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ABSTRACT

This paper studies the equilibrium and welfare implications of quantitative trade models with sector-level external economies of scale (EES). While empirical evidence highlights the importance of EES, prior work has examined their effects only in models without input-output linkages. Focusing on a small open economy, we derive an intuitive condition for equilibrium uniqueness that links trade elasticities, scale elasticities, and input-output coefficients. We further show that this same condition guarantees positive gains from trade, regardless of a country's pattern of specialization.

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A Online Appendix is available at http://www.nber.org/data-appendix/w34258

1 Introduction

Recent evidence points to the importance of sector-level economies of scale (see Costinot *et al.*, 2019; Bartelme *et al.* (2025); Bartelme *et al.*, 2024; (Breinlich *et al.*, 2025), Fajgelbaum *et al.*, 2024; and Lashkaripour and Lugovskyy, 2023), yet standard quantitative trade models assume them away (see Caliendo and Parro, 2014). This paper studies how introducing external economies of scale (EES) into such models affects the set of equilibria and the welfare implications of trade in a small open economy. Kucheryavyy *et al.* (2023, hereafter KLR) examine related questions in gravity models without input-output linkages, while Baqaee and Farhi (2024) analyze the welfare effects of input-output loops in general trade models without EES. To our knowledge, no existing work studies the interaction of EES and the input-output structure in open economy settings. This paper takes a first step in that direction.

Our model builds on the single-factor, multi-sector gravity framework with roundabout production developed by Caliendo and Parro (2014), adapted to a small open economy in the sense of Demidova *et al.* (2024). Production in each sector is Cobb-Douglas, governed by an input share matrix A, and sector-specific trade elasticities are constants $\varepsilon_k > 0$ for $k = 1, \ldots, K$. The small open economy assumption implies that the Home country takes both the prices of foreign varieties and the foreign demand curves for its exports as given. We introduce EES by allowing total factor productivity in each sector to rise with employment, with elasticity θ_k.

Our first result is a sufficient condition for the equilibrium of the economy to exist, be interior and unique. Letting $\mathcal{L} \equiv [\ell_{sk}] = (I - A)^{-1}$ denote the Leontief inverse, the sufficient condition is $\sum_s \theta_s \ell_{sk} \varepsilon_k < 1$ for all k.[1] In the absence of intermediate inputs, this inequality simplifies to $\theta_k \varepsilon_k < 1$ for all k, which is the sufficient condition derived by KLR. Our new condition is stricter because input–output loops amplify the effects of EES: an increase in employment raises productivity, which propagates through the input-output network, boosting demand and potentially triggering further employment increases. Without the tighter bound, such feedback could become explosive, and potentially result in multiple equilibria.

The key step in our proof is to show that, for any given wage, there exists a unique labor allocation $\boldsymbol{L}$ that clears all product markets. Once this is established, it is straightforward to show that the same condition also ensures the existence of a unique wage that clears the labor market. We reformulate the product market–clearing conditions as a fixed-point problem in logs, $\ln \boldsymbol{L} = \ln \boldsymbol{F}(\boldsymbol{L})$, and show that, under our sufficient condi-

[1] The sufficient condition for an equilibrium to be interior is weaker and is given by $\theta_k \varepsilon_k < 1$ for all k.

tion, the Jacobian matrix of $\ln \boldsymbol{F}$ has spectral radius $\rho < 1$ at any point $\boldsymbol{L}$. This implies that $\ln \boldsymbol{F}$ is a local contraction at each of its fixed points, which allows us to apply the Index Theorem to establish uniqueness.[2] By itself, however, local contraction does not imply global contraction, and we show in the Online Appendix that $\ln \boldsymbol{F}$ fails to meet the global contraction conditions of Allen *et al.* (2024). Hence, their uniqueness result does not apply to this particular formulation of fixed-point problem $\ln \boldsymbol{F}$.

Our second set of results concerns the model's welfare implications. As emphasized in Harrison and Rodríguez-Clare (2010) and more recently in KLR, economies with EES may lose from trade if resources shift from sectors with strong spillovers to those with weaker ones — potentially reducing aggregate productivity enough to outweigh the standard gains from trade (Arkolakis *et al.*, 2012). KLR showed that in the absence of intermediate goods, the same condition that ensures uniqueness also guarantees gains from trade. We show that this continues to be true with input-output loops: under our sufficient condition for uniqueness, losses from trade cannot occur. The proof proceeds in three steps: (i) we derive a lower bound for the gains from trade (GT) that excludes beneficial scale effects from exports; (ii) we show that a monotonic transformation of this bound is convex in log domestic trade shares; and (iii) we show that, starting from autarky, any small increase in trade raises this lower bound. Steps (ii) and (iii) then guarantee that any amount of trade increases the lower bound on GT, implying that GT are always positive.

Our paper contributes to the vast quantitative trade literature by providing the first theoretical results on the equilibrium uniqueness and welfare properties of quantitative trade models with intermediate goods and EES. The potential for multiple equilibria was recognized in the older CGE literature that used this type of model; the rule of thumb was that the scale elasticity should be at most half the inverse of the trade elasticity in order to avoid computational problems (Hertel, 2013). More recent papers have dealt with the issue of potential multiplicity with ad-hoc solutions such as smoothing out the IO matrix (Costinot and Rodríguez-Clare, 2014; Bartelme *et al.*, 2025) or simulations to map the parameter space where uniqueness holds (Breinlich *et al.*, 2025). Our paper is a first step towards a more precise understanding of these issues, which have become increasingly relevant with the resurgence of interest in using quantitative models to study trade and industrial policy in an open economy (e.g., (Bartelme *et al.*, 2025); Lashkaripour and Lugovskyy, 2023; Antràs *et al.*, 2024; Choi and Shim (2024)).

[2] Alternatively, $\rho < 1$ at any point $\boldsymbol{L}$ ensures that the Jacobian matrix of the system of equations $\ln \boldsymbol{L} - \ln \boldsymbol{F}(\boldsymbol{L}) = \boldsymbol{0}$ is a P-matrix, so one could also invoke the Gale and Nikaido (1965) theorem to establish uniqueness. We prefer to use the Index Theorem, as it offers the intuitive interpretation that uniqueness requires $\ln \boldsymbol{F}(\boldsymbol{L})$ to be a local contraction at all of its fixed points. This perspective also highlights the contrast with the approach of Allen *et al.* (2024), which relies on $\ln \boldsymbol{F}(\boldsymbol{L})$ being a global contraction.

Our work is also related to Krugman and Venables (1995), who characterize the equilibrium properties of a stylized two-country, two-sector model with an outside good, input-output loops and scale economies that arise from entry in monopolistic competition. Relative to their work, our paper allows for an arbitrary number of sectors and a more general input-output structure, but is restricted to perfect competition and a small open economy. Extending our results to cover settings with monopolistic competition or multiple large countries is an interesting, albeit nontrivial, area for future research.

The remainder of the paper is organized as follows. Section 2 introduces the model. Section 3 defines equilibrium and states our first theorem, which shows that under the sufficient condition, the equilibrium exists, is interior, and unique. Section 4 provides an informal overview of the proof, and Section 5 presents the formal argument. Section 6 states our second theorem, establishing that the gains from trade are positive under the same condition. Section 7 concludes.

2 Model

We consider a small-open economy, Home, with a single factor of production, labor, and K sectors or goods indexed by $k = 1, \ldots, K$. Following the Armington assumption, each of these K goods comes in two varieties: one from Home and one from the rest of the world. The small-open economy assumption implies that Home takes the prices of foreign varieties and the foreign demand curve for its exports as given and dependent only on its own price.[3] This assumption will be important to derive our uniqueness result, but our welfare result goes through even without this assumption. As an additional benefit, we can focus entirely on Home and omit country subscripts.

Production in sector k uses labor and intermediate goods according to a Cobb-Douglas technology,

$$q_k = \left(\alpha_k^{-\alpha_k} \prod_{s=1}^{K} \alpha_{sk}^{-\alpha_{sk}} \right) (T_k L_k)^{\alpha_k} \prod_{s=1}^{K} Q_{sk}^{\alpha_{sk}}, \tag{1}$$

where q_k and L_k are output and employment in sector k, Q_{sk} is the amount of composite intermediate good s (described below) used in sector k, and parameters α_k and α_{sk} satisfy $\alpha_{sk} \in [0, 1]$ for all k and s, $\alpha_k + \sum_s \alpha_{sk} = 1$ for all k, and $\alpha_k > 0$ for all k. These are standard conditions and are assumed to hold throughout the paper.

The labor productivity term T_k is taken as given by individual producers, but it is

[3] Assuming that the rest of the world is one single economy is innocuous given our assumption that Home is a small-open economy. See Demidova *et al.* (2024) on how small open economy can be obtained as a limit of a multi-country economy, although without intermediates.

endogenous with respect to aggregate sectoral employment and is given by

$$T_k \equiv \overline{T}_k^{1/\alpha_k} L_k^{\gamma_k}, \tag{2}$$

with $\overline{T}_k$ exogenous and $\gamma_k \geq 0$. This production structure implies that firms' hiring decisions create external effects on sectoral productivity. The magnitudes of these externalities potentially vary across sectors and are given by $\theta_k \equiv \alpha_k \gamma_k$, which we refer to as scale elasticities (as in Kucheryavyy *et al.*, 2023).

Output from sector k is combined with the corresponding foreign variety to produce the composite good used in consumption or as an intermediate good in production. We assume that this is done via a CES production function with elasticity of substitution $\varepsilon_k + 1 > 1$,

$$Q_{sk} = \left(q_{sk}^{\frac{\varepsilon_k}{\varepsilon_k+1}} + \left[q_{sk}^F \right]^{\frac{\varepsilon_k}{\varepsilon_k+1}} \right)^{\frac{\varepsilon_k+1}{\varepsilon_k}},$$

where q_{sk} and q_{sk}^F represent the quantities of good s used by sector k sourced from Home and Foreign. This implies that the trade elasticity in sector k is ε_k. Letting p_k and p_k^F denote the price of the Home and Foreign variety of good k (with p_k^F exogenous, strictly positive, and finite), Home's domestic expenditure share in sector k is given by

$$\lambda_k = \frac{p_k^{-\varepsilon_k}}{p_k^{-\varepsilon_k} + \left[p_k^F \right]^{-\varepsilon_k}}. \tag{3}$$

Letting $P_k = \left(p_k^{-\varepsilon_k} + \left[p_k^F \right]^{-\varepsilon_k} \right)^{-1/\varepsilon_k}$ be Home's price index of the composite good in sector k, we have

$$P_k = p_k \lambda_k^{1/\varepsilon_k}. \tag{4}$$

We assume that preferences across sectors are Cobb-Douglas and denote sector-level final expenditure shares by e_k. Letting w be the wage and $\bar{L}$ the perfectly inelastic labor supply in Home, total income in Home is $w\bar{L}$. Final demand (in value) for the domestically produced good k in Home is $C_k = \lambda_k e_k w \bar{L}$. We assume isoelastic export demand functions, with demand elasticity $\varepsilon_k - 1$, implying that Home's export revenues in sector k are $X_k = E_k p_k^{-\varepsilon_k}$. The numeraire is implicitly defined by the units in which import prices and export demand shifters are determined, the details of which play no role in our analysis.

Finally, we assume that at least one sector exports, i.e., $E_k > 0$ for at least one k, and we rule out sectors that neither directly nor indirectly contribute to final consumption or exports. Formally, we call a sector k *essential of order* 0 if it is directly used for final consumption or exports, i.e., if $e_k > 0$ or $E_k > 0$. A sector k is *essential of order* $t \geq 1$ if k is not essential of order lower than t and it is used as an input by some sector s (i.e, $\alpha_{ks} > 0$) that is essential of order $t-1$. We say that a sector is *essential* if it is essential of some order, and we assume that all sectors $k = 1, \ldots, K$ are essential.

3 Equilibrium Analysis

We begin with the goods market clearing condition, which for sector k is given by

$$q_k = \frac{\lambda_k}{p_k} e_k w \bar{L} + E_k p_k^{-\varepsilon_k - 1} + \frac{\lambda_k}{p_k} \sum_s P_k Q_{ks}. \tag{5}$$

The right-hand side captures the total demand for good k, which comes from consumption, exports, and intermediate goods use. In the absence of EES ($\gamma_k = 0$ for all k), any equilibrium must feature strictly positive labor allocations across all sectors. It is useful to briefly go through the logic. Suppose employment in some sector k is zero, $L_k = 0$. Then the left-hand side of (5) becomes zero, which can be consistent only with infinite p_k, so that demand for good k is zero. At the same time, without EES, the production function (1) implies that the cost of producing good k is finite.[4] This leads to a contradiction: a producer facing an infinite price and finite cost would earn unbounded profits, which cannot occur in equilibrium.

The situation changes in the presence of EES ($\gamma_k > 0$). The goods market clearing condition still implies infinite price p_k in any sector with zero employment. But now the production function implies that productivity is zero, $T_k = 0$, and so the cost of production becomes infinite as well. As a result, the good-k producer's profit maximization problem is no longer well-defined, and the above logic preventing zero employment no longer applies.

To formalize the producer's behavior in the presence of EES — where the standard profit maximization problem may no longer be well-defined — we follow Kucheryavyy *et al.* (2023) and consider the producer's complementary slackness conditions,

$$\text{VMPL}_k \leq w \perp L_k \geq 0; \tag{6}$$

[4]Foreign varieties have finite prices, so even if Home varieties had infinite prices, the resulting price indices in Home would remain finite. Therefore, the only way for the cost of producing good k to be infinite is if the wage were infinite. However, this would imply zero labor demand, violating the condition of labor market clearing.

$$\text{VMPQ}_{sk} \le P_s \perp Q_{sk} \ge 0, \quad s = 1, \ldots, K; \tag{7}$$

where $\text{VMPL}_k \equiv \alpha_k p_k q_k / L_k$ and $\text{VMPQ}_{sk} \equiv \alpha_{sk} p_k q_k / Q_{sk}$ are the value of the marginal product of labor and of the composite intermediate input from sector s, respectively.[5] These conditions are also not well-defined when $L_k = 0$, as both VMPL_k and VMPQ_{sk} (for $\alpha_{sk} \neq 0$) are undefined in this case.[6] However, this formulation is useful, as it isolates all potential zero and infinite terms on one side of each inequality.

Below we formally define the equilibrium of our economy using conditions (6)-(7), and establish existence and uniqueness. To address the indeterminacy that arises when some labor allocations are zero, we adopt the approach of Kucheryavyy *et al.* (2023), resolving the indeterminacy of conditions (6)-(7) by starting from an allocation with strictly positive employment in all sectors and taking the limit as labor in some sectors goes to zero.

An equilibrium is given by prices $\boldsymbol{p} \equiv (p_1, ..., p_K)^T$ and $\boldsymbol{P} \equiv (P_1, ..., P_K)^T$, quantities $\boldsymbol{q} \equiv (q_1, \ldots, q_K)^T$ and $\{Q_{ks}\}_{k,s=1}^{K}$, domestic trade shares $\boldsymbol{\lambda} \equiv (\lambda_1, ..., \lambda_K)^T$, labor allocations $\boldsymbol{L} = (L_1, ..., L_K)^T$, and a wage w, such that conditions (1)-(7) are satisfied, along with the labor market clearing condition $\sum_k L_k = \bar{L}$. The next theorem presents one of the two main results of this paper.

Theorem 1. *Assume that the following condition holds:*

$$\sum_s \theta_s \ell_{sk} \varepsilon_k < 1 \quad \textit{for all } k, \tag{UC}$$

where $\theta_s \equiv \alpha_s \gamma_s$. Then an equilibrium exists, it is interior, and it is unique.

4 Proof of Theorem 1: Overview

The proof of Theorem 1 proceeds in four steps. In the first step, we take $w > 0$ and a strictly positive labor allocation vector $\boldsymbol{L}$ as given, and show that equations (1)-(4) and (6)-(7) jointly determine implicit functions $p_k(w, \boldsymbol{L})$, with λ_k given by $\lambda_k(p_k(w, \boldsymbol{L}))$ from (3).

In the second step, still taking w and a strictly positive $\boldsymbol{L}$ as given, we use the functions derived in the first step and show that when $\theta_k \varepsilon_k < 1$ for all k, if employment in one or

[5] Condition $x \le x_0 \perp y \ge y_0$ means that $x \le x_0$, $y \ge y_0$, and if $x < x_0$ then $y = y_0$.

[6] If $q_k = 0$ then even with infinite price p_k, the revenue of the producer of good k, given by $p_k q_k$, has to be zero. Thus, if $L_k = 0$ then $Q_{sk} = 0$ for all s. In this case, both VMPL_k and VMPQ_k become indeterminate, as they involve division of zero by zero.

more sectors goes to zero, then VMPL_k diverges to infinity for at least one sector k. This implies that the inequality $\text{VMPL}_k \leq w$ cannot be satisfied for that sector, thereby ruling out any equilibrium with zero labor allocations. Since the condition $\theta_k \varepsilon_k < 1$ for all k is implied by (UC), zero labor allocations are excluded under (UC) as well.

Having ruled out corner solutions with zero labor allocations under (UC), the final two steps focus on strictly positive $\boldsymbol{L}$ (i.e., *interior equilibria*). In the third step, we fix w and use the functions $p_k(w, \boldsymbol{L})$ and $\lambda_k(p_k(w, \boldsymbol{L}))$ to show that there exists a unique $\boldsymbol{L}$ satisfying the goods market clearing condition (5), provided that (UC) holds. This defines an implicit function $\boldsymbol{L}(w)$. In the final step, we substitute $\boldsymbol{L}(w)$ into the labor market clearing condition, reducing the system to a single equation in the single unknown w, and show that this equation has a unique solution, again thanks to (UC).

The critical and most difficult part of the proof of Theorem 1 lies in the third step. Before presenting the formal argument, we first provide an explanation of this step while omitting most of the technical details. Setting aside the issue of zero labor allocations, let us focus on interior allocations and proceed with some key derivations in this case.

