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ABSTRACT

This paper studies the theoretical underpinnings of log-linear models of relative asset demand. It shows that log-linearity provides a poor approximation - in both a global and local sense - to asset demand functions implied by standard portfolio optimization theory. Instead, log-linearity is consistent with a model of portfolio choice in which the investor views the payoffs of different assets, within any state of the world, as being imperfectly substitutable. Equilibria in this model, and all other models that generate log-linear relative asset demand, are generically inconsistent with the existence of a positive stochastic discount factor. I argue that these results imply that the demand system asset pricing approach represents a radically new way to interpret financial market data.

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1 Introduction

Koijen and Yogo (2019) propose modeling the relative demands for financial assets as log-linear functions of their relative prices.[1] This paper examines the theoretical underpinnings for this approach. The first part of the paper shows that standard portfolio optimization theory (SPOT) gives rise to relative asset demands that are poorly approximated - in both a global and local sense - by log-linear functions. The intuition behind this finding is simple. The SPOT framework necessarily implies that asset demand has a *choke* price, at which logged demand and its slope both approach negative infinity. This kind of relationship is poorly approximated by any log-linear function, because it necessarily implies that logged demand and its slope are both finite for any price.

The second part describes an alternative Imperfect Asset Substitution (IAS) model, in which investors view asset payoffs within any given state of the world as imperfectly substitutable. I prove that the IAS model implies that relative asset demand functions are exactly log-linear. I conclude that the log-linear framework of Koijen and Yogo (2019) should be seen as an implementation of the IAS model, not the SPOT model. I show too that in any model which implies that relative asset demands are log-linear, including the IAS model, equilibrium asset prices are generically inconsistent with the existence of a positive stochastic discount factor. In this sense, models like Koijen and Yogo (2019) that are based on log-linear relative asset demands deviate from the large body of asset pricing research which presumes that there are no arbitrage opportunities in financial markets.

The above results differ from the analysis presented in Koijen and Yogo (2019). They consider a SPOT model in which investors have log utility and subjective beliefs. They argue that in this setting, relative asset demands can be well-approximated (at least locally) by log-linear functions. However, they do not prove this assertion formally by examining the properties of the relevant approximation errors.[2] The first part of this paper shows that the

[1] By "relative demand" between two financial assets, I mean to refer to the ratio of the investments in those assets.

[2] More specifically, on page 4, Appendix A, Koijen and Yogo (2019) close their proof of Corollary 1 with

Koijen and Yogo (2019) claim is, in fact, false. Relative asset demands necessarily have a choke price and so cannot be approximated effectively by log-linear functions.[3]

I want to be clear, though, that I do not view the above as a fundamental criticism of the demand system asset pricing (DSAP) framework pioneered by Koijen and Yogo (2019) that has been applied with much success in a variety of contexts. Rather, my purpose is to show that Koijen and Yogo (2019) (and the subsequent literature described in Section 4 of Koijen and Yogo (2025)) have *understated* the novelty of the DSAP approach. My results show that the DSAP model represents an enormous break from the tradition of asset pricing modeling based on the twin assumptions of SPOT and positive SDFs. But (in my view) it seems desirable to contemplate such a change in approach, given the empirical performance of the SPOT/SDF framework over the past few decades.

There is a growing literature concerning identification challenges in estimating the slopes of log-linear relative asset demand functions with multiple risky assets (including Fuchs, et al. (2025), van Binsbergen, et al. (2025), and Haddad, et al. (2025)). The current paper is largely unrelated to this important line of research. The IAS model in the paper builds on the "assets in the utility function" literature, which originates with the work of Sidrauski (1967) and Poterba and Rotemberg (1986). (See An and Huber (2025) for a more recent application.) It is worth noting that (like Koijen and Yogo (2019)) the IAS model in this paper is designed to be applied to individual-level relative asset demands. In this sense, it is distinct from the large literature that studies the properties of *aggregate* asset demand functions induced by the combination of agent heterogeneity and asset market frictions.

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a Taylor series approximation. But they do not study the magnitude of the errors of this approximation.

[3]By way of contrast, the appendix to Koijen and Yogo (2025) is a mathematically correct justification that a *linear* asset demand function is consistent with SPOT (for CARA utility functions). However, most of the literature that builds on Koijen and Yogo (2019) seeks to estimate demand elasticities. This focus is not consistent with linear demand functions, as they imply that demand elasticities are non-constant as a function of price.

2 Standard Portfolio Optimization Theory

This section demonstrates that a log-linear relative asset demand function is inconsistent, even in an approximate sense, with standard portfolio optimization theory (SPOT).

2.1 Standard Portfolio Problem

Suppose an individual is choosing how much to invest in a risky stock and riskfree bond. The riskfree bond has a unit payoff. The investor believes that the payoff of the stock has a cdf F over the open interval (X_{min}, X_{max}), where X_{min} is non-negative and X_{max} may be infinity. The investor maximizes expected utility, with concave and increasing cardinal utility function u. They take the price P of the stock in terms of riskfree bonds as exogenous. The investor has wealth W (denominated in terms of riskfree bonds).

Given the above elements, the portfolio problem of the investor takes the form:

$$\max_{\omega \in \Re} \int_{X_{min}}^{X_{max}} u(\frac{\omega W X}{P} + (1-\omega)W)dF(X) \tag{1}$$

where $\omega^*(P;W)$ is the fraction of wealth that the investor allocates to stocks given their price P. I restrict attention to stock prices that:

- are positive
- lead the investor to choose positive stockholdings and positive bondholdings

Formally, I define $\Pi \equiv \{P | P > 0, 0 < \omega^*(P,W) < 1\}$ and define $\hat{\Pi} = \ln(\Pi)$ to be the domain of logged prices.

I then define the function $g : \hat{\Pi} \to \mathbb{R}$ as the logged relative asset demands of the investor:

$$g(p;W) = \ln(\omega^*(\exp(p);W)) - \ln(1 - \omega^*(\exp(p);W))$$

The investor's local elasticity of relative demand at price $\exp(\bar{p})$ is given by the first derivative

$g'(\bar{p}; W)$. The investor's relative demand function is globally log-linear if the local elasticity $g'(\bar{p}; W)$ is independent of $\bar{p}$.

Fuchs, et al. (2025) suggest a class of objective functions that allow investors to assign distinct weights to the payoffs from different assets (like stocks and bonds):

$$\max_{\omega \in \Re} \int_{X_{min}}^{X_{max}} u(A_{stock} \frac{\omega W X}{P} + A_{bond}(1-\omega)W) dF(X). \tag{2}$$

$$A_{stock}, A_{bond} > 0 \tag{3}$$

All of the results in this section about the non-loglinearity of relative asset demand apply to this broader class of portfolio problems.

