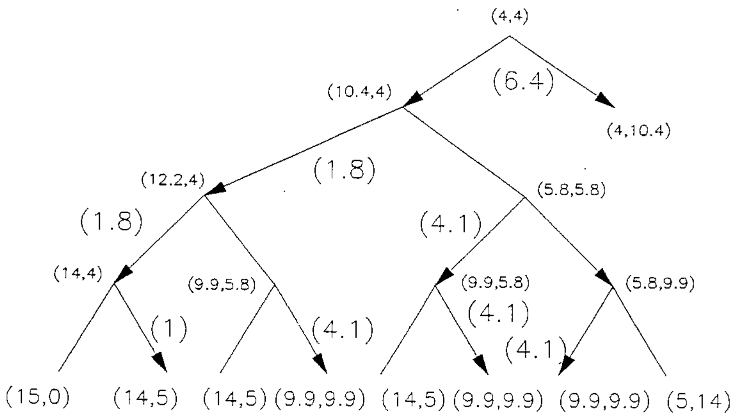
f2(1) −Δf2(1)
f2(1) −Δf2(1)
(Δf2(1))
f2(1)
f2(1)−Δf2(1)
f2(1)−Δf2(1)
f2(1)
(Δf2(1))
(Δf2(1))
f2(2)
f2(0)
f2(1)
f2(1)
f2(1)
f2(1)
f2(0)
f2(2)

Figure 2b



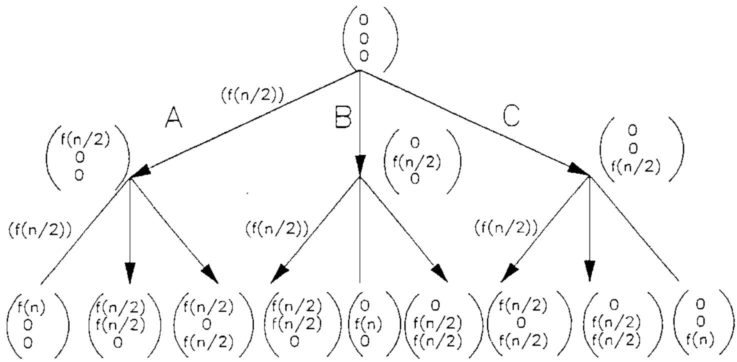
f2(0)
f2(0)
(Δf2(1))
f2(2)−Δf2(0)
f2(0)
f2(0)
f2(2)−Δf2(0)
(Δf2(0))
(Δf2(0))
f2(2)
f2(0)
f2(1)
f2(1)
f2(1)
f2(1)
f2(0)
f2(2)

Figure 3



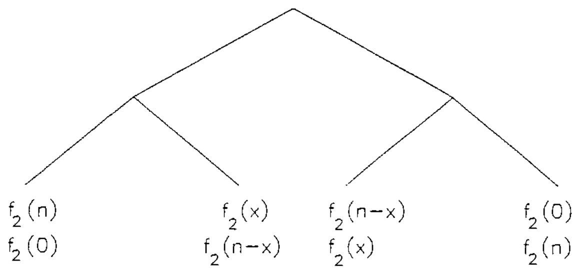
(4,4)
(6.4)
(10.4,4)
(4,10.4)
(12.2,4)
(1.8)
(5.8,5.8)
(1.8)
(4.1)
(14,4)
(9.9,5.8)
(9.9,5.8)
(5.8,9.9)
(1)
(4.1)
(4.1)
(4.1)
(15,0)
(14,5)
(14,5)
(9.9,9.9)
(14,5)
(9.9,9.9)
(9.9,9.9)
(5,14)

Figure 4



0 0 0
(f(n/2))
A
B
C
f(n/2) 0 0
0 f(n/2) 0
0 0 f(n/2)
(f(n/2))
(f(n/2))
(f(n/2))
f(n) 0 0
f(n/2) f(n/2) 0
f(n/2) 0 f(n/2)
f(n/2) f(n/2) 0
0 f(n) 0
0 f(n/2) f(n/2)
f(n/2) 0 f(n/2)
0 f(n/2) f(n/2)
0 0 f(n)

Figure 5



$$p_1 + p_2 = f_n(n) - f_n(0)$$

$$p_2 = \min(\alpha, \beta)$$

Figure 6

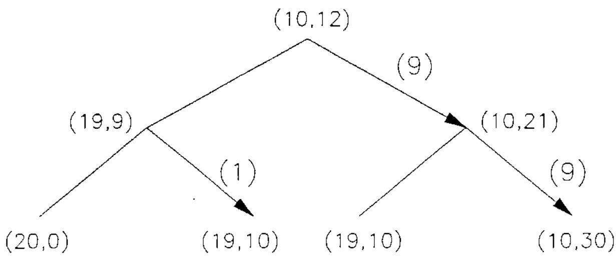


Figure 7