When $\boldsymbol{L}$ is strictly positive, the complementary slackness conditions (6)-(7) bind, turning into equalities: $\text{VMPL}_k = w$ and $\text{VMPQ}_{sk} = P_s$. Substituting these into the goods market clearing condition (5), we obtain

$$L_k/\alpha_k = d_k + \lambda_k \sum_s \alpha_{ks} L_s/\alpha_s, \tag{8}$$

where $d_k \equiv \lambda_k e_k \bar{L} + E_k w^{-1} p_k^{-\varepsilon_k}$ is the final demand for sector k. In addition, combining $\text{VMPL}_k = w$ and $\text{VMPQ}_{sk} = P_s$ with the production function (1) yields

$$p_k = \left(\overline{T}_k L_k^{\theta_k}\right)^{-1} w^{\alpha_k} \prod_s P_s^{\alpha_{sk}}. \tag{9}$$

From here, using $P_k = p_k \lambda_k^{1/\varepsilon_k}$, we can solve for output prices as

$$p_k = w \lambda_k^{-1/\varepsilon_k} \prod_s \left(\overline{T}_s L_s^{\theta_s}\right)^{-\ell_{sk}} \lambda_s^{\ell_{sk}/\varepsilon_s}, \tag{10}$$

where ℓ_{sk} is the (s,k) entry of the $K \times K$ closed-economy (Leontief) linkage matrix $\mathcal{L} \equiv (I - A)^{-1}$, and A is the $K \times K$ matrix with (s,k) entry α_{sk}.[7]

Examining expression (10) for p_k, together with expression (3) for λ_k, we see that for a given wage, labor allocations $\boldsymbol{L}$ affect both prices $\boldsymbol{p}$ and trade shares $\boldsymbol{\lambda}$ solely through

[7] $\mathcal{L}$ is well-defined given that $\sum_s \alpha_{sk} < 1$ for all k.

the productivity terms $T_k \equiv \overline{T}_k^{1/\alpha_k} L_k^{\gamma_k}$. Taking these productivity terms and the wage as given (which also implies that we take the prices p_k and final demand d_k as given), we can use the market clearing condition (8) to solve for the implied labor allocations. This defines a mapping $\boldsymbol{F} : \mathbb{R}_+^K \to \mathbb{R}_+^K$, which maps any strictly positive allocation $\boldsymbol{L}$ to its implied counterpart $\boldsymbol{L}' = \boldsymbol{F}(\boldsymbol{L})$, with components $F_k(\boldsymbol{L}) \equiv \alpha_k \sum_{j=1}^K \tilde{\ell}_{kj} d_j$. The terms $\tilde{\ell}_{kj}$ are elements of the open-economy (Leontief) linkage matrix $\tilde{\mathcal{L}} \equiv (I - D_\lambda A)^{-1}$, where D_λ is a $K \times K$ diagonal matrix with trade shares $\boldsymbol{\lambda}$ on the diagonal. Both $\tilde{\ell}_{kj}$ and d_j depend on $\boldsymbol{L}$ through $\boldsymbol{\lambda}$ and $\boldsymbol{p}$ (which, again, depend on $\boldsymbol{L}$ only through the productivity terms).[8] For notational simplicity, we suppress the dependence of $\tilde{\ell}_{kj}$, d_j, $\boldsymbol{\lambda}$, and $\boldsymbol{p}$ on $\boldsymbol{L}$ here, and reintroduce it explicitly as needed in the formal proofs of the third step.

Any fixed point of $\boldsymbol{F}$ satisfies the market-clearing condition (8) for a given wage w. The core of the third step in our proof is to show that $\boldsymbol{F}$ is a local contraction in a neighborhood of any such fixed point. Once this is established, we invoke the Index Theorem — Theorem 2 below — to conclude that the fixed point is unique. Showing the local contraction property is technically challenging and relies on condition (UC). Here, we offer intuition for the local contraction property, explain the role of condition (UC), and outline the proof.

To prove that $\boldsymbol{F}$ is a local contraction around any of its fixed points, we work with the log-transformed mapping $\ln \boldsymbol{F}$ as a function of $\ln \boldsymbol{L}$. Fixed points are preserved under the log transformation, and, as shown in Appendix A, $\boldsymbol{F}$ is a local contraction at fixed point $\boldsymbol{L}^*$ if and only if $\ln \boldsymbol{F}$ is a local contraction at $\ln \boldsymbol{L}^*$. For notational simplicity, we write $\ln \boldsymbol{F}(\boldsymbol{L})$ while formally working with $\ln \boldsymbol{F}(\exp\{\cdot\})$, and we refer to $\boldsymbol{L}^*$ as a fixed point if $\ln \boldsymbol{F}(\boldsymbol{L}^*) = \ln \boldsymbol{L}^*$. We denote the Jacobian of mapping $\ln \boldsymbol{F}$ at $\boldsymbol{L}$ by $J(\boldsymbol{L}) \equiv \nabla \ln \boldsymbol{F}(\boldsymbol{L})$, with entries $J_{ks}(\boldsymbol{L}) \equiv \partial \ln F_k(\boldsymbol{L}) / \partial \ln L_s$.

The mapping $\ln \boldsymbol{F}$ is a local contraction at a fixed point $\boldsymbol{L}^*$ if there exists a vector norm $\|\cdot\|$ inducing a corresponding matrix norm $|||\cdot|||$, defined by $|||M||| = \max_{\|\boldsymbol{x}\| \neq 0} \dfrac{\|M\boldsymbol{x}\|}{\|\boldsymbol{x}\|}$ for any matrix M, such that $|||J(\boldsymbol{L}^*)||| < 1$.[9] By definition of the matrix norm, the condition $|||J(\boldsymbol{L}^*)||| < 1$ is equivalent to $\|J(\boldsymbol{L}^*)\boldsymbol{x}\| < \|\boldsymbol{x}\|$ for all $\boldsymbol{x} \neq 0$. Thus, for any small perturbation in the labor allocation around $\boldsymbol{L}^*$, $\mathrm{d}\ln \boldsymbol{L} \neq 0$, we have $\|J(\boldsymbol{L}^*)\,\mathrm{d}\ln \boldsymbol{L}\| < \|\mathrm{d}\ln \boldsymbol{L}\|$, and using $J(\boldsymbol{L}^*)\,\mathrm{d}\ln \boldsymbol{L} = \mathrm{d}\ln \boldsymbol{F}(\boldsymbol{L}^*)$, it follows that $\|\mathrm{d}\ln \boldsymbol{F}(\boldsymbol{L}^*)\| < \|\mathrm{d}\ln \boldsymbol{L}\|$. That is, any small change in the labor allocation leads to a strictly smaller change in the implied labor

[8] We can think of $F_k(\boldsymbol{L})$ as a mapping from labor allocation to labor demand: given $\boldsymbol{L}$, we know productivity shifters $\boldsymbol{T}(\boldsymbol{L})$, which, combined with the wage, can be used to solve for labor demand in each sector, $F_k(\boldsymbol{L})$. Without EES, this mapping would be flat, as $\boldsymbol{L}$ would have no effect on any term in $\alpha_k \sum_{j=1}^K \tilde{\ell}_{kj} d_j$.

[9] By continuity of J, it follows that $|||J(\boldsymbol{L})||| \leq \kappa < 1$ for all $\ln \boldsymbol{L}$ in a neighborhood U_{L^*} of $\ln \boldsymbol{L}^*$. A standard mean value theorem argument then establishes that $\ln \boldsymbol{F}$ is a contraction mapping on U_{L^*} with respect to the norm $\|\cdot\|$.

demand, ruling out local amplification of shocks.

From (1)-(2), a change $\mathrm{d}\ln L_s$ in sector s raises log productivity in that sector by $\theta_s\,\mathrm{d}\ln L_s$. Through input-output linkages, this lowers the log price of good r by $\tilde{\ell}_{sr}\theta_s\,\mathrm{d}\ln L_s$ (see Proposition 1 in Section 5.1). Since price changes are the only channel from employment to labor demand, the effect of a shock $(\mathrm{d}\ln L_1,\ldots,\mathrm{d}\ln L_K)$ on labor demand in sector k is

$$\mathrm{d}\ln F_k(\boldsymbol{L}) = \sum_s J_{ks}(\boldsymbol{L})\,\mathrm{d}\ln L_s,$$

with

$$J_{ks}(\boldsymbol{L}) = \frac{\alpha_k}{F_k(\boldsymbol{L})}\sum_{j,r}\frac{\partial\left(\tilde{\ell}_{kj}d_j\right)}{\partial\ln p_r^{-\varepsilon_r}}\cdot\frac{\partial\ln p_r^{-\varepsilon_r}}{\partial\ln L_s}.$$

Expanding the expression for $J_{ks}(\boldsymbol{L})$ yields

$$J_{ks}(\boldsymbol{L}) = \frac{\alpha_k}{F_k(\boldsymbol{L})}\sum_r\Bigg[\underbrace{\sum_j\frac{\partial\tilde{\ell}_{kj}}{\partial\ln p_r^{-\varepsilon_r}}d_j}_{\text{Linkage effect}} + \underbrace{\tilde{\ell}_{kr}\frac{\partial d_r}{\partial\ln p_r^{-\varepsilon_r}}}_{\text{Direct demand effect}}\Bigg]\cdot\underbrace{\varepsilon_r\tilde{\ell}_{sr}\theta_s}_{\text{Price effect}}, \tag{11}$$

where we have used the fact that $\partial d_j/\partial\ln p_r^{-\varepsilon_r} = 0$ for $r\neq j$, by the definition of d_j.

The role of condition (UC) becomes particularly transparent in the extreme case where intermediate goods are non-tradable. Although this case is not formally nested within our model, it is easy to adapt the above expressions to this setting by equating the open- and closed-economy linkage matrices, i.e., by setting $\tilde{\ell}_{kj} = \ell_{kj}$ for all k, j.[10] This implies that the linkage effect in (11) is zero. Using the sup vector norm, $\|\boldsymbol{v}\|_\infty = \max\{|v_1|,|v_2|,\ldots,|v_K|\}$ for $\boldsymbol{v}\in\mathbb{R}^K$, with the associated induced matrix norm, $|||M|||_\infty = \max_k\sum_{s=1}^K|m_{ks}|$ for any $K\times K$ matrix M, we then obtain

$$|||J(\boldsymbol{L})|||_\infty = \max_k\left\{\frac{\alpha_k}{F_k(\boldsymbol{L})}\sum_r\ell_{kr}\frac{\partial d_r}{\partial\ln p_r^{-\varepsilon_r}}\sum_s\varepsilon_r\ell_{sr}\theta_s\right\}.$$

The expression $\sum_s\varepsilon_r\theta_s\ell_{sr}$ captures the maximum potential strength with which employment shocks can propagate through input-output linkages, affect prices, and ultimately influence implied labor demand. Condition (UC), $\sum_s\varepsilon_r\theta_s\ell_{sr} < 1$, ensures that these feedback effects are dampened rather than amplified — exactly what guarantees the local contraction property. Indeed, under this condition, combined with $\partial d_r/\partial\ln p_r^{-\varepsilon_r} =$

[10] With non-tradable intermediate goods, the price of good k is given by $p_k = w\prod_s\left(\overline{T}_s L_s^{\theta_s}\right)^{-\ell_{sk}}$, and expression (8) is replaced by $L_k/\alpha_k = d_k + \sum_s\alpha_{ks}L_s/\alpha_s$, with the definition of d_k unchanged.

$(1-\lambda_r)\,\lambda_r e_r \bar{L} + E_r w^{-1} p_r^{-\varepsilon_r} \le d_r$, and the definition $F_k(\boldsymbol{L}) = \alpha_k \sum_r \ell_{kr} d_r$, it follows that $\||J(\boldsymbol{L})\||_\infty < 1$.[11]

With trade in intermediates, the Jacobian takes a more complex form because productivity changes now affect trade shares in intermediates, which in turn impact the open-economy input-output matrix, $\tilde{\mathcal{L}} \equiv (I - D_\lambda A)^{-1}$. On the one hand, trade in intermediates weakens domestic linkages (i.e., decreases the elements of $\tilde{\mathcal{L}}$), which tends to reduce the entries of the Jacobian through the price and direct demand effects in (11) and make $\ln \boldsymbol{F}$ more likely to be a contraction.[12] On the other hand, higher employment raises productivity, which increases domestic trade shares and strengthens linkages, $\partial \tilde{\ell}_{kj} / \partial \ln p_r^{-\varepsilon_r} \ge 0$, thereby amplifying the induced changes in labor demand through the linkage effect in (11). The tension between these two forces breaks the argument based on the sup norm, and (UC) no longer guarantees $\||J(\boldsymbol{L}^*)\||_\infty < 1$. Thus, to proceed, we work instead with the spectral radius of $J(\boldsymbol{L}^*)$.

A necessary and sufficient condition for $\ln \boldsymbol{F}$ to be a local contraction at $\boldsymbol{L}^*$ is that the spectral radius of the associated Jacobian, denoted $\rho(J(\boldsymbol{L}^*))$, be smaller than one. This condition has a natural interpretation. A shock $\mathrm{d}\ln \boldsymbol{L}$ at a fixed point $\boldsymbol{L}^*$ has first-order impact $J(\boldsymbol{L}^*)\,\mathrm{d}\ln \boldsymbol{L}$, and iterating this response yields $[J(\boldsymbol{L}^*)]^t\,\mathrm{d}\ln \boldsymbol{L}$. By Theorem 5.6.12 in Horn and Johnson (2013), this sequence converges to zero if and only if $\rho(J(\boldsymbol{L}^*)) < 1$, ensuring that shocks decay rather than generate explosive feedback. An alternative interpretation of the spectral radius condition is that there exists a reweighting of sectors under which the sup norm of the transformed Jacobian is strictly below one. Under the regular sup norm, the condition $\||J(\boldsymbol{L}^*)\||_\infty < 1$ implies that a 1% increase in employment across all sectors generates an increase in implied demand of less than 1% in every sector. More generally, when $\rho(J(\boldsymbol{L}^*)) < 1$, we can construct a system of sectoral weights for the sup norm such that a 1% increase in employment across all sectors translates into an increase in implied, reweighted demand of less than 1% in each sector.[13] As shown in Appendix B, proving that such a reweighting exists is equivalent to proving that $\rho(J(\boldsymbol{L}^*)) < 1$.

Our proof that $\rho(J(\boldsymbol{L})) < 1$ for any $\boldsymbol{L}$ proceeds in two parts. In the first part, we replace the trade elasticities ε_r that appear in the price effect term in (11) with their up-

[11] The derivations above in fact show that $\||J(\boldsymbol{L})\||_\infty < 1$ for *any* $\boldsymbol{L}$ when intermediates are non-tradable. This implies that, in this case, $\ln \boldsymbol{F}$ is a *global* contraction under the sup norm, provided the domain is restricted to a subspace of the $\ln \boldsymbol{L}$-space that is mapped into itself by $\ln \boldsymbol{F}$. We do not emphasize this global property because — as discussed below — when intermediates are tradable, there may be no norm under which $\ln \boldsymbol{F}$ is a global contraction.

[12] In the extreme case where all domestic trade shares are zero, we have $\tilde{\mathcal{L}} = I$, and (11) yields $J_{ks}(\boldsymbol{L}) \le \frac{\alpha_k d_k}{F_k(\boldsymbol{L})} \varepsilon_k \theta_k \mathbb{I}_{ks} = \varepsilon_k \theta_k \mathbb{I}_{ks}$, where $\mathbb{I}_{ks} = 1$ if $k = s$ and 0 otherwise. This implies $\||J(\boldsymbol{L})\||_\infty < 1$, where the inequality follows from (UC) since $\sum_s \theta_s \ell_{sk} \varepsilon_k \ge \theta_k \ell_{kk} \varepsilon_k \ge \theta_k \varepsilon_k$.

[13] Formally, if D_ω is a diagonal matrix of sectoral weights, we have $\|D_\omega \mathrm{d}\ln \boldsymbol{F}(\boldsymbol{L}^*)\|_\infty < \|D_\omega \mathrm{d}\ln \boldsymbol{L}\|_\infty$.

per bounds implied by condition (UC), namely, $[\sum_i \ell_{ir}\theta_i]^{-1}$. This yields a "worst-case-scenario" matrix, denoted $\overline{\overline{J}}(\boldsymbol{L})$, for which we prove that $\rho\left(\overline{\overline{J}}(\boldsymbol{L})\right) \leq 1$. Intuitively, just as in the case with non-tradable intermediates, condition (UC) limits the strength with which employment shocks can affect prices, and therefore restricts the resulting feedback into labor demand. At the boundary of this condition, the spectral radius is no greater than 1.

To establish that $\rho\left(\overline{\overline{J}}(\boldsymbol{L})\right) \leq 1$, we first apply an equivalence transformation that turns $\overline{\overline{J}}(\boldsymbol{L})$ into a symmetric matrix $M(\boldsymbol{L})$, preserving the spectral radius. We then show that $I - M(\boldsymbol{L})$ is positive semidefinite, which implies $\rho(M(\boldsymbol{L})) \leq 1$. To that end, we apply a second equivalence transformation to $I - M(\boldsymbol{L})$, yielding a matrix that can be decomposed into a sum of K simpler symmetric matrices. These matrices share common terms, and by combining them in a particular way, we are able to show that all are simultaneously positive semidefinite — thanks to cancellations of the overlapping terms. A crucial element of this argument is the application of the Cauchy–Schwarz inequality.

The second part of the proof that $\rho(J(\boldsymbol{L})) < 1$ proceeds by contradiction. We suppose that $\rho(J(\boldsymbol{L})) = 1$, and show that one can then slightly increase the trade elasticities in the price effect term of (11) — while still satisfying condition (UC) — so that the spectral radius of the resulting perturbed Jacobian exceeds 1. This contradicts the result established in the first part and thereby completes the proof.

While we established $\rho(J(\boldsymbol{L})) < 1$ for any positive vector $\boldsymbol{L}$ — not only for fixed points of $\ln \boldsymbol{F}$, this global property of $\ln \boldsymbol{F}$ does not imply that there exists a norm under which $\ln \boldsymbol{F}$ is a global contraction. Such a norm would need to be independent of $\boldsymbol{L}$, and while condition $\rho(J(\boldsymbol{L})) < 1$ guarantees the existence of a norm $\left|\!\left|\!\left| \cdot \right|\!\right|\!\right|_L$ with $\left|\!\left|\!\left| J(\boldsymbol{L}) \right|\!\right|\!\right|_L < 1$, it generally depends on $\boldsymbol{L}$ and is therefore local. One way to construct an $\boldsymbol{L}$-independent norm $\left|\!\left|\!\left| \cdot \right|\!\right|\!\right|$ is to define a matrix $\overline{J}$ as the element-wise supremum of $J(\boldsymbol{L})$ over all $\boldsymbol{L}$ in the domain of $\boldsymbol{F}$, and then check whether $\rho\left(\overline{J}\right) < 1$. If this condition holds, one obtains a norm $\left|\!\left|\!\left| \cdot \right|\!\right|\!\right|_{\overline{J}}$ that depends only on $\overline{J}$ and satisfies $\left|\!\left|\!\left| J(\boldsymbol{L}) \right|\!\right|\!\right|_{\overline{J}} < 1$ for all $\boldsymbol{L}$. This is the approach taken in Allen *et al.* (2024, AAL).[14] However, it is easy to find examples of our economy in which $\rho(J(\boldsymbol{L})) < 1$ for all $\boldsymbol{L}$ under condition (UC), yet $\rho\left(\overline{J}\right) > 1$.[15] Thus, this global bounding approach does not, in general, yield a norm under which $\ln \boldsymbol{F}$ is a global

[14]In their paper, AAL do not explicitly construct a norm to show global contraction. Instead, they appeal to the Perov Fixed Point Theorem to establish that their system of equations defines a global contraction in a certain *metric* space. In the Online Appendix, we provide an alternative proof of part (i) of Theorem 1 in AAL. Our approach uses AAL's uniqueness conditions to explicitly construct a global norm, and then applies a standard mean-value argument to verify that their system of equations is globally contractive under this norm.

[15]We provide one such typical example in the Online Appendix.

contraction in our setting.[16]

5 Proof of Theorem 1: Formal Analysis

5.1 First Step

Our main result in this step is the following proposition, which is proved in Appendix C.

Proposition 1. *For any $w > 0$ and strictly positive $\boldsymbol{L}$:*

1. *Equations (3) and (10) define an implicit function $p_k(w, \boldsymbol{L}) > 0$ for all k.*
2. *$p_k(w, \boldsymbol{L})$ is non-increasing in L_s for all k and s, with the partial derivatives given by*

$$\partial \ln p_k / \partial \ln L_s = -\theta_s \tilde{\ell}_{sk}. \tag{12}$$

Note that Proposition 1 does not rely on condition (UC). Intuitively, (UC) rules out explosive feedback loops from productivities to prices and labor allocations, then to demand, and back to productivities. However, in Proposition 1, the vector of labor allocations is held fixed, which eliminates these feedback effects and renders (UC) unnecessary.

5.2 Second Step

In this step, we establish the conditions under which any equilibrium is interior.