2.2 Numerical Example

In this subsection, I use a two-state example to illustrate two points:

- the investor's logged relative demand function g is highly nonlinear, as a function of logged prices p.
- the error associated with a first-order approximation to the investor's logged relative demand function is infinite

2.2.1 Two-State Example

Suppose that the cdf F is such that the investor believes X is equally likely to be either a or b, where $a > b > 0$. Suppose too that the investor's utility function $u(C) = \ln(C)$.

Then, the investor's optimal stock share satisfies the first order condition:

$$\frac{a}{a\omega/P + (1-\omega)} + \frac{b}{b\omega/P + (1-\omega)} = \frac{P}{a\omega/P + (1-\omega)} + \frac{P}{b\omega/P + (1-\omega)}$$

The solution to this first order condition takes the form:

$$\omega^*(P) = \frac{a+b-2P}{2(a+b)-2P-2ab/P}$$

(It is independent of wealth W because of the log utility assumption.) As well:

$$(1-\omega^*(P)) = \frac{a+b-2ab/P}{2(a+b)-2P-2ab/P}.$$

It follows that the set of prices that induce positive asset-holdings takes the form:

$$\hat{\Pi} = (\hat{\pi}_{min}, \hat{\pi}_{max})$$

where

$$\hat{\pi}_{max} = \ln(\frac{a+b}{2})$$
$$\hat{\pi}_{min} = \ln(\frac{2ab}{a+b}).$$

As well, logged relative asset demands take the form:

$$g(p) = \ln(a+b-2\exp(p)) - \ln(a+b-2ab\exp(-p)), \tag{4}$$

where p is in $\hat{\Pi}$.

2.2.2 Relative Demand Is Highly Non-Log-linear

This subsection shows that the relative demand function g defined in (4) is highly non-log-linear. It demonstrates this point in three ways:

- the graph of g is highly nonlinear as a function of logged prices
- the derivative of the function g can assume any value in the interval $(-\infty, -\frac{4b^{1/2}/a^{1/2}}{(1-b^{1/2}/a^{1/2})^2})$.

- the sup-norm distance between g and any linear function of p is infinite.

Graph of g

I set $a = 1.2$ and $b = 0.9$. Figure 1 shows that the resulting graph of g is highly nonlinear, as it is becoming increasingly negatively sloped for high values of p in $\hat{\Pi}$. This will turn out to be a general feature for all values of (a, b).

The Derivative of g is Non-Constant

The derivative of the function g is:

$$g'(p) = -\frac{2\exp(p)}{a+b-2\exp(p)} - \frac{2ab\exp(-p)}{a+b-2ab\exp(-p)}.$$

It is straightforward to show that as suggested by Figure 1:

$$\lim_{p \to \hat{\pi}_{max}} g'(p) = -\infty.$$

The maximal value of $g'(p)$ is achieved at $p^* = 0.5(\ln(a) + ln(b))$, and is given by:

$$g'(p^*) = \frac{-4(b^{1/2}/a^{1/2})}{(1 - b^{1/2}/a^{1/2})^2}.$$

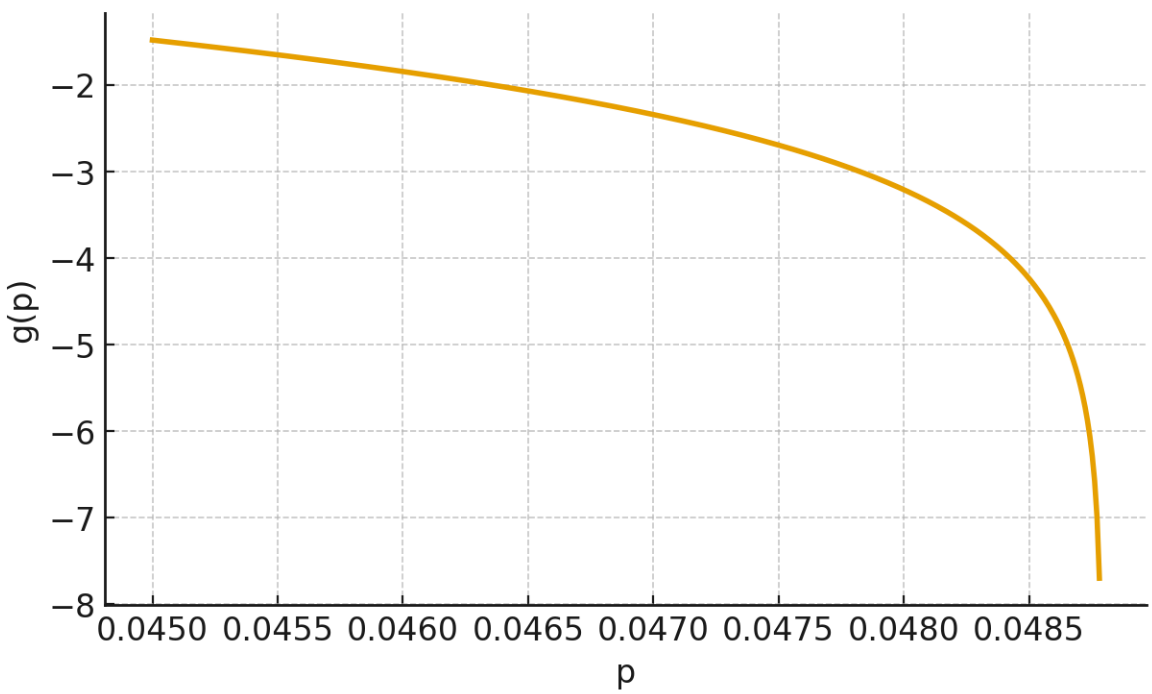
This is finite, since $a > b > 0$. Indeed, it is near zero if (b/a) is near zero. Hence, $g'(p^*)$ takes on a wide range of negative values.

Sup-Norm

To calculate the sup norm distance between g and any linear function $(\alpha + \beta p)$, we consider a sequence $(p_n)_{n=1}^{\infty}$ in $\hat{\Pi}$ such that:

$$\lim_{n \to \infty} p_n = \hat{\pi}_{max} = \ln(\frac{a+b}{2}).$$

Figure 1: Graph of Relative Asset Demands



I used Chat GPT 5.0 to draw this figure. I asked Chat GPT 5.0 to draw a graph of g as defined in the text, with a = 1.2 and b = 0.9.