Fix a wage $w > 0$ and consider a strictly positive $\boldsymbol{L}$. Combining $p_k = \lambda_k^{-1/\varepsilon_k} P_k$ with (9) yields the following expression for domestic trade shares

$$\lambda_k = w^{-\varepsilon_k \alpha_k} P_k^{\varepsilon_k} \left(\prod_s P_s^{-\varepsilon_k \alpha_{sk}} \right) \left(\overline{T}_k L_k^{\theta_k} \right)^{\varepsilon_k}. \tag{13}$$

Thus, conditional on price indices, only the own-sector's productivity affects its domestic trade share. For a strictly positive $\boldsymbol{L}$, the complementary slackness conditions (6)-(7) hold with equality, implying $P_k Q_{ks} = \alpha_{ks} w L_s / \alpha_s$. Substituting this into the goods market clearing condition (5), and then substituting the resulting expression for q_k into the definition

[16] Still, as we establish in Proposition 6 below, under (UC), the fixed-point iteration of $\boldsymbol{F}$ converges to the unique fixed point of $\boldsymbol{F}$ starting from any positive $\boldsymbol{L}$.

$\text{VMPL}_k \equiv \alpha_k p_k q_k / L_k$, and using (13) together with $p_k = \lambda_k^{-1/\varepsilon_k} P_k$ yields

$$\text{VMPL}_k = \alpha_k w^{-\varepsilon_k \alpha_k} P_k^{\varepsilon_k(1-\alpha_{kk})} \left(\prod_{s \neq k} P_s^{-\varepsilon_k \alpha_{sk}} \right) \overline{T}_k^{\varepsilon_k} L_k^{\theta_k \varepsilon_k - 1} \left(e_k w \bar{L} + P_k^{-\varepsilon_k} E_k + \sum_s \alpha_{ks} w L_s / \alpha_s \right). \tag{14}$$

Now suppose that $L_k \to 0$ for some k. Here we sketch the argument that this implies that $\text{VMPL}_s \to \infty$ for some sector s, with the formal proof provided in Appendix D. Given our assumption that p_s^F are exogenous and finite for all s, the price indices P_s are bounded for all labor allocations $\boldsymbol{L}$ and all s. This immediately implies that if sector k is essential of order 0 (i.e., $e_k > 0$ or $E_k > 0$), then $\text{VMPL}_k \to \infty$ whenever $\theta_k \varepsilon_k < 1$. If instead $e_k = 0$ and $E_k = 0$, and VMPL_k remains bounded as $L_k \to 0$, then our assumption that all sectors are essential implies the existence of a downstream sector s that is essential of order 0, relies on intermediates from k (directly or indirectly), and for which $L_s \to 0$. For such a sector, $\text{VMPL}_s \to \infty$ if $\theta_s \varepsilon_s < 1$.

Therefore, if $\theta_k \varepsilon_k < 1$ for all k, then having $L_k \to 0$ for some sector k necessarily implies that either $\text{VMPL}_k \to \infty$ or $\text{VMPL}_s \to \infty$ for a downstream sector s connected to k. As a result, the complementary slackness condition (6) cannot be satisfied in the limit as $L_k \to 0$. In other words, when $\theta_k \varepsilon_k < 1$ for all k, all equilibria must be interior. We summarize this result in the following proposition:

Proposition 2. *Suppose that $\theta_k \varepsilon_k < 1$ for all k. Then any equilibrium is interior.*

The condition $\theta_k \varepsilon_k < 1$ is the same condition as in Kucheryavyy *et al.* (2023), where it both rules out corners and guarantees uniqueness of equilibrium. However, with input-output linkages, the conditions that rule out corners and guarantee uniqueness are different. Intuitively, regardless of the presence of input-output linkages, the condition $\theta_k \varepsilon_k < 1$ ensures that as a sector's employment shrinks, this sector's domestic trade share shrinks slower than its employment ($\lambda_k / L_k \to \infty$ as $L_k \to 0$, which is equivalent to $p_k^{-\varepsilon_k} / L_k \to \infty$ as $L_k \to 0$). This generates an infinite demand pull, causing VMPL_k to shoot off to infinity. The input-output structure does not affect this behavior, because as a sector's employment disappears, this sector becomes effectively disconnected from the rest of the production network. The only thing that matters is the elasticity of domestic trade share in this sector with respect to own employment at the boundary with zero employment. By contrast, the uniqueness condition (UC) is stricter, because it must prevent explosive feedback effects that propagate through the input-output network when employment is perturbed around an interior equilibrium.

5.3 Third Step

In this step, we keep w fixed and focus on strictly positive labor allocations. We prove that (UC) guarantees a unique labor allocation that clears all product markets. Using the results of Proposition 1, we treat p_k as a function of $\boldsymbol{L}$ only. As discussed in Section 4, the goods market clearing conditions (8) for $k = 1, \ldots, K$ define a system of equations $\boldsymbol{L} = \boldsymbol{F}(\boldsymbol{L})$, with elements of $\boldsymbol{F}$ given by $F_k(\boldsymbol{L}) = \alpha_k \sum_{j=1}^{K} \tilde{\ell}_{kj} d_j$, where we suppress the dependence of $\tilde{l}_{kj}$ and d_j on $\boldsymbol{L}$ for brevity of notation.

We begin with the following proposition, which establishes the existence of an infinite family of closed rectangular regions in $\mathbb{R}_{+}^{K}$, each of which is mapped by $\boldsymbol{F}$ into itself and with no fixed points of $\boldsymbol{F}$ outside of these regions.

Proposition 3. *Assume that (UC) holds. Then for any $\underline{\delta} > 0$ and $\overline{\delta} > 0$ with $\underline{\delta} < \overline{\delta}$, there exist $\underline{\delta}_k > 0$ and $\overline{\delta}_k > 0$ for $k = 1, \ldots, K$ with $\max_k \underline{\delta}_k \leq \underline{\delta}$ and $\min_k \overline{\delta}_k \geq \overline{\delta}_k$, such that the closed rectangle $\overline{\mathcal{R}} \equiv [\underline{\delta}_1, \overline{\delta}_1] \times \ldots \times [\underline{\delta}_K, \overline{\delta}_K]$ is mapped by $\boldsymbol{F}$ into the open rectangle $\mathcal{R} \equiv (\underline{\delta}_1, \overline{\delta}_1) \times \ldots \times (\underline{\delta}_K, \overline{\delta}_K)$, that is, $\boldsymbol{F}(\boldsymbol{L}) \in \mathcal{R}$ for all $\boldsymbol{L} \in \overline{\mathcal{R}}$. Moreover, for any $\boldsymbol{L} \notin \overline{\mathcal{R}}$, we have $\boldsymbol{L} \neq \boldsymbol{F}(\boldsymbol{L})$.*

Proposition 3 also applies to $\ln \boldsymbol{F}$: one simply log-transforms the rectangles. The proof of this proposition is provided in Appendix E.1. There we derive the lower bounds $\underline{\delta}_k$ of the self-mapped rectangle under the condition $\theta_k \varepsilon_k < 1$ for all k, which is weaker than (UC) and coincides with the condition in Proposition 2 that rules out corner allocations. Condition (UC) is required only for establishing the upper bounds $\overline{\delta}_k$.

Combined with Brouwer's fixed-point theorem, Proposition 2 guarantees the existence of at least one strictly positive fixed point of $\boldsymbol{F}$. We summarize this result in the following corollary.

Corollary 1. *Assume that (UC) holds. Then there exists a strictly positive fixed point satisfying $\boldsymbol{L} = \boldsymbol{F}(\boldsymbol{L})$.*

As discussed in Section 4, the critical part of Step 3 is to show that the spectral radius of the Jacobian $J(\boldsymbol{L})$ of $\ln \boldsymbol{F}$, evaluated at any point $\boldsymbol{L}$, is less than 1. The result is stated in the next proposition.

Proposition 4. *Assume that (UC) holds. Then $\rho(J(\boldsymbol{L})) < 1$ for any positive vector $\boldsymbol{L}$.*

The proof of Proposition 4 is provided in Appendix E.3. With this result in hand, we next establish the uniqueness of the fixed point of $\ln \boldsymbol{F}$ by applying the following version of the Index Theorem.

Theorem 2 (The Index Theorem). *Let $\overline{\mathcal{R}}$ be a nonempty closed rectangular region of $\mathbb{R}^K$. Let $\mathcal{O}$ be an open set containing $\overline{\mathcal{R}}$ and $\boldsymbol{f} : \mathcal{O} \mapsto \mathbb{R}^K$ be a continuous function which is continuously differentiable at every $\boldsymbol{x} \in \mathcal{S}$, where $\mathcal{S} \equiv \left\{\boldsymbol{x} \in \overline{\mathcal{R}} | \boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x})\right\}$ denotes the set of fixed points of $\boldsymbol{f}$ over $\overline{\mathcal{R}}$. Assume the following: (i) for all $\boldsymbol{x} \in \overline{\mathcal{R}}$, $\boldsymbol{f}(\boldsymbol{x}) \in \overline{\mathcal{R}} \setminus \mathrm{bd}\left(\overline{\mathcal{R}}\right)$, where $\mathrm{bd}\left(\overline{\mathcal{R}}\right)$ is the boundary of $\overline{\mathcal{R}}$; and (ii) for all $\boldsymbol{x} \in \mathcal{S}$, $\det\left(I - \nabla \boldsymbol{f}(\boldsymbol{x})\right) \neq 0$, where $\nabla \boldsymbol{f}(\boldsymbol{x})$ denotes the Jacobian matrix of $\boldsymbol{f}(\boldsymbol{x})$. Then $\mathcal{S}$ is finite and $\sum_{\boldsymbol{x} \in \mathcal{S}} \mathrm{sign}\left(\det\left(I - \nabla \boldsymbol{f}(\boldsymbol{x})\right)\right) = 1$, where $\mathrm{sign}(a) = -1$ if $a < 0$, $\mathrm{sign}(a) = 1$ if $a > 0$, and $\mathrm{sign}(a) = 0$ if $a = 0$.*

Theorem 2 is an implication of the Poincaré-Hopf Theorem (see, e.g., Simsek *et al.*, 2007), and its proof is provided in Appendix E.4. Any log-transformed rectangular region $\ln \overline{\mathcal{R}}$ from Proposition 3 satisfies assumption (i) of the theorem applied to $\mathbf{ln}\, \boldsymbol{F}$. Moreover, Proposition 3 also ensures that $\ln \boldsymbol{F}$ has no fixed points outside $\ln \overline{\mathcal{R}}$. Letting $\mathcal{S} \subset \ln \overline{\mathcal{R}}$ denote the set of all fixed points of $\ln \boldsymbol{F}$, Proposition 4 implies that $\det\left(I - J\left(\boldsymbol{L}^*\right)\right) \neq 0$ for all $\boldsymbol{L}^* \in \mathcal{S}$. Thus, assumption (ii) of Theorem 2 is also satisfied. The theorem therefore implies that $\mathcal{S}$ is finite and that $\sum_{\boldsymbol{L}^* \in \mathcal{S}} \mathrm{sign}\left(\det\left(I - J\left(\boldsymbol{L}^*\right)\right)\right) = 1$. Furthermore, since $\rho\left(J\left(\boldsymbol{L}^*\right)\right) < 1$, it follows that $\det\left(I - J\left(\boldsymbol{L}^*\right)\right) > 0$ for all $\boldsymbol{L}^* \in \mathcal{S}$, and hence $\mathrm{sign}\left(\det\left(I - J\left(\boldsymbol{L}^*\right)\right)\right) = 1$ for all $\boldsymbol{L}^* \in \mathcal{S}$. This implies that $\mathcal{S}$ contains exactly one point. Applying this conclusion to our economy yields the main proposition of Step 3.

Proposition 5. *Assume that (UC) holds. Then for any given $w > 0$ there exist unique $\boldsymbol{\lambda} \in (0,1)^K$, and positive $\boldsymbol{p}$ and $\boldsymbol{L}$ that solve the system of equations (3), (8), and (10).*

Finally, even though Proposition 4 does not deliver a norm under which $\ln \boldsymbol{F}$ (or $\boldsymbol{F}$) is a *global* contraction, the properties of $\boldsymbol{F}$ nevertheless ensure that its fixed point can be obtained by simple iteration from any positive initial vector of labor allocations. This result is stated in the following proposition.

Proposition 6. *Assume that (UC) holds. Then, for any $w > 0$ and any positive vector of labor allocations $\boldsymbol{L}^{(0)}$, the sequence defined by $\boldsymbol{L}^{(t+1)} = \boldsymbol{F}\left(\boldsymbol{L}^{(t)}\right)$ for $t = 0, 1, \ldots$ converges to the unique fixed point of $\boldsymbol{F}$.*

The proof of Proposition 6 is provided in Appendix E.5 and employs a "sandwich" argument based on the monotonicity of $\boldsymbol{F}$. The key insight is that Proposition 3 ensures that for any positive vector $\boldsymbol{L}^{(0)}$, there exists a rectangle $\overline{\mathcal{R}}$ containing $\boldsymbol{L}^{(0)}$ that is mapped into itself by $\boldsymbol{F}$. Since $\boldsymbol{F}$ is non-decreasing given $J_{ks}(\boldsymbol{L}) \geq 0$ for all k, s and any $\boldsymbol{L}$, then iterating $\boldsymbol{F}$ from the lower and upper corners of this rectangle, $\underline{\boldsymbol{L}}^{(0)} \equiv \left(\underline{\delta}_1, \ldots, \underline{\delta}_K\right)$ and $\overline{\boldsymbol{L}}^{(0)} \equiv \left(\overline{\delta}_1, \ldots, \overline{\delta}_K\right)$, produces sequences $\left\{\underline{\boldsymbol{L}}^{(t)}\right\}$ and $\left\{\overline{\boldsymbol{L}}^{(t)}\right\}$ that converge to the unique fixed point of $\boldsymbol{F}$. Moreover, because $\underline{\boldsymbol{L}}^{(0)} \leq \boldsymbol{L}^{(0)} \leq \overline{\boldsymbol{L}}^{(0)}$, monotonicity of $\boldsymbol{F}$ ensures that the iterates $\boldsymbol{L}^{(t)}$ remain sandwiched between $\underline{\boldsymbol{L}}^{(t)}$ and $\overline{\boldsymbol{L}}^{(t)}$ for all t. Therefore $\boldsymbol{L}^{(t)}$ also converges to the unique fixed point of $\boldsymbol{F}$.

5.4 Fourth Step

In this step, we prove the existence of a unique wage that clears the labor market. Given Proposition 5, we can treat $\boldsymbol{\lambda}$, $\boldsymbol{p}$, and $\boldsymbol{L}$ as functions of w, with $\boldsymbol{L}$ being the fixed point of $\boldsymbol{F}$. We begin with the following lemma.

Lemma 1. *Assume that (UC) holds. Then, for each k, the functions $L_k(w)$, $[p_k(w)]^{-\varepsilon_k}$, and $\lambda_k(w)$ are nonincreasing in w.*

The proof of Lemma 1 is provided in Appendix F.1, and proceeds by directly computing the derivatives $\mathrm{d}\ln L_k(w)/\mathrm{d}\ln w$, $\mathrm{d}\ln \lambda_k(w)/\mathrm{d}\ln w$, and $\mathrm{d}\ln [p_k(w)]^{-\varepsilon_k}/\mathrm{d}\ln w$, then showing that each is nonpositive. To establish this nonpositivity, we use the fact that $J(\boldsymbol{L})$ is a nonnegative matrix with $\rho(J(\boldsymbol{L})) < 1$, as shown in Proposition 4.

The next lemma establishes the range of values of $\boldsymbol{L}(w)$ as w varies between 0 and ∞, with the proof provided in Appendix F.2.

Lemma 2. *Assume that (UC) holds. Then $\lim_{w\to 0}\sum_k L_k(w) = \infty$ and $\lim_{w\to\infty}\sum_k L_k(w) < \bar{L}$.*

Lemma 2 is the only point in the proof of Theorem 1 that requires invoking the assumption that $E_k > 0$ for at least one sector k. Intuitively, exports provide the sole means of financing imports, with total export value given by $\sum_k E_k p_k^{-\varepsilon_k}$. If no sector exported ($E_k = 0$ for all k), the model would admit only a degenerate equilibrium with $w = 0$ and $p_k = 0$ for all k, which is not of interest here. If, on the other hand, at least one sector exports, then as wages approach zero the economy generates unbounded demand for that sector's good — and hence for labor.

Lemma 1 implies that $\sum_k L_k(w)$ is a nonincreasing function of w, while Lemma 2 shows that this function ranges from ∞ down to a value strictly below $\bar{L}$ as w increases from 0 and ∞. Therefore, there exists a unique $w^* \in (0,\infty)$ such that $\sum_k L_k(w^*) = \bar{L}$. This establishes the existence and uniqueness of equilibrium in the economy described in Section 2, completing the proof of Theorem 1.

6 Welfare Analysis

In this section we study the implications of our model for the gains from trade, showing in particular that condition (UC) guarantees positive gains from trade.

Given upper-tier Cobb-Douglas preferences with expenditure shares e_k, welfare is $W = w\prod_k P_k^{-e_k} = w\prod_k \lambda_k^{-e_k/\varepsilon_k} p_k^{-e_k}$.[17] Combined with the result in (10), the gains from

[17] In this section, we assume without loss of generality that $e_k > 0$ for all k.

trade are

$$\mathrm{GT} = 1 - \prod_k \lambda_k^{\psi_k/\varepsilon_k} \cdot \prod_k \left(L_k/L_k^A\right)^{-\theta_k \psi_k}, \tag{15}$$

where $\psi_k \equiv \sum_s \ell_{ks} e_s$ is the closed-economy *Domar weight* of sector k, and L_k^A is employment in sector k in the autarky equilibrium.[18] Relative to the case without intermediate goods, this expression replaces expenditure shares with Domar weights. The reason is that a change in the price index of good k now affects welfare not only through the direct channel — lower prices of final consumption captured by e_k — but also through indirect channels. A lower price for good k reduces the cost of goods that use it as an input, which in turn lowers the prices of goods that use those goods as inputs, and so on. The Domar weight ψ_k captures this full cascade of effects on the consumer price index: ℓ_{ks} captures how much a productivity shock in sector k changes the price of sector s through direct and indirect forward input-output linkages; the Domar weight ψ_k takes the average of these effects weighted by final expenditure shares, reflecting households' exposure to sector k through forward linkages (Baqaee and Farhi, 2021).

Although $\prod_k \lambda_k^{\psi_k/\varepsilon_k} < 1$, the gains from trade could in principle be negative if $\prod_k \left(L_k/L_k^A\right)^{-\theta_k \psi_k} > 1$. In words, if trade causes employment to fall in sectors with high Domar-weighted scale elasticities, it would then lead to what we will refer to as EES-induced productivity losses. These losses could outweigh the gains associated with accessing foreign goods. The question is whether the net effect from trade can ever be negative. Our second main result in the paper (along with Theorem 1) states that this cannot happen if condition (UC) holds.

Theorem 3. *Condition (UC) ensures positive gains from trade for any trade pattern,* $\mathrm{GT} > 0$*.*

The remainder of this section presents the proof, which proceeds in three steps. First, we introduce a lower bound for GT, denoted as GT^*. Second, we show that the monotonic transformation of GT^* given by $-\ln\left(1 - \mathrm{GT}^*\right)$ is a convex function of the log of trade shares, $\ln \boldsymbol{\lambda}$. Finally, we demonstrate that starting from autarky, any small increase in trade leads to an increase in GT^*. Together, these steps establish that $\mathrm{GT} > 0$.