Then:

$$\begin{aligned} &\lim_{n\to\infty} g(p_n) - \alpha - \beta p_n \\ = -\alpha + \beta \ln(\frac{a+b}{2}) + &\lim_{n\to\infty} g(p_n) \\ &= -\infty \end{aligned}$$

It follows that:

$$\sup_{p\in\hat{\Pi}} |g(p) - \alpha - \beta p| = \infty.$$

The (sup-norm) distance between g and any linear function is infinite.

2.2.3 Unbounded First-Order Approximation Error

The above subsection investigated the global fit of a log-linear approximation to asset demand g. Of course, we typically think of linear or log-linear approximations as intended to capture local properties. In this subsection, we shall see that a log-linear approximation also fares poorly from this more limited perspective.

Pick $h \in (0, \hat{\pi}_{max} - \hat{\pi}_{min})$. Let $p^* \in (\hat{\pi}_{min} + h, \hat{\pi}_{max})$. Consider the following first-order approximation to $g^{ex}(p^* - h)$:

$$(g(p^*) - g'(p^*)h)$$

The error associated with this approximation is given by:

$$e(p^*, h) = |g(p^* - h) - g(p^*) + g'(p^*)h|$$

The supremal error (in absolute value) for a given value of h is bounded from below by the limit of the error as p^* converges to $\hat{\pi}^{max}$:

$$\sup_{\hat{\pi}_{max} > p^* > \hat{\pi}_{min} + h} e(p^*, h) \geq \lim_{p^* \to \hat{\pi}^{max}} e(p^*, h).$$

Using algebra that is available upon request, the limit can be shown to be infinity in the two-state example and so the absolute error associated with the first order approximation is unbounded from above.

Figure 2 illustrates this result, for $a = 1.2$ and $b = 0.9$. It depicts the line drawn tangent to g at p^*, for three values of p^*. The tangent line - which is the first order approximation to g - increasingly diverges from g itself as p^* converges to its maximal value $(\ln(a/2 + b/2) = \ln(1.05))$.

2.2.4 Aside On (Locally) Inelastic Demand

It is often argued that standard portfolio optimization theory generates overly elastic asset demand (relative to what's in the data). For that reason, it's worth noting that the derivative of g lies in the interval:

$$(-\infty, -\frac{4(b^{1/2}/a^{1/2})}{(1 - b^{1/2}/a^{1/2})^2}).$$

Hence, the derivative of g can be near zero in absolute value (if b/a is small). There is nothing intrinsic in standard portfolio optimization theory that forces elasticities to be large in absolute value.[4]

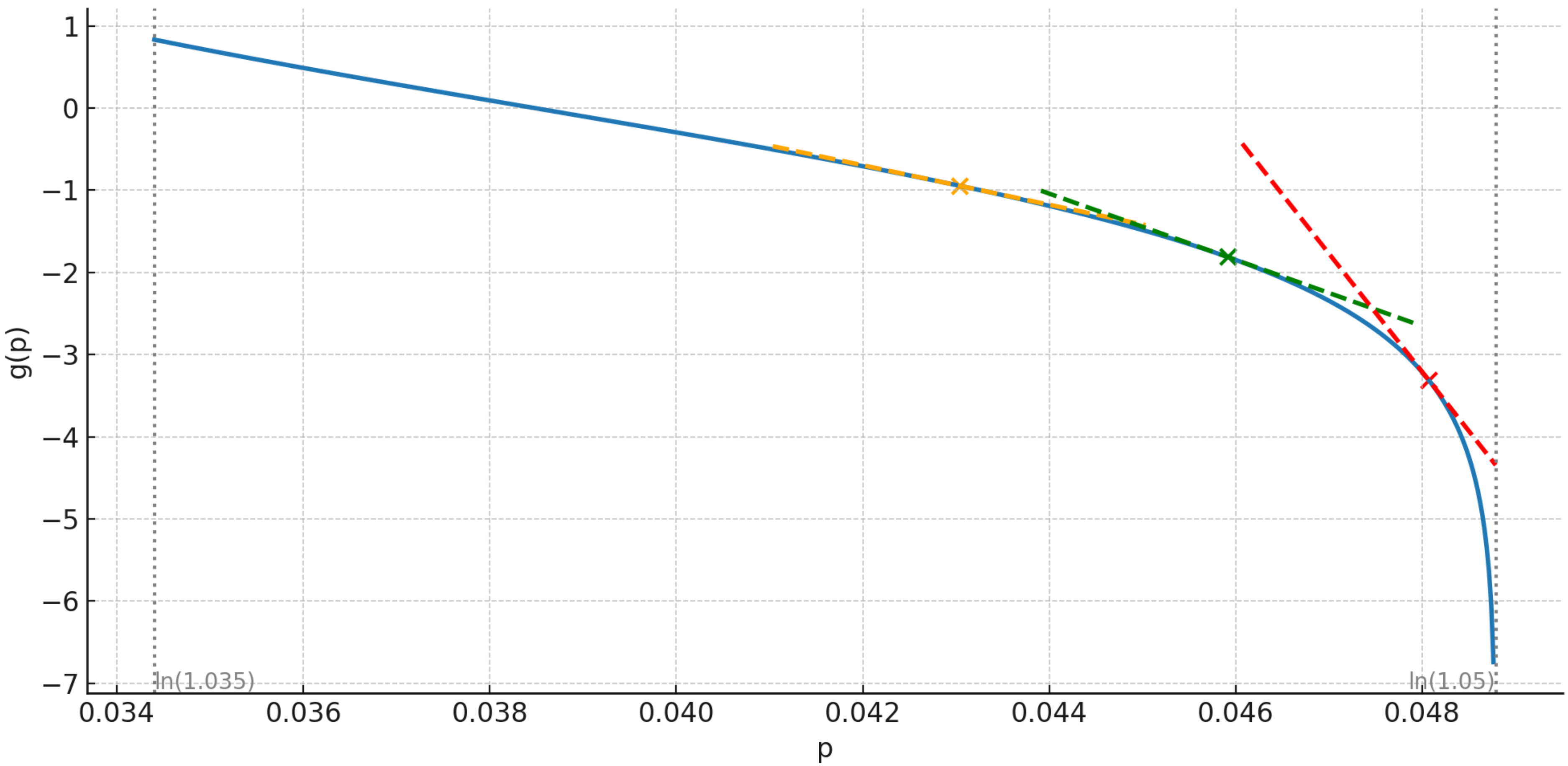
2.3 General Inconsistency Theorems

In this subsection, I prove two general results (for arbitrary F and u) which build on the above example. The first result is that the (sup-norm) distance between the logged relative

[4]The own elasticity of demand for *shares* of the risky asset with respect to the risky asset price P is equal to:

$$g'(p)(1 - \omega) - 1.$$

Figure 2: Unbounded First-Order Approximation Error



This figure was constructed using Chat-GPT 5.0. I asked Chat-GPT 5.0 to draw a graph of g as defined in the text for a = 1.2 and b = 0.9, with three tangent lines.

demand function g and any linear function of logged price p is infinite. This shows that the relative demand function cannot be seen as being close to globally loglinear. The second result is that the error associated with a first-order approximation to g becomes unbounded as P nears $E_F(X)$ (the expectation of X with respect to the cdf F).