Step 1. Consider the following function of trade shares:

$$\mathrm{GT}^*\left(\boldsymbol{\lambda}\right) \equiv 1 - \prod_k \lambda_k^{\psi_k/\varepsilon_k} \cdot \left(\frac{\sum_r \tilde{\ell}_{kr}\left(\boldsymbol{\lambda}\right) \lambda_r e_r}{\psi_k}\right)^{-\theta_k \psi_k},$$

[18] As in Costinot and Rodríguez-Clare (2014), we define the gains from trade as the negative of the percentage change in real income arising from a move to autarky, $\mathrm{GT} \equiv 1 - W^A/W$.

where we have used $wL_k^A = \psi_k w\bar{L}$.[19] The expression GT^* differs from GT by ignoring employment associated with exports. In particular, since

$$wL_k = \sum_r \tilde{\ell}_{kr}\left[\lambda_r e_r w\bar{L} + E_r p_r^{-\varepsilon_r}\right] \geq \sum_r \tilde{\ell}_{kr}\lambda_r e_r w\bar{L},$$

it follows that $\mathrm{GT} \geq \mathrm{GT}^*$, and hence a sufficient condition for positive gains from trade is $\mathrm{GT}^* > 0$ for all $\boldsymbol{\lambda}$.[20] The reason is that GT^* omits the EES-induced productivity gains from exports, while still capturing the EES-induced productivity losses from import competition.

Step 2. Let $\boldsymbol{y} \equiv \ln\boldsymbol{\lambda}$ and consider the function $f(\boldsymbol{y}) \equiv -\ln\left(1 - \mathrm{GT}^*(\exp\boldsymbol{y})\right)$.[21] As shown in Appendix G, $f(\boldsymbol{y})$ is convex. Intuitively, the standard gains-from-trade component is log-linear in trade shares while the EES-induced productivity losses from increased imports are attenuated by weaker input-output linkages in the open economy.

Step 3. Simple differentiation reveals that

$$\frac{\partial f(\mathbf{0})}{\partial y_s} = \psi_s \cdot \left(\sum_k \theta_k \ell_{ks} - 1/\varepsilon_s\right).$$

Our condition (UC) immediately implies $\partial f\left(\mathbf{0}\right)/\partial y_s < 0$.

Finally, for any $\tilde{\boldsymbol{y}} \leq \mathbf{0}$ such that $\tilde{y}_k \neq 0$ for at least some k, consider $g\left(t\right) \equiv f\left(t\tilde{\boldsymbol{y}}\right)$ for $t \in [0,1]$. We have $g'\left(t\right) = \sum_s \tilde{y}_s \partial f\left(t\tilde{\boldsymbol{y}}\right)/\partial y_s$, and so $g'\left(0\right) > 0$. Moreover, since $g''\left(t\right) = \tilde{\boldsymbol{y}}^T H_f \tilde{\boldsymbol{y}}$, where H_f is the Hessian of f evaluated at $t\tilde{\boldsymbol{y}}$, and since f is convex, it follows that $g''\left(t\right) \geq 0$ for all $t \in [0,1]$. Given that $g'\left(0\right) > 0$, this implies $g'\left(t\right) > 0$ for all $t \in [0,1]$. As $g\left(0\right) = 0$, it follows that $g\left(t\right) > 0$ for all $t \in (0,1]$, and in particular $f\left(\tilde{\boldsymbol{y}}\right) = g\left(1\right) > 0$. Hence $\mathrm{GT} \geq \mathrm{GT}^* > 0$ for any $\boldsymbol{\lambda}$.

[19] We formally define the function GT^* on the domain $\boldsymbol{\lambda} \in \left(0,\overline{\lambda}\right)^K$, where $\overline{\lambda} > 1$ is such that matrix $\tilde{\mathcal{L}}\left(\boldsymbol{\lambda}\right)$ is well-defined for $\boldsymbol{\lambda} = \left(\overline{\lambda},\ldots,\overline{\lambda}\right)$. The argument in Steps 2 and 3 below relies on differentiation of $\mathrm{GT}^*\left(\boldsymbol{\lambda}\right)$ at any point $\boldsymbol{\lambda} \in (0,1]^K$, which formally requires considering an open set containing $(0,1]^K$. Since $\sum_s \alpha_{sk} < 1$ for all k, we can always choose a small enough $\overline{\lambda} > 1$ such that $\sum_s \overline{\lambda}\alpha_{sk} < 1$ for all k, so that matrix $I - D_\lambda A$ is invertible for $\boldsymbol{\lambda} = \left(\overline{\lambda},\ldots,\overline{\lambda}\right)$.

[20] The converse is not necessarily true: in principle we could have $\mathrm{GT}^* < 0$ while having positive gains from trade, $\mathrm{GT} > 0$.

[21] The function f is defined for $\boldsymbol{y} \in \left(-\infty, \ln\overline{\lambda}\right)^K$.

7 Final Remarks

We have shown that in quantitative trade models with sector-level external economies of scale and input-output linkages, a simple condition — linking trade elasticities, scale elasticities, and input-output coefficients — ensures both uniqueness of equilibrium and positive gains from trade. This result generalizes prior work by allowing for roundabout production and highlighting the dual role of EES in shaping equilibrium and welfare.

Whether sufficient condition (UC) holds empirically remains an open question. Existing estimates of scale elasticities, such as those in Bartelme *et al.* (2025) and Lashkaripour and Lugovskyy (2023), are based on models where EES depend on gross output rather than value added, and thus are not directly applicable to our framework. Still, we can get a sense of the magnitude of the upper bound on scale elasticities implied by condition (UC). Since $\sum_s \alpha_s \ell_{sk} = 1$ this condition can be rewritten as $\varepsilon_k \bar{\gamma}_k < 1$ for all k, where $\bar{\gamma}_k \equiv \sum_s \gamma_s \omega_{sk}$ is an average scale elasticity with weights $\omega_{sk} \equiv \alpha_s \ell_{sk}$.[22] With a common scale elasticity across sectors (i.e., $\gamma_k = \gamma$ for all k), this condition reduces to $\gamma \varepsilon_k < 1$ for all k. If the trade elasticity is equal to 5 in all sectors — a standard assumption in quantitative trade models, see for instance Cappariello *et al.* (2020) and (Galle *et al.*, 2023) — then this implies an upper bound of 0.2 for γ.

It is important to emphasize that condition (UC) is sufficient but not necessary for uniqueness. A weaker condition, namely $\varepsilon_k \gamma_k < 1$ for all k, already rules out the most pathological case of multiplicity: that a sector may have zero employment in some equilibria and positive employment in others. This weaker condition is even sufficient to ensure uniqueness when domestic trade shares in intermediates are zero (see footnote 12). However, high domestic trade shares imply that amplification through domestic input-output linkages can be stronger, and ensuring uniqueness then requires the stricter condition (UC).

Our analysis has been conducted under the small open economy (SOE) assumption, which simplifies matters in two key ways. First, it allows us to isolate the determination of labor allocations in Home, abstracting from the interdependence between labor allocations in Home and abroad that arises in the general case. Second, the SOE assumption reduces labor market clearing to solving for a single wage in Home. Uniqueness of this equilibrium is assured by showing that aggregate labor demand is downward-sloping. Outside the SOE, however, one must solve for the full vector of wages across countries simultaneously. KLR tackled this problem in a model without intermediate goods and were able to prove uniqueness only in the case of two countries. With input-output loops, even

[22] $\sum_s \omega_{sk} = 1$ since $\{\sum_s \alpha_s \ell_{sk}\} = \left[(I-A)^{-1}\right]^T \left[I - A^T\right] \iota = \iota$, where ι is the $K \times 1$ vector of ones.

this limited result seems mathematically difficult to replicate.

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Appendices

A Local Contraction in Logs versus Levels

Here we show that the local contraction property of the mapping $\boldsymbol{F}$ is equivalent to that of its log-transformation $\ln \boldsymbol{F}$. This can be established in two ways: via the spectral radius and via vector norms.

Spectral radius argument. Let $\nabla \boldsymbol{F}(\boldsymbol{L}^*)$ be the Jacobian matrix of $\boldsymbol{F}(\boldsymbol{L})$ evaluated at its fixed point $\boldsymbol{L}^*$, and — following the notation in the main text — let $J(\boldsymbol{L}^*)$ be the corresponding Jacobian matrix of $\ln \boldsymbol{F}$ at the same point. The mapping $\boldsymbol{F}$ is a local contraction at $\boldsymbol{L}^*$ if $\rho(\nabla \boldsymbol{F}(\boldsymbol{L}^*)) < 1$, and $\ln \boldsymbol{F}$ is a local contraction at $\ln \boldsymbol{L}^*$ if $\rho(J(\boldsymbol{L}^*)) < 1$.

Let $D_{\boldsymbol{F}(\boldsymbol{L}^*)}$ and $D_{\boldsymbol{L}^*}$ denote diagonal matrices with diagonal entries $F_k(\boldsymbol{L}^*)$ and L_k^*, respectively. By definition, $\nabla \boldsymbol{F}(\boldsymbol{L}^*) = D_{\boldsymbol{F}(\boldsymbol{L}^*)} J(\boldsymbol{L}^*) D_{\boldsymbol{L}^*}^{-1}$. Since $\boldsymbol{L}^*$ is a fixed point of $\boldsymbol{F}$, we have $D_{\boldsymbol{F}(\boldsymbol{L}^*)} = D_{\boldsymbol{L}^*}$, so $\nabla \boldsymbol{F}(\boldsymbol{L}^*) = D_{\boldsymbol{L}^*} J(\boldsymbol{L}^*) D_{\boldsymbol{L}^*}^{-1}$. This is a similarity transformation, which leaves the spectral radius unchanged, implying $\rho(\nabla \boldsymbol{F}(\boldsymbol{L}^*)) = \rho(J(\boldsymbol{L}^*))$. Thus, $\rho(\nabla \boldsymbol{F}(\boldsymbol{L}^*)) < 1$ if and only if $\rho(J(\boldsymbol{L}^*)) < 1$.

Norm-based argument. As stated in Section 4, $\ln \boldsymbol{F}$ is a local contraction around $\boldsymbol{L}^*$ if there exists a vector norm $\|\cdot\|$ with induced matrix norm $\|\|\cdot\|\|$ such that $\|\|J(\boldsymbol{L}^*)\|\| < 1$. Similarly, $\boldsymbol{F}$ is a local contraction if there exists a vector norm $\|\cdot\|$ with induced matrix norm $\|\|\cdot\|\|$ such that $\|\|\nabla \boldsymbol{F}(\boldsymbol{L}^*)\|\| < 1$.

Suppose $\ln \boldsymbol{F}$ is a local contraction around $\boldsymbol{L}^*$ with respect to some vector norm $\|\cdot\|_a$ and associated matrix norm $\|\|\cdot\|\|_a$. Define a new vector norm by $\|v\|_b \equiv \left\| D_{\boldsymbol{L}^*}^{-1} v \right\|_a$ for $v \in \mathbb{R}^K$, with the corresponding induced matrix norm $\|\|M\|\|_b \equiv \left\|\left\| D_{\boldsymbol{L}^*}^{-1} M D_{\boldsymbol{L}^*} \right\|\right\|_a$ for any $K \times K$ matrix M (see Theorem 5.6.7 in Horn and Johnson, 2013). With these definitions, $\|\|\nabla \boldsymbol{F}(\boldsymbol{L}^*)\|\|_b = \|\|J(\boldsymbol{L}^*)\|\|_a$. Therefore, $\|\|\nabla \boldsymbol{F}(\boldsymbol{L}^*)\|\|_b < 1$ if and only if $\|\|J(\boldsymbol{L}^*)\|\|_a < 1$. Moreover, we have $\|\mathrm{d}\boldsymbol{F}(\boldsymbol{L}^*)\|_b = \|\mathrm{d}\ln \boldsymbol{F}(\boldsymbol{L}^*)\|_a$ and $\|\mathrm{d}\boldsymbol{L}\|_b = \|\mathrm{d}\ln \boldsymbol{L}^*\|_a$, and thus $\|\mathrm{d}\boldsymbol{F}(\boldsymbol{L}^*)\|_b < \|\mathrm{d}\boldsymbol{L}\|_b$ is equivalent to $\|\mathrm{d}\ln \boldsymbol{F}(\boldsymbol{L}^*)\|_a < \|\mathrm{d}\ln \boldsymbol{L}^*\|_a$.

B Reweighted Sup Norms and the Spectral Radius Condition

Let M be a nonnegative matrix. If there exists a norm $\|\|\cdot\|\|$ such that $\|\|M\|\| < 1$, then $\rho(M) < 1$ (Theorem 5.6.9 in Horn and Johnson, 2013).

Conversely, suppose that $\rho(M) < 1$. Choose any $\tilde{\rho}$ such that $\rho(M) < \tilde{\rho} < 1$, and define $\boldsymbol{x} \equiv (\tilde{\rho} I - M)^{-1} \iota$, where $\iota \equiv (1, \ldots, 1)^T$. Since $\tilde{\rho} > \rho(M)$, the matrix $(\tilde{\rho} I - M)^{-1}$ is nonnegative, so all entries of $\boldsymbol{x}$ are positive. Moreover, $M\boldsymbol{x} = \tilde{\rho}\boldsymbol{x} - \iota < \tilde{\rho}\boldsymbol{x}$. Let D_x be the diagonal matrix with $\boldsymbol{x}$ on its diagonal, and define the matrix norm $\|\|M\|\|_M \equiv \left\|\left\| D_x^{-1} M D_x \right\|\right\|_\infty$. We have

$$\|\|M\|\|_M = \max_i \frac{1}{x_i} \sum_j M_{ij} x_j < \max_i \frac{1}{x_i} \tilde{\rho} x_i = \tilde{\rho} < 1.$$

The vector norm that induces $\|\|\cdot\|\|_M$ is given by $\|\boldsymbol{v}\|_M \equiv \left\| D_x^{-1} \boldsymbol{v} \right\|_\infty$ for any $\boldsymbol{v} \in \mathbb{R}^K$, and can be interpreted as a reweighting of entries of $\boldsymbol{v}$. Thus, the condition $\rho(M) < 1$ is equivalent to the existence of a norm $\|\|\cdot\|\|_M$ under which $\|\|M\|\|_M < 1$.

C First Step: Proof of Proposition 1

To begin, note that for any fixed positive labor allocation $\boldsymbol{L}$, equations (3) and (10), taken in isolation, could admit solutions with $\lambda_k = 0$ and $p_k^{-\varepsilon_k} = 0$, or with $\lambda_k = 1$ and $p_k^{-\varepsilon_k} = \infty$. Such solutions are not of interest because, via the goods market-clearing condition (8), they are consistent only with $L_k = 0$ or $L_k = \infty$. Indeed, from (3), $\lambda_k = 0$ implies $p_k^{-\varepsilon_k} = 0$, and thus $L_k = 0$ by (8). Likewise, $\lambda_k = 1$ only if $p_k^{-\varepsilon_k} \to \infty$, which in turn implies $L_k \to \infty$. Since we restrict attention to finite, positive L_k, we therefore look for solutions to (3) and (10) only with $\lambda_k \in (0, 1)$ and finite, positive $p_k^{-\varepsilon_k}$ for all k.

Given $\lambda_k \in (0, 1)$, we can solve for p_k from (3) and substitute the result into (10) to get

$$(1 - \lambda_k) \prod_s \lambda_s^{-b_{sk}} = B_k, \quad k = 1, \ldots, K; \tag{16}$$

where $b_{sk} \equiv \varepsilon_k \ell_{sk} / \varepsilon_s$ and $B_k \equiv \left[p_k^F\right]^{-\varepsilon_k} w^{\varepsilon_k} \prod_s \left(\overline{T}_s L_s^{\theta_s}\right)^{-\varepsilon_k \ell_{sk}}$. Given that p_k^F and L_k are finite and positive for all k, B_k is finite and positive for all k.

Existence of a Solution. We can write (16) as $f_k(\boldsymbol{\lambda}) = 0$ for $k = 1, \ldots, K$, where $f_k(\boldsymbol{\lambda}) \equiv \lambda_k + B_k \prod_s \lambda_s^{b_{sk}} - 1$. Since $b_{sk} \geq 0$ and $b_{ss} > 0$ for all k and s, we have $f_k(\boldsymbol{\lambda}) = -1 \leq 0$ if $\lambda_k = 0$ and $f_k(\boldsymbol{\lambda}) = B_k \prod_{s \neq k} [\lambda_s]^{b_{sk}} \geq 0$ if $\lambda_k = 1$. By the Poincaré-Miranda theorem (Kulpa, 1997), applied to the cube $[0, 1]^K$, there exists a point $\boldsymbol{\lambda} \in [0, 1]^K$ at which $f_k(\boldsymbol{\lambda}) = 0$ for all k, giving a solution to the system. This solution is positive since $f_k(\boldsymbol{\lambda}) = -1 \neq 0$ if $\lambda_k = 0$ for any k, and it satisfies $\lambda_k < 1$ for all k because $f_k(\boldsymbol{\lambda}) > 0$ if $\lambda_k = 1$, given that $\lambda_s > 0$ for all s. Therefore, the original system (16) has an interior solution.

Uniqueness. Define $x_k \equiv \ln \lambda_k$ and $\boldsymbol{x} \equiv (x_1, \ldots, x_K)^T$. Taking logs on both sides of (16) and rearranging terms, we obtain a system of nonlinear equations $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{0}$ for $\boldsymbol{x} \in (-\infty, 0)^K$, where $\boldsymbol{g}(\boldsymbol{x}) \equiv (g_1(\boldsymbol{x}), \ldots, g_k(\boldsymbol{x}))^T$ and $g_k(\boldsymbol{x}) \equiv -\ln(1 - \exp\{x_k\}) + \sum_s b_{sk} x_s + \ln B_k$. The Jacobian matrix of this system, $\{\partial g_k(\boldsymbol{x}) / \partial x_r\}$, is given by

$$\{\partial g_k(\boldsymbol{x}) / \partial x_r\} = (I - D_\lambda)^{-1} D_\lambda + D_\varepsilon \mathcal{L}^T D_\varepsilon^{-1},$$

where we used $b_{rk} = \varepsilon_k \ell_{rk} / \varepsilon_r$. Here D_λ and D_ε are diagonal matrices with elements λ_k and ε_k on the diagonals, correspondingly. The matrix $\mathcal{L}$ is an inverse M-matrix, which is a special case of a P-matrix. Transposing a P-matrix, multiplying by positive diagonal matrices from left or right, and adding a positive diagonal matrix results in a P-matrix. Hence, $\{\partial g_k(\boldsymbol{x}) / \partial x_r\}$ is a P-matrix.

Supposing $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{0}$ has multiple solutions, let $\Omega \subset (0, 1)^K$ be a closed box containing more than one solution. The argument above establishes that the Jacobian matrix of $\boldsymbol{g} : \Omega \to \mathbb{R}^K$ is a P-matrix in all points of Ω. By Theorem 4 of Gale and Nikaido (1965), $\boldsymbol{g}$ is therefore a univalent mapping in Ω, contradicting the supposition of multiple solutions for $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{0}$ in Ω. Thus, $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{0}$ has a unique solution.

The uniqueness of a solution to the system of equations (3) and (10) in λ_k and p_k for $k = 1, \ldots, K$ implies that these equations define the implicit functions $\boldsymbol{\lambda}(w, \boldsymbol{L})$ and $\boldsymbol{p}(w, \boldsymbol{L})$. This completes the proof of part 1 of Proposition 1.

Derivatives. Treating at λ_k and p_k as functions of $\boldsymbol{L}$ and totally differentiating (3) and (10), we get $-\varepsilon_k \mathrm{d}\ln p_k = (1-\lambda_k)^{-1} \mathrm{d}\ln\lambda_k$ and

$$-\varepsilon_k \mathrm{d}\ln p_k = \mathrm{d}\ln\lambda_k + \sum_s \varepsilon_k \ell_{sk}\theta_s \mathrm{d}\ln L_s - \sum_s \frac{\varepsilon_k \ell_{sk}}{\varepsilon_s} \mathrm{d}\ln\lambda_s,$$

where we drop the dependence of λ_k and p_k on $\boldsymbol{L}$ for brevity. Substituting the first equation into the second and using matrix notation, we get

$$\left(D_\lambda (I - D_\lambda)^{-1} + D_\varepsilon \mathcal{L}^T D_\varepsilon^{-1}\right) \mathrm{d}\ln\boldsymbol{\lambda} = D_\varepsilon \mathcal{L}^T D_\theta \mathrm{d}\ln\boldsymbol{L},$$

where $\mathrm{d}\ln\boldsymbol{\lambda} \equiv (\mathrm{d}\ln\lambda_1, \ldots, \mathrm{d}\ln\lambda_K)^T$, $\mathrm{d}\ln\boldsymbol{L} \equiv (\mathrm{d}\ln L_1, \ldots, \mathrm{d}\ln L_K)^T$, and D_θ is a diagonal matrix with elements θ_k on the diagonal. Observe that

$$\left(D_\lambda (I - D_\lambda)^{-1} + D_\varepsilon \mathcal{L}^T D_\varepsilon^{-1}\right)^{-1} D_\varepsilon \mathcal{L}^T = (I - D_\lambda) D_\varepsilon \tilde{\mathcal{L}}^T,$$

which yields $\mathrm{d}\ln\boldsymbol{\lambda} = (I - D_\lambda) D_\varepsilon \tilde{\mathcal{L}}^T D_\theta \mathrm{d}\ln\boldsymbol{L}$. Combining this with $\mathrm{d}\ln\boldsymbol{p} = -D_\varepsilon^{-1}(I - D_\lambda)^{-1} \mathrm{d}\ln\boldsymbol{\lambda}$, we get $\mathrm{d}\ln\boldsymbol{p} = -\tilde{\mathcal{L}}^T D_\theta \mathrm{d}\ln\boldsymbol{L}$, and so the partial derivatives are given by $\partial\ln p_k/\partial\ln L_s = -\theta_s\tilde{\ell}_{sk}$. This completes the proof of part 2 of Proposition 1.

D Second Step: Proof of Proposition 2

Suppose that $\theta_k\varepsilon_k < 1$ for all k. Fix a wage $w > 0$, and consider any sequence of labor allocations $\{\boldsymbol{L}_t\}_{t=1}^\infty$, where $\boldsymbol{L}_t \equiv (L_{1,t}, \ldots, L_{K,t})$ with $L_{k,t} > 0$ for all k and t. Suppose this sequence converges to a finite limit $\boldsymbol{L}_\infty \equiv (L_{1,\infty}, \ldots, L_{K,\infty})$, with $L_{k_0,\infty} = 0$ for some k_0.

Let $p_{k,t}$ and $\lambda_{k,t}$ be the values of prices and trade shares for each $\boldsymbol{L}_t$, as implied by Proposition 1 from the first step. For convenience, we restate expression (14) for VMPL_k:

$$\mathrm{VMPL}_k = \alpha_k w^{-\varepsilon_k\alpha_k} P_k^{\varepsilon_k(1-\alpha_{kk})} \left(\prod_{s\neq k} P_s^{-\varepsilon_k\alpha_{sk}}\right) \overline{T}_k^{\varepsilon_k} L_k^{\theta_k\varepsilon_k - 1} \left(e_k w\bar{L} + P_k^{-\varepsilon_k} E_k + \sum_s \alpha_{ks} w L_s/\alpha_s\right).$$

From $P_{k,t} = \left(p_{k,t}^{-\varepsilon_k} + \left[p_k^F\right]^{-\varepsilon_k}\right)^{-1/\varepsilon_k}$ we have $P_{k,t} \leq p_k^F$, and hence $\lim_{t\to\infty} P_{k,t}$ is finite given our assumption that p_k^F is exogenous and finite. Therefore, $\lim_{t\to\infty} P_{k,t}^{-\varepsilon_k} \geq \left[p_k^F\right]^{-\varepsilon_k} > 0$ for all k, where, with slight abuse of notation, we also allow the possibility that $\lim_{t\to\infty} P_{k,t}^{-\varepsilon_k} = \infty$. Simlarly, since $\alpha_{kk} \geq 1$ and $\alpha_{sk} \geq 0$, we have $\lim_{t\to\infty} P_{k,t}^{\varepsilon_k(1-\alpha_{kk})} > 0$ and $\lim_{t\to\infty} P_{s,t}^{-\varepsilon_k\alpha_{sk}} > 0$ for all k and s.[23] Given expression (14), it follows immediately that if sector k_0 is essential of order 0 (i.e., $e_{k_0} > 0$ or $E_{k_0} > 0$) then $\lim_{t\to\infty} \mathrm{VMPL}_{k_0,t} = \infty$ under the assumption $\theta_{k_0}\varepsilon_{k_0} < 1$.

Now suppose k_0 is essential of order greater than 0. Under our assumption that all sectors are essential, there exists at least one sector $s \neq k_0$ such that $\alpha_{k_0 s} > 0$ and s is essential of strictly lower order than k_0. Let $\mathcal{E}_{k_0}$ be the set of all such sectors. If $L_{s,\infty} \neq 0$ for some $s \in \mathcal{E}_{k_0}$, then again $\lim_{t\to\infty} \mathrm{VMPL}_{k_0,t} = \infty$. Otherwise, if $L_{s,\infty} = 0$ for all $s \in \mathcal{E}_{k_0}$, choose any $k_1 \in \mathcal{E}_{k_0}$ with lower essentiality order than k_0. We then apply the same analysis to sector k_1, which also satisfies $L_{k_1,\infty} = 0$. This process continues recursively: if k_1 is essential of order 0, or if $L_{s,\infty} \neq 0$ for some $s \in \mathcal{E}_{k_1}$, then $\lim_{t\to\infty} \mathrm{VMPL}_{k_1,t} = \infty$. Otherwise, choose any $k_2 \in \mathcal{E}_{k_1}$ with lower essentiality order than k_1 and repeat. Since the number of sectors is finite and each iteration moves to a sector of strictly lower essentiality order, this process must terminate. Therefore, we eventually reach a sector k' with $L_{k',\infty} = 0$ and $\lim_{t\to\infty} \mathrm{VMPL}_{k',t} = \infty$.

E Proofs for the Third Step

E.1 Proof of Proposition 3

In the proof of Proposition 3, we make the dependence of p_k, λ_k, d_k, and the matrix $\tilde{\mathcal{L}}$ on the labor allocations vector $\boldsymbol{L}$ explicit by writing $p_k(\boldsymbol{L})$, $\lambda_k(\boldsymbol{L})$, $d_k(\boldsymbol{L})$, and $\tilde{\mathcal{L}}(\boldsymbol{L})$.

The following lemma establishes the monotonicity of $\boldsymbol{F}$, which will be used to show the existence of both the lower and upper bounds $\underline{\delta}_k > 0$ and $\overline{\delta}_k > 0$ in Proposition 3.

Lemma 3. *Let $\boldsymbol{L}'$ and $\boldsymbol{L}''$ be two positive vectors of labor allocations. If $\boldsymbol{L}' \geq \boldsymbol{L}''$, then $\boldsymbol{F}(\boldsymbol{L}') \geq \boldsymbol{F}(\boldsymbol{L}'')$.*

Proof. From Proposition 1 and the definition (3) of λ_k, both $[p_k(\boldsymbol{L})]^{-\varepsilon_k}$ and $\lambda_k(\boldsymbol{L})$ are non-decreasing in L_s for all k and s. This implies $d_k(\boldsymbol{L}') \geq d_k(\boldsymbol{L}'')$ and $D_\lambda(\boldsymbol{L}') \geq D_\lambda(\boldsymbol{L}'')$. By the Neumann series expansion of $\tilde{\mathcal{L}}(\boldsymbol{L})$, we then have $\tilde{\mathcal{L}}(\boldsymbol{L}') \geq \tilde{\mathcal{L}}(\boldsymbol{L}'')$, and hence $\boldsymbol{F}(\boldsymbol{L}') \geq \boldsymbol{F}(\boldsymbol{L}'')$. $\square$

[23] The facts that $\alpha_{kk} \geq 1$ and $\alpha_{sk} \geq 0$ follow from the Neumann series expansion of $\mathcal{L} = (I-A)^{-1} = \sum_{t=0}^{\infty} A^t$.

E.1.1 Lower bound

For this part of Proposition 3, we assume that $\varepsilon_k \theta_k < 1$ for all k. This is the same assumption as in Proposition 2 and is weaker than condition (UC).

Let $\delta > 0$ and $\boldsymbol{\Delta}_\delta \equiv (\delta^{\epsilon_1}, \ldots, \delta^{\epsilon_K})$, where $\epsilon_1, \ldots, \epsilon_K$ are positive constants. We begin with the following auxiliary result.

Lemma 4. *If $\varepsilon_k \theta_k < 1$ for all k, then positive exponents $\{\epsilon_k\}_{k=1}^K$ can be chosen such that $\lim_{\delta \to 0} \dfrac{F_k(\boldsymbol{\Delta}_\delta)}{\delta^{\epsilon_k}} = \infty$ for all k.*

Proof. Let $\mathcal{H}_r$ be the set of sectors of essentiality order r, and let $R \geq 0$ be the maximal essentiality order. We assign the same value ϕ_r to ϵ_k for all sectors $k \in \mathcal{H}_r$, starting with $\phi_R = 1$ and then proceeding recursively for $r = R-1, \ldots, 0$. Specifically, once $\{\phi_{r+1}, ..., \phi_R\}$ have been chosen, we set $\phi_r = 0.5\,(1 - \overline{m})\,\phi_{r+1}$, where $\overline{m} \equiv \max_k \varepsilon_k \theta_k$. Since $\varepsilon_k \theta_k < 1$ for all k by assumption, this ensures $\phi_r > 0$.

Now consider any sector $k_r \in \mathcal{H}_r$ for $r \in \{1, ..., R-1\}$. There exists a chain of sectors $k_r \to k_{r-1} \to \cdots \to k_0$, where each k_i has essentiality order i and $\alpha_{k_i k_{i-1}} > 0$ for all $i = 1, \ldots, r$. Taking the $t = r$ term of the Neumann series expansion $\tilde{\mathcal{L}}(\boldsymbol{\Delta}_\delta) = \sum_{t=0}^{\infty} [D_\lambda(\boldsymbol{\Delta}_\delta)\, A]^t$, yields the lower bound

$$F_{k_r}(\boldsymbol{\Delta}_\delta) \geq \underline{F}_{k_r}(\boldsymbol{\Delta}_\delta) \equiv C_{k_r} \cdot \left(\prod_{i=1}^{r} \lambda_{k_i}(\boldsymbol{\Delta}_\delta) \right) \cdot d_{k_0}(\boldsymbol{\Delta}_\delta),$$

with $C_{k_r} \equiv \alpha_{k_r} \prod_{i=1}^{r} \alpha_{k_i k_{i-1}}$. If $r = 0$, the lower bound is simply $\underline{F}_{k_r}(\boldsymbol{\Delta}_\delta) \equiv C_{k_0} \cdot d_{k_0}(\boldsymbol{\Delta}_\delta)$ with $C_{k_0} \equiv \alpha_{k_0}$. To understand this, note that d_{k_0} directly requires $\lambda_{k_1} \alpha_{k_1 k_0} d_{k_0}$ of input k_1 (in value). In turn, this means d_{k_0} directly requires $\lambda_{k_2} \alpha_{k_2 k_1} \lambda_{k_1} \alpha_{k_1 k_0} d_{k_0}$ of input k_2 (in value). Following this progression reveals that d_{k_0} directly requires $\underline{F}_{k_r}(\boldsymbol{\Delta}_\delta)$ of input k_r (in value).

From (13), each $\lambda_{k_i}(\boldsymbol{\Delta}_\delta)$ can be written as $\bar{\lambda}_{k_i}(\boldsymbol{\Delta}_\delta) \cdot \delta^{\epsilon_{k_i} \varepsilon_{k_i} \theta_{k_i}}$, where

$$\bar{\lambda}_{k_i}(\boldsymbol{\Delta}_\delta) \equiv \overline{T}_{k_i}^{\varepsilon_{k_i}} w^{-\varepsilon_{k_i} \alpha_{k_i}} \left[P_{k_i}(\boldsymbol{\Delta}_\delta) \right]^{\varepsilon_{k_i}(1 - \alpha_{k_i k_i})} \left(\prod_{s \neq k_i} [P_s(\boldsymbol{\Delta}_\delta)]^{-\varepsilon_{k_i} \alpha_{s k_i}} \right).$$

Similarly, from the definition of d_k and equation 4, $d_{k_0}(\boldsymbol{\Delta}_\delta)$ can be written as $\bar{\lambda}_{k_0}(\boldsymbol{\Delta}_\delta)\, \bar{d}_{k_0}(\boldsymbol{\Delta}_\delta)\, \delta^{\epsilon_{k_0} \varepsilon_{k_0} \theta_{k_0}}$, where

$$\bar{d}_k(\boldsymbol{\Delta}_\delta) \equiv e_k \bar{L} + E_k w^{-1} [P_k(\boldsymbol{\Delta}_\delta)]^{-\varepsilon_k}.$$

Hence,

$$\frac{\underline{F}_{k_r}\left(\boldsymbol{\Delta}_\delta\right)}{\delta^{\epsilon_{k_r}}}=C_{k_r}\cdot\left(\prod_{i=0}^{r}\bar{\lambda}_{k_i}\left(\boldsymbol{\Delta}_\delta\right)\right)\bar{d}_{k_0}\left(\boldsymbol{\Delta}_\delta\right)\delta^{\mathcal{E}_r},$$

where $\mathcal{E}_r\equiv\sum_{i=0}^{r}\epsilon_{k_i}\varepsilon_{k_i}\theta_{k_i}-\epsilon_{k_r}$. The terms $\bar{\lambda}_{k_i}\left(\boldsymbol{\Delta}_\delta\right)$ and $\bar{d}_{k_0}\left(\boldsymbol{\Delta}_\delta\right)$ depend on $\boldsymbol{\Delta}_\delta$ only through price indices $P_1\left(\boldsymbol{\Delta}_\delta\right),\ldots,P_K\left(\boldsymbol{\Delta}_\delta\right)$ in non-positive powers. As $\delta\to0$, any such price index in non-positive power either converges to a positive finite value or diverges to infinity. Thus, $C_{k_r}\cdot\left(\prod_{i=0}^{r}\bar{\lambda}_{k_i}\left(\boldsymbol{\Delta}_\delta\right)\right)\bar{d}_{k_0}\left(\boldsymbol{\Delta}_\delta\right)$ either converges to a positive finite limit or diverges to infinity.

We next show that our choice of $\{\epsilon_k\}_{k=1}^{K}$ implies that $\mathcal{E}_r<0$ for any $r\in\{0,...,R\}$. This will establish that $\lim_{\delta\to0}\underline{F}_{k_r}\left(\boldsymbol{\Delta}_\delta\right)/\delta^{\epsilon_{k_r}}=\infty$, and hence also $\lim_{\delta\to0}F_{k_r}\left(\boldsymbol{\Delta}_\delta\right)/\delta^{\epsilon_{k_r}}=\infty$.

Our assignment of values to ε_k implies that $\mathcal{E}_r=\sum_{i=0}^{r}\phi_i\varepsilon_{k_i}\theta_{k_i}-\phi_r$. Since $\phi_i=0.5\left(1-\overline{m}\right)\phi_{i+1}$ for each $i=0,\ldots,r-1$, we get $\phi_i=\left[0.5\left(1-\overline{m}\right)\right]^{r-i}\phi_r$. Combined with $\varepsilon_{k_i}\theta_{k_i}\leq\overline{m}$ for all i, this then implies

$$\mathcal{E}_r\leq\phi_r\left(\overline{m}\sum_{i=0}^{r}\left[0.5\left(1-\overline{m}\right)\right]^{r-i}-1\right)<\phi_r\left(\overline{m}\sum_{j=0}^{\infty}\left[0.5\left(1-\overline{m}\right)\right]^{j}-1\right)=\phi_r\frac{\overline{m}-1}{\overline{m}+1}<0.$$

□

Now, applying Lemma 4, choose exponents $\{\epsilon_k\}_{k=1}^{K}$ such that $\lim_{\delta\to0}F_k\left(\boldsymbol{\Delta}_\delta\right)/\delta^{\epsilon_k}=\infty$ for all k. Then for any $B>1$ and each k, there exists $\underline{\chi}_k>0$ such that $F_k\left(\boldsymbol{\Delta}_\delta\right)\geq B\delta^{\epsilon_k}$ for all $\delta\in\left(0,\underline{\chi}_k\right]$. For any $\underline{\delta}>0$, let $\underline{\chi}\equiv\min\left\{0.5,\min_k\underline{\delta}^{1/\epsilon_k},\min_k\underline{\chi}_k\right\}$. We have $\boldsymbol{F}\left(\boldsymbol{\Delta}_\delta\right)\geq B\boldsymbol{\Delta}_\delta>\boldsymbol{\Delta}_\delta$ for all $\delta\in\left(0,\underline{\chi}\right]$. This yields two results. First, if $\boldsymbol{L}\geq\boldsymbol{\Delta}_{\underline{\chi}}$, then by Lemma 3, $\boldsymbol{F}\left(\boldsymbol{L}\right)\geq\boldsymbol{F}\left(\boldsymbol{\Delta}_{\underline{\chi}}\right)>\boldsymbol{\Delta}_{\underline{\chi}}$. Second, if $\boldsymbol{L}\leq\boldsymbol{\Delta}_{\underline{\chi}}$ then $\boldsymbol{F}\left(\boldsymbol{L}\right)\neq\boldsymbol{L}$. To see this, let $\delta\equiv\min_k L_k^{1/\epsilon_k}$. This implies $\boldsymbol{\Delta}_\delta\leq\boldsymbol{L}$ and hence by Lemma 3 we have (i) $\boldsymbol{F}\left(\boldsymbol{\Delta}_\delta\right)\leq\boldsymbol{F}\left(\boldsymbol{L}\right)$. Moreover, $\boldsymbol{L}\leq\boldsymbol{\Delta}_{\underline{\chi}}$ implies $\delta\leq\underline{\chi}$, and hence (ii) $\boldsymbol{F}\left(\boldsymbol{\Delta}_\delta\right)>\boldsymbol{\Delta}_\delta$. Results (i) and (ii) imply $\boldsymbol{F}\left(\boldsymbol{L}\right)>\boldsymbol{\Delta}_\delta$. Since for at least one k we have $L_k=\delta^{\epsilon_k}$, it follows that we cannot have $\boldsymbol{L}>\boldsymbol{\Delta}_\delta$. Combined with $\boldsymbol{F}\left(\boldsymbol{L}\right)>\boldsymbol{\Delta}_\delta$, this implies $\boldsymbol{F}\left(\boldsymbol{L}\right)\neq\boldsymbol{L}$.