The key to both of these results is that stock demand is zero at a finite price.

Proposition 1. *Suppose that $P^* = E_F(X) = \int_0^\infty X dF(X)$ is finite. Then:*

$$\omega^*(P^*; W) = 0$$

and:

$$\omega^*(P; W) > 0$$

for P sufficiently near to, but less than, P^*.

Proof. Define $\omega^*_\mu \equiv \omega^*(P^*; W)$. It satisfies:

$$\int_{X_{min}}^{X_{max}} u'(\omega^*_\mu \frac{WX}{E(X)} + (1 - \omega^*_\mu)W)(\frac{X}{E(X)} - 1)dF(X) = 0.$$

It is straightforward to verify that $\omega^*_\mu = 0$ is the unique solution to this first order condition.

Now consider the general first order condition:

$$\int_{X_{min}}^{X_{max}} u'(\omega \frac{WX}{P} + (1 - \omega)W)(\frac{X}{P} - 1)dF(X) = 0$$

We use comparative statics at $\omega_\mu = 0$ and $P = P^*$ by totally differentiating the above first order condition:

$$0 = ((\int_{X_{min}}^{X_{max}} u''(\omega\frac{WX}{P} + (1-\omega)W)(\frac{X}{P} - 1)^2 dF(X))d\omega$$
$$+((\int_{X_{min}}^{X_{max}} u''(\omega\frac{WX}{P} + (1-\omega)W)(\frac{X}{P} - 1)(-\frac{\omega WX}{P^2})dF(X))dP$$
$$+((\int_{X_{min}}^{X_{max}} u'(\omega\frac{WX}{P} + (1-\omega)W)(-\frac{X}{P^2})dF(X))dP$$

It follows that since $\omega^*(P^*; W) = 0$:

$$\frac{d\omega}{dP}|_{P=P^*} = \frac{u'(W)E(X)}{u''(W)Var(X)} < 0.$$

Hence, $\omega^*(P; W) > 0$ for P slightly below $E_F(X)$. □

In standard portfolio optimization problems, agents do not view any particular asset as being essential. Rather, assets are merely substitutable ways to generate payoffs in different states. As a consequence, there is a finite choke price at which an investor's holdings of a given asset falls to zero. As we shall see, this property creates a significant problem for log-linear approximations.

2.3.1 Logged Relative Demand is Highly Nonlinear

We prove first that the sup-norm distance between the logged relative demand function g and any linear function of logged prices is infinite. The idea of the proof is simple: the logged relative demand function becomes an arbitrarily large negative number as the price of the asset converges from below to the mean of X.

Proposition 2. *Suppose that $E_F(X)$ is finite. For any (ψ_0, ψ_1) :*

$$\sup_{p \in \hat{\Pi}} |g(p) - \psi_0 - \psi_1 p| = \infty.$$

Proof. Proposition 1 implies that there is an open interval:

$$(p_-, \ln(E_F(X))$$

such that:

$$(p_-, \ln(E_F(X)) \subseteq \hat{\Pi}.$$

Take a sequence $(p_n)_{n=1}^{\infty}$ such that $p_n \in (p_-, \ln(E_F(X))$ for all n, and such that;

$$\lim_{n\to\infty} p_n = \ln(E_F(X)).$$

Together with Proposition 1, the first order condition:

$$\int_{X_{min}}^{X_{max}} u'(\omega \exp(-p_n)WX + (1-\omega)W)(X\exp(-p_n) - 1)dF(X) = 0$$

implies that:

$$\lim_{n\to\infty} \omega^*(\exp(p_n); W) = 0.$$

Hence for any (ψ_0, ψ_1) :

$$\begin{aligned} &\lim_{n\to\infty}(g(p_n) - \psi_0 - \psi_1 p_n) \\ = -\psi_0 - \psi_1 \ln(E_F(X)) + \lim_{n\to\infty} \ln(\omega^*(\exp(p_n); W)) - &\lim_{n\to\infty} \ln(1 - \omega^*(\exp(p_n)); W) \\ &= -\infty. \end{aligned}$$

It follows that:

$$\sup_{p\in\hat{\Pi}} |g(p) - \psi_0 - \psi_1 p| = \infty.$$

□

2.3.2 First-Order Approximations Have Unbounded Error

Proposition 2 showed that any *fixed* linear approximation of g has infinite error over the set of all possible prices. But what if we consider a *local* linear approximation? The following proposition shows that the error of such a first-order approximation grows without bound as the price p nears $\ln(E_F(X))$.

Proposition 3. *Suppose* $E_F(X) < \infty$ *and define* $\hat{\pi}_{max} = \ln(E_F(X))$. *Consider an interval* $(\hat{\pi}_{min}, \hat{\pi}_{max}) \subseteq \hat{\Pi}$, *and let* $h \in (0, \hat{\pi}_{max} - \hat{\pi}_{min})$. *For any* $p \in (\hat{\pi}_{min} + h, \hat{\pi}_{max})$, *define the first-order approximation error to the relative asset demand function* g *as:*

$$e(p, h) = |g(p - h) - g(p) + g'(p)h|.$$

Then for any $h \in (0, \hat{\pi}_{max} - \hat{\pi}_{min})$:

$$\sup_{p^* \in (\hat{\pi}_{min}+h, \hat{\pi}_{max})} e(p, h) = \infty.$$

Proof. From the proof of Proposition 2, we know that there is an interval $(\hat{\pi}_{min}, \hat{\pi}_{max})$ that is a subset of $\hat{\Pi}$. We prove the proposition by showing that:

$$\lim_{p \to \hat{\pi}^{max}} e(p, h) = \infty.$$

for any $h \in (0, \hat{\pi}_{max} - \hat{\pi}_{min})$. Note first that:

$$\begin{aligned} e(p, h) &= |g(p - h) - g(p) + g'(p)h| \\ &= |\ln(\frac{\omega^*(\exp(p - h))}{1 - \omega^*(\exp(p - h))}) - \ln(\frac{\omega^*(\exp(p))}{1 - \omega^*(\exp(p))}) + \frac{1 - \omega^*(\exp(p))}{\omega^*(\exp(p))}\omega^{*\prime}(\exp(p))\exp(p)h| \end{aligned}$$