Finally, setting $\underline{\delta}_k\equiv\underline{\chi}^{\epsilon_k}$ establishes the statement of Proposition 3 for the lower bound of the rectangular region $\overline{\mathcal{R}}$ of $\mathbb{R}^K$ mapped into itself by $\boldsymbol{F}$. Since $\underline{\chi}\leq\underline{\delta}^{1/\epsilon_k}$ and $\epsilon_k>0$ for all k, we have $\underline{\delta}_k=\underline{\chi}^{\epsilon_k}\leq\underline{\delta}$ for all k, as required by Proposition 3. Moreover, since $\underline{\chi}\leq0.5$, we have $\underline{\delta}_k<1$ for all k, which will be used later to argue that the rectangle $\overline{\mathcal{R}}$ is non-empty.

E.1.2 Upper bound

For this part of Proposition 3, we rely on condition (UC). Define $\boldsymbol{\Delta}_\delta \equiv (\delta, \ldots, \delta)$. We begin with the following result.

Lemma 5. *Assume that (UC) holds. Then* $\lim_{\delta\to\infty} \frac{F_k(\boldsymbol{\Delta}_\delta)}{\delta} = 0$ *for all* k.

Proof. From (3) and (10), we can write $\lambda_k(\boldsymbol{\Delta}_\delta) = 1 - \Lambda_k(\boldsymbol{\Delta}_\delta)$, where

$$\Lambda_k(\boldsymbol{\Delta}_\delta) \equiv \left[p_k^F\right]^{-\varepsilon_k} \left(w \prod_s \overline{T}_s^{-\ell_{sk}}\right)^{\varepsilon_k} \delta^{-\sum_s \varepsilon_k \ell_{sk}\theta_s} \prod_s \left[\lambda_s(\boldsymbol{\Delta}_\delta)\right]^{\varepsilon_k \ell_{sk}/\varepsilon_s}.$$

Since $\lambda_s(\boldsymbol{\Delta}_\delta) \in [0,1]$ for all s, the term $\prod_s [\lambda_s(\boldsymbol{\Delta}_\delta)]^{\varepsilon_k\ell_{sk}/\varepsilon_s}$ converges to a finite limit as $\delta \to \infty$. If $\sum_s \varepsilon_k \ell_{sk}\theta_s > 0$, then $\delta^{-\sum_s \varepsilon_k\ell_{sk}\theta_s} \to 0$, implying $\Lambda_k(\boldsymbol{\Delta}_\delta) \to 0$ and hence $\lambda_k(\boldsymbol{\Delta}_\delta) \to 1$. Consider the case when $\sum_s \varepsilon_k \ell_{sk}\theta_s = 0$. If we had $\lambda_k(\boldsymbol{\Delta}_\delta) \to 0$, then because $\ell_{kk} \geq 1$, we would also have $[\lambda_k(\boldsymbol{\Delta}_\delta)]^{\ell_{kk}} \to 0$. But this would force $\Lambda_k(\boldsymbol{\Delta}_\delta) \to 0$, contradicting $\lambda_k = 1 - \Lambda_k$. Thus, if $\sum_s \varepsilon_k \ell_{sk}\theta_s = 0$ then $\lim_{\delta\to\infty} \lambda_k(\boldsymbol{\Delta}_\delta) > 0$. Combining the two cases, we conclude that $\lim_{\delta\to\infty} \lambda_k(\boldsymbol{\Delta}_\delta) \in (0,1]$ for any $\sum_s \varepsilon_k \ell_{sk}\theta_s \geq 0$.

Next, using expression (10) for p_k, we obtain

$$\frac{p_k(\boldsymbol{\Delta}_\delta)^{-\varepsilon_k}}{\delta} = w^{-\varepsilon_k} \left(\prod_s \overline{T}_s^{\varepsilon_k \ell_{sk}}\right) \left[\lambda_k(\boldsymbol{\Delta}_\delta)\right]^{1-\ell_{kk}} \left(\prod_{s\neq k} \left[\lambda_s(\boldsymbol{\Delta}_\delta)\right]^{-\varepsilon_k \ell_{sk}/\varepsilon_s}\right) \delta^{\sum_s \theta_s \varepsilon_k \ell_{sk} - 1}.$$

Since each $\lambda_s(\boldsymbol{\Delta}_\delta)$ has a positive limit, condition (UC) ensures that $\lim_{\delta\to\infty} p_k(\boldsymbol{\Delta}_\delta)^{-\varepsilon_k}/\delta = 0$. It then follows that

$$\lim_{\delta\to\infty} \frac{d_k(\boldsymbol{\Delta}_\delta)}{\delta} = e_k \bar{L} \lim_{\delta\to\infty} \frac{\lambda_k(\boldsymbol{\Delta}_\delta)}{\delta} + E_k w^{-1} \lim_{\delta\to\infty} \frac{[p_k(\boldsymbol{\Delta}_\delta)]^{-\varepsilon_k}}{\delta} = 0.$$

In turn, this implies

$$\lim_{\delta\to\infty} \frac{F_k(\boldsymbol{\Delta}_\delta)}{\delta} \leq \alpha_k \sum_{s=1}^{K} \ell_{ks} \lim_{\delta\to\infty} \frac{d_s(\boldsymbol{\Delta}_\delta)}{\delta} = 0,$$

where we used $\tilde{\mathcal{L}}(\boldsymbol{\Delta}_\delta) \leq \mathcal{L}$ for any δ. Thus, $\lim_{\delta\to\infty} F_k(\boldsymbol{\Delta}_\delta)/\delta = 0$ for all k. □

Invoking Lemma 5, we get that for any $b \in (0,1)$ and each k, there exists $\overline{\chi}_k > 0$ such that $F_k(\boldsymbol{\Delta}_\delta) \leq b\delta$ for all $\delta \geq \overline{\chi}_k$. Letting $\overline{\chi} \equiv \max\{1, \overline{\delta}, \max_k \overline{\chi}_k\}$, we obtain $\boldsymbol{F}(\boldsymbol{\Delta}_\delta) \leq b\boldsymbol{\Delta}_\delta < \boldsymbol{\Delta}_\delta$ for all $\delta \geq \overline{\chi}$. This yields two results. First, if $\boldsymbol{L} \leq \boldsymbol{\Delta}_{\overline{\chi}}$, then by Lemma 3, $\boldsymbol{F}(\boldsymbol{L}) \leq \boldsymbol{F}(\boldsymbol{\Delta}_{\overline{\chi}}) < \boldsymbol{\Delta}_{\overline{\chi}}$. Second, if $\boldsymbol{L} \geq \boldsymbol{\Delta}_{\overline{\chi}}$ then $\boldsymbol{F}(\boldsymbol{L}) \neq \boldsymbol{L}$. To see this, let $\delta \equiv \max_k L_k$. This implies $\boldsymbol{\Delta}_{\delta} \geq \boldsymbol{L}$ and hence by Lemma 3 we have (i) $\boldsymbol{F}(\boldsymbol{\Delta}_{\delta}) \geq \boldsymbol{F}(\boldsymbol{L})$. Moreover, $\boldsymbol{L} \geq \boldsymbol{\Delta}_{\overline{\chi}}$ implies $\delta \geq \overline{\chi}$, and hence (ii) $\boldsymbol{F}(\boldsymbol{\Delta}_{\delta}) < \boldsymbol{\Delta}_{\delta}$. Results (i) and (ii) imply $\boldsymbol{F}(\boldsymbol{L}) < \boldsymbol{\Delta}_{\delta}$. Since for at least one k we have $L_k = \delta$, it follows that we cannot have $\boldsymbol{L} > \boldsymbol{\Delta}_{\delta}$. Combined with $\boldsymbol{F}(\boldsymbol{L}) < \boldsymbol{\Delta}_{\delta}$, this implies $\boldsymbol{F}(\boldsymbol{L}) \neq \boldsymbol{L}$.

Finally, setting $\overline{\delta}_k \equiv \overline{\chi}$ establishes the statement of Proposition 3 for the upper bound of the rectangular region $\overline{\mathcal{R}}$ of $\mathbb{R}^K$ mapped into itself by $\boldsymbol{F}$. Since $\overline{\chi} \geq \overline{\delta}$, we have $\overline{\delta}_k = \overline{\chi} \geq \overline{\delta}$ for all k, as required by Proposition 3. Moreover, since $\overline{\chi} \geq 1$, we have $\overline{\delta}_k \geq 1$ for all k.

E.1.3 Combining the Bounds

To complete the proof of Proposition 3, we need to make sure that $\underline{\delta}_k < \overline{\delta}_k$ for all k. This is indeed the case, since our choice of $\underline{\delta}_k$ and $\overline{\delta}_k$ implies that $\underline{\delta}_k < 1 \leq \overline{\delta}_k$.

E.2 Derivative of the Linkage Matrix $\tilde{\mathcal{L}}$ With Respect to Prices

The matrix $\tilde{\mathcal{L}}$ depends on the labor allocations vector $\boldsymbol{L}$ only through the prices $p_r^{-\varepsilon_r}$. Below, we compute the partial derivatives $\partial \tilde{\ell}_{kj} / \partial \ln p_r^{-\varepsilon_r}$. We have

$$\begin{aligned}\frac{\partial \tilde{\mathcal{L}}}{\partial \ln p_r^{-\varepsilon_r}} &= \frac{\partial (I - D_\lambda A)^{-1}}{\partial \ln p_r^{-\varepsilon_r}} = -(I - D_\lambda A)^{-1} \frac{\partial (I - D_\lambda A)}{\partial \ln p_r^{-\varepsilon_r}} (I - D_\lambda A)^{-1} \\ &= \tilde{\mathcal{L}} \frac{\partial (D_\lambda A)}{\partial \ln p_r^{-\varepsilon_r}} \tilde{\mathcal{L}} = \left(\frac{\partial \ln \lambda_r}{\partial \ln p_r^{-\varepsilon_r}} \lambda_r \right) \tilde{\mathcal{L}} A_{[r\cdot]} \tilde{\mathcal{L}},\end{aligned}$$

where $A_{[r\cdot]}$ is a matrix with row r equal to the rth row of matrix A and zeros everywhere else. Using $\partial \ln \lambda_r / \partial \ln p_r^{-\varepsilon_r} = 1 - \lambda_r$, we then get

$$\frac{\partial \tilde{\ell}_{kj}}{\partial \ln p_r^{-\varepsilon_r}} = (1 - \lambda_r) \lambda_r \tilde{\ell}_{kr} \sum_i \alpha_{ri} \tilde{\ell}_{ij}.$$

Given the definition $\tilde{\mathcal{L}} = (I - D_\lambda A)^{-1}$, we have $(I - D_\lambda A) \tilde{\mathcal{L}} = I$, which gives $D_\lambda A \tilde{\mathcal{L}} = \tilde{\mathcal{L}} - I$. The (r, j)-th element of this matrix equality is given by $\lambda_r \sum_i \alpha_{ri} \tilde{\ell}_{ij} = \tilde{\ell}_{rj} - \mathbb{I}_{rj}$, where $\mathbb{I}_{rj} = 1$ if $r = j$ and 0 otherwise. Using this, we get

$$\frac{\partial \tilde{\ell}_{kj}}{\partial \ln p_r^{-\varepsilon_r}} = (1 - \lambda_r) \tilde{\ell}_{kr} \left(\tilde{\ell}_{rj} - \mathbb{I}_{rj} \right).$$

E.3 Proof of Proposition 4

Assume that (UC) holds. Throughout, as elsewhere in the paper, let D_x denote a diagonal matrix with elements x_k on the diagonal. Denote $f_k \equiv \sum_j \tilde{\ell}_{kj} d_j$ and recall that $F_k(\boldsymbol{L}) = \alpha_k f_k$. We study the Jacobian matrix of $\ln \boldsymbol{F}$ with respect to $\ln \boldsymbol{L}$, the entries of which are given by expression (11) and restated here for convenience,

$$J_{ks} = \frac{1}{f_k} \sum_r \left[\sum_j \frac{\partial \tilde{\ell}_{kj}}{\partial \ln p_r^{-\varepsilon_r}} d_j + \tilde{\ell}_{kr} \frac{\partial d_r}{\partial \ln p_r^{-\varepsilon_r}} \right] \cdot \varepsilon_r \tilde{\ell}_{sr} \theta_s,$$

where we drop the dependence of J_{ks} on $\boldsymbol{L}$ for brevity. From Appendix E.2, we have the derivative of the open-economy Leontief matrix,

$$\frac{\partial \tilde{\ell}_{kj}}{\partial \ln p_r^{-\varepsilon_r}} = (1 - \lambda_r) \tilde{\ell}_{kr} \left(\tilde{\ell}_{rj} - \mathbb{I}_{rj} \right),$$

where $\mathbb{I}_{rj} = 1$ if $r = j$ and $\mathbb{I}_{rj} = 0$ otherwise. From the definition of $d_r = \lambda_r e_r \bar{L} + E_r w^{-1} p_r^{-\varepsilon_r}$, we can bound its derivative as

$$\frac{\partial d_r}{\partial \ln p_r^{-\varepsilon_r}} = (1 - \lambda_r) \lambda_r e_r \bar{L} + E_r w^{-1} p_r^{-\varepsilon_r} \leq d_r.$$

Substituting the above derivatives into the expression for J_{ks} yields an element-wise upper bound,

$$J_{ks} \leq \bar{J}_{ks} \equiv \frac{1}{f_k} \sum_r \tilde{\ell}_{kr} \Big[(1 - \lambda_r) f_r + \lambda_r d_r \Big] \cdot \varepsilon_r \tilde{\ell}_{sr} \theta_s. \tag{17}$$

In matrix form, this inequality is

$$J \leq \bar{J} \equiv D_f^{-1} \tilde{\mathcal{L}} \left[(I - D_\lambda) D_f + D_\lambda D_d \right] D_\varepsilon \tilde{\mathcal{L}}^T D_\theta. \tag{18}$$

Since both J and $\bar{J}$ are nonnegative matrices and $J \leq \bar{J}$, Corollary 8.1.19 in Horn and Johnson (2013) implies $\rho(J) \leq \rho(\bar{J})$. The task now reduces to proving that $\rho(\bar{J}) < 1$. The proof proceeds in two main stages: we first prove that $\rho(\bar{J}) \leq 1$, and then we show that this inequality must be strict.

E.3.1 Proving the Spectral Radius is Bounded by One

We first prove that $\rho\left(\bar{J}\right) \leq 1$. We begin with the case where all scale elasticities θ_k and own-sector input shares α_{kk} are positive, and then generalize using a limit argument.

Part A: Upper bound $\boldsymbol{\rho\left(\bar{J}\right) \leq 1}$ under $\boldsymbol{\theta_k > 0}$ and $\boldsymbol{\alpha_{kk} > 0}$. Assume $\theta_k > 0$ and $\alpha_{kk} > 0$ for all k. Condition (UC) implies an upper bound on each trade elasticity: $\varepsilon_k \leq \bar{\varepsilon}_k \equiv \left[\sum_s \ell_{sk}\theta_s\right]^{-1}$. Since $\alpha_{kk} > 0$, the Neumann series expansion of $\mathcal{L}$ implies $\ell_{kk} > 1$, and so $\bar{\varepsilon}_k$ are well defined and satisfy $\bar{\varepsilon}_k < \theta_k^{-1}$.

Define a second upper-bound matrix (the "worst-case scenario"), $\bar{\bar{J}}$, by replacing D_ε in the definition of $\bar{J}$ with the diagonal matrix $D_{\bar{\varepsilon}}$,

$$\bar{J} \leq \bar{\bar{J}} \equiv D_f^{-1}\tilde{\mathcal{L}}\left[\left(I - D_\lambda\right)D_f + D_\lambda D_d\right]D_{\bar{\varepsilon}}\tilde{\mathcal{L}}^T D_\theta.$$

By Corollary 8.1.19 in Horn and Johnson (2013), we have $\rho\left(\bar{J}\right) \leq \rho\left(\bar{\bar{J}}\right)$. We will show that $\rho\left(\bar{\bar{J}}\right) \leq 1$.

Consider a similarity transform of $\bar{\bar{J}}$ given by

$$\begin{aligned}\widetilde{J} &\equiv \left(D_\theta^{1/2}D_f^{1/2}\right)\cdot\bar{\bar{J}}\cdot\left(D_\theta^{1/2}D_f^{1/2}\right)^{-1}\\ &= D_\theta^{1/2}D_f^{-1/2}\tilde{\mathcal{L}}\left[\left(I - D_\lambda\right)D_f + D_\lambda D_d\right]D_{\bar{\varepsilon}}\tilde{\mathcal{L}}^T D_f^{-1/2}D_\theta^{1/2}.\end{aligned}$$

Matrix $\widetilde{J}$ has the same eigenvalues as $\bar{\bar{J}}$, and therefore $\rho\left(\bar{\bar{J}}\right) = \rho\left(\widetilde{J}\right)$. Moreover, $\widetilde{J}$ is symmetric, hence the condition $\rho\left(\widetilde{J}\right) \leq 1$ is equivalent to the matrix $I - \widetilde{J}$ being positive semidefinite. In turn, $I - \widetilde{J}$ is positive semidefinite if and only if

$$\begin{aligned}M &\equiv \left(D_\theta^{-1/2}D_f^{1/2}\right)\cdot\left(I - \widetilde{J}\right)\cdot\left(D_\theta^{-1/2}D_f^{1/2}\right)\\ &= D_\theta^{-1}D_f - \tilde{\mathcal{L}}\left[\left(I - D_\lambda\right)D_f + D_\lambda D_d\right]D_{\bar{\varepsilon}}\tilde{\mathcal{L}}^T\end{aligned}$$

is positive semidefinite.

Given that $f_k \equiv \sum_i \tilde{\ell}_{ki}d_i$, we can write $D_f = \sum_i d_i D_f^{(i)}$ with $D_f^{(i)} \equiv \text{diag}\left\{\left(\tilde{\ell}_{1i}, \ldots, \tilde{\ell}_{Ki}\right)\right\}$. Also, $D_d = \sum_i d_i E^{(i)}$, where $E^{(i)}$ is a matrix with 1 in position (i,i) and zeros elsewhere. Therefore, we have $M = \sum_i d_i M^{(i)}$, where

$$M^{(i)} \equiv D_\theta^{-1}D_f^{(i)} - \tilde{\mathcal{L}}\left[\left(I - D_\lambda\right)D_f^{(i)} + D_\lambda E^{(i)}\right]D_{\bar{\varepsilon}}\tilde{\mathcal{L}}^T.$$

If we can show that each $M^{(i)}$ is positive semidefinite, then M must also be positive semidefinite.

Proving $M^{(i)}$ is positive semidefinite for all i is equivalent to showing $z_i \equiv \boldsymbol{x}^T M^{(i)} \boldsymbol{x} \geq 0$ for any vector $\boldsymbol{x} \neq \mathbf{0}$. In turn, the condition $z_i \geq 0$ for $i = 1, \ldots, K$ is satisfied if

$$\sum_{i=1}^{K} \alpha_{ij} \lambda_i z_i \leq z_j \quad \text{for } j = 1, \ldots, K. \tag{19}$$

Indeed, (19) can be written in matrix form as $A^T D_\lambda \boldsymbol{z} \leq \boldsymbol{z}$ with $\boldsymbol{z} \equiv (z_1, \ldots, z_K)^T$, which is equivalent to $\left(I - A^T D_\lambda\right) \boldsymbol{z} \geq \mathbf{0}$. Since $\left(I - A^T D_\lambda\right)^{-1}$ is a nonnegative matrix with a positive diagonal, we can multiply both sides of $\left(I - A^T D_\lambda\right) \boldsymbol{z} \geq \mathbf{0}$ on $\left(I - A^T D_\lambda\right)^{-1}$ to get $\boldsymbol{z} \geq \mathbf{0}$.

Denoting $\boldsymbol{y} \equiv \tilde{\mathcal{L}}^T \boldsymbol{x}$, we can write

$$z_i = \sum_{r=1}^{K} \tilde{\ell}_{ri} \frac{x_r^2}{\theta_r} - \sum_{r=1}^{K} (1 - \lambda_r) \, \tilde{\ell}_{ri} \bar{\varepsilon}_r y_r^2 - \lambda_i \bar{\varepsilon}_i y_i^2.$$

We have $A^T D_\lambda \tilde{\mathcal{L}}^T = \tilde{\mathcal{L}}^T - I$, with the (j, r)-th component of this matrix equality given by $\sum_{i=1}^{K} \alpha_{ij} \lambda_i \tilde{\ell}_{ri} = \tilde{\ell}_{rj} - \mathbb{I}_{rj}$, where $\mathbb{I}_{rj} = 1$ if $r = j$ and $\mathbb{I}_{rj} = 0$ otherwise. Using this result, we can calculate for $j = 1, \ldots, K$,

$$\begin{aligned}
\sum_{i=1}^{K} \alpha_{ij} \lambda_i z_i &= \sum_{r=1}^{K} \left(\sum_{i=1}^{K} \alpha_{ij} \lambda_i \tilde{\ell}_{ri} \right) \frac{x_r^2}{\theta_r} - \sum_{r=1}^{K} (1 - \lambda_r) \left(\sum_{i=1}^{K} \alpha_{ij} \lambda_i \tilde{\ell}_{ri} \right) \bar{\varepsilon}_r y_r^2 - \sum_{i=1}^{K} \alpha_{ij} \bar{\varepsilon}_i \lambda_i^2 y_i^2 \\
&= \sum_{r=1}^{K} \left(\tilde{\ell}_{rj} - \mathbb{I}_{rj} \right) \frac{x_r^2}{\theta_r} - \sum_{r=1}^{K} (1 - \lambda_r) \left(\tilde{\ell}_{rj} - \mathbb{I}_{rj} \right) \bar{\varepsilon}_r y_r^2 - \sum_{i=1}^{K} \alpha_{ij} \bar{\varepsilon}_i \lambda_i^2 y_i^2 \\
&= z_j + g_j,
\end{aligned}$$

where $g_j \equiv \bar{\varepsilon}_j y_j^2 - \theta_j^{-1} x_j^2 - \sum_{i=1}^{K} \alpha_{ij} \bar{\varepsilon}_i \lambda_i^2 y_i^2$. Then to show (19), we must prove that $g_j \leq 0$.

Using the Cauchy-Schwarz inequality $\left(\sum_i a_{ij} b_{ij}\right)^2 \leq \left(\sum_i a_{ij}^2\right) \left(\sum_i b_{ij}^2\right)$ with $a_{ij} = \sqrt{\alpha_{ij} \bar{\varepsilon}_i^{-1}}$ and $b_{ij} = \sqrt{\alpha_{ij} \bar{\varepsilon}_i \lambda_i^2 y_i^2}$, we get

$$\left(\sum_i \alpha_{ij} \lambda_i y_i \right)^2 \leq \left(\sum_i \alpha_{ij} \bar{\varepsilon}_i^{-1} \right) \left(\sum_i \alpha_{ij} \bar{\varepsilon}_i \lambda_i^2 y_i^2 \right). \tag{20}$$

Given that $\bar{\varepsilon}_k = \left[\sum_s \ell_{sk} \theta_s\right]^{-1}$, we have $\bar{\varepsilon}_k^{-1} = \sum_s \ell_{sk} \theta_s$, which in matrix form can be written as $D_{\bar{\varepsilon}}^{-1} \iota = \mathcal{L}^T D_\theta \iota$ with $\iota \equiv (1, \ldots, 1)^T$. Using the definition $\mathcal{L} \equiv (I - A)^{-1}$, we get $A^T D_{\bar{\varepsilon}}^{-1} \iota = D_{\bar{\varepsilon}}^{-1} \iota - D_\theta \iota$, with the j-th component of this vector equality given by

$\sum_i \alpha_{ij}\bar{\varepsilon}_i^{-1} = \bar{\varepsilon}_j^{-1} - \theta_j$. Also, the definition of $\boldsymbol{y} = \tilde{\mathcal{L}}^T \boldsymbol{x}$ implies $\sum_i \alpha_{ij}\lambda_i y_i = y_j - x_j$. Thus, we can write (20) as

$$\sum_i \alpha_{ij}\bar{\varepsilon}_i\lambda_i^2 y_i^2 \geq \left(\bar{\varepsilon}_j^{-1} - \theta_j\right)^{-1}\left(y_j - x_j\right)^2.$$

Substituting this into g_j, yields

$$g_j \leq \bar{\varepsilon}_j y_j^2 - \theta_j^{-1} x_j^2 - \frac{\left(y_j - x_j\right)^2}{\bar{\varepsilon}_j^{-1} - \theta_j} = -\frac{\left(x_j/\sqrt{\theta_j\bar{\varepsilon}_j} - y_j\sqrt{\theta_j\bar{\varepsilon}_j}\right)^2}{\bar{\varepsilon}_j^{-1} - \theta_j} \leq 0.$$

The final inequality holds because $\bar{\varepsilon}_j^{-1} > \theta_j$ by construction. Therefore, (19) holds, which confirms that M is positive semidefinite and establishes that $\rho\left(\bar{J}\right) \leq \rho\left(\bar{\bar{J}}\right) \leq 1$ for the case with $\theta_k > 0$ and $\alpha_{kk} > 0$ for all k.

Part B: Allowing for $\theta_k = 0$ and/or $\alpha_{kk} = 0$. Consider now the case with $\theta_k = 0$ or $\alpha_{kk} = 0$ for some or all k. For $\epsilon > 0$ define perturbed parameters $\theta_k^{(\epsilon)} \equiv \theta_k + \epsilon$ if $\theta_k = 0$, else $\theta_k^{(\epsilon)} = \theta_k$; and $\alpha_{kk}^{(\epsilon)} \equiv \epsilon$ if $\alpha_{kk} = 0$, else $\alpha_{kk}^{(\epsilon)} \equiv \alpha_{kk}$. Let $A^{(\epsilon)}$ be the matrix obtained from A by replacing α_{kk} with $\alpha_{kk}^{(\epsilon)}$. Define $\mathcal{L}^{(\epsilon)} \equiv \left(I - A^{(\epsilon)}\right)^{-1}$ and $\tilde{\mathcal{L}}^{(\epsilon)} \equiv \left(I - D_\lambda A^{(\epsilon)}\right)^{-1}$, with their elements denoted by $\ell_{ks}^{(\epsilon)}$ and $\tilde{\ell}_{ks}^{(\epsilon)}$. Also, denote $f_k^{(\epsilon)} \equiv \sum_j \tilde{\ell}_{kj}^{(\epsilon)} d_j$ and let $D_f^{(\epsilon)}$ and $D_\theta^{(\epsilon)}$ be matrices with elements $f_k^{(\epsilon)}$ and $\theta_k^{(\epsilon)}$, respectively.

Let $\{\epsilon_t\}_{t=0}^{\infty}$ be a sequence with $\lim_{t\to\infty}\epsilon_t = 0$ such that $\epsilon_t > 0$, $\alpha_{kk}^{(\epsilon_t)} + \sum_{s\neq k}\alpha_{sk} < 1$, and $\varepsilon_k \sum_s \ell_{sk}^{(\epsilon_t)}\theta_s^{(\epsilon_t)} \leq 1$ for all k and t.[24] Then matrices $\mathcal{L}^{(\epsilon_t)}$ and $\tilde{\mathcal{L}}^{(\epsilon_t)}$ are well-defined for all t. Also, the Neumann expansions of $\tilde{\mathcal{L}}^{(\epsilon_t)}$ and $\tilde{\mathcal{L}}$ imply $\tilde{\mathcal{L}}^{(\epsilon_t)} \geq \tilde{\mathcal{L}}$, and thus $f_k^{(\epsilon_t)} \geq f_k > 0$ for t.

Define $\bar{J}^{(\epsilon_t)}$ as in (18) with $D_f^{(\epsilon_t)}$, $D_\theta^{(\epsilon_t)}$, and $\tilde{\mathcal{L}}^{(\epsilon_t)}$, i.e.,

$$\bar{J}^{(\epsilon_t)} \equiv \left[D_f^{(\epsilon_t)}\right]^{-1}\tilde{\mathcal{L}}^{(\epsilon_t)}\left[\left(I - D_\lambda\right)D_f^{(\epsilon_t)} + D_\lambda D_d\right]D_\varepsilon\left[\tilde{\mathcal{L}}^{(\epsilon_t)}\right]^T D_\theta^{(\epsilon_t)}.$$

Repeating the proof from Appendix E.3.1 for $\bar{J}^{(\epsilon_t)}$ and $\bar{\varepsilon}^{(\epsilon_t)} \equiv \left[\sum_s \ell_{sk}^{(\epsilon_t)}\theta_s^{(\epsilon_t)}\right]^{-1}$ gives $\rho\left(\bar{J}^{(\epsilon_t)}\right) \leq 1$ for all t. Since the entries of $\bar{J}^{(\epsilon_t)}$ depend on ϵ_t continuously, we have $\bar{J}^{(\epsilon_t)} \to \bar{J}$ entry-wise. Then by continuity of the spectral radius in matrix entries, $\rho\left(\bar{J}\right) = \lim_{t\to\infty}\rho\left(\bar{J}^{(\epsilon_t)}\right) \leq 1$.

[24] Recall that we assume $\alpha_k > 0$, implying $\sum_s \alpha_{sk} < 1$. Thus, for all small enough $\epsilon_t > 0$ we have $\alpha_{kk}^{(\epsilon_t)} + \sum_{s\neq k}\alpha_{sk} < 1$.

E.3.2 Proving the Strict Inequality

In Appendix E.3.1, we established that $\rho\left(\bar{J}\right) \leq 1$. Here, we strengthen this result to a strict inequality, $\rho\left(\bar{J}\right) < 1$. The key observation is that, holding everything else fixed, the trade elasticities $\{\varepsilon_r\}$ that directly enter the expression (17) for $\bar{J}$ can be treated as free parameters, provided each ε_r lies strictly below its (UC)-implied upper bound $\left[\sum_s \ell_{sr}\theta_s\right]^{-1}$. For any such admissible set of $\{\varepsilon_r\}$, the argument from Appendix E.3.1 applies unchanged and yields $\rho\left(\bar{J}\right) \leq 1$. We exploit this in a proof by contradiction, supposing that $\rho\left(\bar{J}\right) = 1$ for some $\{\varepsilon_r\}$ and constructing a perturbed matrix $\hat{J}$ with slightly larger ε_r that still satisfy (UC). This perturbation pushes the spectral radius strictly above one, contradicting the result from Appendix E.3.1, which also applies to $\hat{J}$ and thus yields $\rho\left(\hat{J}\right) \leq 1$.

The formal argument is as follows. If all $\theta_k = 0$, then $\bar{J} = 0$ and $\rho\left(\bar{J}\right) = 0 < 1$, so there is nothing to prove. Otherwise, assume at least one $\theta_k > 0$ and suppose, for the sake of contradiction, that $\rho\left(\bar{J}\right) = 1$. Since $\bar{J}$ is a nonnegative matrix, the Perron-Frobenius theorem (see Horn and Johnson, 2013, Theorem 8.3.1) guarantees the existence of a nonnegative, nonzero eigenvector $\boldsymbol{x}$ (the Perron eigenvector) corresponding to the spectral radius, with $\bar{J}\boldsymbol{x} = \rho\left(\bar{J}\right)\boldsymbol{x} = \boldsymbol{x}$. The k-th component of this vector equation can be written using the expression for $\bar{J}_{ks}$ from (17) as

$$\left(\bar{J}\boldsymbol{x}\right)_k = \frac{1}{f_k}\sum_r \tilde{\ell}_{kr}\left[\left(1-\lambda_r\right)f_r + \lambda_r d_r\right] \cdot \varepsilon_r \sum_s \tilde{\ell}_{sr}\theta_s x_s.$$

Define $\hat{\varepsilon}_r \equiv 0.5\left(\varepsilon_r + \left[\sum_s \ell_{sr}\theta_s\right]^{-1}\right)$ if $\sum_s \ell_{sr}\theta_s > 0$, and $\hat{\varepsilon}_r \equiv \varepsilon_r$ otherwise. By construction $\hat{\varepsilon}_r \sum_s \ell_{sr}\theta_s < 1$ for all r. Moreover, since $\ell_{ss} \geq 1$ for all s and $\theta_s \neq 0$ for at least one s, we have $\sum_s \ell_{sr}\theta_s > 0$ for at least one r. For any such r, we have $\hat{\varepsilon}_r > \varepsilon_r$. Let $\hat{J}$ be the matrix obtained by replacing each ε_r with $\hat{\varepsilon}_r$ in the definition of $\bar{J}$.

Let us compare the components $\left(\hat{J}\boldsymbol{x}\right)_k$ and $\left(\bar{J}\boldsymbol{x}\right)_k$. If $\left(\hat{J}\boldsymbol{x}\right)_k \neq 0$, there exists at least one index r such that $\tilde{\ell}_{kr} \neq 0$ and $\sum_s \tilde{\ell}_{sr}\theta_s x_s \neq 0$. (Note that $\left(1-\lambda_r\right)f_r + \lambda_r d_r > 0$ because $\lambda_r \in (0,1)$ and $f_r > 0$ for all r.) The condition $\sum_s \tilde{\ell}_{sr}\theta_s x_s \neq 0$ implies that $\tilde{\ell}_{sr}\theta_s \neq 0$ for at least one s. Since $\lambda_k \in (0,1)$ for all k, the Neumann expansions of $\tilde{\mathcal{L}}$ and $\mathcal{L}$ show that $\tilde{\ell}_{sr} \neq 0$ if and only if $\ell_{sr} \neq 0$. Hence $\ell_{sr}\theta_s \neq 0$ for at least one s, and therefore $\sum_s \ell_{sr}\theta_s \neq 0$. For this index r, our construction then implies $\hat{\varepsilon}_r > \varepsilon_r$, which in turn gives $\left(\hat{J}\boldsymbol{x}\right)_k > \left(\bar{J}\boldsymbol{x}\right)_k$. If instead $\left(\hat{J}\boldsymbol{x}\right)_k = 0$, then for each r either $\tilde{\ell}_{kr} = 0$ or $\sum_s \tilde{\ell}_{sr}\theta_s x_s = 0$. In this case, changing ε_r has no effect, so $\left(\bar{J}\boldsymbol{x}\right)_k = 0$ as well.

Thus, $\left(\hat{J}\boldsymbol{x}\right)_k > \left(\bar{J}\boldsymbol{x}\right)_k = x_k$ for any $x_k > 0$, and $\left(\hat{J}\boldsymbol{x}\right)_k = \left(\bar{J}\boldsymbol{x}\right)_k = 0$ for any $x_k = 0$. Let

$$\epsilon = \min_{k: x_k > 0} \left\{ \frac{\left(\hat{J}\boldsymbol{x}\right)_k}{x_k} - 1 \right\} > 0.$$

Then for all k with $x_k > 0$ we have $\left(\hat{J}\boldsymbol{x}\right)_k \geq (1+\epsilon)\, x_k$, and for k with $x_k = 0$ we also have $\left(\hat{J}\boldsymbol{x}\right)_k \geq 0 = (1+\epsilon)\, x_k$. Hence, $\hat{J}\boldsymbol{x} \geq (1+\epsilon)\, \boldsymbol{x}$. By Theorem 8.3.2 in Horn and Johnson (2013), this implies $\rho\left(\hat{J}\right) \geq 1 + \epsilon > 1$. However, by construction, each $\hat{\varepsilon}_r$ satisfies $\hat{\varepsilon}_r \sum_s \ell_{sr}\theta_s < 1$. Therefore, the argument from Appendix E.3.1 applies to $\hat{J}$ and yields $\rho\left(\hat{J}\right) \leq 1$, a contradiction. We conclude that our supposition $\rho\left(\bar{J}\right) = 1$ is false, and thus $\rho\left(\bar{J}\right) < 1$.

E.4 Proof of Theorem 2

Theorem 2 is an implication of the Poincaré-Hopf Theorem (see, e.g., Simsek *et al.*, 2007). The original Poincaré-Hopf Theorem is formulated in terms of zeros of function $\boldsymbol{g}(\boldsymbol{x}) \equiv \boldsymbol{x} - \boldsymbol{f}(\boldsymbol{x})$, which results in $\sum_{\boldsymbol{x}\in\mathcal{S}} \text{sign}\left(\det\left(\nabla \boldsymbol{g}(\boldsymbol{x})\right)\right) = \chi(\mathcal{R})$, where $\chi(\mathcal{R})$ is the Euler characteristic of $\mathcal{R}$. Noting that $\nabla \boldsymbol{g}(\boldsymbol{x}) = I - \nabla \boldsymbol{f}(\boldsymbol{x})$ and $\chi(\mathcal{R}) = 1$ for any nonempty rectangular region $\mathcal{R}$ of $\mathbb{R}^K$ (Simsek *et al.*, 2007, p. 193), we get the expression from the statement of Theorem 2, $\sum_{\boldsymbol{x}\in\mathcal{S}} \text{sign}\left(\det\left(I - \nabla \boldsymbol{f}(\boldsymbol{x})\right)\right) = 1$.

Assumption (i) in Theorem 2 replaces the original Poincaré-Hopf Theorem's requirement that function $\boldsymbol{g}(\boldsymbol{x})$ points outward on the boundary of $\mathcal{R}$. Formally, this means that for any $\boldsymbol{x} \in \text{bd}(\mathcal{R})$, there exists a sequence $\epsilon_i \downarrow 0$ such that $\boldsymbol{x} + \epsilon_i \boldsymbol{G}(\boldsymbol{x}) \notin \mathcal{R}$ for all $i = 1, 2, \ldots$. To complete the proof, we verify that this holds under assumption (i).

Let $\mathcal{R} = [\underline{r}_1, \bar{r}_1] \times \cdots \times [\underline{r}_K, \bar{r}_K]$ with $\underline{r}_k < \bar{r}_k$ for all k. Consider $\boldsymbol{x} \in \text{bd}(\mathcal{R})$ and any $\epsilon_i \downarrow 0$. For any k such that $x_k = \underline{r}_k$, we have $x_k + \epsilon_i g_k(\boldsymbol{x}) = \underline{r}_k + \epsilon_i\left(\underline{r}_k - F_k(\boldsymbol{x})\right)$. By assumption (i), $f_k(\boldsymbol{x}) > \underline{r}_k$, so $x_k + \epsilon_i g_k(\boldsymbol{x}) < \underline{r}_k$ for $\epsilon_i > 0$, implying $x_k + \epsilon_i g_k(\boldsymbol{x}) \notin \mathcal{R}$ for all i. Similarly, if $x_k = \bar{r}_k$, then $x_k + \epsilon_i g_k(\boldsymbol{x}) = \bar{r}_k + \epsilon_i\left(\bar{r}_k - f_k(\boldsymbol{x})\right)$. Assumption (i) implies $f_k(\boldsymbol{x}) < \bar{r}_k$, so $x_k + \epsilon_i g_k(\boldsymbol{x}) > \bar{r}_k$ for $\epsilon_i > 0$, and thus $x_k + \epsilon_i g_k(\boldsymbol{x}) \notin \mathcal{R}$ for all i.