As in the proof of Proposition 1, we can show:

$$\lim_{p \to \hat{\pi}_{max}} \omega^{*\prime}(\exp(p)) = -\frac{u'(W)E(X)}{Var(X)u''(W)} < 0$$

Hence:

$$\begin{aligned}\lim_{p\to\hat{\pi}_{max}} e(p,h) &= \lim_{p\to\hat{\pi}_{max}} |-ln(\omega^*(\exp(p)) - h\frac{E(X)^2}{\omega^*(\exp(p))}\frac{u'(W)}{Var(X)u''(W)}| \\ &= \lim_{z\to 0} |-\ln(z) - \frac{h}{z}\frac{E(X)^2}{Var(X)}\frac{u'(W)}{u''(W)}| \\ &= \infty\end{aligned}$$

Here, the first equality drops terms in the definition of the error e that remain finite as p converges to $\hat{\pi}_{max}$. □

2.4 Comments

This subsection provides an explanation of what goes wrong with the approximation argument in Koijen and Yogo (2019), and argues that the approximation errors are likely to be significantly larger in investment situations with many risky assets.

2.4.1 Comment 1: What Goes Wrong With the Approximation Argument in Koijen and Yogo

Koijen and Yogo (2019)'s approximation argument proceeds in two steps. The first step (equation 7, p. 1481) relates the wealth share of an asset to moments of its logged returns (using the approach of Campbell and Viceira (2002)). The above discussion does not reflect on this (relatively standard) approximation. However, in a second step (p. 4 of Appendix A), Koijen and Yogo (2019) apply a Taylor approximation to re-write the Campbell and Viceira fomula as an expression with *logged* relative wealth shares on the left-hand side. This step is not valid, as the log-linear approximation becomes arbitrarily inaccurate when relative wealth shares are near zero (but still positive).

As pointed out by footnote 3, the above criticism does not apply to the appendix in Koijen and Yogo (2025), which shows that a version of SPOT with CARA utility is consistent with *linear* asset demand functions. But Koijen and Yogo (2019) and the subsequent literature focuses on estimating demand elasticities. This emphasis is inconsistent with linear demand, which (as is well-known) implies that the elasticity of demand ranges over all of the negative reals as a function of price.

2.4.2 Comment 2: Approximation Error with Multiple Risky Assets

The above analysis focuses on the case of a single risky asset. However, the problem becomes (much) worse with a large number of risky assets. In that case, the wealth share invested in each asset is near zero (assuming that the wealth share in the riskfree asset is positive). Hence, the Taylor approximation in Appendix A of Koijen and Yogo (2019) is necessarily highly inaccurate.

3 A Foundation for Log-Linear Relative Demand

In this section, I construct a model in which agents view the payoffs from different assets as being imperfectly substitutable. I show that this imperfect asset substitution (IAS) model implies that relative demand functions are globally log-linear. I demonstrate too that, for a generic specification of asset supplies, the equilibrium asset prices are not consistent with the existence of a positive stochastic discount factor. This last characterization is valid for all models with log-linear relative demands.

3.1 A Two-Asset IAS Model

Suppose that, as in Section 2, the investor is seeking to split their wealth optimally between stocks and bonds. The investor believes that the stock's payoffs have a cdf F over the open interval (X_{min}, X_{max}), where F is such that the mean of the payoff of X is finite. The bond has a unit payoff, and the investor has wealth W (in terms of riskfree bonds). The investor treats the stock price P (in terms of riskfree bonds) as exogenous.

Suppose the investor chooses to allocate a share ω of their wealth into stocks. If the stock's payoff has realization X, then the investor receives a payment $\omega XW/P$ from their stockholdings and a payment $(1-\omega)W$ from their bondholdings. It is typical to assume that the investor views these payments as perfectly substitutable (dollars or consumption). Instead, we will assume that the investor sees these payments as being imperfectly substitutable. Specifically, given $A_{stock}, A_{bond} > 0$ and $\alpha \in (0,1)$, the investor's ex-post utility is:

$$\frac{1}{1-\rho}(\alpha A_{stock}^{1-\rho}(\frac{\omega XW}{P})^{1-\rho} + (1-\alpha)A_{bond}^{1-\rho}(1-\omega))^{1-\rho}W^{1-\rho}), \rho > 0, \rho \neq 1$$
$$\text{or } \alpha \ln(\frac{A_{stock}\omega XW}{P}) + (1-\alpha)\ln(A_{bond}(1-\omega)W)$$

when the realization of the stock's payoff is X. Given their beliefs about X, the investor's ex-ante expected utility takes the form

$$\frac{1}{1-\rho}\int_{X_{min}}^{X_{max}} (\alpha(A_{stock}\frac{\omega XW}{P})^{1-\rho} + (1-\alpha)(A_{bond}(1-\omega)W)^{1-\rho})dF(X), \rho > 0, \rho \neq 1 \quad (5)$$
$$\text{or } \int_{X_{min}}^{X_{max}} \alpha \ln(\frac{A_{stock}\omega XW}{P}) + (1-\alpha)\ln(A_{bond}(1-\omega)W))dF(X) \quad (6)$$

if they choose to invest a share ω of their wealth in stocks.

In this ex-ante objective, the parameter ρ plays two roles. As noted above, it governs the within-state substitutability of asset payoffs. But it also determines cross-state substitutability (that is, risk aversion). The appendix develops a broader class of objective functions which parameterizes these two aspects of preferences separately.

3.2 Globally Log-Linear Relative Demand

Investors seek to maximize the objective (5)-(6) by choosing the stock share $\omega \in \mathbb{R}$. It is straightforward to show that the optimal $\omega^*(P)$ satisfies the first order condition:

$$\int_{X_{min}}^{X_{max}} \alpha(A_{stock}\frac{\omega^*(P)X}{P})^{-\rho}A_{stock}\frac{X}{P}dF(X) = (1-\alpha)(A_{bond}(1-\omega^*(P))^{-\rho}A_{bond}$$

where $\rho = 1$ corresponds to the log utility case. It follows that:

$$\frac{\omega^*(P)}{1-\omega^*(P)} = (\frac{(1-\alpha)}{\alpha} \frac{(\frac{A_{bond}}{A_{stock}})^{1-\rho} P^{1-\rho}}{\int_{X_{min}}^{X_{max}} X^{1-\rho} dF(X)})^{-1/\rho}$$
$$= (\frac{(1-\alpha)}{\alpha})^{-1/\rho} (\frac{A_{stock}}{A_{bond}})^{1/\rho - 1} (\int_{X_{min}}^{X_{max}} X^{1-\rho} dF(X))^{1/\rho} P^{1-1/\rho}$$

and that:

$$\ln(\frac{\omega^*(P)}{1-\omega^*(P)}) = \rho^{-1}(\ln(\alpha) - \ln(1-\alpha)) \quad (7)$$
$$+ (1 - 1/\rho)\ln(P) \quad (8)$$
$$+ (\rho^{-1} - 1)(\ln(A_{stock}) - \ln(A_{bond})) + \rho^{-1}\ln(E(X^{1-\rho})), \rho > 0 \quad (9)$$