E.5 Proof of Proposition 6

Let $\boldsymbol{L}^{(0)}$ be any positive vector of labor allocations. By Proposition 3, there exists a rectangle $\overline{\mathcal{R}} \equiv \left[\underline{\delta}_1, \bar{\delta}_1\right] \times \ldots \times \left[\underline{\delta}_K, \bar{\delta}_K\right]$ that contains $\boldsymbol{L}^{(0)}$ and is mapped into itself by $\boldsymbol{F}$. Define the bounding vectors $\underline{\boldsymbol{L}}^{(0)} \equiv \left(\underline{\delta}_1, \ldots, \underline{\delta}_K\right)$ and $\overline{\boldsymbol{L}}^{(0)} \equiv \left(\bar{\delta}_1, \ldots, \bar{\delta}_K\right)$, and let $\underline{\boldsymbol{L}}^{(t+1)} = \boldsymbol{F}\left(\underline{\boldsymbol{L}}^{(t)}\right)$ and $\overline{\boldsymbol{L}}^{(t+1)} = \boldsymbol{F}\left(\overline{\boldsymbol{L}}^{(t)}\right)$ for $t = 0, 1, \ldots$ denote the iterates of $\boldsymbol{F}$ starting from these endpoints.

Since by Proposition 3, $\boldsymbol{F}(\boldsymbol{L}) \in \mathcal{R}$ for any $\boldsymbol{L} \in \overline{\mathcal{R}}$, we have $\boldsymbol{F}\left(\underline{\boldsymbol{L}}^{(0)}\right) > \underline{\boldsymbol{L}}^{(0)}$ and $\boldsymbol{F}\left(\overline{\boldsymbol{L}}^{(0)}\right) < \overline{\boldsymbol{L}}^{(0)}$. This gives us $\underline{\boldsymbol{L}}^{(1)} > \underline{\boldsymbol{L}}^{(0)}$ and $\overline{\boldsymbol{L}}^{(1)} < \overline{\boldsymbol{L}}^{(0)}$. Since $\boldsymbol{F}$ is non-decreasing (Lemma 3 in Appendix E.1), we can iteratively apply the mapping to preserve these inequalities. Specifically, if $\underline{\boldsymbol{L}}^{(t+1)} \geq \underline{\boldsymbol{L}}^{(t)}$, then $\boldsymbol{F}\left(\underline{\boldsymbol{L}}^{(t+1)}\right) \geq \boldsymbol{F}\left(\underline{\boldsymbol{L}}^{(t)}\right)$, which implies $\underline{\boldsymbol{L}}^{(t+2)} \geq \underline{\boldsymbol{L}}^{(t+1)}$. The same logic applies to the upper sequence. Therefore, the sequence $\left\{\underline{\boldsymbol{L}}^{(t)}\right\}_{t=0}^{\infty}$ is nondecreasing and $\left\{\overline{\boldsymbol{L}}^{(t)}\right\}_{t=0}^{\infty}$ is nonincreasing, both bounded within $\overline{\mathcal{R}}$.

Since every monotone bounded sequence in $\mathbb{R}^K$ converges coordinate-wise (see Rudin, 1976, Theorem 3.14), both $\left\{\underline{\boldsymbol{L}}^{(t)}\right\}_{t=0}^{\infty}$ and $\left\{\overline{\boldsymbol{L}}^{(t)}\right\}_{t=0}^{\infty}$ converge. Let $\underline{\boldsymbol{L}}^{(\infty)} = \lim_{t\to\infty} \underline{\boldsymbol{L}}^{(t)}$ and $\overline{\boldsymbol{L}}^{(\infty)} = \lim_{t\to\infty} \overline{\boldsymbol{L}}^{(t)}$. Since, $\underline{\boldsymbol{L}}^{(t+1)} = \boldsymbol{F}\left(\underline{\boldsymbol{L}}^{(t)}\right)$ and $\overline{\boldsymbol{L}}^{(t+1)} = \boldsymbol{F}\left(\overline{\boldsymbol{L}}^{(t)}\right)$ for all t, continuity of $\boldsymbol{F}$ implies that $\underline{\boldsymbol{L}}^{(\infty)} = \boldsymbol{F}\left(\underline{\boldsymbol{L}}^{(\infty)}\right)$ and $\overline{\boldsymbol{L}}^{(\infty)} = \boldsymbol{F}\left(\overline{\boldsymbol{L}}^{(\infty)}\right)$. Thus both $\underline{\boldsymbol{L}}^{(\infty)}$ and $\overline{\boldsymbol{L}}^{(\infty)}$ are fixed points of $\boldsymbol{F}$. Since $\boldsymbol{F}$ has a unique fixed point $\boldsymbol{L}^*$, we conclude that $\underline{\boldsymbol{L}}^{(\infty)} = \overline{\boldsymbol{L}}^{(\infty)} = \boldsymbol{L}^*$.

Finally, consider the sequence $\boldsymbol{L}^{(t+1)} = \boldsymbol{F}\left(\boldsymbol{L}^{(t)}\right)$ starting from our original vector $\boldsymbol{L}^{(0)}$. By construction of $\overline{\mathcal{R}}$, we have $\underline{\boldsymbol{L}}^{(0)} \leq \boldsymbol{L}^{(0)} \leq \overline{\boldsymbol{L}}^{(0)}$. Monotonicity of $\boldsymbol{F}$ preserves this ordering at each iteration: $\underline{\boldsymbol{L}}^{(t)} \leq \boldsymbol{L}^{(t)} \leq \overline{\boldsymbol{L}}^{(t)}$ for all t. Taking the limits of these inequalities as $t \to \infty$, gives $\lim_{t\to\infty} \boldsymbol{L}^{(t)} = \boldsymbol{L}^*$.

F Proofs for the Fourth Step

F.1 Proof of Lemma 1

In the proof of this lemma, we treat $\boldsymbol{\lambda}$, $\boldsymbol{p}$, and $\boldsymbol{L}$ as functions of w implicitly defined by equations (3), (8), and (10). For brevity, we do not write the dependence on w below.

Differentiating the log of equation (3) with respect to $\ln w$, we get

$$\frac{\mathrm{d}\ln\lambda_k}{\mathrm{d}\ln w} = (1-\lambda_k)\,\frac{\mathrm{d}\ln p_k^{-\varepsilon_k}}{\mathrm{d}\ln w}. \tag{21}$$

Differentiating the log of (10) with respect to $\ln w$, and substituting (21), yields

$$\frac{\mathrm{d}\ln p_k^{-\varepsilon_k}}{\mathrm{d}\ln w} = -\varepsilon_k + (1-\lambda_k)\,\frac{\mathrm{d}\ln p_k^{-\varepsilon_k}}{\mathrm{d}\ln w} - \sum_s \frac{\varepsilon_k \ell_{sk}}{\varepsilon_s}\,(1-\lambda_s)\,\frac{\mathrm{d}\ln p_s^{-\varepsilon_s}}{\mathrm{d}\ln w} + \varepsilon_k \sum_s \ell_{sk}\theta_s \frac{\mathrm{d}\ln L_s}{\mathrm{d}\ln w}.$$

Bringing all $\mathrm{d}\ln p_s^{-\varepsilon_s}/\mathrm{d}\ln w$ terms to the left-hand side, the resulting expression can be written in matrix form as

$$\left(D_\lambda + \mathcal{L}^T \left(I - D_\lambda\right)\right) D_\varepsilon^{-1} \left\{ \frac{\mathrm{d} \ln p_k^{-\varepsilon_k}}{\mathrm{d} \ln w} \right\} = -\iota + \mathcal{L}^T D_\theta \left\{ \frac{\mathrm{d} \ln L_k}{\mathrm{d} \ln w} \right\},$$

where $\iota \equiv (1, \ldots, 1)^T$, and $\left\{ \mathrm{d} \ln p_k^{-\varepsilon_k} / \mathrm{d} \ln w \right\}$ and $\left\{ \mathrm{d} \ln L_k / \mathrm{d} \ln w \right\}$ are column-vectors with, correspondingly, elements $\mathrm{d} \ln p_k^{-\varepsilon_k} / \mathrm{d} \ln w$ and $\mathrm{d} \ln L_k / \mathrm{d} \ln w$ in position k. Multiplying both sides of the above expression on $\mathcal{L}^{-T}$, using $\mathcal{L}^{-T} D_\lambda + (I - D_\lambda) = \tilde{\mathcal{L}}^{-T}$ and $\mathcal{L}^{-T} \iota = D_\alpha \iota$, and solving for $\left\{ \mathrm{d} \ln p_k / \mathrm{d} \ln w \right\}$, we get

$$\left\{ \frac{\mathrm{d} \ln p_k^{-\varepsilon_k}}{\mathrm{d} \ln w} \right\} = D_\varepsilon \tilde{\mathcal{L}}^T \left(-D_\alpha \iota + D_\theta \left\{ \frac{\mathrm{d} \ln L_k}{\mathrm{d} \ln w} \right\} \right).$$

The rth entry of this vector equality is given by

$$\frac{\mathrm{d} \ln p_r^{-\varepsilon_r}}{\mathrm{d} \ln w} = -\varepsilon_r \sum_s \tilde{\ell}_{sr} \alpha_s + \varepsilon_r \sum_s \tilde{\ell}_{sr} \theta_s \frac{\mathrm{d} \ln L_s}{\mathrm{d} \ln w} \tag{22}$$

Next, since $\tilde{\ell}_{kj}$ depends on w only through prices $p_r^{-\varepsilon_r}$, we have

$$\frac{\mathrm{d} \tilde{\ell}_{kj}}{\mathrm{d} \ln w} = \sum_r \frac{\partial \tilde{\ell}_{kj}}{\partial \ln p_r^{-\varepsilon_r}} \cdot \frac{\mathrm{d} \ln p_r^{-\varepsilon_r}}{\mathrm{d} \ln w}.$$

Also, using the definition of $d_j = \lambda_j e_j \bar{L} + E_j w^{-1} p_j^{-\varepsilon_j}$, we obtain

$$\frac{\mathrm{d} d_j}{\mathrm{d} \ln w} = \frac{\partial d_j}{\partial \ln p_j^{-\varepsilon_j}} \cdot \frac{\mathrm{d} \ln p_j^{-\varepsilon_j}}{\mathrm{d} \ln w} - E_j w^{-1} p_j^{-\varepsilon_j}.$$

Then, using these expressions and differentiating the log of $L_k = F_k(\boldsymbol{L}) = \alpha_k \sum_j \tilde{\ell}_{kj} d_j$ with respect to $\ln w$, we obtain

$$\frac{\mathrm{d} \ln L_k}{\mathrm{d} \ln w} = \frac{\alpha_k}{F_k(\boldsymbol{L})} \sum_r \left(\sum_j \frac{\partial \tilde{\ell}_{kj}}{\partial \ln p_r^{-\varepsilon_r}} d_j + \tilde{\ell}_{kr} \frac{\partial d_r}{\partial \ln p_r^{-\varepsilon_r}} \right) \frac{\mathrm{d} \ln p_r^{-\varepsilon_r}}{\mathrm{d} \ln w} - \frac{\alpha_k}{F_k(\boldsymbol{L})} \sum_r \tilde{\ell}_{kr} E_r w^{-1} p_r^{-\varepsilon_r}.$$

Substituting expression (22) for $\mathrm{d} \ln p_r^{-\varepsilon_r} / \mathrm{d} \ln w$ into the above, yields

$$\frac{\mathrm{d} \ln L_k}{\mathrm{d} \ln w} = -b_k + \sum_s J_{ks}(\boldsymbol{L}) \frac{\mathrm{d} \ln L_s}{\mathrm{d} \ln w}, \tag{23}$$

where

$$b_k \equiv \frac{\alpha_k}{F_k(\boldsymbol{L})} \sum_r \left(\sum_j \frac{\partial \tilde{\ell}_{kj}}{\partial \ln p_r^{-\varepsilon_r}} d_j + \tilde{\ell}_{kr} \frac{\partial d_r}{\partial \ln p_r^{-\varepsilon_r}} \right) \varepsilon_r \sum_s \tilde{\ell}_{sr} \alpha_s + \frac{\alpha_k}{F_k(\boldsymbol{L})} \sum_r \tilde{\ell}_{kr} E_r w^{-1} p_r^{-\varepsilon_r}.$$

Here we also used the expression for the entries of the Jacobian matrix $J(\boldsymbol{L})$, given in (11),

$$J_{ks}(\boldsymbol{L}) \equiv \frac{\partial \ln F_k(\boldsymbol{L})}{\partial \ln L_s} = \frac{\alpha_k}{F_k(\boldsymbol{L})} \sum_r \left(\sum_j \frac{\partial \tilde{\ell}_{kj}}{\partial \ln p_r^{-\varepsilon_r}} d_j + \tilde{\ell}_{kr} \frac{\partial d_r}{\partial \ln p_r^{-\varepsilon_r}} \right) \varepsilon_r \tilde{\ell}_{sr} \theta_s.$$

Solving for $\mathrm{d} \ln L_k / \mathrm{d} \ln w$ from (23), we obtain

$$\left\{ \frac{\mathrm{d} \ln L_k}{\mathrm{d} \ln w} \right\} = -\left(I - J(\boldsymbol{L})\right)^{-1} \boldsymbol{b},$$

where $\boldsymbol{b} \equiv (b_1, \ldots, b_K)^T$.

Matrix $J(\boldsymbol{L})$ is nonnegative, and Proposition 4 implies that $\rho(J(\boldsymbol{L})) < 1$ under the condition (UC). Therefore, $I - J(\boldsymbol{L})$ is an M-matrix, and hence its inverse is well defined and nonnegative. Moreover, given that $\partial d_r / \partial \ln p_r^{-\varepsilon_r} \geq 0$ and $\partial \tilde{\ell}_{kj} / \partial \ln p_r^{-\varepsilon_r} \geq 0$, it follows that $b_k \geq 0$ for all k. Hence, $\mathrm{d} \ln L_k / \mathrm{d} \ln w \leq 0$ for all k. Substituting this result into expression (22), we also obtain $\mathrm{d} \ln p_k^{-\varepsilon_k} / \mathrm{d} \ln w \leq 0$ for all k. Combined with (21), this further implies that $\mathrm{d} \ln \lambda_k / \mathrm{d} \ln w \leq 0$ for all k.

F.2 Proof of Lemma 2

Limit as $w \to 0$. Consider a sector k with $E_k > 0$ (by assumption, at least one such sector exists). As established in Lemma 1, $[p_k(w)]^{-\varepsilon_k}$ is nonincreasing function of w. Since $p_k(w) > 0$ for any $w > 0$, it follows that either $\lim_{w \to 0} [p_k(w)]^{-\varepsilon_k}$ is a finite positive number, or $[p_k(w)]^{-\varepsilon_k}$ diverges to infinity as $w \to 0$. In either case, we have $\lim_{w \to 0} E_k \cdot [p_k(w)]^{-\varepsilon_k} / w = \infty$. Writing the goods market clearing condition for sector k as

$$L_k(w) = \alpha_k \sum_s \tilde{\ell}_{ks}(w) \left(\lambda_s(w) e_s \bar{L} + w^{-1} E_s [p_s(w)]^{-\varepsilon_s} \right), \tag{24}$$

and noting that $\tilde{\ell}_{kk}(w) \geq 1$ for any w, we obtain $\lim_{w \to 0} L_k(w) = \infty$. Therefore, $\lim_{w \to 0} \sum_s L_s(w) = \infty$.

Limit as $w \to \infty$. By Lemma 1, both $[p_k(w)]^{-\varepsilon_k}$ and $\lambda_k(w)$ are nonincreasing functions of w. Since $[p_k(w)]^{-\varepsilon_k} > 0$ and $\lambda_k(w) \in (0, 1)$ for any $w > 0$, it follows that

$\lim_{w\to\infty}[p_k(w)]^{-\varepsilon_k}$ is finite, and $\lim_{w\to\infty}\lambda_k(w)<1$. Consequently, we have $\lim_{w\to\infty}E_k[p_k(w)]^{-\varepsilon_k}/w=0$ for all k.

For any $w>0$, the Neumann expansions of $\tilde{\mathcal{L}}$ and $\mathcal{L}$ imply $\tilde{\mathcal{L}}\leq\mathcal{L}$. Substituting this bound into (24) and summing over k, we obtain

$$\sum_k L_k(w)\leq\sum_k\alpha_k\sum_s\ell_{ks}\left(\lambda_s(w)e_s\bar{L}+w^{-1}E_s[p_s(w)]^{-\varepsilon_s}\right).$$

Taking the limit as $w\to\infty$ and using $\lim_{w\to\infty}E_k[p_k(w)]^{-\varepsilon_k}/w=0$ together with $\sum_k\alpha_k\ell_{ks}=1$ for all s, we get

$$\lim_{w\to\infty}\sum_k L_k(w)\leq\sum_s\left(\lim_{w\to\infty}\lambda_s(w)\right)e_s\bar{L}.$$

If $e_s=0$ for all s, then $\lim_{w\to\infty}\sum_k L_k(w)=0<\bar{L}$. If instead $e_s>0$ for at least one s, then $\lim_{w\to\infty}\sum_k L_k(w)<\sum_s e_s\bar{L}=\bar{L}$, where the strict inequality follows from the fact that $\lim_{w\to\infty}\lambda_s(w)<1$ for all s.

G Proof of Convexity of $f(\boldsymbol{y})$ in Section 6

We have

$$f(\boldsymbol{y})\equiv-\ln[1-\mathrm{GT}^*(\exp\boldsymbol{y})]=-\sum_k\frac{\psi_k}{\varepsilon_k}y_k+\sum_k\theta_k\psi_k\ln\left(\sum_s\tilde{\ell}_{ks}e_s\exp y_s\right)+\text{constant},$$

where we omit the explicit dependence of $\tilde{\ell}_{ks}$ on $\exp\boldsymbol{y}$ for brevity. Given that the first term above is linear in $\boldsymbol{y}$, we only need to show that the second term is convex. Since the sum of convex functions is convex, it is sufficient to show that $\sum_s\tilde{\ell}_{ks}e_s\exp y_s$ is log-convex for each k. To prove this, we apply the fact that both products and sums of log-convex functions are log-convex.[25] Thus, to show that $\sum_s\tilde{\ell}_{ks}e_s\exp y_s$ is log-convex, it suffices to show that $\tilde{\ell}_{ks}e_s\exp y_s$ is log-convex for all s. Since $\exp y_s$ is log-linear, we need only to show that $\tilde{\ell}_{ks}$ is log-convex for all s.

By definition, $\tilde{\ell}_{ks}$ are the entries of $\tilde{\mathcal{L}}=\sum_{t=0}^{\infty}B^{(t)}$, where $B^{(t)}\equiv(\mathcal{D}\{\exp y_k\}A)^t$ with elements $b_{ij}^{(t)}$ for $t=0,1,\ldots$. Observing that $b_{ij}^{(0)}$ is constant and $b_{ij}^{(1)}=\alpha_{ij}\exp\{y_i\}$ is log-linear in $\boldsymbol{y}$, we conclude that both $B^{(0)}$ and $B^{(1)}$ are log-convex. For $B^{(t)}$ with $t>1$, we proceed by induction. Suppose that $B^{(t-1)}$ is log-convex for some $t>1$. We can

[25] The fact that the product of log-convex functions is log-convex is trivial. The proof of the fact that a sum of twice differentiable log-convex functions is log-convex can be found in, for example, Boyd and Vandenberghe (2004, pp. 105-106).

represent $B^{(t)}$ as $B^{(1)}B^{(t-1)}$ and so $b_{ij}^{(t)} = \sum_r b_{ir}^{(1)} b_{rj}^{(t-1)}$, which is log-convex since it is a sum of products of log-convex functions. Therefore, $\sum_{t=0}^{\tau} B^{(t)}$ is log-convex for any finite τ, and by taking the limit as $\tau \to \infty$, we conclude that $\tilde{\mathcal{L}}$ is indeed log-convex.