(Note that the last term is finite because $E(X^{1-\rho}) < E(X)^{1-\rho}$ through Jensen's inequality.) It follows that relative demand is globally log-linear. The relative demand elasticity is constant at β, where $\beta \equiv (1 - 1/\rho)$ and can take on any value in the set $(-\infty, 1)$.

It may seem puzzling that a demand elasticity can be positive. However, as in Koijen and Yogo (2019), β represents the percentage response of relative expenditures on the two assets in response to a 1% change in their relative price. This can be positive, zero, or negative, depending on the degree of substitutability of the two assets in the agent's preferences. We could instead consider a quantity elasticity $\eta(P)$ which measures the percentage change in the number of shares of the risky asset demanded in response to a percentage change in its price. (This quantity elasticity is a function of the price P even when the relative demand elasticity is not.) It is readily shown that such a quantity elasticity (evaluated at any given P) is related to the constant elasticity β by:

$$\eta(P) = \beta(1 - \omega^*(P)) - 1 < 0$$

if $\beta < 1$ and $\omega^*(P) > 0$.

It will be useful to also allow for the possibility that the relative demand elasticity $\beta = 1$, as Koijen and Yogo (2019) argue that this parameterization can be seen as corresponding to the behavior of an index fund. If $X_{min} > 0$, then the limit of (7) as ρ converges to ∞ is given by:

$$\ln(\frac{\omega^*(P)}{1-\omega^*(P)}) = \ln(P) - \ln(X_{min}) + \ln(\frac{A_{bond}}{A_{stock}})$$

It is easily seen that this demand function describes optimal behavior for investors who have an ex-ante objective function:

$$\min_X \min W\{\frac{\omega X}{P} A_{stock}, (1-\omega)A_{bond}\}$$

that is simultaneously Leontief over asset payoffs and state-contingent utilities.

3.3 Equilibrium

This section defines equilibrium in an IAS model economy with heterogeneous agents and a single risky asset. It proves existence for arbitrary asset supplies.

3.3.1 Heterogeneous Investors

Suppose there are I heterogeneous investors. Agent i has an ex-ante objective function of the form.

$$\frac{1}{1-\rho_i}\int_{X_{min}}^{X_{max}} (\alpha_i(A^i_{stock}\frac{\omega^i}{P}X)^{1-\rho_i} + (1-\alpha_i)(A^i_{bond}(1-\omega^i))^{1-\rho_i})dF_i(X), \rho_i > 0, \rho_i \neq 1$$
$$\text{or } \int_{X_{min}}^{X_{max}} (\alpha_i \ln(A^i_{stock}\frac{\omega^i}{P}X) + (1-\alpha_i)\ln(A^i_{bond}(1-\omega^i))dF_i(X)$$

over risky asset shares $(\omega^i_k)_{k=1}^K$. Each agent is allowed to have distinct beliefs and distinct attitudes toward the various assets. They may also differ in their elasticities of substitution between the two assets. It follow from the analysis in the prior subsection that each agent has log-linear relative demands, so that for $\rho_i > 0$:

$$\ln(\omega^{i*}) - \ln(1-\omega^{i*}) = Z^i + \beta_i \ln(P)$$
$$Z^i = \rho_i^{-1}(\ln(\frac{\alpha}{1-\alpha}) + (1/\rho_i - 1)\ln(\frac{A_{stock}}{A_{bond}}) + \rho_i^{-1}\ln(E_i(X^{1-\rho_i}))$$

where $\beta_i = 1 - 1/\rho_i$, and E_i corresponds to the expectation with respect to investor i's possibly subjective beliefs F_i. As discussed above, if $X_{min} > 0$, we can also rationalize log-linear relative asset demand with $\rho_i = \infty$:

$$\ln(\omega^{i*}) - \ln(1-\omega^{i*}) = \ln(P) - \ln(X_{min}) + \ln(A_{bond}/A_{stock})$$

if the investor's objective function is Leontief.

3.3.2 Asset Supply and Definition of Equilibrium

This subsection defines equilibrium. To do so, we need to have a notion of asset supply. I assume that there is an outside supply of $\bar{S}$ units of the risky asset available for purchase by the investors. Suppose investor i has wealth W^i denominated in terms of bonds. Then, market-clearing for the risky asset means that the total demand for that asset, denominated in terms of bonds, is equal to the value of the outstanding supply:

$$\sum_{h=1}^{I} W^h \frac{exp(Z^h)P^{\beta_h}}{1 + exp(Z^h)P^{\beta_h}} = P\bar{S}$$

or equivalently:

$$\sum_{h=1}^{I} \frac{W^h}{P} \frac{exp(Z^h)P^{\beta_h}}{1 + exp(Z^h)P^{\beta_h}} = \bar{S} \tag{10}$$

Then, we define an equilibrium as a price P^* such that the market for the risky asset clears - that is, (10) is satisfied.

3.3.3 Existence of Equilibrium

The following proposition shows that, as long as some investor's β_i is strictly smaller than 1, there is an equilibrium.

Proposition 4. *Suppose all investors' relative demands are globally log-linear, with elasticity $\beta_h \leq 1$ for all h and at least one elasticity $\beta_i < 1$. Then there exists a unique equilibrium.*

Proof. Define excess demand for the risky asset as:

$$\begin{aligned}\Upsilon(P) &= \sum_{h=1}^{I} \frac{W^h}{P} \frac{exp(Z^h)P^{\beta_h}}{1 + exp(Z^h)P^{\beta_h}} = \bar{S}. \\ &= \sum_{h=1}^{I} \frac{W_h}{exp(-Z^h)P^{1-\beta_h} + P} - \bar{S}\end{aligned}$$

Since $1 - \beta_h \geq 0$ for all h :

$$\lim_{P\to\infty} \Upsilon(P) = -\bar{S} < 0.$$

As well, since $(1 - \beta_i) > 0$ for some i :

$$\lim_{P\to 0} \Upsilon(P) = \infty > 0.$$

It follows from the Intermediate Value Theorem that there is an equilibrium. Since $(1-\beta_h) \geq 0$ for all h, Υ is strictly decreasing and the equilibrium is unique. □

3.4 Non-Existence of an SDF

In this subsection, I show that in a model with log-linear relative demand functions, equilibrium is inconsistent with the existence of a positive stochastic discount factor for a generic specification of asset supplies.

I assume that, while investors can have different subjective beliefs, they are all absolutely continuous with respect to a common measure (meaning that they agree on what events have zero probability). Formally, I assume that there exists a cumulative distribution function G

over (X_{min}, X_{max}) such that for all i :

$$dF_i(X) = f_i(X)dG(X)$$

where f_i is a positive continuous function over (X_{min}, X_{max}). Then, an equilibrium is consistent with the existence of a stochastic discount factor if there exists a positive continuous function q over (X_{min}, X_{max}) such that:

$$P^* = \int_{X_{min}}^{X_{max}} q(X)XdG(X)$$
$$1 = \int_{X_{min}}^{X_{max}} q(X)dG(X)$$

The following proposition shows that if the supply of assets is sufficiently small, then the unique IAS equilibrium is necessarily inconsistent with the existence of a stochastic discount factor.

Proposition 5. *Suppose $X_{max} < \infty$. There exists $\bar{S}_{min}$ such that if $\bar{S} < \bar{S}_{min}$, the unique equilibrium price is inconsistent with the existence of a stochastic discount factor.*

Proof. Define $\bar{S}_{min}$ by:

$$\sum_{h=1}^{I} \frac{W^h}{X_{max}} \frac{exp(Z^h)X_{max}^{\beta_h}}{1 + exp(Z^h)X_{max}^{\beta_h}} = \bar{S}_{min}.$$

Then, the equilibrium price when asset supply equals $\bar{S}_{min}$ is given by X_{max}. Suppose asset supply $\bar{S}$ is lower than $\bar{S}_{min}$. Then, the proof of Proposition 4 implies that the equilibrium price P^* is higher than X_{max}.

Now suppose there is a stochastic discount factor. Then:

$$P^* = \int_{X_{min}}^{X_{max}} q(X)XdG(X)$$
$$1 = \int_{X_{min}}^{X_{max}} q(X)dG(X).$$

Divide the first equation by P^*. We get:

$$
\begin{aligned}
1 &= \int_{X_{min}}^{X_{max}} q(X)\frac{X}{P^*}dG(X) \\
&< \int_{X_{min}}^{X_{max}} q(X)\frac{X}{X_{max}}dG(X) \\
&\leq \int_{X_{min}}^{X_{max}} q(X)dG(X) \\
&= 1
\end{aligned}
$$

which is a contradiction. Here, the inequalities both make use of the fact that q is positive. $\square$

The proof of the proposition is not special to the IAS model, as it only makes use of globally log-linear relative demand. The basic idea is that, when relative demand is globally log-linear, then given any positive price, we can find a specification of asset supply that induces that price as an equilibrium. But that will necessarily include prices that are sufficiently high to imply the existence of an arbitrage opportunity.

4 Multiple Risky Assets

This section extends the IAS model to multiple risky assets and uses it to provide a foundation for characteristics-based demand.

4.1 Setup

There are N states. An investor assigns (possibly subjective) positive probabilities to those states $(\pi_n)_{n=1}^N$. They consider how to split their wealth W among K risky assets and one riskfree asset. Risky asset k has a stochastic payoff $(X_{kn})_{n=1}^N$, while the riskfree asset has a unit payoff. The investor takes the K risky asset prices $(P_1, ..., P_K)$, in terms of the riskfree bond, as given.

As in the two-asset case, within a given state, the investor cares which asset generates the payoff from their portfolio. More specifically, if the individual invests wealth shares $(\omega_1, ..., \omega_K)$ in the K risky assets and wealth share ω_0 in the riskfree asset, their state n utility is given by:

$$\frac{M_n}{1-\rho}(\alpha_0(a_0 W\omega_0)^{1-\rho} + \sum_{k=1}^{K} \alpha_k (a_k \frac{W\omega_k}{P_k} X_{kn})^{1-\rho}), \rho > 0, \rho \neq 1$$
$$\text{or } M_n(\alpha_0 \ln(a_0 W\omega_0) + \sum_{k=1}^{K} \alpha_k \ln(a_k \frac{W\omega_k}{P_k} X_{kn}))$$

Here, $\alpha_k > 0$ and $a_k > 0$ for all k. As well, $\sum_{k=1}^{K} \alpha_k = 1$. Without loss of generality, I assume that:

$$\sum_{n=1}^{N} \pi_n M_n = 1$$
$$M_n > 0 \text{ for all } n$$

4.2 Portfolio Problem

Given this objective, the agent's portfolio problem takes the form:

$$\max_{(\omega_1,...,\omega_K)\in\mathbb{R}^K} (1-\rho)^{-1} \sum_{n=1}^{N} \pi_n M_n (\alpha_0 a_0^{1-\rho} (1 - \sum_{k=1}^{K} \omega_k)^{1-\rho} + \sum_{k=1}^{K} \alpha_k a_k^{1-\rho} (\frac{\omega_k}{P_k} X_{kn})^{1-\rho}) \tag{11}$$

The following proposition shows that, given this portfolio problem, the agent's asset demand functions are globally log-linear.

Proposition 6. *Given a vector of exogenous risky asset prices, the solution to the portfolio problem (11) is a vector $(\omega_1^*, ..., \omega_K^*)$ of asset shares such that:*

$$ln(\omega_k^*) - ln(\omega_0^*) = Z_k + \beta ln(P_k),$$

where:

$$\beta = 1 - 1/\rho$$
$$Z_k = \rho^{-1}(\ln(\frac{\alpha_k}{\alpha_0}) + (1-\rho)\ln(\frac{a_k}{a_0}) + \ln(E(MX_k^{1-\rho}))$$
$$\omega_0^* = 1 - \sum_{k=1}^{K} \omega_k^*$$

Proof. The FOC to this optimization problem are:

$$-\sum_{n=1}^{N} M_n \pi_n \alpha_0 a_0^{1-\rho}(1 - \sum_{k=1}^{K} \omega_k)^{-\rho} + \alpha_k a_k^{1-\rho}(\frac{\omega_k}{P_k} X_{kn})^{-\rho} \frac{X_{kn}}{P_k}) = 0, k = 1, ..., K.$$

We can rewrite these as:

$$\alpha_0 a_0^{1-\rho} \omega_0^{-\rho} = \alpha_k a_k^{1-\rho} \omega_k^{-\rho} P_k^{\rho-1} \sum_{n=1}^{N} \pi_n M_n X_{kn}^{1-\rho}.$$

Taking logs:

$$\ln(\omega_k) - \ln(\omega_0) = \frac{\rho - 1}{\rho} \ln(P_k) + \rho^{-1}(\ln(\frac{\alpha_k}{\alpha_0}) + (1-\rho)\ln(\frac{a_k}{a_0}) + \ln(\sum_{n=1}^{N} \pi_n M_n X_{kn}^{1-\rho}))$$
$$= \beta \ln(P_k) + \ln(Z_k)$$

which proves the proposition. □

The elasticity of relative demand is equal to $\beta = 1 - 1/\rho$, and so is determined by the substitutability of the payoffs from the various assets in investor preferences.

4.3 Characteristics-Based Demand

So far, we have treated each risky asset as distinct. We now compress their differences through the notion of characteristics described in Koijen and Yogo (2019). In particular, I define a function $\chi : \{1, 2, ..., K\} \rightarrow \Phi$, where Φ is the space of possible asset characteristics. The following assumption implies that the investor's attitude toward different assets is determined by their characteristics:

$$\chi(k) = \chi(k') \Rightarrow a_k = a_{k'}, \alpha_k = \alpha_{k'}, \text{ and } (M, X_k) \text{ have the same joint cdf as } (M, X_{k'}) \quad (12)$$

Proposition 5 then implies that, under the above assumption, the level of demand is a function only of characteristics.

Corollary 1. *Suppose (12) is satisfied. Then there exists* $\{Z_\phi\}_{\phi \in \Phi}$ *such that:*

$$\ln(\omega_k) - \ln(\omega_0) = Z_{\chi(k)} - \beta \ln(P_k).$$

The demand for a given asset k, relative to the baseline demand for riskfree assets, is an arbitrary function of characteristics $\phi(= \chi(k))$ and a linear function of logged prices

5 Conclusion

The main message of this paper is that the DSAP model pioneered by Koijen and Yogo (2019) represents a sharp break from the past asset pricing literature which relies on SPOT (at the individual level) and absence of arbitrage (in equilibrium). Instead, because it views relative asset demands as log-linear, the DSAP framework can be seen as treating asset payoffs, within a given state of the world, as imperfectly substitutable.

The IAS model described in this paper is one way to capture this imperfect substitutability. But it raises at least two key questions:

- what economic factors drive this imperfect substitution between asset payoffs and its magnitude?
- what kinds of policy and environmental interventions will leave these magnitudes unchanged?

The answers to these questions are outside the scope of this paper, but seem likely to be highly context-dependent.

Appendix

This appendix presents a model which nests the SPOT setup (from Section 2) and the IAS setup (from Section 3).

We again consider an agent who is splitting wealth W (denominated in bonds) between a risky stock with a random payoff X and a riskfree bond with payoff 1. Suppose that, if the agent allocates a share ω of their wealth to stocks, then the agent's ex-ante objective takes the form:

$$\frac{1}{1-\gamma}\int_{X_{min}}^{X_{max}}(\alpha(A_{stock}\frac{X\omega}{P}W)^{(1-\rho)}+(1-\alpha)(A_{bond}(1-\omega)W)^{1-\rho})^{\frac{(1-\gamma)}{1-\rho}}dF(X),\rho>0,\gamma>0,\gamma\neq 1 \quad (13)$$

$$\text{OR }\frac{1}{1-\gamma}\int_{X_{min}}^{X_{max}}((A_{stock}\frac{X\omega W}{P})^{\alpha}(A_{bond}(1-\omega)W)^{1-\alpha})^{1-\gamma}dF(X). \quad (14)$$

where the latter branch corresponds to setting $\rho=1$. This objective function is concave and increasing.

The function (13) nests the conventional expected utility objective of the kind in SPOT and the CES function in the IAS model. Thus, if $\rho=0$, the objective function (13) becomes:

$$\frac{1}{1-\gamma}\int_{X_{min}}^{X_{max}}(\frac{X\omega}{P}W+(1-\omega)W)^{1-\gamma}dF(X)$$

if $A_{stock}=A_{bond}=1$ and $\alpha=0.5$. The agent views the payoffs from the two assets as perfect substitutes, and so the investor has standard CRRA expected utility. On the other hand, if

$\rho = \gamma$, then (13) becomes:

$$\frac{1}{1-\rho}\int_{X_{min}}^{X_{max}}(\alpha(A_{stock}\frac{X\omega}{P}W)^{(1-\rho)} + (1-\alpha)(A_{bond}(1-\omega)W)^{1-\rho})dF(X),$$
$$\text{OR} \int_{X_{min}}^{X_{max}}(\alpha\ln(A_{stock}\frac{X\omega}{P}W) + (1-\alpha)\ln(A_{bond}(1-\omega)W))dF(X)$$

which is the objective used in the IAS model.[5]

References

[1] An, Y., and Huber, A., 2025, "Demand Propagation Through Traded Risk Factors," Wharton School working paper.

[2] Campbell, J., and Viceira, L., 2002, *Strategic Asset Allocation: Portfolio Choice for Long-Term Investors,* Oxford: Oxford Academic.

[3] van Binsbergen, J., David, B., and Opp, C., 2025, "How (Not) to Identify Demand Elasticities in Asset Markets," Wharton School working paper.

[4] Fuchs, W., Fukuda, S., and Neuhaun, D., 2025, "Demand-System Asset Pricing: Theoretical Foundations," University of Texas working paper.

[5] Haddad, V., He, Z., Huebner, P., Kondor, P., and Loualiche, E., 2025, "Causal Inference for Asset Pricing," UCLA working paper.

[6] Koijen, R., and Yogo, M., 2019, "A Demand System Approach to Asset Pricing," *Journal of Political Economy* 127, 1475-1515.

[7] Koijen, R., and Yogo, M., 2025, "On the Theory and Econometrics of (Demand System) Asset Pricing," The University of Chicago working paper.

[5]It may seem that that the IAS model should emerge if we set $\gamma = 0$, so that the investor is risk neutral. But the objective function (13) raises each state-contingent utility level to the power $(1-\rho)$ before applying an expectation operator. As a result, it implies different ex-ante preferences than the objective in the IAS model (which doesn't have this power).

[8] Poterba, J., and Rotemberg, J., 1986, “Money in the Utility Function: An Empirical Implementation,” NBER working paper.1796.

[9] Sidrauski, M., 1967, “Inflation and Economic Growth,” *Journal of Political Economy* 75, 796-810.